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Isabelle Chalendar, Lucas Oger, Jonathan R. Partington

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LINEAR ISOMETRIES ON THE ANNULUS: DESCRIPTION AND SPECTRAL PROPERTIES

I. CHALENDAR, L. OGER, AND J. R. PARTINGTON

ABSTRACT. We give a complete characterisation of the linear isometries of $\text{Hol}(\Omega)$, where Ω is the half-plane, the complex plane or an annulus centered at 0 and symmetric to the unit circle. Moreover, we introduce new techniques to describe the holomorphic maps on the annulus that preserve the unit circle, and we finish by proving results about the spectra of the linear isometries on the annulus.

1. INTRODUCTION

In order to study a metric linear space, and in particular its geometry, one of the main tools is an analysis of the linear isometries of this space. For spaces of holomorphic functions, there is a strong connection between isometries and weighted composition operators.

This link can be traced back to Banach himself [3] in the space $\mathcal{C}(K)$ of continuous real-valued functions on a compact metric space K . For most well-known spaces, such as the Hardy, Bergman, Bloch spaces or the disc algebra, the linear isometries have been completely described [6, 8, 9, 11, 13, 16, 19, 20].

This paper can be considered as a sequel to [7], where the authors characterised completely the linear isometries V on the space $\text{Hol}(\mathbb{D})$ of holomorphic functions on the unit disc, relative to one of the two classical distances. We showed there that V is of the form

$$V(f)(z) = \alpha f(\beta z) =: T_{\alpha, \beta}(f)(z), \quad \alpha, \beta \in \mathbb{T},$$

where \mathbb{T} is the unit circle of \mathbb{C} . Those V are determined by being isometric for any two seminorms of $\text{Hol}(\mathbb{D})$, defined by $\|f\|_{\infty, r} := \sup_{z \in K_r} |f(z)|$, with $r \in [0, 1)$ and $K_r = r\overline{\mathbb{D}}$.

It is natural to consider the same problem when replacing the unit disc \mathbb{D} with other domains, such as the half-plane \mathbb{H} or the whole plane \mathbb{C} . Those two cases are easily obtained from the results of [7].

★ For the half-plane \mathbb{H} , let us define the *Cayley transform* $\tau : \mathbb{D} \rightarrow \mathbb{H}$ by

$$\tau(z) = \frac{1+z}{1-z}.$$

The map τ is biholomorphic. For all $r \in [0, 1)$, let us denote $L_r = \tau(K_r)$. We aim to find a linear isometry V of $\text{Hol}(\mathbb{H})$ for the seminorms associated with the sets L_r . To do so, note that for all $F \in \text{Hol}(\mathbb{H})$, the map $f = F \circ \tau$ is in $\text{Hol}(\mathbb{D})$. Hence, for every linear isometry V of $\text{Hol}(\mathbb{H})$,

$$\sup_{w \in K_r} |f(z)| = \sup_{z \in L_r} |F(z)| = \sup_{z \in L_r} |(VF)(z)| = \sup_{w \in K_r} |(C_\tau \circ V \circ C_{\tau^{-1}})(f)(w)|.$$

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Therefore, $C_\tau \circ V \circ C_{\tau^{-1}}$ is a linear isometry of $\text{Hol}(\mathbb{D})$, so we obtain that V is of the form $C_{\tau^{-1}} \circ T_{\alpha, \beta} \circ C_\tau$ for some $\alpha, \beta \in \mathbb{T}$.

- ★ For the plane \mathbb{C} , assume that $\|Tf\|_{\infty, \rho} = \|f\|_{\infty, \rho}$ for all $f \in \text{Hol}(\mathbb{C})$ and $\rho \geq 0$. Then, it is in particular true for all f polynomial. By density, for all $f \in \text{Hol}(\mathbb{D})$ and $0 \leq \rho < 1$, $\|Tf\|_{\infty, \rho} = \|f\|_{\infty, \rho}$. Thus, T is an isometry of $\text{Hol}(\mathbb{D})$, and $T = T_{\alpha, \beta}$ for some $\alpha, \beta \in \mathbb{T}$.

We now consider a doubly-connected domain, that is an annulus. We fix $R > 1$ and define $\mathbb{A} = \{z \in \mathbb{C} : 1/R < |z| < R\}$. Our main goal is to obtain a similar characterisation as in [7].

In Sections 2 and 3, we give preliminary results about distances, seminorms, composition operators and powers of z . These will be present in the proof of the Main Theorem (Section 4), which says that if we consider annuli symmetric to the unit circle, then the only linear isometries of $\text{Hol}(\mathbb{A})$ are rotations or inversions.

In [7], we also used a description of the Blaschke products to obtain a characterisation of linear isometries for one seminorm associated to one of the K_r . In Section 5, we describe the annulus version of Blaschke products, that is, unimodular functions on the unit circle. Results of Fatou [12] and Bourgain [5] will be in the spotlight in this section.

Finally, Section 6 is devoted to the spectral study of the linear isometries of $\text{Hol}(\mathbb{A})$. In particular, depending on the angle of the rotation, we give examples of rotations such that the spectrum is different from the point spectrum or the unit circle.

2. FIRST RESULTS

This section is devoted to proving the main ingredients in the proof of the main result, that is, the description of linear isometries on the Fréchet space of analytic functions on an annulus.

The content of this section may be of independent interest.

2.1. Algebraic characterisation of composition operators.

From now on we write e_n (where $n \in \mathbb{Z}$) for the function defined by $e_n(z) = z^n$.

Lemma 2.1. *Let $L : \text{Hol}(\mathbb{A}) \rightarrow \mathbb{C}$ be non-zero, linear, continuous and multiplicative.*

Then there exists $z_0 \in \mathbb{A}$ such that for all $f \in \text{Hol}(\mathbb{A})$, $L(f) = f(z_0)$.

Proof. The proof uses the same ideas as in [1, Lemma 2.2]. For the sake of completeness, we detail them. Note that for all $f \in \text{Hol}(\mathbb{A})$,

$$L(f) = L(fe_0) = L(f)L(e_0).$$

Choosing f such that $L(f) \neq 0$, we have $L(e_0) = 1$. Denote by $z_0 = L(e_1)$. Thus,

- Assume that $z_0 \notin \mathbb{A}$. Then $g = (e_1 - z_0 e_0)^{-1} \in \text{Hol}(\mathbb{A})$ and

$$(e_1 - z_0 e_0)g = e_0 \implies 0 = [L(e_1) - z_0 L(e_0)]L(g) = L(e_0) = 1,$$

a contradiction. Hence, $z_0 \in \mathbb{A}$.

- Let $f \in X$ such that $f(z_0) = 0$. There exists $h \in \text{Hol}(\mathbb{A})$ such that $f = (e_1 - z_0 e_0)h$, so $L(f) = [L(e_1) - z_0 L(e_0)]L(h) = [z_0 - z_0]L(h) = 0$.
- If $f(z_0) \neq 0$, then the function $g = f - f(z_0)e_0 \in \text{Hol}(\mathbb{A})$ satisfies $g(z_0) = 0$. Hence, $L(g) = 0$, and we obtain $L(f) = L(f(z_0)e_0) = f(z_0)$. \square

Along the same lines as in [1], Lemma 2.1 combined with Runge's theorem [18], provides an algebraic characterisation of composition operators, whose proof is omitted.

Proposition 2.2. *Let $T : \text{Hol}(\mathbb{A}) \rightarrow \text{Hol}(\mathbb{A})$ be a continuous linear operator. Then T is a composition operator if and only if for all $n \in \mathbb{Z}$, $Te_n = (Te_1)^n$, where $e_n(z) = z^n$.*

2.2. From distances to seminorms. The two next results are contained in [7].

Lemma 2.3 ([7, Lemma 1.1]). *Let X be a Fréchet space, and $(\|\cdot\|_k)_{k \geq 1}$ an associated increasing family of semi-norms. A linear operator $T : X \rightarrow X$ is isometric, considering the distance d_X defined by*

$$d_X(x, y) = \sum_{k=1}^{\infty} 2^{-k} \min(1, \|x - y\|_k),$$

if and only if for all $x \in X$ and $k \in \mathbb{N}$, $\|Tx\|_k = \|x\|_k$.

Lemma 2.4 ([7, Lemma 1.2]). *Let X be a Fréchet space, and $(\|\cdot\|_k)_{k \geq 1}$ the associated increasing family of semi-norms. A linear operator $T : X \rightarrow X$ is isometric, considering the distance d'_X defined by*

$$d'_X(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|x - y\|_k}{1 + \|x - y\|_k},$$

if and only if for all $x \in X$ and $k \in \mathbb{N}$, $\|Tx\|_k = \|x\|_k$.

In particular, we consider here the seminorms

$$\|f\|_{\infty, n} := \sup_{z \in K_n} |f(z)|, \quad K_n = \{z \in \mathbb{C} : R_n^{\frac{1}{n}-1} \leq |z| \leq R_n^{1-\frac{1}{n}}\}.$$

Clearly, K_1 is the unit circle \mathbb{T} . In the following, denote $R_n = R^{1-\frac{1}{n}}$.

3. THREE USEFUL CHARACTERISATIONS

We first recall the famous Hadamard three-circle theorem. Its detailed proof can be found for example in [17]. However, this paper deals with regular functions on an annulus rather than a half-line. Since we work with holomorphic maps on the entire annulus, we can be more precise in the case of equality.

Theorem 3.1.

Let $f \in \text{Hol}(\mathbb{A})$, and for $r \in (1/R, R)$, let $M(r) = \sup\{|f(z)| : |z| = r\}$. Then, the map $r \mapsto \log(M(r))$ is convex, that is for all $1/R < r_1 < r_2 < r_3 < R$,

$$(1) \quad \log\left(\frac{r_3}{r_1}\right) \log(M(r_2)) \leq \log\left(\frac{r_3}{r_2}\right) \log(M(r_1)) + \log\left(\frac{r_2}{r_1}\right) \log(M(r_3)).$$

Moreover, we have equality if and only if $f(z) = cz^n$ for some $c \in \mathbb{C}$ and $n \in \mathbb{Z}$.

We can use this result to show that the analytic functions on an annulus \mathbb{A} preserving three circles of \mathbb{A} are the obvious ones.

Corollary 3.2. *Let $f \in \text{Hol}(\mathbb{A})$, and $1/R < r_1 < r_2 < r_3 < R$. Assume that $f(r_j\mathbb{T}) \subset r_j\mathbb{T}$ for all $j = 1, 2, 3$. Then, there exists $c \in \mathbb{T}$ such that $f(z) = cz$.*

Proof. Taking the notations of Theorem 3.1, for all $j = 1, 2, 3$, $M(r_j) = r_j$. If the map f is not of the form cz^n , with $c \in \mathbb{C}$ and $n \in \mathbb{Z}$, then

$$\begin{aligned} \log\left(\frac{r_3}{r_1}\right) \log(r_2) &< \log\left(\frac{r_3}{r_2}\right) \log(r_1) + \log\left(\frac{r_2}{r_1}\right) \log(r_3) \\ \iff \log(r_3) \log(r_2) - \log(r_1) \log(r_2) &< \log(r_2) \log(r_3) - \log(r_2) \log(r_1) \\ \iff 0 &< 0. \end{aligned}$$

It follows that $f(z) = cz^n$ for some $c \in \mathbb{C}$ and $n \in \mathbb{Z}$. Then, $f(r_j\mathbb{T}) \subset |c|r_j^n\mathbb{T}$. Thus,

$$r_j = |c|r_j^n, \quad j = 1, 2, 3,$$

which means that $|c| = 1$ and $n = 1$. □

We now present and prove the last tool we use in the proof of our main result. The ideas are very similar to the ones involved in [7, Theorem 1.4].

Theorem 3.3. *Let $f \in \text{Hol}(\mathbb{A})$ such that for some $r \in (1/R, R)$ and $\rho > 0$,*

$$\sup_{|z|=r} |f(z)| = \rho.$$

Then the set $\Xi = \{\xi \in r\mathbb{T} : |f(\xi)| = \rho\} = f^{-1}(\rho\mathbb{T}) \cap r\mathbb{T}$ is either equal to $r\mathbb{T}$, or a finite set.

Proof. Assume that $\Xi \neq r\mathbb{T}$. Since f is continuous on \mathbb{A} and $f^{-1}(\rho\mathbb{T}) \neq \emptyset$, we know that $f^{-1}(\rho\mathbb{T})$ is a closed subset of \mathbb{A} . Hence, Ξ is a closed subset of $r\mathbb{T}$.

Assume that Ξ is not finite. Then, the set $r\mathbb{T} \setminus \Xi$ is an open subset of $r\mathbb{T}$, which is a union of an infinite number of open intervals of $r\mathbb{T}$, and one of the edges is a limit point of Ξ , denoted in the following as z_0 .

This point satisfies $|f(z_0)| = \rho$, and there exist two sequences $(u_n) \subset \Xi$ and $(v_n) \subset r\mathbb{T} \setminus \Xi$ (it is sufficient to take points from the chosen interval) which tend to z_0 , such that $|f(u_n)| = \rho$ and $|f(v_n)| \neq \rho$.

We set $\phi_1 : r\mathbb{T} \rightarrow \mathbb{R} \cup \{\infty\}$ and $\phi_2 : \rho\mathbb{T} \rightarrow \mathbb{R} \cup \{\infty\}$ two conformal maps defined by

$$\phi_1(z) = i \frac{z - z_0}{z + z_0}, \quad \phi_2(z) = i \frac{z - f(z_0)}{z + f(z_0)}.$$

They satisfy $\phi_1(z_0) = 0$ and $\phi_2(f(z_0)) = 0$. Denote $h = \phi_2 \circ f \circ \phi_1^{-1}$.

Since ϕ_1^{-1} is holomorphic near 0, as is ϕ_2 near $f(z_0)$, the function h is holomorphic around 0, satisfies $h(0) = 0$ and there exist two sequences $(a_n), (b_n) \subset \mathbb{R}$ (corresponding to (u_n) and $(v_n) \subset r\mathbb{T}$) such that $a_n \rightarrow 0, b_n \rightarrow 0, h(a_n) \in \mathbb{R}$ and $h(b_n) \notin \mathbb{R}$. However, all the derivatives of h at 0 must be real, so $h(b_n) \in \mathbb{R}$ (since $b_n \in \mathbb{R}$). Impossible.

Finally, we must have $r\mathbb{T} \setminus \Xi = \emptyset$, i.e., $\Xi = r\mathbb{T}$, that is $f(r\mathbb{T}) \subset \rho\mathbb{T}$. □

4. MAIN RESULT

Theorem 4.1. *Let $T : \text{Hol}(\mathbb{A}) \rightarrow \text{Hol}(\mathbb{A})$ a linear and continuous operator such that for all $f \in \text{Hol}(\mathbb{A})$ and $n \in \mathbb{N}$,*

$$\|T(f)\|_{\infty, n} = \|f\|_{\infty, n} = \sup_{z \in K_n} |f(z)|.$$

Then there exist two constants $\alpha, \beta \in \mathbb{T}$ such that $T = T_{\alpha, \beta}$ or $S_{\alpha, \beta}$, defined by

$$(2) \quad T_{\alpha, \beta}(f)(z) = \alpha f(\beta z), \quad S_{\alpha, \beta}(f)(z) = \alpha f(\beta/z).$$

Proof. The steps of the proof go along the same lines as in [7, Theorem 2.1], with important technical changes since our domain is not simply connected. In the following, denote by $e_k(z) = z^k$ and $f_k = Te_k$, for $k \in \mathbb{Z}$.

Step 1: We begin by showing that $Te_0 = \alpha e_0$, with $|\alpha| = 1$. Indeed, since the operator T is isometric for the seminorms $\|\cdot\|_{\infty,2}$ and $\|\cdot\|_{\infty,3}$, we obtain

$$\|f_0\|_{\infty,2} = \|f_0\|_{\infty,3} = 1.$$

Using the maximum modulus principle, the map f_0 is constant and unimodular. Denote by α this constant. Without loss of generality, we will assume that $\alpha = 1$.

Step 2: We show that f_k is a rotation or an inversion, for all $k \in \mathbb{Z}$. Note that for $n \in \mathbb{N}$,

$$\|f_k\|_{\infty,n} = \|e_k\|_{\infty,n} = (R_n)^k.$$

The maximum is attained at the boundary of K_n , which is $R_n\mathbb{T} \cup R_n^{-1}\mathbb{T}$. Hence,

$$\sup_{|z| \in \{R_n, 1/R_n\}} |f_k(z)| = (R_n)^k.$$

Now, remark that

$$(R_n)^k\mathbb{T} \subset f_k(K_n) =: L.$$

Indeed, assume that there exists $\xi = (R_n)^k e^{i\theta}$ such that $\xi \notin L$, which is a compact set. Then $\delta = d(\xi, L) > 0$, so $L \subset (R_n)^k \mathbb{D} \setminus D(\xi, \delta)$. We set $g = e_0 + e^{-i\theta} e_k$. Hence, $Tg = e_0 + e^{-i\theta} f_k$. Now, compare the seminorms: we have $\|g\|_{\infty,n} = 1 + (R_n)^k$ and

$$\|Tg\|_{\infty,n} = \sup_{z \in K_n} |1 + e^{-i\theta} f_k(z)| = \sup_{w \in L} |1 + e^{-i\theta} w| < 1 + (R_n)^k.$$

This is impossible, since T is isometric for $\|\cdot\|_{\infty,n}$. Therefore, $(R_n)^k\mathbb{T} \subset L$.

Assume that $\Xi_1 = \{\xi \in R_n\mathbb{T} : |f_k(\xi)| = (R_n)^k\}$ and $\Xi_2 = \{\xi \in R_n^{-1}\mathbb{T} : |f_k(\xi)| = (R_n)^k\}$ are both finite sets. Since $f_k(K_n) \subset (R_n)^k\mathbb{D}$ and $f_k(\Xi_1 \cup \Xi_2)$ is finite, we obtain a contradiction because $(R_n)^k\mathbb{T} \subset L$. Therefore, $\Xi_1 = R_n\mathbb{T}$ or $\Xi_2 = R_n^{-1}\mathbb{T}$, that is

$$(E_1) : f_k(R_n\mathbb{T}) \subset (R_n)^k\mathbb{T} \quad \text{or} \quad (E_2) : f_k(R_n^{-1}\mathbb{T}) \subset (R_n)^k\mathbb{T}.$$

In particular, for $n = 1$, we obtain $f_k(\mathbb{T}) \subset \mathbb{T}$. Moreover, since there are two choices, there are at least two different values of n among 2, 3 and 4 satisfying both (E_1) , or both (E_2) . Without loss of generality, we assume that $n = 2$ and $n = 3$ satisfy the same inclusion.

- ★ If $f_k(R_n\mathbb{T}) \subset (R_n)^k\mathbb{T}$ and $g_k(z) = f_k(z)/z^{k-1}$, then $g_k(R_n\mathbb{T}) \subset R_n\mathbb{T}$ for $n = 1, 2, 3$. Hence, by Corollary 3.2, $g_k(z) = c_k z$, with $|c_k| = 1$, so $f_k(z) = c_k z^k$.
- ★ If $f_k(R_n^{-1}\mathbb{T}) \subset (R_n)^k\mathbb{T}$ and $g_k(z) = z^{k+1} f_k(z)$, then $g_k(R_n^{-1}\mathbb{T}) \subset R_n^{-1}\mathbb{T}$ for $n = 1, 2, 3$. Hence, by Corollary 3.2, $g_k(z) = c_k z$, with $|c_k| = 1$, so $f_k(z) = c_k/z^k$.

We have shown that for all $k \in \mathbb{Z}$, $f_k = c_k e_k$ or $c_k e_{-k}$, for some $c_k \in \mathbb{T}$.

Step 3: Assume that $f_1 = c_1 e_1$. If there exists $k \in \mathbb{Z}$ such that $f_k = c_k e_{-k}$, then for all $n \in \mathbb{N}$,

$$R_n + (R_n)^k = \|e_1 + e_k\|_{\infty,n} = \|f_1 + f_k\|_{\infty,n} = \|c_1 e_1 + c_k e_{-k}\|_{\infty,n}.$$

Note that

$$\|c_1 e_1 + c_k e_{-k}\|_{\infty,n} = \sup_{z \in K_n} \left| c_1 z + \frac{c_k}{z^k} \right| = \sup_{z \in K_n} \left| z + \frac{c_k}{c_1} \frac{1}{z^k} \right|.$$

By the maximum modulus principle, the maximum is attained at the boundary of K_n , which is $R_n\mathbb{T} \cup R_n^{-1}\mathbb{T}$. We consider two possibilities:

★ If the maximum is attained in $R_n\mathbb{T}$, then

$$\begin{aligned} R_n + (R_n)^k &= \sup_{|z|=R_n} \left| z + \frac{c_k}{c_1} \frac{1}{z^k} \right| \\ &= \frac{1}{(R_n)^k} \sup_{|z|=R_n} \left| z^{k+1} + \frac{c_k}{c_1} \right| = \frac{(R_n)^{k+1} + 1}{(R_n)^k} = R_n + \frac{1}{(R_n)^k}. \end{aligned}$$

★ If the maximum is attained in $R_n^{-1}\mathbb{T}$, then

$$\begin{aligned} R_n + (R_n)^k &= \sup_{|z|=1/R_n} \left| z + \frac{c_k}{c_1} \frac{1}{z^k} \right| \\ &= (R_n)^k \sup_{|z|=1/R_n} \left| z^{k+1} + \frac{c_k}{c_1} \right| = (R_n)^k \left(\frac{1}{(R_n)^{k+1}} + 1 \right) = \frac{1}{R_n} + (R_n)^k. \end{aligned}$$

Both of the cases lead to a contradiction, since $R_n \neq 1$ for $n \geq 2$. Hence, $f_k = c_k e_k$ for some $|c_k| = 1$. In the same way, if $f_1 = c_1 e_{-1}$, then $f_k = c_k e_{-k}$ for all $k \in \mathbb{Z}$.

Step 4: In the following, we assume that $f_1 = c_1 e_1$. Let $k \in \mathbb{N}$. Note that

$$\begin{aligned} \sup_{z \in \partial K_n} |(f_k + f_{k+1} + f_{k+2})(z)| &= \sup_{z \in \partial K_n} |(c_k e_k + c_{k+1} e_{k+1} + c_{k+2} e_{k+2})(z)| \\ &= \sup_{z \in \partial K_n} |c_k z^k + c_{k+1} z^{k+1} + c_{k+2} z^{k+2}| \\ &= \sup_{z \in \partial K_n} |z^k + z^{k+1} + z^{k+2}| = (R_n)^k + (R_n)^{k+1} + (R_n)^{k+2}. \end{aligned}$$

Recall that $\partial K_n = R_n\mathbb{T} \cup R_n^{-1}\mathbb{T}$. Hence, for $n \geq 2$ and $k \in \mathbb{N}$,

$$\begin{aligned} \sup_{|z|=1/R_n} |c_k z^k + c_{k+1} z^{k+1} + c_{k+2} z^{k+2}| &\leq \frac{1}{(R_n)^k} + \frac{1}{(R_n)^{k+1}} + \frac{1}{(R_n)^{k+2}} \\ &< (R_n)^k + (R_n)^{k+1} + (R_n)^{k+2}. \end{aligned}$$

In the same way,

$$\begin{aligned} \sup_{|z|=R_n} |c_k z^{-k} + c_{k+1} z^{-k-1} + c_{k+2} z^{-k-2}| &\leq \frac{1}{(R_n)^k} + \frac{1}{(R_n)^{k+1}} + \frac{1}{(R_n)^{k+2}} \\ &< (R_n)^k + (R_n)^{k+1} + (R_n)^{k+2}. \end{aligned}$$

Thus,

$$(R_n)^k + (R_n)^{k+1} + (R_n)^{k+2} = \begin{cases} \sup_{|z|=R_n} |c_k z^k + c_{k+1} z^{k+1} + c_{k+2} z^{k+2}|, & k \in \mathbb{N} \\ \sup_{|z|=1/R_n} |c_k z^{-k} + c_{k+1} z^{-k-1} + c_{k+2} z^{-k-2}|, & k \in \mathbb{N}. \end{cases}$$

Up to considering $w = 1/z$, we only have to consider k positive. Dividing by z^k , we have

$$\sup_{|z|=R_n} |c_k + c_{k+1}z + c_{k+2}z^2| = 1 + R_n + (R_n)^2.$$

Dividing by c_k , since $|c_k| = 1$, we have

$$\sup_{|z|=R_n} \left| 1 + \frac{c_{k+1}}{c_k} z + \frac{c_{k+2}}{c_k} z^2 \right| = 1 + R_n + (R_n)^2.$$

Using the triangle inequality, for all z satisfying $|z| = R_n$,

$$\left| 1 + \frac{c_{k+1}}{c_k} z + \frac{c_{k+2}}{c_k} z^2 \right| \leq 1 + R_n + (R_n)^2,$$

with equality if and only if

$$0 = \arg(1) \equiv \arg\left[\frac{c_{k+1}}{c_k} z\right] \equiv \arg\left[\frac{c_{k+2}}{c_k} z^2\right] \pmod{2\pi}.$$

Denoting $\theta = \arg[z]$, this gives

$$0 \equiv \arg[c_{k+1}] - \arg[c_k] + \theta \equiv \arg[c_{k+2}] - \arg[c_k] + 2\theta \pmod{2\pi}.$$

Finally,

$$\begin{aligned} & \arg[c_k] - 2\arg[c_{k+1}] + \arg[c_{k+2}] \\ &= (\arg[c_{k+2}] - \arg[c_k] + 2\theta) - 2(\arg[c_{k+1}] - \arg[c_k] + \theta) \equiv 0 \pmod{2\pi}. \end{aligned}$$

We show by induction that for all $k \in \mathbb{N}_0$, $\arg[c_k] \equiv k \arg[c_1] \pmod{2\pi}$.

- $k = 0$ or 1 : It is immediate (see Step 1 for $k = 0$).
- Assume that the formula is valid for some nonnegative integers k and $k + 1$. Since $\arg[c_k] - 2\arg[c_{k+1}] + \arg[c_{k+2}] \equiv 0 \pmod{2\pi}$, using induction hypothesis, we have

$$k \arg[c_1] - 2(k + 1) \arg[c_1] + \arg[c_{k+2}] \equiv 0 \pmod{2\pi}.$$

After rearranging the terms, we obtain $\arg[c_{k+2}] \equiv (k + 2) \arg[c_1] \pmod{2\pi}$.

In the same way, for all $k \in \mathbb{N}$, $\arg[c_{-k}] \equiv k \arg[c_{-1}] \pmod{2\pi}$.

Step 5: We have shown that under the assumption $f_1 = c_1 e_1$, for some $|c_1| = 1$,

- ★ For all $k \in \mathbb{Z}$, $f_k = c_k e_k$ (Step 3),
- ★ For all $k \in \mathbb{N}$, $c_k = c_1^k$ and $c_{-k} = c_{-1}^k$ (Step 4).

The only thing left to show is that $c_{-1} = 1/c_1$. Indeed,

$$\begin{aligned} \sup_{z \in \partial K_n} |(f_{-1} + f_0 + f_1)(z)| &= \sup_{z \in \partial K_n} |(c_{-1} e_{-1} + e_0 + c_1 e_1)(z)| \\ &= \sup_{z \in \partial K_n} \left| \frac{c_{-1}}{z} + 1 + c_1 z \right| = \sup_{z \in \partial K_n} \left| \frac{1}{z} + 1 + z \right| = \frac{1}{R_n} + 1 + R_n. \end{aligned}$$

We have two choices. If the maximum is attained on $R_n \mathbb{T}$, then

$$\sup_{|z|=R_n} \left| \frac{c_{-1}}{z} + 1 + c_1 z \right| = \frac{1}{R_n} + 1 + R_n.$$

However, we have equality if and only if $\arg(c_{-1}/z) \equiv \arg(1) \equiv \arg(c_1 z) \pmod{2\pi}$, that is

$$0 \equiv \arg(c_{-1}) - \arg(z) \equiv \arg(c_1) + \arg(z) \pmod{2\pi}.$$

Hence, $\arg(z) \equiv \arg(c_{-1}) \equiv -\arg(c_1) \pmod{2\pi}$, so $c_{-1} = 1/c_1$. It is the same if the maximum is attained on $R_n^{-1} \mathbb{T}$.

Conclusion: For all $k \in \mathbb{Z}$, $f_k = c_1^k e_k = (c_1 e_1)^k = f_1^k$. Hence, T is a composition operator, with symbol $\varphi = f_1 = c_1 e_1$ or $c_1 e_{-1}$, by Proposition 2.2. Finally, multiplying by the α of Step 1, for all $f \in \text{Hol}(\mathbb{A})$,

$$(Tf)(z) = \alpha f(c_1 z) \text{ or } \alpha f(c_1/z). \quad \square$$

5. DESCRIPTION OF UNIMODULAR FUNCTIONS ON THE UNIT CIRCLE

There exist many approximation results for measurable unimodular functions on the unit circle such as the Helson–Sarason Theorem [15] which says that given $\varepsilon > 0$ and f unimodular and continuous on \mathbb{T} , there exist finite Blaschke products B_1 and B_2 such that

$$\left\| f - \frac{B_1}{B_2} \right\|_{\infty} < \varepsilon.$$

In this section we investigate factorisation results instead of approximation results for unimodular function on the unit circle. The first one is due to Fatou.

Theorem 5.1 ([12]). *If f is analytic on \mathbb{D} and $\lim_{|z| \rightarrow 1} |f(z)| = 1$, then f is a finite Blaschke product. In particular, if f is a function in the disc algebra $A(\mathbb{D})$, then f is a finite Blaschke product whenever f is unimodular on \mathbb{T} .*

There is also a meromorphic version of this.

Corollary 5.2 ([14]). *Suppose that f is meromorphic on \mathbb{D} and extends continuously to \mathbb{T} , the unit circle. If f is unimodular on \mathbb{T} , then f is a quotient of two finite Blaschke products.*

In 1986, Bourgain solved a problem of factorization posed by Douglas and Rudin.

Theorem 5.3 ([5, 4]). *Let f be a bounded measurable function on the unit circle \mathbb{T} . Then the following assertions are equivalent:*

- (i) *The function $\log |f|$ belongs to $L^1(\mathbb{T})$,*
- (ii) *There exist $g, h \in H^\infty$ such that $f = g\bar{h}$.*

In particular, any measurable function unimodular almost everywhere on the unit circle is of the form $f = g\bar{h}$, for some functions $g, h \in H^\infty$.

The following theorem is a complete description of unimodular functions on the unit circle that are analytic in a neighbourhood of \mathbb{T} .

Theorem 5.4. *Let f be an analytic map on \mathbb{A} . Then the following assertions are equivalent:*

- (i) *The function f is unimodular on \mathbb{T} ,*
- (ii) *there exists $s \in (1, R]$, $n \in \mathbb{Z}$ and an outer function g_0^s in H^∞ which extends analytically on $s\mathbb{D}$ such that*

$$f(z) = z^n \frac{g_0^s(z)}{g_0^s(1/\bar{z})} \quad \text{for all } z \in \mathbb{A}_s := \{z \in \mathbb{C} : 1/s < |z| < s\}.$$

For example, a Blaschke factor $f(z) = (z - \alpha)/(1 - \bar{\alpha}z)$ with $|\alpha| < 1$ arises on taking $g_0^s(z) = 1/(1 - \bar{\alpha}z)$ with $n = 1$ and $1 < s < 1/|\alpha|$.

Here is an example showing that the set of unimodular functions on \mathbb{T} and analytic in a neighborhood of \mathbb{T} is much larger than the set of quotients of finite Blaschke products.

Let $f(z) = \exp(z - 1/z)$. Then $f(\xi) = \exp(\xi - \bar{\xi}) = \exp(2i\Im(\xi))$ for all $\xi \in \mathbb{T}$. It follows that f is unimodular on \mathbb{T} and analytic on \mathbb{C}^* . In this case we have $s = R$, $n = 0$ and $g_0(z) = \exp(z)$.

Our proof relies on Theorem 5.3 and the following lemma.

Lemma 5.5. *Let $s > 1$. Suppose that g is analytic and does not vanish in the annulus $\mathbb{A}_s = \{z \in \mathbb{C} : 1/s < |z| < s\}$. Then there exist $n \in \mathbb{Z}$ and h analytic in \mathbb{A}_s such that*

$$g(z) = z^n \exp(h(z)) \quad \text{for all } z \in \mathbb{A}_s.$$

Proof. Since \mathbb{A}_s is not simply connected, the existence of an analytic map h in \mathbb{A}_s such that $g(z) = \exp(h(z))$ is not necessarily true. Now recall that the existence of h such that $g = \exp(h)$ on a domain Ω is equivalent to

$$\int_C \frac{g'(z)}{g(z)} dz = 0 \quad \text{for all Jordan curves } C \text{ in } \Omega.$$

For $\Omega = \mathbb{A}_s$, this is equivalent to

$$\int_{C_t} \frac{g'(z)}{g(z)} dz = 0 \quad \text{for all } t \in (1/s, s), \text{ where } C_t = t\mathbb{T}.$$

Note that $\int_{C_t} g'(z)/g(z) dz \in 2i\pi\mathbb{Z}$, and therefore there exists $n \in \mathbb{Z}$ such that

$$\int_{C_t} \frac{g'(z)}{g(z)} dz = 2i\pi n.$$

It follows that $\tilde{g}(z) := g(z)/z^n$ satisfies

$$\int_{C_t} \frac{\tilde{g}'(z)}{\tilde{g}(z)} dz = 0 \quad \text{for all } t \in (1/s, s).$$

Finally, $g(z) = z^n \exp(h(z))$ for all $z \in \mathbb{A}_s$. □

Proof of Theorem 5.4. Since f is analytic in \mathbb{A} and unimodular on the unit circle, there exists $s \in (1, R]$ such that $f(z) \neq 0$ for all $z \in \mathbb{A}_s$. By Lemma 5.5, there exist $n \in \mathbb{Z}$ and h analytic in \mathbb{A}_s such that

$$f(z) = z^n \exp(h(z)) \quad \text{for all } z \in \mathbb{A}_s.$$

Note that $h(z) = h_+(z) + h_-(z)$, where h_+ is analytic on $s\mathbb{D}$, h_- is analytic on $\mathbb{C} \setminus s^{-1}\mathbb{D}$ and $h_+(0) = h(0)$. It follows that $g_0 := \exp(h_+)$ is an outer function which extends analytically on $s\mathbb{D}$. Indeed, $\exp(h_+)$ has no zero in \mathbb{D} and moreover, since it extends analytically on $s\mathbb{D}$, its singular inner part is trivial. In addition,

$$g_1(z) := \begin{cases} \overline{\exp(h_-(1/\bar{z}))}, & 0 < |z| < s \\ 1, & z = 0 \end{cases}$$

is also analytic on $s\mathbb{D}$ and outer. Therefore, we have

$$f(z) = z^n g_0(z) \overline{g_1(1/\bar{z})}, \quad z \in s\mathbb{D}.$$

Since f is unimodular on \mathbb{T} , we get $|g_0(\xi)| |g_1(\xi)| = 1$ for all $\xi \in \mathbb{T}$. It follows that the outer functions g_0 and g_1 satisfy $g_1(z) = 1/g_0(z)$, so for all $z \in \mathbb{A}_s$,

$$f(z) = z^n \frac{g_0(z)}{g_0(1/\bar{z})}.$$

The reverse implication is immediate, since for all $z \in \mathbb{T}$, $1/\bar{z} = z$. □

6. SPECTRAL STUDY OF LINEAR ISOMETRIES ON THE ANNULUS

In this section we consider the spectrum σ and point spectrum σ_p of the operators $T_{\alpha,\beta}$ and $S_{\alpha,\beta}$ defined in (2). It will be convenient to begin with the simplest of these, namely $S_{\alpha,\beta}$, where $S_{\alpha,\beta}f(z) = \alpha f(\beta/z)$.

Theorem 6.1. *For $|\alpha| = |\beta| = 1$ we have $\sigma(S_{\alpha,\beta}) = \sigma_p(S_{\alpha,\beta}) = \{\alpha, -\alpha\}$. Moreover, $\ker(S_{\alpha,\beta} - \lambda \text{Id})$ is infinite-dimensional for $\lambda \in \{\alpha, -\alpha\}$.*

Proof. For $f(z) = z^k \pm \beta^k/z^k$ with $k \in \mathbb{N}$ we have

$$S_{\alpha,\beta}f(z) = \alpha\beta^k/z^k \pm \alpha\beta^k/(\beta^k/z^k) = \pm\alpha f(z),$$

and so we see that $\{\alpha, -\alpha\} \subseteq \sigma_p(S_{\alpha,\beta})$ and the eigenspaces are infinite-dimensional.

Moreover, $S_{\alpha,\beta}^2 = \alpha^2 \text{Id}$, and so

$$(S_{\alpha,\beta} - \lambda \text{Id})(S_{\alpha,\beta} + \lambda \text{Id}) = S_{\alpha,\beta}^2 - \lambda^2 \text{Id} = (\alpha^2 - \lambda^2) \text{Id},$$

and thus $S_{\alpha,\beta} - \lambda \text{Id}$ is invertible if $\lambda^2 \neq \alpha^2$. That is, $\sigma_p(S_{\alpha,\beta}) = \sigma(S_{\alpha,\beta}) = \{\alpha, -\alpha\}$. \square

We now turn our attention to $T_{\alpha,\beta}$.

Theorem 6.2. *Let $\alpha, \beta \in \mathbb{T}$ and $T_{\alpha,\beta} : \text{Hol}(\mathbb{A}) \rightarrow \text{Hol}(\mathbb{A})$ defined by*

$$T(f)(z) = \alpha f(\beta z), \quad f \in \text{Hol}(\mathbb{A}).$$

(i) *If there exists $n \in \mathbb{N}$ such that $\beta^n = 1$, then*

$$\sigma_p(T_{\alpha,\beta}) = \sigma(T_{\alpha,\beta}) = \{\alpha\beta^k : 0 \leq k \leq n-1\}.$$

Moreover, for all $\lambda \in \sigma_p(T_{\alpha,\beta})$, $\text{Ker}(T_{\alpha,\beta} - \lambda \text{Id})$ has infinite dimension.

(ii) *If for all $n \in \mathbb{Z}$, $\beta^n \neq 1$, then*

$$\sigma_p(T_{\alpha,\beta}) = \{\alpha\beta^k : k \in \mathbb{Z}\} \subset \sigma(T_{\alpha,\beta}) \subset \mathbb{T}.$$

Moreover, for all $\lambda \in \sigma_p(T_{\alpha,\beta})$, $\text{Ker}(T_{\alpha,\beta} - \lambda \text{Id})$ has dimension 1.

Proof.

(i) Let $e_k(z) = z^k$, $0 \leq k \leq n-1$. Then,

$$T_{\alpha,\beta}(e_k)(z) = \alpha e_k(\beta z) = \alpha(\beta z)^k = \alpha\beta^k z^k = \alpha\beta^k e_k(z).$$

Therefore, $\alpha\beta^k \in \sigma_p(T_{\alpha,\beta})$, and $e_k \in \text{Ker}(T_{\alpha,\beta} - \alpha\beta^k \text{Id})$. In addition, it is easy to show that for all $\ell \in \mathbb{Z}$, $e_{k+n\ell} \in \text{Ker}(T_{\alpha,\beta} - \alpha\beta^k \text{Id})$, since $\beta^{k+n\ell} = \beta^k$, so the eigenspaces have infinite dimension.

Moreover, note that $T_{\alpha,\beta}^n = \alpha^n \text{Id}$, so

$$(\alpha^n - \lambda^n) \text{Id} = T_{\alpha,\beta}^n - \lambda^n \text{Id} = (T_{\alpha,\beta} - \lambda \text{Id}) \sum_{k=1}^n \lambda^{k-1} T_{\alpha,\beta}^{n-k}.$$

Hence, $T_{\alpha,\beta} - \lambda \text{Id}$ is invertible if and only if $\lambda^n \neq \alpha^n$, i.e. $\lambda \notin \{\alpha\beta^k : 0 \leq k \leq n-1\}$ since $\beta^n = 1$.

(ii) Let $f \in \text{Hol}(\mathbb{A})$. Note that we can write

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k = a_0 + \sum_{k \geq 1} a_k z^k + \sum_{k \geq 1} a_{-k} z^{-k},$$

where the first series converges for $|z| < R$ and the second one converges for $|z| > 1/R$. Then,

$$(T_{\alpha, \beta} - \lambda \text{Id})(f)(z) = \sum_{k \in \mathbb{Z}} (\alpha \beta^k - \lambda) a_k z^k.$$

Let $g \in \text{Hol}(\mathbb{A})$. If $\lambda \notin \mathbb{T}$, then consider, for $k \in \mathbb{Z}$, $a_k = \frac{b_k}{\alpha \beta^k - \lambda}$. We obtain

$$\frac{|b_k|}{1 + |\lambda|} < |a_k| < \frac{|b_k|}{|1 - |\lambda||}.$$

Hence, the power series determined by (a_k) , (b_k) , (a_{-k}) and (b_{-k}) have again the same radius of convergence, namely R . This means that $f = \sum_{k \in \mathbb{Z}} a_k z^k \in \text{Hol}(\mathbb{A})$. Moreover, by construction, $(T_{\alpha, \beta} - \lambda \text{Id})(f) = g$, so $(T_{\alpha, \beta} - \lambda \text{Id})$ is invertible: $\lambda \notin \sigma(T_{\alpha, \beta})$.

Now, let $e_k(z) = z^k$, $k \in \mathbb{Z}$. Then, $T_{\alpha, \beta}(e_k)(z) = \alpha e_k(\beta z) = \alpha \beta^k e_k(z)$, so we have $\alpha \beta^k \in \sigma_p(T_{\alpha, \beta})$, and $e_k \in \text{Ker}(T_{\alpha, \beta} - \alpha \beta^k \text{Id})$.

Let $h \in \text{Ker}(T_{\alpha, \beta} - \alpha \beta^k \text{Id})$. Then, $\alpha h(\beta z) = \alpha \beta^k h(z) \iff h(\beta z) = \beta^k h(z)$. This implies that $h(z) \neq 0$ for all $z \in \mathbb{A}$. Indeed, if $h(z_0) = 0$ for some $z_0 \in \mathbb{A}$, then for all $k \in \mathbb{N}$, $h(\beta^k z_0) = 0$. Since $\{\beta^k z_0 : k \in \mathbb{N}\}$ is a dense subset of $|z_0| \mathbb{T}$, we obtain $h \equiv 0$.

By Lemma 5.5, $h(z) = z^j \exp(\theta(z))$ for some $j \in \mathbb{Z}$ and $\theta \in \text{Hol}(\mathbb{A})$. Let us denote $\beta = e^{i\gamma}$. Then, $\beta^j z^j \exp(\theta(\beta z)) = \beta^k z^j \exp(\theta(z))$. Dividing by z^j and letting $z \rightarrow 0$, we obtain $\beta^j = \beta^k$. Hence, $j = k$ since $\beta^n \neq 1$ for all $n \in \mathbb{Z}$. Thus,

$$h(z) = z^k \exp(\theta(z)).$$

The equality $h(\beta z) = \beta^k h(z)$ becomes $\exp(\theta(\beta z)) = \exp(\theta(z))$. Therefore, there exists $\ell(z) \in \mathbb{Z}$ such that

$$\theta(\beta z) = \theta(z) + 2i\pi \ell(z).$$

By continuity, ℓ is a constant map, and for $z = 0$, $\ell(0) = 0$. Thus, for all $z \in \mathbb{A}$,

$$\theta(\beta z) = \theta(z).$$

Iterating this equality, for all $m \in \mathbb{N}$, $\theta(\beta^m z) = \theta(z)$. But $\{\beta^m z : m \in \mathbb{N}\}$ is a dense subset of $|z| \mathbb{T}$, so θ is a constant map. Finally, $h \in \text{span}(e_k)$, so $\text{Ker}(T_{\alpha, \beta} - \alpha \beta^k \text{Id})$ has dimension 1. This finishes the proof. \square

The spectrum of $T_{\alpha, \beta}$ can be different from \mathbb{T} if β is aperiodic. We take our inspiration from the results on the disc [2].

Example 6.3. Let $\tau > 2$. The set of *Diophantine numbers of order τ* is defined by

$$\mathcal{D}(\tau) = \left\{ \xi \in \mathbb{R} : \exists \gamma > 0, \forall p \in \mathbb{Z}, \forall q \in \mathbb{N}, \left| \frac{p}{q} - \xi \right| \geq \frac{\gamma}{q^\tau} \right\}$$

Let $\xi \in \mathcal{D}(\tau)$, and $\beta = e^{2i\pi\xi}$. Then, we show that for $r \in \mathbb{Q} \setminus \mathbb{Z}$, $\lambda = e^{2i\pi r} \notin \sigma(T_{1, \beta})$. Since multiplying by α only rotates our problem, this is also true for $T_{\alpha, \beta}$, $\alpha \in \mathbb{T}$.

Indeed, let $\gamma > 0$ such that $|\xi - p/q| \geq \gamma q^{-\tau}$, for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Let $r = p_0/q_0$ and $k \in \mathbb{N}$. Choose $p \in \mathbb{Z}$ such that $|\xi k - r - p| \leq 1/2$. Then,

$$\begin{aligned} |\beta^k - \lambda| &= |e^{2i\pi\xi k} - e^{2i\pi r}| \\ &= |e^{2i\pi\xi k} - e^{2i\pi(r+p)}| \\ &= |e^{i\pi(\xi k - r - p)} - e^{-i\pi(\xi k - r - p)}| = 2 \sin |\pi(\xi k - r - p)| \geq 2 \cdot \frac{2}{\pi} \cdot \pi |\xi k - r - p|, \end{aligned}$$

since the sine function is concave on $[0, \pi/2]$ and $|\pi(\xi k - r - p)| \in [0, \pi/2]$. Consequently,

$$\frac{1}{|\beta^k - \lambda|} \leq \frac{1}{4|\xi k - r - p|} = \frac{1}{4k} \frac{1}{\left| \xi - \frac{p_0 + pq_0}{q_0 k} \right|} \leq \frac{1}{4k} \frac{(q_0 k)^\tau}{\gamma} = ck^{\tau-1}, \quad c := \frac{q_0^\tau}{4\gamma}.$$

Finally,

$$1 = \lim_{k \rightarrow +\infty} \frac{1}{2^{1/k}} \leq \liminf_{k \rightarrow +\infty} \frac{1}{|\beta^k - \lambda|^{1/k}} \leq \limsup_{k \rightarrow +\infty} \frac{1}{|\beta^k - \lambda|^{1/k}} \leq \limsup_{k \rightarrow +\infty} c^{1/k} k^{(\tau-1)/k} \leq 1.$$

Hence, if $g(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in \text{Hol}(\mathbb{A})$, then

$$f(z) := \sum_{n \in \mathbb{Z}} \frac{a_n}{\beta^n - \lambda} z^n \in \text{Hol}(\mathbb{A}),$$

and $f(\beta z) - \lambda f(z) = g(z)$. Finally, $T_{1,\beta} - \lambda \text{Id}$ is surjective, and injective by Theorem 6.2. Thus, $\sigma(T_{1,\beta}) \subset \{e^{2i\pi s} : s \notin \mathbb{Q}\} \cup \{1\}$.

Likewise, it is possible for $\sigma_p(T_{\alpha,\beta})$ to be strictly contained in $\sigma(T_{\alpha,\beta})$.

Example 6.4. For our example, we take $\alpha = 1$.

Let $p_1 = 1$ and for $n \geq 1$, $p_{n+1} = 2^{n^{p_n}}$. Then, the map g defined by $g(z) = \sum_{n=1}^{\infty} (z/R)^{p_n}$ is in $\text{Hol}(R\mathbb{D}) \subset \text{Hol}(\mathbb{A})$, where $\mathbb{A} = \{1/R < |z| < R\}$. Moreover,

- ★ For all $n \geq 1$, $p_{n+1} = 2^{n^{p_n}} \geq 2^{p_n} \geq p_n$.
- ★ For all $n \geq 1$, p_n is a power of two, hence an integer.
- ★ For all $n \geq 1$, denote $q_n = 2^{q_n}$. Then, $q_{n+1} = n^{p_n} \geq p_n \geq q_n$, so

$$\frac{p_{n+1}}{p_n} = 2^{q_{n+1} - q_n} \in \mathbb{N}.$$

Moreover, $q_{n+1} - q_n = n^{p_n} - q_n \geq np_n - q_n \geq (n-1)p_n \geq n-1$, so $p_{n+1}/p_n \geq 2^{n-1}$.

Now we take

$$\beta = \exp(i\pi\theta), \quad \text{where } \theta = \sum_{n=1}^{\infty} \frac{1}{p_n}.$$

Note that for all $n \in \mathbb{N}$, we may write

$$\theta = \frac{\ell_n}{p_n} + \varepsilon_n,$$

where ℓ_n is an integer and

$$0 < \varepsilon_n = \sum_{k \geq n+1} \frac{1}{p_k} = \frac{1}{p_{n+1}} \sum_{k \geq n+1} \frac{p_{n+1}}{p_k} \leq \frac{1}{p_{n+1}} \sum_{k \geq n+1} \frac{1}{2^{k-2}} \leq \frac{2}{p_{n+1}}.$$

In addition, $p_{n+1} \geq p_n^{n+1}$ (this is immediate for $n = 1, 2$; when $n \geq 3$, this comes from the fact that if $p_n = 2^{q_n}$, then $(n + 1)q_n \leq (n + 1)p_n \leq n^{p_n} = q_{n+1}$). Therefore

$$0 < \varepsilon_n \leq \frac{2}{p_n^{n+1}} \leq \frac{1}{p_n^n}.$$

This means that θ is a Liouville (hence, irrational) number. Thus, $-1 \notin \sigma_p(T_{\alpha,\beta})$.

We claim that $-1 \in \sigma(T_{\alpha,\beta})$. To solve $f(\beta z) + f(z) = g(z)$, with $g(z) = \sum_{k \geq 1} (z/R)^k$, we require $f(z) = \sum_{k \geq 1} \frac{1}{1+\beta^k} (z/R)^k$. However,

$$\beta^{p_n} = \exp(i\pi p_n \theta) = \exp\left(i\pi \sum_{k \geq n} \frac{p_n}{p_k}\right) = -\exp\left(i\pi \sum_{k \geq n+1} \frac{p_n}{p_k}\right),$$

so we obtain

$$|1 + \beta^{p_n}| = 2 \left| \sin\left(\pi \sum_{k \geq n+1} \frac{p_n}{p_k}\right) \right| \underset{n \rightarrow \infty}{\sim} 2\pi \sum_{k \geq n+1} \frac{p_n}{p_k} \underset{n \rightarrow \infty}{\sim} 2\pi \frac{p_n}{p_{n+1}} \underset{n \rightarrow \infty}{\sim} 2\pi \frac{p_n}{2^{n^{p_n}}}.$$

Looking at the coefficient $c_n = \frac{1}{\beta^{p_{n+1}}}$ of $(z/R)^{p_n}$ in f , we see that

$$|c_n|^{1/p_n} \underset{n \rightarrow \infty}{\sim} \frac{2^{\frac{1}{p_n} n^{p_n}}}{p_n^{1/p_n}} \xrightarrow{n \rightarrow \infty} \infty.$$

The radius of convergence of the series for f is then 0, which implies that -1 is in the spectrum. Note that if we take $R = 1$ in the above estimations, then we obtain an example of composition operator induced by a rotation on the disc for which the point spectrum is different from the spectrum, thus providing an example that complements the results of [2].

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ISABELLE CHALENDAR, UNIVERSITÉ GUSTAVE EIFFEL, LAMA, (UMR 8050), UPEM, UPEC, CNRS, F-77454, MARNE-LA-VALLÉE (FRANCE)

Email address: isabelle.chalendar@univ-eiffel.fr

LUCAS OGER, UNIVERSITÉ GUSTAVE EIFFEL, LAMA, (UMR 8050), UPEM, UPEC, CNRS, F-77454, MARNE-LA-VALLÉE (FRANCE)

Email address: lucas.oger@univ-eiffel.fr

JONATHAN R. PARTINGTON, SCHOOL OF MATHEMATICS, UNIVERSITY OF LEEDS, LEEDS LS2 9JT, YORKSHIRE, U.K.

Email address: j.r.partington@leeds.ac.uk