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# ON MODULAR RIGIDITY FOR $GL_n$

by

Nadir Matringe, Alberto Mínguez & Vincent Sécherre

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**Abstract.** — Let  $k$  be a global field and  $\mathbb{A}_k$  be its ring of adèles. Let  $\ell$  be a prime number and fix a field isomorphism from  $\mathbb{C}$  to  $\overline{\mathbb{Q}}_\ell$ . Let  $\Pi_1, \Pi_2$  be cuspidal automorphic representations of  $GL_n(\mathbb{A}_k)$  for some integer  $n \geq 1$ . In this paper, we study the following question: assuming that there is a finite set  $S$  of places of  $k$  containing all Archimedean places and all finite places above  $\ell$  such that, for all  $v \notin S$ , the local components  $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  and  $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  are unramified and their Satake parameters are congruent mod  $\ell$ , are the local components  $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  and  $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  integral, and do their reductions mod  $\ell$  share an irreducible factor for all non-Archimedean places  $w$  not dividing  $\ell$ ? We show that, under certain conditions on  $\Pi_1, \Pi_2$ , the answer is yes. We also give a simple proof when  $k$  is a function field.

Keywords and Phrases: Automorphic forms, Congruences mod  $\ell$ , Satake parameters, Whittaker models, Automorphic representations

## 1. Introduction

**1.1.** Let  $k$  be a number field and  $\mathbb{A}_k$  be its ring of adèles. Let  $\Pi_1$  and  $\Pi_2$  be cuspidal automorphic representations of  $GL_n(\mathbb{A}_k)$  for some integer  $n \geq 1$ . The rigidity (or strong multiplicity 1) theorem asserts that, if there is a finite set  $S$  of places of  $k$  containing all Archimedean places such that, for all  $v \notin S$ , the local components  $\Pi_{1,v}$  and  $\Pi_{2,v}$  are unramified and have the same Satake parameter, then  $\Pi_1$  and  $\Pi_2$  are isomorphic ([17, 3, 10, 11]). A similar result holds over function fields.

**1.2.** Now fix a field isomorphism  $\iota$  from  $\mathbb{C}$  to an algebraic closure  $\overline{\mathbb{Q}}_\ell$  of the field of  $\ell$ -adic numbers for some prime number  $\ell$ , and consider the collections of irreducible smooth  $\overline{\mathbb{Q}}_\ell$ -representations of  $GL_n(k_v)$  defined by

$$(1.1) \quad \pi_{i,v} = \Pi_{i,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell, \quad i \in \{1, 2\},$$

where the tensor product is taken with respect to  $\iota$ ,  $v$  runs over all finite places of  $k$  and  $k_v$  is the completion of  $k$  at  $v$ . As  $\Pi_1$  and  $\Pi_2$  are cuspidal, these representations are generic (see §2.2).

Suppose that there exists a finite set  $S$  of places of  $k$  containing all Archimedean places and all finite places above  $\ell$  such that, for all  $v \notin S$ , the following are satisfied:

- (1) the representations  $\pi_{1,v}$  and  $\pi_{2,v}$  are unramified representations of  $GL_n(k_v)$ ,

(2) the Satake parameters  $\sigma_{1,v}$  and  $\sigma_{2,v}$  of these unramified representations, considered as conjugacy classes of semisimple elements of  $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ , have their characteristic polynomials  $P_{1,v}(X)$  and  $P_{2,v}(X)$  in  $\overline{\mathbb{Z}}_\ell[X]$ , where  $\overline{\mathbb{Z}}_\ell$  is the ring of integers of  $\overline{\mathbb{Q}}_\ell$ ,

(3) the reductions of  $P_{1,v}(X)$  and  $P_{2,v}(X)$  in  $\overline{\mathbb{F}}_\ell[X]$  are equal,  $\overline{\mathbb{F}}_\ell$  being the residue field of  $\overline{\mathbb{Z}}_\ell$ .

Assumption 2 is equivalent to saying that the unramified representations  $\pi_{1,v}$  and  $\pi_{2,v}$  are integral, that is, their  $\overline{\mathbb{Q}}_\ell$ -vector spaces contain  $\mathrm{GL}_n(k_v)$ -stable  $\overline{\mathbb{Z}}_\ell$ -lattices (see §2.4). One can then consider their reductions mod  $\ell$ , denoted  $\mathbf{r}_\ell(\pi_{1,v})$  and  $\mathbf{r}_\ell(\pi_{2,v})$ , which are finite length, semisimple smooth  $\overline{\mathbb{F}}_\ell$ -representations of  $\mathrm{GL}_n(k_v)$  (see Section 2 for a precise definition of reduction mod  $\ell$ ). Assumption 3 is then equivalent to saying that the representations  $\mathbf{r}_\ell(\pi_{1,v})$  and  $\mathbf{r}_\ell(\pi_{2,v})$  are equal (see Remarks 4.2 and 4.3).

Now let  $w$  be a finite place of  $k$  not dividing  $\ell$ . Our first question is

**Question 1.1.** — *Are the irreducible representations  $\pi_{1,w}$  and  $\pi_{2,w}$  integral?*

Assume that this is the case. One can then form  $\mathbf{r}_\ell(\pi_{1,w})$  and  $\mathbf{r}_\ell(\pi_{2,w})$ . These representations may not be equal (see Remark 4.4 for an example), but one may address the following question.

**Question 1.2.** — *Do  $\mathbf{r}_\ell(\pi_{1,w})$  and  $\mathbf{r}_\ell(\pi_{2,w})$  have an irreducible component in common?*

If  $k$  is a totally real (respectively, CM) number field, and if  $\Pi_1, \Pi_2$  are algebraic regular, self-dual (respectively, conjugate-selfdual) cuspidal automorphic representations, then [16] Theorem 8.2 says that the answers to Questions 1.1 and 1.2 are yes. More precisely:

- the representations  $\pi_{1,w}$  and  $\pi_{2,w}$  are integral for all finite places  $w$  of  $k$  not dividing  $\ell$ ,
- their reductions mod  $\ell$  have a unique generic irreducible component in common,
- this unique common generic irreducible component occurs with multiplicity 1.

Such a result, which can be thought of as a modular rigidity theorem, has been used in [16] in order to study the behavior of local transfer for cuspidal  $\overline{\mathbb{Q}}_\ell$ -representations of quasi-split classical groups with respect to congruences mod  $\ell$ .

More generally, thanks to the results of [6, 20, 23], one can make the argument of the proof of [16] Theorem 8.2 work with no duality assumption on  $\Pi_1$  and  $\Pi_2$ : if  $k$  is a totally real or CM number field, and if  $\Pi_1, \Pi_2$  are algebraic regular, cuspidal automorphic representations, the answers to Questions 1.1 and 1.2 are still yes; more precisely, the three properties above still hold. (See §4.2 below for a detailed argument, which relies on the existence of a correspondence from algebraic regular cuspidal automorphic representation to Galois representations with local-global compatibility at all finite places not dividing  $\ell$ .)

It is natural to ask whether the ‘totally real or CM’ assumption on  $k$ , or the ‘algebraic regular’ assumption on the representations  $\Pi_1$  and  $\Pi_2$ , or the cuspidality assumption, can be removed. We will not investigate these questions in the present article.

It is also natural to seek an elementary, purely automorphic proof of such a modular rigidity theorem, avoiding the use of Galois representations and local-global compatibility theorems. We will study this question in the case of function fields, which is easier since there are no Archimedean places.

**1.3.** We now assume that  $k$  is a function field of characteristic  $p$ , with ring of adèles  $\mathbb{A}_k$ . Recall that we have fixed a field isomorphism  $\iota$  from  $\mathbb{C}$  to  $\overline{\mathbb{Q}}_\ell$  for some prime number  $\ell$  which we assume to be different from  $p$ . In this article, we prove the following theorem (see Theorem 4.5).

**Theorem 1.3.** — *Let  $\Pi_1$  and  $\Pi_2$  be cuspidal automorphic representations of  $\mathrm{GL}_n(\mathbb{A}_k)$ . Associated with them, there are the representations  $\pi_{i,v}$  defined by (1.1). Suppose that there exists a finite set  $S$  of places of  $k$  such that, for all  $v \notin S$ , one has:*

- (1) *the representations  $\pi_{1,v}$  and  $\pi_{2,v}$  are unramified,*
- (2) *the characteristic polynomials of their Satake parameters are in  $\overline{\mathbb{Z}}_\ell[X]$  and have the same reduction in  $\overline{\mathbb{F}}_\ell[X]$ .*

*Let  $w$  be a place of  $k$ . Then*

- *the representations  $\pi_{1,w}$  and  $\pi_{2,w}$  are integral,*
- *their reductions mod  $\ell$  share a generic irreducible component,*
- *and such a generic component is unique and occurs with multiplicity 1 in both reductions.*

This theorem can be easily deduced from L. Lafforgue’s global Langlands correspondence [13] (see Remark 4.9). Our purpose is to give a simple proof of Theorem 1.3 which does not rely on the Langlands correspondence for function fields. Our argument, inspired from Piatetski-Shapiro’s proof of the classical rigidity theorem [17, 3] is described below. We currently do not know how to extend our argument to number fields.

**1.4.** Before explaining the proof of Theorem 1.3, we introduce our main local ingredients. Let  $F$  be a non-Archimedean locally compact field of residue characteristic  $p$  (and characteristic 0 or  $p$ ). Fix a non-trivial smooth  $\overline{\mathbb{Q}}_\ell$ -character  $\vartheta$  of  $F$ .

**Proposition 1.4.** — *Let  $\pi_1$  and  $\pi_2$  be integral generic irreducible  $\overline{\mathbb{Q}}_\ell$ -representations of  $\mathrm{GL}_n(F)$ . Suppose that there are functions  $W_1$  and  $W_2$  in the Whittaker models of  $\pi_1$  and  $\pi_2$  with respect to  $\vartheta$  satisfying the following conditions:*

- (1)  *$W_1$  and  $W_2$  are  $\overline{\mathbb{Z}}_\ell$ -valued and  $W_1(1) = W_2(1) = 1$ ,*
- (2) *the reductions of  $W_1(g)$  and  $W_2(g)$  in  $\overline{\mathbb{F}}_\ell$  are equal for all  $g \in G$ .*

*Then  $\mathbf{r}_\ell(\pi_1)$  and  $\mathbf{r}_\ell(\pi_2)$  share a generic irreducible component, such a generic irreducible component is unique and it occurs with multiplicity 1.*

Let  $P$  be the mirabolic subgroup of  $\mathrm{GL}_n$ , made of all matrices with last row  $(0 \dots 0 \ 1)$ , and  $N$  be its unipotent radical. We have the following remarkable integrality criterion (see Proposition 2.5), which follows from [8] and [14].

**Proposition 1.5.** — *Let  $\pi$  be a generic irreducible  $\overline{\mathbb{Q}}_\ell$ -representation of  $\mathrm{GL}_n(F)$ . The following assertions are equivalent.*

- (1) *The representation  $\pi$  is integral.*
- (2) *Given any function in the Whittaker model of  $\pi$  with respect to  $\vartheta$  whose restriction to  $P(F)$  is compactly supported mod  $N(F)$ , this function is  $\overline{\mathbb{Z}}_\ell$ -valued on  $\mathrm{GL}_n(F)$  if and only if it is  $\overline{\mathbb{Z}}_\ell$ -valued on  $P(F)$ .*

**1.5.** We now introduce our main global ingredients. Fix a continuous unitary complex character  $\psi$  of  $\mathbb{A}_k$ , and consider a cuspidal automorphic form  $\varphi$  on  $\mathrm{GL}_n(\mathbb{A}_k)$ . Associated with it by (3.3), with respect to the choice of  $\psi$ , there is a Whittaker function  $W$  on  $\mathrm{GL}_n(\mathbb{A}_k)$ . Note that  $\psi$  is valued in the group  $\mu_p$  of complex  $p$ th roots of 1. Let  $Z$  be the centre of  $\mathrm{GL}_n$ .

Given any sub- $\mathbb{Z}[\mu_p]$ -module  $I$  of  $\mathbb{C}$ , we prove in Section 3 that:

- if  $W$  takes values in  $I$  on  $P(\mathbb{A}_k)$ , then  $\varphi$  takes values in  $I$  on  $P(\mathbb{A}_k)$  (Theorem 3.1),
- if  $\varphi$  takes values in  $I$  on  $Z(\mathbb{A}_k)P(\mathbb{A}_k)$ , then  $W$  takes values in  $I$  on  $\mathrm{GL}_n(\mathbb{A}_k)$  (Theorem 3.7).

**1.6.** We now consider two cuspidal automorphic representations  $\Pi_1, \Pi_2$  of  $\mathrm{GL}_n(\mathbb{A}_k)$  as in Theorem 1.3. Let  $A$  be the local ring  $\iota^{-1}(\overline{\mathbb{Z}}_\ell)$  and  $\mathfrak{m}$  be its maximal ideal. Fix a place  $w \in S$ .

We first observe that the central characters of  $\Pi_1$  and  $\Pi_2$  are  $A$ -valued and congruent mod  $\mathfrak{m}$  (see Proposition 4.8), thanks to the information we have at all places  $v \notin S$ .

For each place  $v$  of  $k$ , let  $\psi_v$  be the local component of  $\psi$  at  $v$ . It is a smooth character of  $k_v$ , which we may assume, for all  $v \notin S$ , to be trivial on the ring of integers of  $k_v$  but not on the inverse of its maximal ideal.

Let  $W_{i,v}$  be any function in the Whittaker model of  $\Pi_{i,v}$  with respect to  $\psi_v$  such that:

- (1) if  $v \notin S$ , then  $W_{i,v}$  is  $\mathrm{GL}_n(\mathcal{O}_v)$ -invariant (see §2.5),
- (2) if  $v \in S$ , then  $W_{1,v}$  and  $W_{2,v}$  coincide on  $P(k_v)$ , and their restriction to  $P(k_v)$  is compactly supported mod  $N(k_v)$  and  $A$ -valued (see §2.2),
- (3) and  $W_{1,v}(1) = W_{2,v}(1) = 1$  for all  $v \neq w$ .

The tensor product of the  $W_{i,v}$  is a function  $W_i$  in the Whittaker model of  $\Pi_i$  with respect to  $\psi$ . First, it follows from the Shintani formula (see [21, 3] and Proposition 2.4) that, for all  $v \notin S$ :

- the functions  $W_{1,v}$  and  $W_{2,v}$  are  $A$ -valued,
- the difference  $W_{1,v} - W_{2,v}$  is  $\mathfrak{m}$ -valued.

The functions  $W_1$  and  $W_2$  are thus  $A$ -valued and  $W_1 - W_2$  is  $\mathfrak{m}$ -valued on  $Z(\mathbb{A}_k)P(\mathbb{A}_k)$ .

Since  $A$  and  $\mathfrak{m}$  are sub- $\mathbb{Z}[\mu_p]$ -modules of  $\mathbb{C}$ , we may apply the result of Paragraph 1.5, from which we deduce that the functions  $W_1$  and  $W_2$  are  $A$ -valued and  $W_1 - W_2$  is  $\mathfrak{m}$ -valued on the whole of  $\mathrm{GL}_n(\mathbb{A}_k)$ . Consequently, thanks to Assumption (3) above, we get:

- (★) the functions  $W_{1,w}$  and  $W_{2,w}$  are  $A$ -valued,
- (★★) the difference  $W_{1,w} - W_{2,w}$  is  $\mathfrak{m}$ -valued.

Starting with any function in the Whittaker model of  $\pi_{i,w}$  with respect to  $\psi_w$  whose restriction to  $P_w$  is compactly supported mod  $N_w$  and  $\overline{\mathbb{Z}}_\ell$ -valued, we thus proved that this function is  $\overline{\mathbb{Z}}_\ell$ -valued (see (★) above). Applying Proposition 1.5, we deduce that  $\pi_{1,w}$  and  $\pi_{2,w}$  are integral, proving the first assertion of Theorem 1.3.

Now assume that  $W_{1,w}$  and  $W_{2,w}$  satisfy the additional condition  $W_{1,w}(1) = W_{2,w}(1) = 1$ . The remaining two assertions of Theorem 1.3 then follow from (★) and (★★) by Proposition 1.4.

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## 2. Local considerations

In this section,  $F$  denotes a locally compact non-Archimedean field of residue characteristic  $p$  and  $n \geq 1$  is a positive integer. We write  $\mathcal{O}_F$  for the ring of integers of  $F$  and  $\mathfrak{p}_F$  for its maximal ideal. We also write  $G$  for the locally profinite group  $GL_n(F)$ .

Let  $\ell$  be a prime number different from  $p$ . We write  $\overline{\mathbb{Q}}_\ell$  for an algebraic closure of the field of  $\ell$ -adic integers,  $\overline{\mathbb{Z}}_\ell$  for its ring of integers and  $\overline{\mathbb{F}}_\ell$  for its residue field.

Let  $\psi : F \rightarrow \overline{\mathbb{Q}}_\ell^\times$  be a non-trivial smooth character. It defines a non-degenerate character

$$x \mapsto \psi(x_{1,2} + \cdots + x_{n-1,n})$$

of  $N$ , the subgroup of upper triangular unipotent matrices of  $G$ , still denoted  $\psi$ . Note that this character takes values in  $\overline{\mathbb{Z}}_\ell$ , and even more precisely in the group of roots of unity in  $\overline{\mathbb{Q}}_\ell$  whose order is a power of  $p$ .

Let  $P$  denote the mirabolic subgroup of  $G$ , made of all matrices whose last row is  $(0 \dots 0 \ 1)$ .

The representations we will consider will be smooth representations of locally profinite groups with coefficients in  $\mathbb{Z}[1/p]$ -algebras.

**2.1.** A  $\overline{\mathbb{Q}}_\ell$ -representation of finite length  $\pi$  of  $G$  is said to be integral if its vector space  $V$  contains a  $G$ -stable  $\overline{\mathbb{Z}}_\ell$ -lattice. (A  $G$ -stable  $\overline{\mathbb{Z}}_\ell$ -lattice is a  $G$ -stable free  $\overline{\mathbb{Z}}_\ell$ -module generated by a basis of  $V$  or, equivalently, an admissible  $\overline{\mathbb{Z}}_\ell[G]$ -module containing a basis of  $V$ .)

If this is the case, and if  $L$  is such a  $G$ -stable  $\overline{\mathbb{Z}}_\ell$ -lattice, the representation of  $G$  on  $L \otimes \overline{\mathbb{F}}_\ell$  is smooth and has finite length, and its semisimplification does not depend on the choice of  $L$  (see [26] Theorem 1). This semisimplified  $\overline{\mathbb{F}}_\ell$ -representation is called the reduction mod  $\ell$  of  $\pi$  and is denoted  $\mathbf{r}_\ell(\pi)$ .

An irreducible  $\overline{\mathbb{Q}}_\ell$ -representation  $\pi$  which embeds in the parabolic induction of some cuspidal irreducible representation  $\rho$  of some Levi subgroup  $M$  of  $G$  is integral if and only if the central character of  $\rho$  is  $\overline{\mathbb{Z}}_\ell$ -valued (see [25] II.4.12, II.4.14 and [4] Proposition 6.7).

**2.2.** In this paragraph,  $\pi$  is a generic irreducible  $\overline{\mathbb{Q}}_\ell$ -representation of  $G$ , that is, its vector space  $V$  carries a non-zero  $\overline{\mathbb{Q}}_\ell$ -linear form  $\Lambda$  such that  $\Lambda(\pi(u)v) = \psi(u)\Lambda(v)$  for all  $u \in N$ ,  $v \in V$ . Let

$$(2.1) \quad \mathcal{W}(\pi, \psi) \subseteq \text{Ind}_N^G(\psi)$$

denote its Whittaker model with respect to  $\psi$ , where  $\text{Ind}_N^G$  denotes smooth induction from  $N$  to  $G$ . Let  $\mathcal{K}(\pi, \psi)$  denote the Kirillov model of  $\pi$ , that is, the space of smooth  $\overline{\mathbb{Q}}_\ell$ -valued functions

on  $P$  which extend to a function in  $\mathcal{W}(\pi, \psi)$ . By Kirillov's theory, restriction from  $G$  to  $P$  induces a  $P$ -equivariant isomorphism from  $\mathcal{W}(\pi, \psi)$  to  $\mathcal{K}(\pi, \psi)$ , and one has the containments

$$\mathrm{ind}_N^P(\psi) \subseteq \mathcal{K}(\pi, \psi) \subseteq \mathrm{Ind}_N^P(\psi)$$

where  $\mathrm{ind}_N^P$  denotes compact induction from  $N$  to  $P$  (see [1]).

**2.3.** In this paragraph,  $\pi_1$  and  $\pi_2$  are integral generic irreducible  $\overline{\mathbb{Q}}_\ell$ -representations of  $G$ . Let  $\mathfrak{m}_\ell$  denote the maximal ideal of  $\overline{\mathbb{Z}}_\ell$ .

**Proposition 2.1.** — *Suppose there are Whittaker functions  $W_1 \in \mathcal{W}(\pi_1, \psi)$  and  $W_2 \in \mathcal{W}(\pi_2, \psi)$  with values in  $\overline{\mathbb{Z}}_\ell$  such that:*

- (1)  $W_1(1) = W_2(1) = 1$ ,
- (2)  $W_1(g)$  and  $W_2(g)$  are congruent mod  $\mathfrak{m}_\ell$  for all  $g \in G$ .

*Then the reductions mod  $\ell$  of  $\pi_1$  and  $\pi_2$  share a generic irreducible component, such a generic irreducible component is unique and it occurs with multiplicity 1.*

*Proof.* — We will need the following result.

**Lemma 2.2.** — *Let  $\pi$  be an integral generic irreducible  $\overline{\mathbb{Q}}_\ell$ -representation of  $G$ . Then its reduction mod  $\ell$  contains a unique irreducible generic factor, occurring with multiplicity 1.*

*Proof.* — The existence of an irreducible generic factor follows from the fact that any non-zero linear form in  $\mathrm{Hom}_N(\pi, \psi)$  is non-zero on any  $G$ -stable  $\overline{\mathbb{Z}}_\ell$ -lattice of  $\pi$ . Its uniqueness follows for instance from [15] Proposition 8.4 applied to the representation parabolically induced from the cuspidal support of  $\pi$ .  $\square$

Let  $i \in \{1, 2\}$ . By [26] Theorem 2, the  $\overline{\mathbb{Z}}_\ell$ -module  $L_i$  made of all  $\overline{\mathbb{Z}}_\ell$ -valued Whittaker functions in  $\mathcal{W}(\pi_i, \psi)$  is a  $G$ -stable  $\overline{\mathbb{Z}}_\ell$ -lattice. Let  $\Lambda_i$  be the  $\overline{\mathbb{Z}}_\ell[G]$ -module generated by  $W_i$  in  $\mathcal{W}(\pi_i, \psi)$ . It contains a  $\overline{\mathbb{Q}}_\ell$ -basis of  $\mathcal{W}(\pi_i, \psi)$  since  $\pi_i$  is irreducible, and it is contained in  $L_i$ . It follows that it is a  $G$ -stable  $\overline{\mathbb{Z}}_\ell$ -lattice in  $\mathcal{W}(\pi_i, \psi)$ . Let  $M_i$  denote the submodule of  $L_i$  made of all  $\mathfrak{m}_\ell$ -valued functions. The containment of  $\mathfrak{m}_\ell \Lambda_i$  in  $\Lambda_i \cap M_i$  implies that we have morphisms:

$$\Lambda_i \otimes \overline{\mathbb{F}}_\ell \rightarrow \Lambda_i / (\Lambda_i \cap M_i) \simeq (\Lambda_i + M_i) / M_i \subseteq L_i / M_i \rightarrow \mathrm{Ind}_N^G(\vartheta \otimes \overline{\mathbb{F}}_\ell)$$

where the left hand side morphism  $\alpha_i$  is surjective, the right hand side morphism  $\beta_i$  is injective, and  $\vartheta \otimes \overline{\mathbb{F}}_\ell$  denotes the  $\overline{\mathbb{F}}_\ell$ -character of  $N$  obtained by reducing  $\vartheta$  mod  $\mathfrak{m}_\ell$ .

As  $W_1$  and  $W_2$  are congruent mod  $\mathfrak{m}_\ell$  on  $G$  and take 1 to 1, the intersection:

$$(2.2) \quad \beta_1((\Lambda_1 + M_1) / M_1) \cap \beta_2((\Lambda_2 + M_2) / M_2)$$

is non-zero in  $\mathrm{Ind}_N^G(\vartheta \otimes \overline{\mathbb{F}}_\ell)$  for it contains the function  $\beta_1(W_1 \bmod M_1) = \beta_2(W_2 \bmod M_2)$  and the latter is non-zero. The socle of (2.2), denoted  $\Sigma$ , is made of generic irreducible  $\overline{\mathbb{F}}_\ell$ -representations appearing in both the reductions mod  $\ell$  of  $\pi_1$  and  $\pi_2$ . By Lemma 2.2, the reduction mod  $\ell$  of  $\pi_i$  contains a unique irreducible generic factor  $\rho_i$ . The socle  $\Sigma$  is thus irreducible, reduced to  $\rho_i$ . It follows that  $\rho_1$  and  $\rho_2$  are isomorphic.  $\square$

**2.4.** Let  $K$  be the maximal compact subgroup  $GL_n(\mathcal{O}_F)$ . In this paragraph,  $\pi$  is an unramified irreducible  $\overline{\mathbb{Q}}_\ell$ -representation of  $G$ , that is,  $\pi$  has a non-zero  $K$ -fixed vector. It defines a conjugacy class of semisimple elements in  $GL_n(\overline{\mathbb{Q}}_\ell)$ , called its Satake parameter. The characteristic polynomial of this conjugacy class is denoted  $\chi(\pi)$ . This is a polynomial of degree  $n$  in  $\overline{\mathbb{Q}}_\ell[X]$ .

**Lemma 2.3.** — *The unramified representation  $\pi$  is integral if and only if the polynomial  $\chi(\pi)$  has all its coefficients in  $\overline{\mathbb{Z}}_\ell$ .*

*Proof.* — Since  $\overline{\mathbb{Z}}_\ell$  is integrally closed,  $\chi(\pi)$  has all its coefficients in  $\overline{\mathbb{Z}}_\ell$  if and only if its roots are in  $\overline{\mathbb{Z}}_\ell$ , that is, if and only if  $\pi$  is parabolically induced from an integral unramified character of the diagonal torus of  $GL_n(F)$ . The lemma then follows from Paragraph 2.1.  $\square$

**2.5.** Now assume that  $\pi$  is a generic unramified irreducible  $\overline{\mathbb{Q}}_\ell$ -representation of  $G$ , and that  $\psi$  is trivial on  $\mathcal{O}_F$  but not on  $\mathfrak{p}_F^{-1}$ . Its Whittaker model  $\mathcal{W}(\pi, \psi)$  contains a unique Whittaker function  $W_\pi$  such that:

- (1) one has  $W_\pi(gk) = W_\pi(g)$  for all  $g \in G$  and  $k \in K$ ,
- (2) and  $W_\pi(1) = 1$ .

Let us recall the Shintani–Casselman–Shalika formula [21, 3], which gives the values of  $W_\pi$  at diagonal elements in terms of the Satake parameter of  $\pi$ .

Fix a representative  $(\mu_1, \dots, \mu_n) \in \overline{\mathbb{Q}}_\ell^{\times n}$  of the Satake parameter of  $\pi$ . If we write

$$\chi(\pi) = X^n + c_1(\pi)X^{n-1} + \dots + c_n(\pi) \in \overline{\mathbb{Q}}_\ell[X],$$

then

$$(-1)^r c_r(\pi) = \sum_{1 \leq i_1 < \dots < i_r \leq n} \mu_{i_1} \dots \mu_{i_r}$$

for all  $r \in \{1, \dots, n\}$ . Let  $q$  be the cardinality of the residue field of  $F$ . Fix a uniformizer  $\varpi \in F$  and let  $\Delta$  be the subgroup of  $G$  made of all diagonal matrices whose eigenvalues are integral powers of  $\varpi$ . The Iwasawa decomposition  $G = N\Delta K$  shows that  $W_\pi$  is entirely determined by its restriction to  $\Delta$ . Given  $a \in \mathbb{Z}^n$ , write  $\varpi^a$  for the diagonal matrix whose  $i$ th eigenvalue is  $\varpi^{a_i}$ .

One has the formula:

$$(2.3) \quad W_\pi(\varpi^a) = q^{\sum_{j=1}^n a_j(j-(n+1)/2)} \cdot \frac{\det((\mu_j^{a_l+n-l})_{j,l})}{\prod_{j < l} (\mu_j - \mu_l)} \quad \text{if } a_1 \geq \dots \geq a_n$$

and  $W_\pi$  vanishes at  $\varpi^a$  otherwise.

**2.6.** Formula (2.3) has the following application.

**Proposition 2.4.** — *Let  $\pi_1$  and  $\pi_2$  be integral generic unramified irreducible representations of  $G$ . Assume that the polynomials  $\chi(\pi_1)$  and  $\chi(\pi_2)$  have the same reduction mod  $\mathfrak{m}_\ell$  in  $\overline{\mathbb{F}}_\ell[X]$ . Then the Whittaker functions  $W_{\pi_1}$  and  $W_{\pi_2}$  are  $\overline{\mathbb{Z}}_\ell$ -valued on  $G$  and one has:*

$$(2.4) \quad W_{\pi_1}(g) \equiv W_{\pi_2}(g) \pmod{\mathfrak{m}_\ell}$$

for all  $g \in G$ .



*Proof.* — Notice that  $\chi(\pi_1)$  and  $\chi(\pi_2)$  have coefficients in  $\overline{\mathbb{Z}}_\ell$  by Lemma 2.3, thus the reduction of  $\chi(\pi_1)$  and  $\chi(\pi_2)$  mod  $\mathfrak{m}_\ell$  is well-defined.

It suffices to prove that  $W_{\pi_1}$  and  $W_{\pi_2}$  are  $\overline{\mathbb{Z}}_\ell$ -valued on  $\Delta$  and that the relation (2.4) is satisfied for all  $g \in \Delta$ . Fix a representative  $\boldsymbol{\mu}_i = (\mu_{i,1}, \dots, \mu_{i,n}) \in \overline{\mathbb{Q}}_\ell^{\times n}$  of the Satake parameter of  $\pi_i$  for  $i = 1, 2$ . The scalars  $\mu_{i,1}, \dots, \mu_{i,n}$  are the roots of  $\chi(\pi_i)$ . They are thus in  $\overline{\mathbb{Z}}_\ell$ . The fact that the polynomials  $\chi(\pi_1)$  and  $\chi(\pi_2)$  are congruent mod  $\mathfrak{m}_\ell$  ensures that, up to reordering, we may assume that  $\mu_{1,j}$  and  $\mu_{2,j}$  are congruent mod  $\mathfrak{m}_\ell$  for all  $j = 1, \dots, n$ . The lemma now follows from the Shintani formula (2.3).  $\square$

**2.7.** In this paragraph,  $\pi$  is a generic irreducible  $\overline{\mathbb{Q}}_\ell$ -representation of  $G$ . We have the following remarkable integrality criterion, based on [8] and [14].

**Proposition 2.5.** — *Let  $\pi$  be a generic irreducible  $\overline{\mathbb{Q}}_\ell$ -representation of  $G$ . The following assertions are equivalent:*

- (1) *The representation  $\pi$  is integral.*
- (2) *Given any  $\overline{\mathbb{Z}}_\ell$ -valued function  $f \in \text{ind}_N^P(\psi)$ , the Whittaker function in  $\mathcal{W}(\pi, \psi)$  extending  $f$  is  $\overline{\mathbb{Z}}_\ell$ -valued.*

**Remark 2.6.** — Assertion 2 can be restated as follows: given any function  $W \in \mathcal{W}(\pi, \psi)$  whose restriction to  $P$  is compactly supported mod  $N$ , the function  $W$  is  $\overline{\mathbb{Z}}_\ell$ -valued on  $G$  if and only if it is  $\overline{\mathbb{Z}}_\ell$ -valued on  $P$ .

*Proof.* — That the first assertion implies the second one follows from [14] Corollary 4.3 (which gives an even stronger result: it says that a Whittaker function  $W \in \mathcal{W}(\pi, \psi)$  is  $\overline{\mathbb{Z}}_\ell$ -valued on  $G$  if and only if it is  $\overline{\mathbb{Z}}_\ell$ -valued on  $P$ ).

Let us prove that the second assertion implies the first one. We will use [8] Theorem 3.2, which is stated for Noetherian algebras over the ring  $\mathbb{W}_\ell$  of Witt vectors of  $\overline{\mathbb{F}}_\ell$ . Let us explain how it applies to a generic  $\overline{\mathbb{Q}}_\ell$ -representation  $\pi$  satisfying Assertion 2.

Let  $V$  be the  $\overline{\mathbb{Q}}_\ell$ -vector space of  $\pi$ . By [25] II.4.9, there exists a finite extension  $E$  of  $\mathbb{Q}_\ell^{\text{ur}}$ , the maximal unramified extension of  $\mathbb{Q}_\ell$  in  $\overline{\mathbb{Q}}_\ell$ , such that  $\pi$  is defined over  $E$ , that is,  $V$  contains a  $G$ -stable  $E$ -vector space  $V_E$  such that  $V = V_E \otimes_E \overline{\mathbb{Q}}_\ell$ . Let  $\pi_E$  denote the  $E$ -representation of  $G$  on  $V_E$ . If  $\psi_E$  denotes the character  $\psi$  considered as being valued in  $E$ , then  $\pi_E$  is generic with respect to  $\psi_E$ . Let  $K$  be the completion of  $E$ . Then  $\pi_K = \pi_E \otimes_E K$  is generic with respect to the character  $\psi_K = \psi_E \otimes_E K$ . Since the complete discrete valuation ring  $\mathbb{W}_\ell$  is isomorphic to the completion of the ring of integers of  $\mathbb{Q}_\ell^{\text{ur}}$ , the ring  $\mathcal{O}$  of integers of  $K$  is a Noetherian  $\mathbb{W}_\ell$ -algebra. Let us show that  $\pi_K$  satisfies the analogue of Assertion 2 for the ring  $\mathcal{O}$ .

**Lemma 2.7.** — *Given any  $\mathcal{O}$ -valued function  $f \in \text{ind}_N^P(\psi_K)$ , there exists an  $\mathcal{O}$ -valued function in  $\mathcal{W}(\pi_K, \psi_K)$  extending  $f$ .*

*Proof.* — This  $f$  can be written  $a_1 f_1 + \dots + a_r f_r$  with  $a_1, \dots, a_r \in \mathcal{O}$ , and where the functions  $f_1, \dots, f_r \in \text{ind}_N^G(\psi)$  are  $\overline{\mathbb{Z}}_\ell$ -valued. By assumption on  $\pi$ , the function  $W_i \in \mathcal{W}(\pi, \psi)$  extending  $f_i$  is  $\overline{\mathbb{Z}}_\ell$ -valued. Thus  $a_1 W_1 + \dots + a_r W_r$  is in  $\mathcal{W}(\pi_K, \psi_K)$ , it extends  $f$  and it is  $\mathcal{O}$ -valued.  $\square$

Let us collect some results from [7] about the category  $\mathbf{Rep}_{\mathbb{W}_\ell}(G)$  of all smooth  $\mathbb{W}_\ell$ -representations of  $G$ . This category decomposes into a product of blocks indexed by inertial classes  $\Omega$  of supercuspidal  $\overline{\mathbb{F}}_\ell$ -representations of  $G$ . Associated with each block, there is its centre  $\mathfrak{z}_\Omega$ , which is a finitely generated commutative  $\mathbb{W}_\ell$ -algebra, and its *universal co-Whittaker* module  $\mathcal{W}_\Omega$ , which is an admissible  $\mathfrak{z}_\Omega[G]$ -module.

The representation  $\pi_K$  is absolutely irreducible and generic. It is thus a co-Whittaker  $K[G]$ -module in the sense of [8] Definition 2.1. Also, by Schur's lemma, if  $\Omega$  is the inertial class associated with it, the action of the centre  $\mathfrak{z}_\Omega$  on  $\pi_K$  defines a morphism of  $\mathbb{W}_\ell$ -algebras  $\chi : \mathfrak{z}_\Omega \rightarrow K$ . By [7] Theorem 6.3, the representation  $\pi_K$  is a quotient of  $\mathcal{W}_\Omega \otimes_{\mathfrak{z}_\Omega} K$ .

We now apply [8] Theorem 3.2 to  $\pi_K$  (with  $A = K$  and  $A' = \mathcal{O}$ ), which says that, thanks to Lemma 2.7,  $\chi$  is valued in  $\mathcal{O}$ , which makes  $\mathcal{O}$  into a  $\mathfrak{z}_\Omega$ -algebra. By [7] Lemma 6.4 (or more precisely its proof), the image  $L$  of  $\mathcal{W}_\Omega \otimes_{\mathfrak{z}_\Omega} \mathcal{O}$  in  $\pi_K$  is an  $\mathcal{O}$ -torsion free co-Whittaker  $\mathcal{O}[G]$ -module such that  $L \otimes_{\mathcal{O}} K = \pi_K$ . By [8] Definition 2.1, this  $\mathcal{O}[G]$ -module  $L$  is admissible. The representation  $\pi_K$  is thus  $\mathcal{O}$ -integral.

Fix a parabolic subgroup  $Q$  of  $G$  with Levi subgroup  $M$ , and a cuspidal irreducible representation  $\rho$  of  $M$  such that  $\pi$  embeds in the parabolic induction  $i_Q^G(\rho)$ . We may and will choose  $E$  so that  $\rho$  is also defined over  $E$ : we thus have an  $E$ -representation  $\rho_E$  such that  $\rho_E \otimes_E \overline{\mathbb{Q}}_\ell = \rho$  and  $\pi_E$  embeds in  $i_Q^G(\rho_E)$ . Thus  $\pi_K$  embeds in  $i_Q^G(\rho_K)$ , where  $\rho_K = \rho_E \otimes_E K$  is cuspidal and absolutely irreducible. Note that the central character  $\omega$  of  $\rho_K$  takes values in  $E$ .

Since  $L$  is admissible, [4] Proposition 6.7 implies that the Jacquet module  $r_Q^G(L)$  is admissible, thus  $\rho_K$  is  $\mathcal{O}$ -integral. Its central character  $\omega$  thus takes values in  $\mathcal{O}_E$ , thus the central character of  $\rho$  takes values in  $\overline{\mathbb{Z}}_\ell$ . It follows (see Paragraph 2.1) that  $\pi$  is integral, as expected.  $\square$

### 3. Global considerations

**3.1.** Let  $k$  be a global field, that is, either a finite extension of  $\mathbb{Q}$  or the field of rational functions over a smooth irreducible projective curve  $X$  defined over a finite field of cardinality  $q$ . Let  $\mathbb{A}$  be the ring of adèles of  $k$ .

Given an integer  $n \geq 2$ , let  $N = N_n$  be the subgroup of upper triangular unipotent matrices of  $GL_n$  and  $P = P_n$  be its mirabolic subgroup, made of all matrices whose last row is  $(0 \dots 0 \ 1)$ . More generally, for  $m \in \{0, \dots, n\}$ , let  $N_{m, n-m}$  denote the unipotent radical of the parabolic subgroup of  $GL_n$  generated by upper triangular matrices and the Levi subgroup  $GL_m \times GL_{n-m}$ .

Let  $\psi : \mathbb{A} \rightarrow \mathbb{C}^\times$  be a non-trivial continuous character trivial on  $k$ . It defines in the usual way a non-degenerate character of  $N(k) \backslash N(\mathbb{A})$ , namely

$$u \mapsto \psi(u_{1,2} + \dots + u_{n-1,n})$$

for all  $u \in N(\mathbb{A})$ , which we still denote by  $\psi$ .

For any place  $v$  of  $k$ , let  $k_v$  denote the completion of  $k$  at  $v$ . If  $v$  is finite, we write  $\mathcal{O}_v$  for the ring of integers of  $k_v$  and  $\mathfrak{p}_v$  for its maximal ideal. The character  $\psi$  decomposes as

$$(3.1) \quad \psi = \bigotimes_v \psi_v$$

where  $\psi_v$  is a non-trivial continuous character of  $k_v$ , trivial on  $\mathcal{O}_v$  but not on  $\mathfrak{p}_v^{-1}$  for almost all finite  $v$ .

**3.2.** Let us fix a Haar measure  $du$  on  $N(k)\backslash N(\mathbb{A})$ . Given a cuspidal irreducible automorphic representation  $\Pi$  of  $\mathrm{GL}_n(\mathbb{A})$ , the linear form

$$(3.2) \quad \varphi \mapsto \int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1} \varphi(u) \, du$$

on  $\Pi$  is known to be well-defined and non-zero (see [3] Theorem 1.1 or (3.4) below).

Associated to  $\varphi \in \Pi$ , there is a Whittaker function  $W_\varphi$  defined by:

$$(3.3) \quad W_\varphi(g) = \int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1} \varphi(ug) \, du$$

for all  $g \in \mathrm{GL}_n(\mathbb{A})$ . The map  $\varphi \mapsto W_\varphi$  is a morphism from  $\Pi$  to its Whittaker model  $\mathcal{W}(\Pi, \psi)$ .

If we choose for  $du$  the Haar measure giving measure 1 to the compact group  $N(k)\backslash N(\mathbb{A})$ , one also has a converse expansion:

$$(3.4) \quad \varphi(g) = \sum_{\gamma \in N_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_\varphi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

for all  $g \in \mathrm{GL}_n(\mathbb{A})$ , with absolute and uniform convergence on compact subsets (see for instance [5] Theorem 13.5.4 or [3] Theorem 1.1).

**3.3.** From now on, assume that  $k$  is a function field of characteristic  $p$ . There is thus no Archimedean place. Let  $I$  be any sub- $\mathbb{Z}[\mu_p]$ -module of  $\mathbb{C}$ , where  $\mu_p$  denotes the subgroup of  $p$ th roots of unity in  $\mathbb{C}$ . Note that  $\psi$  takes values in  $\mu_p$ .

**Theorem 3.1.** — *Let  $\phi : P_n(\mathbb{A}) \rightarrow \mathbb{C}$  be a smooth function such that:*

- (1) *one has  $\phi(\gamma g) = \phi(g)$  for all  $\gamma \in P_n(k)$  and all  $g \in P_n(\mathbb{A})$ ,*
- (2) *the function  $\phi$  is cuspidal in the sense that*

$$\int_{N_{m,n-m}(k)\backslash N_{m,n-m}(\mathbb{A})} \phi(ug) \, du = 0$$

*for all  $g \in P_n(\mathbb{A})$  and all  $m \in \{1, \dots, n-1\}$ ,*

- (3) *one has*

$$\int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1} \phi(ug) \, du \in I$$

*for all  $g \in P_n(\mathbb{A})$ .*

*Then  $\phi(g) \in I$  for all  $g \in P_n(\mathbb{A})$ .*

We will prove this theorem by induction on  $n \geq 2$ . Given a function  $\phi$  as in Theorem 3.1, it is useful to define a function  $W_\phi$  on  $P_n(\mathbb{A})$  by setting

$$W_\phi(g) = \int_{N(k) \backslash N(\mathbb{A})} \psi(u)^{-1} \phi(ug) \, du$$

for all  $g \in P_n(\mathbb{A})$ . We will often use the fact that, by Assumption 3, it takes values in  $I$ .

**3.4.** We first treat the case where  $n = 2$ . We will need the following lemma.

**Lemma 3.2.** — *For any smooth functions  $f, g \in \mathcal{C}^\infty(\mathbb{A}/k, \mathbb{C})$ , we have*

$$\int_{\mathbb{A}/k} f(x)g(x) \, dx = \sum_{\gamma \in k} \int_{\mathbb{A}/k} \psi^{-1}(\gamma x) f(x) \, dx \cdot \int_{\mathbb{A}/k} \psi(\gamma x) g(x) \, dx.$$

*Proof.* — Start with the Fourier expansion formula

$$f(y) = \sum_{\gamma \in k} \psi(\gamma y) \cdot \int_{\mathbb{A}/k} \psi^{-1}(\gamma x) f(x) \, dx$$

for  $y \in \mathbb{A}/k$ . Then multiply by  $g(y)$  and integrate over  $\mathbb{A}/k$ . □

Let  $f \in \mathcal{C}^\infty(\mathbb{A}/k, \mathbb{C})$ . For any  $g \in P_2(\mathbb{A})$ , we thus get

$$(3.5) \quad \int_{\mathbb{A}/k} \phi \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) f(u) \, du = \sum_{\gamma \in k} \Phi(\gamma, g) F(\gamma)$$

where

$$\Phi(\gamma, g) = \int_{\mathbb{A}/k} \psi^{-1}(\gamma u) \phi \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) \, du \quad \text{and} \quad F(\gamma) = \int_{\mathbb{A}/k} \psi(\gamma u) f(u) \, du.$$

Therefore, we have  $\Phi(0, g) = 0$  by cuspidality of  $\phi$  and, if  $\gamma \neq 0$ , we have

$$\begin{aligned} \Phi(\gamma, g) &= \int_{\mathbb{A}/k} \psi^{-1}(u) \phi \left( \begin{pmatrix} 1 & \gamma^{-1}u \\ 0 & 1 \end{pmatrix} g \right) \, du \\ &= \int_{\mathbb{A}/k} \psi^{-1}(u) \phi \left( \begin{pmatrix} \gamma & u \\ 0 & 1 \end{pmatrix} g \right) \, du \\ &= W_\phi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right) \end{aligned}$$

where the first equality follows from the fact that the module of  $\gamma$  is 1 by the product formula, and the second one follows from the fact that  $\phi$  is  $P_2(k)$ -invariant.

Now let  $U = U(\phi, g)$  be a compact open subgroup of  $\mathbb{A}/k$  such that

$$\phi \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) = \phi(g) \quad \text{for all } u \in U.$$

Let  $f$  be the characteristic function of  $U$ . Thus

$$F(\gamma) = \int_U \psi(\gamma u) \, du.$$

On the one hand, we have

$$\int_{\mathbb{A}/k} \phi \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) f(u) \, du = \phi(g) \cdot |U|$$

(where  $|U|$  is the volume of  $U$  with respect to  $du$ ). On the other hand, we have

$$\int_{\mathbb{A}/k} \phi \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) f(u) \, du = \sum_{\gamma \in k^\times} W_\phi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right) \cdot \int_U \psi(\gamma u) \, du,$$

which gives the identity

$$\phi(g) = \sum_{\gamma \in k^\times} W_\phi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right) \cdot \int_U \psi(\gamma u) \frac{du}{|U|}.$$

As  $W_\phi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right) \in I$ , for all  $\gamma \in k^\times$ , and

$$\int_U \psi(\gamma u) \frac{du}{|U|} = \begin{cases} 0 & \text{if } \psi \text{ is non-trivial on } \gamma U, \\ 1 & \text{otherwise,} \end{cases}$$

it only remains to prove that the sum over  $\gamma$  is finite, that is, there are only finitely many  $\gamma \in k^\times$  such that  $\gamma U \subseteq \text{Ker}(\psi)$ . Assume that

$$U = \prod_{v \in S} \mathfrak{p}_v^{m_v} \times \prod_{v \notin S} \mathcal{O}_v$$

for some finite set  $S$  of places of  $k$  and some integers  $m_v \in \mathbb{Z}$ . Recall that we have the decomposition (3.1) of  $\psi$ . By taking a bigger  $S$  if necessary, we may (and will) assume that the character  $\psi_v$  is trivial on  $\mathcal{O}_v$  but not on  $\mathfrak{p}_v^{-1}$  for all  $v \notin S$ . Thus  $\gamma U \subseteq \text{Ker}(\psi)$  if and only if  $\gamma \mathcal{O}_v \subseteq \text{Ker}(\psi_v)$  for all  $v \notin S$  and  $\gamma \mathfrak{p}_v^{m_v} \subseteq \text{Ker}(\psi_v)$  for all  $v \in S$ . Equivalently, this means that  $\gamma$  belongs to the space of  $f \in k$ , considered as rational functions on the curve  $X$  defining the field  $k$ , such that

- $f$  has no pole at  $v \notin S$ ,
- $f$  has a pole of order  $\geq -m_v$  at  $v \in S$ .

The expected finiteness result now follows from the fact that these  $f$  form a finite dimensional vector space over  $\mathbb{F}_q$  (see for instance [22] Proposition 1.4.9).

**3.5.** We now assume that  $n \geq 3$ , and that Theorem 3.1 has been proved for  $P_{n-1}(\mathbb{A})$ .

We fix an arbitrary  $g \in P_n(\mathbb{A})$  and define a function  $\phi' = \phi'_g$  on  $P_{n-1}(\mathbb{A})$  by setting

$$\phi'(h) = \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \phi \left( \begin{pmatrix} h & u \\ 0 & 1 \end{pmatrix} g \right) \, du$$

for all  $h \in P_{n-1}(\mathbb{A})$ , where  $\eta = (0, \dots, 0, 1) \in k^{n-1}$  and  $\alpha \cdot u = \alpha_1 u_1 + \dots + \alpha_{n-1} u_{n-1} \in (\mathbb{A}/k)^{n-1}$  for any  $\alpha \in k^{n-1}$  and  $u \in (\mathbb{A}/k)^{n-1}$ . It has the following properties.

**Lemma 3.3.** — *The function  $\phi'$  is cuspidal on  $P_{n-1}(\mathbb{A})$ , that is, one has*

$$\int_{N_{m,n-1-m}(k) \backslash N_{m,n-1-m}(\mathbb{A})} \phi'(vh) \, dv = 0$$

for all  $h \in P_{n-1}(\mathbb{A})$  and all  $m \in \{1, \dots, n-2\}$ .

*Proof.* — Let us fix an  $h \in P_{n-1}(\mathbb{A})$  and an  $m \in \{1, \dots, n-2\}$ . Then

$$(3.6) \quad \int_{N_{m,n-1-m}(k) \backslash N_{m,n-1-m}(\mathbb{A})} \phi'(vh) \, dv = \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \Omega_{g,h}(u) \, du$$

where

$$\Omega_{g,h}(u) = \int_{N_{m,n-1-m}(k) \backslash N_{m,n-1-m}(\mathbb{A})} \phi \left( \begin{pmatrix} vh & u \\ 0 & 1 \end{pmatrix} g \right) \, dv$$

and the right hand side of (3.6) is equal to

$$\int_{(\mathbb{A}/k)^{n-1-m}} \psi^{-1}(\eta \cdot u_2) \int_{(\mathbb{A}/k)^m} \Omega_{g,h} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \, du_1 \, du_2 = \int_{(\mathbb{A}/k)^{n-1-m}} \psi^{-1}(\eta \cdot u_2) \Lambda_{g,h}(u_2) \, du_2$$

where

$$\Lambda_{g,h}(u_2) = \int_{N_{m,n-m}(k) \backslash N_{m,n-m}(\mathbb{A})} \phi \left( w \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g \right) \, dw$$

and this quantity is equal to 0 thanks to the fact that  $\phi$  is cuspidal.  $\square$

**Lemma 3.4.** — *One has*

$$\phi'(\alpha h) = \phi'(h)$$

for all  $\alpha \in P_{n-1}(k)$  and  $h \in P_{n-1}(\mathbb{A})$ .

*Proof.* — Let us fix an  $\alpha \in P_{n-1}(k)$ . Thanks to the fact that  $\phi$  is  $P_n(k)$ -invariant, one has

$$\begin{aligned} \phi'(\alpha h) &= \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \phi \left( \begin{pmatrix} \alpha h & u \\ 0 & 1 \end{pmatrix} g \right) \, du \\ &= \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \alpha \cdot u) \phi \left( \begin{pmatrix} h & u \\ 0 & 1 \end{pmatrix} g \right) \, du. \end{aligned}$$

Since  $\alpha \in P_{n-1}(k)$ , we get  $\eta \alpha = \eta$ , thus  $\phi'(\alpha h) = \phi'(h)$ .  $\square$

**Lemma 3.5.** — *One has*

$$\int_{N_{n-1}(k) \backslash N_{n-1}(\mathbb{A})} \psi^{-1}(v) \phi'(vh) \, dv \in I$$

for all  $h \in P_{n-1}(\mathbb{A})$ .

*Proof.* — It suffices to notice that

$$\begin{aligned} \int_{N_{n-1}(k) \backslash N_{n-1}(\mathbb{A})} \psi^{-1}(v) \phi'(vh) \, dv &= \int_{N_n(k) \backslash N_n(\mathbb{A})} \psi^{-1}(w) \phi \left( w \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g \right) \, dw \\ &= W_\phi \left( \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g \right) \end{aligned}$$

which takes values in  $I$  for all  $g \in P_n(\mathbb{A})$  and  $h \in P_{n-1}(\mathbb{A})$ . □

Applying now the inductive hypothesis to the function  $\phi' = \phi'_g$ , we deduce that

$$(3.7) \quad \phi'_g(h) \in I, \quad \text{for all } g \in P_n(\mathbb{A}) \text{ and all } h \in P_{n-1}(\mathbb{A}).$$

We can do even better.

**Lemma 3.6.** — *For all  $g \in P_n(\mathbb{A})$  and all  $g' \in \text{GL}_{n-1}(\mathbb{A})$ , we have*

$$(3.8) \quad \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \phi \left( \begin{pmatrix} g' & u \\ 0 & 1 \end{pmatrix} g \right) \, du \in I.$$

*Proof.* — Indeed, we have

$$\int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \phi \left( \begin{pmatrix} g' & u \\ 0 & 1 \end{pmatrix} g \right) \, du = \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \phi \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix} g \right) \, du$$

which is equal to  $\phi'_x(1)$  with

$$x = \begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix} g \in P_n(\mathbb{A}).$$

The lemma thus follows from (3.7) applied to the function  $\phi'_x$ . □

We now extend  $\phi' = \phi'_g$  to  $\text{GL}_{n-1}(\mathbb{A})$  by setting

$$\phi'(g') = \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \phi \left( \begin{pmatrix} g' & u \\ 0 & 1 \end{pmatrix} g \right) \, du$$

for all  $g' \in \text{GL}_{n-1}(\mathbb{A})$ . By Lemma 3.6, it takes values in  $I$  on  $\text{GL}_{n-1}(\mathbb{A})$  for all  $g \in P_n(\mathbb{A})$ .

Now, by Fourier analysis on the compact Abelian group  $(\mathbb{A}/k)^{n-1}$ , we have

$$\begin{aligned} \phi \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) &= \sum_{\beta \in k^{n-1}} \psi(\beta \cdot u) \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\beta \cdot x) \phi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \, dx \\ &= \sum_{\rho \in P_{n-1}(k) \backslash \text{GL}_{n-1}(k)} \psi(\eta \rho \cdot u) \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot x) \phi \left( \begin{pmatrix} \rho & x \\ 0 & 1 \end{pmatrix} g \right) \, dx \\ &= \sum_{\rho \in P_{n-1}(k) \backslash \text{GL}_{n-1}(k)} \psi(\eta \rho \cdot u) \phi'(\rho). \end{aligned}$$

Multiplying by  $f(u)$  for some function  $f \in \mathcal{C}^\infty((\mathbb{A}/k)^{n-1}, \mathbb{C})$  and integrating, we get

$$\int_{(\mathbb{A}/k)^{n-1}} \phi \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) f(u) \, du = \sum_{\rho \in P_{n-1}(k) \backslash \mathrm{GL}_{n-1}(k)} \phi'(\rho) \int_{(\mathbb{A}/k)^{n-1}} \psi(\eta\rho \cdot u) f(u) \, du.$$

Now let  $U = U(\phi, g)$  be a compact open subgroup of  $\mathbb{A}/k$  such that

$$\phi \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) = \phi(g) \quad \text{for all } u \in U^{n-1} \subseteq (\mathbb{A}/k)^{n-1}.$$

Now take for  $f$  the characteristic function of  $U^{n-1}$ . We get

$$(3.9) \quad \phi(g) = \sum_{\rho \in P_{n-1}(k) \backslash \mathrm{GL}_{n-1}(k)} \phi'(\rho) \int_{U^{n-1}} \psi(\eta\rho \cdot u) \frac{du}{|U|^{n-1}}.$$

For all  $\rho$ , we have

$$\int_{U^{n-1}} \psi(\eta\rho \cdot u) \frac{du}{|U|^{n-1}} = \begin{cases} 0 & \text{if } \psi \text{ is non-trivial on } \eta\rho \cdot U^{n-1}, \\ 1 & \text{otherwise.} \end{cases}$$

A coset  $\rho \in P_{n-1}(k) \backslash \mathrm{GL}_{n-1}(k)$  satisfies  $\eta\rho \cdot U^{n-1} \subseteq \mathrm{Ker}(\psi)$  if and only if the vector

$$\beta = (\beta_1, \dots, \beta_{n-1}) = \eta\rho \in k^{n-1}$$

satisfies  $\beta \cdot U^{n-1} \subseteq \mathrm{Ker}(\psi)$ , that is,  $\beta_i U \subseteq \mathrm{Ker}(\psi)$  for all  $i$ . But it follows from the case where  $n = 2$  that there are finitely many  $\beta_i \in k$  such that  $\beta_i U \subseteq \mathrm{Ker}(\psi)$ . There are thus finitely many cosets  $\rho$  contributing to the sum (3.9).

Moreover,  $\phi'(\rho) \in I$  for all  $g \in P_n(\mathbb{A})$  and all  $\rho \in P_{n-1}(k) \backslash \mathrm{GL}_{n-1}(k)$ . It follows that  $\phi(g) \in I$  for all  $g \in P_n(\mathbb{A})$ . This finishes the proof of Theorem 3.1.

**3.6.** As in Paragraph 3.3,  $k$  is a function field of characteristic  $p$  and  $I$  is a sub- $\mathbb{Z}[\mu_p]$ -module of  $\mathbb{C}$ . Let  $Z = Z_n$  denote the centre of  $\mathrm{GL}_n$ . We will prove the following result.

**Theorem 3.7.** — *Let  $\phi : \mathrm{GL}_n(\mathbb{A}) \rightarrow \mathbb{C}$  be a smooth function such that:*

- (1) *one has  $\phi(\gamma g) = \phi(g)$  for all  $\gamma \in \mathrm{GL}_n(k)$  and all  $g \in \mathrm{GL}_n(\mathbb{A})$ ,*
- (2) *the function  $\phi$  is cuspidal in the sense that*

$$\int_{N_{m,n-m}(k) \backslash N_{m,n-m}(\mathbb{A})} \phi(ug) \, du = 0$$

*for all  $g \in \mathrm{GL}_n(\mathbb{A})$  and all  $m \in \{1, \dots, n-1\}$ ,*

- (3) *one has  $\phi(g) \in I$  for all  $g \in Z(\mathbb{A})P(\mathbb{A})$ .*

*Then*

$$\int_{N(k) \backslash N(\mathbb{A})} \psi(u)^{-1} \phi(ug) \, du \in I$$

*for all  $g \in \mathrm{GL}_n(\mathbb{A})$ .*

We first prove the following lemma.



**Lemma 3.8.** — *The image of  $\mathrm{GL}_n(k)Z(\mathbb{A})P(\mathbb{A})$  in  $\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})$  is dense.*

*Proof.* — Since the projective space  $\mathbb{P}^{n-1}$  is  $k$ -isomorphic to  $\mathrm{GL}_n/ZP$ , it is equivalent to saying that the image of the diagonal embedding of  $\mathbb{P}^{n-1}(k)$  in  $\mathbb{P}^{n-1}(\mathbb{A})$  is dense, that is,  $\mathbb{P}^{n-1}$  satisfies the strong approximation property. We thus have to prove that, given any finite set  $S$  of places of  $k$  and, for  $v \in S$ , any non-empty open subset  $U_v$  of  $\mathbb{P}^{n-1}(k_v)$ , the intersection

$$\left( \prod_{v \in S} U_v \times \prod_{v \notin S} \mathbb{P}^{n-1}(\mathcal{O}_v) \right) \cap \mathbb{P}^{n-1}(k)$$

is non-empty. As  $\mathbb{P}^{n-1}(\mathcal{O}_v) = \mathbb{P}^{n-1}(k_v)$  for all  $v$  (given any point  $[x_1 : \dots : x_n] \in \mathbb{P}^{n-1}(k_v)$ , one may multiply the  $x_i$  by a suitable power of a uniformizer so that the coordinates are in  $\mathcal{O}_v$ ), this is equivalent to proving the weak approximation property, that is, proving that there is a point  $P = [x_1 : \dots : x_n] \in \mathbb{P}^{n-1}(k)$  such that  $P \in U_v$  for all  $v \in S$ . Restricting to the affine open subspace made of all points whose last coordinate is non-zero, we are reduced to proving the weak approximation property for an affine space. This follows from the weak approximation theorem for  $k$  (see for instance [22] Theorem 1.3.1).  $\square$

By Assumptions 1 and 3, the function  $\phi$  takes values in  $I$  on  $\mathrm{GL}_n(k)Z(\mathbb{A})P(\mathbb{A})$ . Since  $\phi$  is locally constant on  $\mathrm{GL}_n(\mathbb{A})$ , it follows from Lemma 3.8 that  $\phi$  takes values in  $I$  on  $\mathrm{GL}_n(\mathbb{A})$ .

We now prove the theorem. Let us apply the map  $\phi \mapsto W_\phi$  of Paragraph 3.2 defined by (3.3). This defines a function:

$$W_\phi(g) = \int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1} \phi(ug) \, du, \quad g \in \mathrm{GL}_n(\mathbb{A}).$$

Let us prove that  $W_\phi$  takes values in  $I$  on  $\mathrm{GL}_n(\mathbb{A})$ . Since  $N(k)\backslash N(\mathbb{A})$  is compact, there exists a compact open subgroup  $C$  of  $N(k)\backslash N(\mathbb{A})$  such that  $N(k)\backslash N(\mathbb{A})$  is the union of finitely many  $u_i C$  and the function  $u \mapsto \psi(u)^{-1} \phi(ug)$  is constant on these cosets. This gives us

$$W_\phi(g) = |C| \cdot \sum_{i=1}^r \psi^{-1}(u_i) \phi(u_i g).$$

It thus remains to prove that  $|C|$  is in  $\mathbb{Z}[\mu_p]$ .

The measure  $du$  gives measure 1 to the compact group  $N(k)\backslash N(\mathbb{A})$ . Thus  $|C|$  is the index of  $C$  in  $N(k)\backslash N(\mathbb{A})$ . Let us prove that  $|C|$  is a  $p$ -power. For this, it suffices to prove that  $N(k)\backslash N(\mathbb{A})$  is a pro- $p$ -group. For this, by dévissage, it suffices to prove that  $\mathbb{A}/k$  is a pro- $p$ -group.

By [19] Theorem 5.8, there are a finite set  $S$  of places of  $k$  and integers  $m_v \geq 0$  for  $v \in S$  such that  $\mathbb{A} = k + U$  for some compact open subgroup

$$U = \prod_{v \in S} \mathfrak{p}_v^{m_v} \times \prod_{v \notin S} \mathcal{O}_v$$

thus  $\mathbb{A}/k$  is a quotient of  $U$ . But  $U$  is clearly a pro- $p$ -group, as it is a product of pro- $p$ -groups.

#### 4. Modular rigidity

Let  $k$  be a global field as in Paragraph 3.1. Fix a prime number  $\ell$  different from the characteristic of  $k$  and a field isomorphism  $\iota : \mathbb{C} \rightarrow \overline{\mathbb{Q}}_\ell$ . Let  $\mathbb{A} = \mathbb{A}_k$  be its ring of adèles.

Given any place  $v$  of  $k$ , write  $G_v = GL_n(k_v)$  and  $N_v = N(k_v)$ , and  $P_v = P(k_v)$  for the mirabolic subgroup of  $G_v$ .

Recall that  $\mathfrak{m}_\ell$  is the maximal ideal of  $\overline{\mathbb{Z}}_\ell$ .

**4.1.** Let us state the following conjecture.

**Conjecture 4.1.** — *Let  $\Pi_1, \Pi_2$  be cuspidal automorphic representations of  $GL_n(\mathbb{A})$  with central characters  $\Omega_1, \Omega_2$ , respectively. Let  $\iota$  be a field isomorphism from  $\mathbb{C}$  to  $\overline{\mathbb{Q}}_\ell$  for some prime number  $\ell$  different from the characteristic of  $k$ . Suppose that:*

- (1) *the characters  $\Omega_1 \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  and  $\Omega_2 \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  are  $\overline{\mathbb{Z}}_\ell$ -valued and congruent mod  $\mathfrak{m}_\ell$ ,*
- (2) *there exists a finite set  $S$  of places of  $k$ , containing all Archimedean places and all finite places above  $\ell$ , such that, for all  $v \notin S$ , one has:*

- (a) *the local components  $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  and  $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  are unramified,*
- (b) *the characteristic polynomials of their Satake parameters belong to  $\overline{\mathbb{Z}}_\ell[X]$  and have the same reduction mod  $\mathfrak{m}_\ell$  in  $\overline{\mathbb{F}}_\ell[X]$ .*

*Let  $w$  be a finite place not dividing  $\ell$  such that the representations  $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  and  $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  are integral. Then the reductions mod  $\ell$  of these representations have a common generic irreducible component, and such a generic component is unique and occurs with multiplicity 1.*

**Remark 4.2.** — Note that the case where  $w \notin S$  is easy. Indeed, if  $w \notin S$ , write

$$\pi_1 = \Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell, \quad \pi_2 = \Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell.$$

These representations are generic (as  $\Pi_{1,w}$  and  $\Pi_{2,w}$  are local components of cuspidal automorphic representations) and unramified. For each  $i$ , there is thus an unramified character  $\omega_i$  of the diagonal torus  $T_w$  of  $G_w$  whose parabolic induction is isomorphic to  $\pi_i$ . By Lemma 2.3, these representations are integral, that is, the character  $\omega_i$  takes values in  $\overline{\mathbb{Z}}_\ell^\times$ . By [16] Proposition 6.2, the fact that the characteristic polynomials of their Satake parameters are congruent implies that the reductions mod  $\mathfrak{m}_\ell$  of  $\omega_1$  and  $\omega_2$  are conjugate by the normalizer of  $T_w$  in  $G_w$ . It follows that  $\mathbf{r}_\ell(\pi_1)$  and  $\mathbf{r}_\ell(\pi_2)$  are equal, and the conclusion follows from Lemma 2.2. We will thus concentrate on the case where  $w \in S$ .

**Remark 4.3.** — The reader should be aware that there are integral unramified irreducible  $\overline{\mathbb{Q}}_\ell$ -representations of  $GL_n(k_w)$  whose Satake parameters have congruent characteristic polynomials, but whose reductions mod  $\ell$  are unequal. (For instance, this is the case for the trivial  $\overline{\mathbb{Q}}_\ell$ -character and any integral unramified principal series  $\overline{\mathbb{Q}}_\ell$ -representation whose Satake parameter has a characteristic polynomial congruent to that of the Satake parameter of the trivial  $\overline{\mathbb{Q}}_\ell$ -character.) However, this phenomenon does not appear for *generic* unramified representations.

**Remark 4.4.** — The reductions mod  $\ell$  of  $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_{\ell}$  and  $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_{\ell}$  won't be equal in general for  $w \in S$ . Here is an example. Start with a unitary group  $\mathbf{G}$  of rank 2 with respect to a totally imaginary quadratic extension  $l$  of a totally real number field  $k$ . Suppose that:

- the group  $\mathbf{G}(k_v)$  is compact for all Archimedean places  $v$ ,
- there is a finite place  $w$  of  $k$  above a prime number  $p \neq \ell$  such that  $\mathbf{G}(k_w) \simeq \mathrm{GL}_2(k_w)$  and  $q$ , the cardinality of the residue field of  $\mathcal{O}_w$ , has order 2 mod  $\ell$ .

Thanks to our assumption on  $q$ , the  $\overline{\mathbb{F}}_{\ell}$ -representation induced from the trivial  $\overline{\mathbb{F}}_{\ell}$ -character of a Borel subgroup of  $\mathrm{GL}_2(k_w)$  has length 3: its irreducible subquotients are the trivial character, the unramified character of order 2 and a cuspidal subquotient denoted  $\rho$  (see [24] Théorème 3, Corollaire 5). Let  $\pi$  be a cuspidal lift of  $\rho$  to  $\overline{\mathbb{Q}}_{\ell}$ , that is,  $\pi$  is an integral cuspidal  $\overline{\mathbb{Q}}_{\ell}$ -representation of  $\mathrm{GL}_2(k_w)$  such that  $\mathbf{r}_{\ell}(\pi) = \rho$ . (The existence of such a  $\pi$  is granted by [25] III.5.10.)

Now realize  $\pi$  as the local component at  $w$  of some automorphic representation  $\Pi_1$  of  $\mathbf{G}(\mathbb{A}_k)$  which is trivial at infinity, and whose local component at another place  $u \neq w$  where  $\mathbf{G}$  splits is a given cuspidal representation  $\eta$  of  $\mathrm{GL}_2(k_u)$ .

We now follow [16] Section 3. Let  $K_w$  be the maximal compact subgroup  $\mathrm{GL}_2(\mathcal{O}_w)$  and  $\mathbb{F}_q$  be the residue field of  $k_w$ . Let:

- $\kappa_1$  be the inflation to  $K_w$  of the cuspidal irreducible representation of  $\mathrm{GL}_2(\mathbb{F}_q)$  occurring in the parabolic induction of the trivial  $\overline{\mathbb{F}}_{\ell}$ -character of a Borel subgroup (thus the restriction of  $\pi$  to  $K_w$  contains  $\kappa_1$ ),
- $\kappa_2$  be the inflation to  $K_w$  of the Steinberg representation of  $\mathrm{GL}_2(\mathbb{F}_q)$ .

Since the reduction mod  $\ell$  of  $\kappa_2$  contains that of  $\kappa_1$ , we get an automorphic representation  $\Pi_2$  of  $\mathbf{G}(\mathbb{A}_k)$  such that:

- the representation  $\Pi_2$  is trivial at infinity,
- the representation  $\Pi_{2,u}$  is isomorphic to  $\eta$ ,
- the restriction of  $\Pi_{2,w}$  to  $K_w$  contains  $\kappa_2$  (thus  $\Pi_{2,w}$  has non-zero Iwahori fixed vectors),
- there is a finite set of places  $S$  of  $k$ , containing all Archimedean places and  $u, w$ , such that, for all  $v \notin S$ , the representations  $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_{\ell}$  and  $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_{\ell}$  are unramified and the characteristic polynomials of their Satake parameters have coefficients in  $\overline{\mathbb{Z}}_{\ell}$  and have the same reduction.

Using [12], we transfer  $\Pi_1$  and  $\Pi_2$  to algebraic regular, conjugate-selfdual, cuspidal automorphic representations  $\tilde{\Pi}_1$  and  $\tilde{\Pi}_2$  of  $\mathrm{GL}_2(\mathbb{A}_l)$ . Applying [16] Theorem 8.2, and as the local transfer at  $w$  is the identity since the group  $\mathbf{G}$  splits at  $w$ , we deduce that the representations  $\mathbf{r}_{\ell}(\pi) = \rho$  and  $\mathbf{r}_{\ell}(\Pi_{2,w})$  share a generic irreducible component. Since  $\mathbf{r}_{\ell}$  commutes to parabolic restriction (by [4] Proposition 6.7), proving that  $\mathbf{r}_{\ell}(\Pi_{2,w}) \neq \rho$  reduces to proving that  $\Pi_{2,w}$  is not cuspidal. But this follows from the fact that  $\Pi_{2,w}$  has non-zero Iwahori fixed vectors.

**4.2.** An instance of Conjecture 4.1 is provided by [16] Theorem 8.2. More generally, the results of [6, 20, 23] imply the conjecture in the case when  $k$  is a totally real or imaginary CM number field and  $\Pi_1, \Pi_2$  are algebraic regular, by passing to the Galois side and using a density argument. In that case, note that:

- Assumption 1 on central characters is unnecessary,
- the representations  $\Pi_{i,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_{\ell}$  are automatically integral for all finite  $v$  not dividing  $\ell$ .

More precisely, assume that  $k$  is a totally real or imaginary CM number field and let  $\Pi_1, \Pi_2$  be algebraic regular cuspidal automorphic representation of  $GL_n(\mathbb{A})$ . Assume that there exists a finite set  $S$  of places of  $k$ , containing all Archimedean places and all finite places dividing  $\ell$ , such that, for all  $v \notin S$ , one has:

- (1) the local components  $\Pi_{1,v}, \Pi_{2,v}$  are unramified,
- (2) the characteristic polynomials of the conjugacy classes of semisimple elements in  $GL_n(\overline{\mathbb{Q}}_\ell)$  associated with  $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  and  $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  have coefficients in  $\overline{\mathbb{Z}}_\ell$  and are congruent mod  $\mathfrak{m}_\ell$ .

Associated with  $\Pi_i$  in [6] and [20], there is a continuous  $\ell$ -adic Galois representation

$$\Sigma_i : \text{Gal}(\overline{\mathbb{Q}}/k) \rightarrow GL_n(\overline{\mathbb{Q}}_\ell)$$

(depending on  $\iota : \mathbb{C} \rightarrow \overline{\mathbb{Q}}_\ell$ ) for  $i = 1, 2$ . For any finite place  $v$  of  $k$  not dividing  $\ell$ , fix a decomposition subgroup  $\Gamma_v$  of  $\text{Gal}(\overline{\mathbb{Q}}/k)$ . The Weil-Deligne representation associated with  $\Sigma_i|_{\Gamma_v}$  is made of a smooth  $\ell$ -adic representation  $\rho_{i,v}$  together with a nilpotent operator on the space of  $\rho_{i,v}$ . On the other hand, the Weil-Deligne representation associated with  $\Pi_{i,v} \otimes |\det|_v^{(1-n)/2}$  by the local Langlands correspondence is made of a semisimple smooth complex representation  $\sigma_{i,v}$  together with a nilpotent operator on the space of  $\sigma_{i,v}$ . By [23], for any finite place  $v$  of  $k$  not dividing  $\ell$ , one has

$$\rho_{i,v}^{\text{ss}} \simeq \sigma_{i,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$$

(where  $\rho_{i,v}^{\text{ss}}$  stand for the semisimplification of  $\rho_{i,v}$ ). Arguing as in [16] 8.2, we deduce that, for any finite place  $v$  of  $k$  not dividing  $\ell$ , the representations  $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  and  $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  are integral, their reductions mod  $\ell$  share a generic irreducible component, which occurs with multiplicity 1.

**4.3.** *From now on, and until the end of this section,  $k$  is a function field of characteristic  $p$ . We are going to prove Conjecture 4.1 in this case. We will actually prove a stronger result.*

**Theorem 4.5.** — *Let  $\Pi_1, \Pi_2$  be cuspidal automorphic representations of  $GL_n(\mathbb{A})$ . Let  $\iota$  be a field isomorphism from  $\mathbb{C}$  to  $\overline{\mathbb{Q}}_\ell$  for some prime number  $\ell$  different from  $p$ . Suppose that there is a finite set  $S$  of places of  $k$  such that, for all  $v \notin S$ , one has:*

- (1) *the local components  $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  and  $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  are unramified,*
- (2) *the characteristic polynomials of their Satake parameters belong to  $\overline{\mathbb{Z}}_\ell[X]$  and have the same reduction mod  $\mathfrak{m}_\ell$  in  $\overline{\mathbb{F}}_\ell[X]$ .*

*Let  $w$  be a finite place. Then:*

- *the representations  $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  and  $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  are integral*
- *the reductions mod  $\ell$  of these representations have a common generic irreducible component,*
- *and such a generic component is unique and occurs with multiplicity 1.*

**Remark 4.6.** — Note that, by taking a bigger  $S$ , we may (and will) assume that the character  $\psi_v$  is trivial on  $\mathcal{O}_v$  but not on  $\mathfrak{p}_v^{-1}$  for all  $v \notin S$ .

First, let us prove that, under the assumptions of Theorem 4.5, for all  $v$ , the central characters of  $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  and  $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  take values in  $\overline{\mathbb{Z}}_\ell^\times$  and are congruent mod  $\mathfrak{m}_\ell$ .

**Lemma 4.7.** — *Let  $\chi$  be an automorphic character of  $\mathbb{A}^\times/k^\times$  and  $U$  be a subgroup of  $\mathbb{C}^\times$ . Assume that there is a finite set  $S$  of places of  $k$  such that, for all  $v \notin S$ , the local component  $\chi_v$  is unramified and takes values in  $U$ . Then, for all  $v$ , the character  $\chi_v$  takes values in  $U$ .*

*Proof.* — If  $S$  is empty, there is nothing to prove. We thus assume that there is a place  $w \in S$ . Let  $x \in k_w^\times$ . Define an idèle  $x' \in \mathbb{A}^\times$  by

$$x'_v = \begin{cases} x & \text{if } v = w, \\ 1 & \text{otherwise.} \end{cases}$$

The weak approximation theorem implies that there is a  $y \in k^\times$  such that  $y \in \text{Ker}(\chi_v)$  if  $v \in S$  and  $v \neq w$ , and  $yx \in \text{Ker}(\chi_w)$ . We have

$$\chi_w(x) = \chi(x') = \chi(yx') = \chi_w(xy) \cdot \prod_{\substack{v \in S \\ v \neq w}} \chi_v(y) \cdot \prod_{v \notin S} \chi_v(y).$$

Thanks to the conditions given by the weak approximation theorem, this is equal to the product of  $\chi_v(y)$  for all  $v \notin S$ . (Note that this is a product of finitely many terms, since  $y$  is a unit in the ring of integers of  $k_v$  for almost all  $v \notin S$ .) The result follows from the fact that, for such  $v$ , one has  $\chi_v(y) \in U$ .  $\square$

**Proposition 4.8.** — *Let  $\chi_1$  and  $\chi_2$  be automorphic characters of  $\mathbb{A}^\times/k^\times$ , and fix a field isomorphism  $\iota : \mathbb{C} \rightarrow \overline{\mathbb{Q}_\ell}$ . Assume there is a finite set  $S$  of places of  $k$  such that, for all  $v \notin S$ :*

- (1) *the characters  $\chi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}_\ell}$  and  $\chi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}_\ell}$  are unramified and take values in  $\overline{\mathbb{Z}_\ell}^\times$ ,*
- (2) *the reductions mod  $\ell$  of these characters are equal.*

*Then, for all places  $v$ , the characters  $\chi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}_\ell}$  and  $\chi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}_\ell}$  take values in  $\overline{\mathbb{Z}_\ell}^\times$  and are congruent mod  $\mathfrak{m}_\ell$ .*

*Proof.* — For Assertion 1 of the proposition, apply Lemma 4.7 to  $\chi_i$  and  $U = \iota^{-1}(A^\times)$ . For Assertion 2, apply Lemma 4.7 to  $\chi = \chi_1\chi_2^{-1}$  and  $U = 1 + \iota^{-1}(\mathfrak{m}_\ell)$ .  $\square$

**Remark 4.9.** — Note that Theorem 4.5 follows from [13] Théorème VI.9 by a global argument in the spirit of §4.2 (see also [9] IV.1.6).

**4.4.** The remainder of this section is devoted to the proof of Theorem 4.5. By Remark 4.2, we may and will assume that  $w \in S$ .

Let  $A$  denote the image of  $\overline{\mathbb{Z}_\ell}$  by  $\iota^{-1}$  and  $\mathfrak{m}$  denote the image of  $\mathfrak{m}_\ell$  by  $\iota^{-1}$ . Thus  $A$  contains the complex  $p$ th roots of unity and the character  $\psi$  of Paragraph 3.1 takes values in  $A^\times$ . Notice that  $A$  and  $\mathfrak{m}$  are sub- $\mathbb{Z}[\mu_p]$ -modules of  $\mathbb{C}$ .

For any place  $v$  of  $k$  and  $i \in \{1, 2\}$ , let  $W_{i,v}$  be a function in the Whittaker model  $\mathcal{W}(\Pi_{i,v}, \psi_v)$  satisfying the conditions:

- if  $v \notin S$ , then  $W_{i,v}$  is the unique  $\text{GL}_n(\mathcal{O}_v)$ -invariant function such that  $W_{i,v}(1) = 1$  (see Paragraph 2.5),
- if  $v \in S$ , we fix an arbitrary  $A$ -valued function  $f_v \in \text{ind}_{N_v}^P(\psi_v)$  and let  $W_{i,v} \in \mathcal{W}(\Pi_{i,v}, \psi_v)$  be the unique function extending  $f_v$  to  $G_v$  (see Paragraph 2.2),
- for all  $v \in S$  such that  $v \neq w$ , we further assume that  $f_v(1) = 1$ .

For  $i \in \{1, 2\}$ , we consider the global Whittaker function

$$W_i = \bigotimes_v W_{i,v} \in \mathcal{W}(\Pi_i, \psi).$$

For  $x \in P(\mathbb{A})$ , we thus have

$$W_i(x) = \prod_{v \in S} f_v(x_v) \cdot \prod_{v \notin S} W_{i,v}(x_v).$$

It follows from Proposition 2.4 that  $W_1$  and  $W_2$  take values in  $A$  and  $W_1 - W_2$  takes values in  $\mathfrak{m}$  on  $P(\mathbb{A})$ . Let  $\varphi_i \in \Pi_i$  be the automorphic form corresponding to  $W_i$  via (3.4), that is:

$$\varphi_i(g) = \sum_{\gamma \in N_{n-1}(k) \backslash GL_{n-1}(k)} W_i \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

for all  $g \in GL_n(\mathbb{A})$ . By Theorem 3.1, the functions  $\varphi_1$  and  $\varphi_2$  take values in  $A$  and  $\varphi_1 - \varphi_2$  takes values in  $\mathfrak{m}$  on  $P(\mathbb{A})$ .

Thanks to Proposition 4.8, the central characters of  $\Pi_{1,v}$  and  $\Pi_{2,v}$  take values in  $A^\times$  and are congruent mod  $\mathfrak{m}$  for all  $v$ . It follows that  $\varphi_1$  and  $\varphi_2$  take values in  $A$  and  $\varphi_1 - \varphi_2$  takes values in  $\mathfrak{m}$  on  $Z(\mathbb{A})P(\mathbb{A})$ , where  $Z$  is the centre of  $GL_n$ . Applying Theorem 3.7, we deduce that  $W_1$  and  $W_2$  take values in  $A$  and  $W_1 - W_2$  takes values in  $\mathfrak{m}$  on  $GL_n(\mathbb{A})$ .

Now let us consider the place  $w$ . For  $i = 1, 2$  and  $g \in G_w \subseteq G(\mathbb{A})$ , one has:

$$\begin{aligned} W_i(g) &= \prod_v W_{i,v}(g_v) \\ &= W_{i,w}(g) \cdot \prod_{v \neq w} W_{i,v}(1) \\ &= W_{i,w}(g). \end{aligned}$$

It follows that  $W_{1,w}$  and  $W_{2,w}$  take values in  $A$ , and that  $W_{1,w} - W_{2,w}$  takes values in  $\mathfrak{m}$  on  $G_w$ . We thus proved that, given any  $A$ -valued function  $f_w \in \text{ind}_{N_w}^{P_w}(\psi_w)$ , the functions  $W_{1,w}$  and  $W_{2,w}$  extending  $f_w$  are  $A$ -valued. Proposition 2.5 thus implies that  $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  and  $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$  are integral. Now assume further that  $f_w(1) = 1$ . Then Theorem 4.5 follows from Proposition 2.1.

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NADIR MATRINGE, Institute of Mathematical Sciences, NYU Shanghai, 3663 Zhongshan Road North Shanghai, 200062, China & Institut de Mathématiques de Jussieu-Paris Rive Gauche, Université Paris Cité, 75205, Paris, France • *E-mail* : [nrm6864@nyu.edu](mailto:nrm6864@nyu.edu), [matringe@img-prg.fr](mailto:matringe@img-prg.fr)

ALBERTO MÍNGUEZ, University of Vienna, Fakultät für Mathematik, Oskar-Morgenstern-Platz 1, 1090 Wien  
*E-mail* : [alberto.minguez@univie.ac.at](mailto:alberto.minguez@univie.ac.at)

VINCENT SÉCHERRE, Laboratoire de Mathématiques de Versailles, UVSQ, CNRS, Université Paris-Saclay, 78035 Versailles, France • *E-mail* : [vincent.secherre@uvsq.fr](mailto:vincent.secherre@uvsq.fr)