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ON MODULAR RIGIDITY FOR GL_n

by

Nadir Matringe, Alberto Mínguez & Vincent Sécherre

Abstract. — Let k be a global field and \mathbb{A}_k be its ring of adèles. Let ℓ be a prime number and fix a field isomorphism from \mathbb{C} to $\overline{\mathbb{Q}}_\ell$. Let Π_1, Π_2 be cuspidal automorphic representations of $GL_n(\mathbb{A}_k)$ for some integer $n \geq 1$. In this paper, we study the following question: assuming that there is a finite set S of places of k containing all Archimedean places and all finite places above ℓ such that, for all $v \notin S$, the local components $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are unramified and their Satake parameters are congruent mod ℓ , are the local components $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ integral, and do their reductions mod ℓ share an irreducible factor for all non-Archimedean places w not dividing ℓ ? We show that, under certain conditions on Π_1, Π_2 , the answer is yes. We also give a simple proof when k is a function field.

Keywords and Phrases: Automorphic forms, Congruences mod ℓ , Satake parameters, Whittaker models, Automorphic representations

1. Introduction

1.1. Let k be a number field and \mathbb{A}_k be its ring of adèles. Let Π_1 and Π_2 be cuspidal automorphic representations of $GL_n(\mathbb{A}_k)$ for some integer $n \geq 1$. The rigidity (or strong multiplicity 1) theorem asserts that, if there is a finite set S of places of k containing all Archimedean places such that, for all $v \notin S$, the local components $\Pi_{1,v}$ and $\Pi_{2,v}$ are unramified and have the same Satake parameter, then Π_1 and Π_2 are isomorphic ([17, 3, 10, 11]). A similar result holds over function fields.

1.2. Now fix a field isomorphism ι from \mathbb{C} to an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field of ℓ -adic numbers for some prime number ℓ , and consider the collections of irreducible smooth $\overline{\mathbb{Q}}_\ell$ -representations of $GL_n(k_v)$ defined by

$$(1.1) \quad \pi_{i,v} = \Pi_{i,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell, \quad i \in \{1, 2\},$$

where the tensor product is taken with respect to ι , v runs over all finite places of k and k_v is the completion of k at v . As Π_1 and Π_2 are cuspidal, these representations are generic (see §2.2).

Suppose that there exists a finite set S of places of k containing all Archimedean places and all finite places above ℓ such that, for all $v \notin S$, the following are satisfied:

- (1) the representations $\pi_{1,v}$ and $\pi_{2,v}$ are unramified representations of $GL_n(k_v)$,

(2) the Satake parameters $\sigma_{1,v}$ and $\sigma_{2,v}$ of these unramified representations, considered as conjugacy classes of semisimple elements of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, have their characteristic polynomials $P_{1,v}(X)$ and $P_{2,v}(X)$ in $\overline{\mathbb{Z}}_\ell[X]$, where $\overline{\mathbb{Z}}_\ell$ is the ring of integers of $\overline{\mathbb{Q}}_\ell$,

(3) the reductions of $P_{1,v}(X)$ and $P_{2,v}(X)$ in $\overline{\mathbb{F}}_\ell[X]$ are equal, $\overline{\mathbb{F}}_\ell$ being the residue field of $\overline{\mathbb{Z}}_\ell$.

Assumption 2 is equivalent to saying that the unramified representations $\pi_{1,v}$ and $\pi_{2,v}$ are integral, that is, their $\overline{\mathbb{Q}}_\ell$ -vector spaces contain $\mathrm{GL}_n(k_v)$ -stable $\overline{\mathbb{Z}}_\ell$ -lattices (see §2.4). One can then consider their reductions mod ℓ , denoted $\mathbf{r}_\ell(\pi_{1,v})$ and $\mathbf{r}_\ell(\pi_{2,v})$, which are finite length, semisimple smooth $\overline{\mathbb{F}}_\ell$ -representations of $\mathrm{GL}_n(k_v)$ (see Section 2 for a precise definition of reduction mod ℓ). Assumption 3 is then equivalent to saying that the representations $\mathbf{r}_\ell(\pi_{1,v})$ and $\mathbf{r}_\ell(\pi_{2,v})$ are equal (see Remarks 4.2 and 4.3).

Now let w be a finite place of k not dividing ℓ . Our first question is

Question 1.1. — *Are the irreducible representations $\pi_{1,w}$ and $\pi_{2,w}$ integral?*

Assume that this is the case. One can then form $\mathbf{r}_\ell(\pi_{1,w})$ and $\mathbf{r}_\ell(\pi_{2,w})$. These representations may not be equal (see Remark 4.4 for an example), but one may address the following question.

Question 1.2. — *Do $\mathbf{r}_\ell(\pi_{1,w})$ and $\mathbf{r}_\ell(\pi_{2,w})$ have an irreducible component in common?*

If k is a totally real (respectively, CM) number field, and if Π_1, Π_2 are algebraic regular, self-dual (respectively, conjugate-selfdual) cuspidal automorphic representations, then [16] Theorem 8.2 says that the answers to Questions 1.1 and 1.2 are yes. More precisely:

- the representations $\pi_{1,w}$ and $\pi_{2,w}$ are integral for all finite places w of k not dividing ℓ ,
- their reductions mod ℓ have a unique generic irreducible component in common,
- this unique common generic irreducible component occurs with multiplicity 1.

Such a result, which can be thought of as a modular rigidity theorem, has been used in [16] in order to study the behavior of local transfer for cuspidal $\overline{\mathbb{Q}}_\ell$ -representations of quasi-split classical groups with respect to congruences mod ℓ .

More generally, thanks to the results of [6, 20, 23], one can make the argument of the proof of [16] Theorem 8.2 work with no duality assumption on Π_1 and Π_2 : if k is a totally real or CM number field, and if Π_1, Π_2 are algebraic regular, cuspidal automorphic representations, the answers to Questions 1.1 and 1.2 are still yes; more precisely, the three properties above still hold. (See §4.2 below for a detailed argument, which relies on the existence of a correspondence from algebraic regular cuspidal automorphic representation to Galois representations with local-global compatibility at all finite places not dividing ℓ .)

It is natural to ask whether the ‘totally real or CM’ assumption on k , or the ‘algebraic regular’ assumption on the representations Π_1 and Π_2 , or the cuspidality assumption, can be removed. We will not investigate these questions in the present article.

It is also natural to seek an elementary, purely automorphic proof of such a modular rigidity theorem, avoiding the use of Galois representations and local-global compatibility theorems. We will study this question in the case of function fields, which is easier since there are no Archimedean places.

1.3. We now assume that k is a function field of characteristic p , with ring of adèles \mathbb{A}_k . Recall that we have fixed a field isomorphism ι from \mathbb{C} to $\overline{\mathbb{Q}}_\ell$ for some prime number ℓ which we assume to be different from p . In this article, we prove the following theorem (see Theorem 4.5).

Theorem 1.3. — *Let Π_1 and Π_2 be cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_k)$. Associated with them, there are the representations $\pi_{i,v}$ defined by (1.1). Suppose that there exists a finite set S of places of k such that, for all $v \notin S$, one has:*

- (1) *the representations $\pi_{1,v}$ and $\pi_{2,v}$ are unramified,*
- (2) *the characteristic polynomials of their Satake parameters are in $\overline{\mathbb{Z}}_\ell[X]$ and have the same reduction in $\overline{\mathbb{F}}_\ell[X]$.*

Let w be a place of k . Then

- *the representations $\pi_{1,w}$ and $\pi_{2,w}$ are integral,*
- *their reductions mod ℓ share a generic irreducible component,*
- *and such a generic component is unique and occurs with multiplicity 1 in both reductions.*

This theorem can be easily deduced from L. Lafforgue’s global Langlands correspondence [13] (see Remark 4.9). Our purpose is to give a simple proof of Theorem 1.3 which does not rely on the Langlands correspondence for function fields. Our argument, inspired from Piatetski-Shapiro’s proof of the classical rigidity theorem [17, 3] is described below. We currently do not know how to extend our argument to number fields.

1.4. Before explaining the proof of Theorem 1.3, we introduce our main local ingredients. Let F be a non-Archimedean locally compact field of residue characteristic p (and characteristic 0 or p). Fix a non-trivial smooth $\overline{\mathbb{Q}}_\ell$ -character ϑ of F .

Proposition 1.4. — *Let π_1 and π_2 be integral generic irreducible $\overline{\mathbb{Q}}_\ell$ -representations of $\mathrm{GL}_n(F)$. Suppose that there are functions W_1 and W_2 in the Whittaker models of π_1 and π_2 with respect to ϑ satisfying the following conditions:*

- (1) *W_1 and W_2 are $\overline{\mathbb{Z}}_\ell$ -valued and $W_1(1) = W_2(1) = 1$,*
- (2) *the reductions of $W_1(g)$ and $W_2(g)$ in $\overline{\mathbb{F}}_\ell$ are equal for all $g \in G$.*

Then $\mathbf{r}_\ell(\pi_1)$ and $\mathbf{r}_\ell(\pi_2)$ share a generic irreducible component, such a generic irreducible component is unique and it occurs with multiplicity 1.

Let P be the mirabolic subgroup of GL_n , made of all matrices with last row $(0 \dots 0 \ 1)$, and N be its unipotent radical. We have the following remarkable integrality criterion (see Proposition 2.5), which follows from [8] and [14].

Proposition 1.5. — *Let π be a generic irreducible $\overline{\mathbb{Q}}_\ell$ -representation of $\mathrm{GL}_n(F)$. The following assertions are equivalent.*

- (1) *The representation π is integral.*
- (2) *Given any function in the Whittaker model of π with respect to ϑ whose restriction to $P(F)$ is compactly supported mod $N(F)$, this function is $\overline{\mathbb{Z}}_\ell$ -valued on $\mathrm{GL}_n(F)$ if and only if it is $\overline{\mathbb{Z}}_\ell$ -valued on $P(F)$.*

1.5. We now introduce our main global ingredients. Fix a continuous unitary complex character ψ of \mathbb{A}_k , and consider a cuspidal automorphic form φ on $\mathrm{GL}_n(\mathbb{A}_k)$. Associated with it by (3.3), with respect to the choice of ψ , there is a Whittaker function W on $\mathrm{GL}_n(\mathbb{A}_k)$. Note that ψ is valued in the group μ_p of complex p th roots of 1. Let Z be the centre of GL_n .

Given any sub- $\mathbb{Z}[\mu_p]$ -module I of \mathbb{C} , we prove in Section 3 that:

- if W takes values in I on $P(\mathbb{A}_k)$, then φ takes values in I on $P(\mathbb{A}_k)$ (Theorem 3.1),
- if φ takes values in I on $Z(\mathbb{A}_k)P(\mathbb{A}_k)$, then W takes values in I on $\mathrm{GL}_n(\mathbb{A}_k)$ (Theorem 3.7).

1.6. We now consider two cuspidal automorphic representations Π_1, Π_2 of $\mathrm{GL}_n(\mathbb{A}_k)$ as in Theorem 1.3. Let A be the local ring $\iota^{-1}(\overline{\mathbb{Z}}_\ell)$ and \mathfrak{m} be its maximal ideal. Fix a place $w \in S$.

We first observe that the central characters of Π_1 and Π_2 are A -valued and congruent mod \mathfrak{m} (see Proposition 4.8), thanks to the information we have at all places $v \notin S$.

For each place v of k , let ψ_v be the local component of ψ at v . It is a smooth character of k_v , which we may assume, for all $v \notin S$, to be trivial on the ring of integers of k_v but not on the inverse of its maximal ideal.

Let $W_{i,v}$ be any function in the Whittaker model of $\Pi_{i,v}$ with respect to ψ_v such that:

- (1) if $v \notin S$, then $W_{i,v}$ is $\mathrm{GL}_n(\mathcal{O}_v)$ -invariant (see §2.5),
- (2) if $v \in S$, then $W_{1,v}$ and $W_{2,v}$ coincide on $P(k_v)$, and their restriction to $P(k_v)$ is compactly supported mod $N(k_v)$ and A -valued (see §2.2),
- (3) and $W_{1,v}(1) = W_{2,v}(1) = 1$ for all $v \neq w$.

The tensor product of the $W_{i,v}$ is a function W_i in the Whittaker model of Π_i with respect to ψ . First, it follows from the Shintani formula (see [21, 3] and Proposition 2.4) that, for all $v \notin S$:

- the functions $W_{1,v}$ and $W_{2,v}$ are A -valued,
- the difference $W_{1,v} - W_{2,v}$ is \mathfrak{m} -valued.

The functions W_1 and W_2 are thus A -valued and $W_1 - W_2$ is \mathfrak{m} -valued on $Z(\mathbb{A}_k)P(\mathbb{A}_k)$.

Since A and \mathfrak{m} are sub- $\mathbb{Z}[\mu_p]$ -modules of \mathbb{C} , we may apply the result of Paragraph 1.5, from which we deduce that the functions W_1 and W_2 are A -valued and $W_1 - W_2$ is \mathfrak{m} -valued on the whole of $\mathrm{GL}_n(\mathbb{A}_k)$. Consequently, thanks to Assumption (3) above, we get:

- (\star) the functions $W_{1,w}$ and $W_{2,w}$ are A -valued,
- ($\star\star$) the difference $W_{1,w} - W_{2,w}$ is \mathfrak{m} -valued.

Starting with any function in the Whittaker model of $\pi_{i,w}$ with respect to ψ_w whose restriction to P_w is compactly supported mod N_w and $\overline{\mathbb{Z}}_\ell$ -valued, we thus proved that this function is $\overline{\mathbb{Z}}_\ell$ -valued (see (\star) above). Applying Proposition 1.5, we deduce that $\pi_{1,w}$ and $\pi_{2,w}$ are integral, proving the first assertion of Theorem 1.3.

Now assume that $W_{1,w}$ and $W_{2,w}$ satisfy the additional condition $W_{1,w}(1) = W_{2,w}(1) = 1$. The remaining two assertions of Theorem 1.3 then follow from (\star) and ($\star\star$) by Proposition 1.4.

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2. Local considerations

In this section, F denotes a locally compact non-Archimedean field of residue characteristic p and $n \geq 1$ is a positive integer. We write \mathcal{O}_F for the ring of integers of F and \mathfrak{p}_F for its maximal ideal. We also write G for the locally profinite group $GL_n(F)$.

Let ℓ be a prime number different from p . We write $\overline{\mathbb{Q}}_\ell$ for an algebraic closure of the field of ℓ -adic integers, $\overline{\mathbb{Z}}_\ell$ for its ring of integers and $\overline{\mathbb{F}}_\ell$ for its residue field.

Let $\psi : F \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be a non-trivial smooth character. It defines a non-degenerate character

$$x \mapsto \psi(x_{1,2} + \cdots + x_{n-1,n})$$

of N , the subgroup of upper triangular unipotent matrices of G , still denoted ψ . Note that this character takes values in $\overline{\mathbb{Z}}_\ell$, and even more precisely in the group of roots of unity in $\overline{\mathbb{Q}}_\ell$ whose order is a power of p .

Let P denote the mirabolic subgroup of G , made of all matrices whose last row is $(0 \dots 0 \ 1)$.

The representations we will consider will be smooth representations of locally profinite groups with coefficients in $\mathbb{Z}[1/p]$ -algebras.

2.1. A $\overline{\mathbb{Q}}_\ell$ -representation of finite length π of G is said to be integral if its vector space V contains a G -stable $\overline{\mathbb{Z}}_\ell$ -lattice. (A G -stable $\overline{\mathbb{Z}}_\ell$ -lattice is a G -stable free $\overline{\mathbb{Z}}_\ell$ -module generated by a basis of V or, equivalently, an admissible $\overline{\mathbb{Z}}_\ell[G]$ -module containing a basis of V .)

If this is the case, and if L is such a G -stable $\overline{\mathbb{Z}}_\ell$ -lattice, the representation of G on $L \otimes \overline{\mathbb{F}}_\ell$ is smooth and has finite length, and its semisimplification does not depend on the choice of L (see [26] Theorem 1). This semisimplified $\overline{\mathbb{F}}_\ell$ -representation is called the reduction mod ℓ of π and is denoted $\mathbf{r}_\ell(\pi)$.

An irreducible $\overline{\mathbb{Q}}_\ell$ -representation π which embeds in the parabolic induction of some cuspidal irreducible representation ρ of some Levi subgroup M of G is integral if and only if the central character of ρ is $\overline{\mathbb{Z}}_\ell$ -valued (see [25] II.4.12, II.4.14 and [4] Proposition 6.7).

2.2. In this paragraph, π is a generic irreducible $\overline{\mathbb{Q}}_\ell$ -representation of G , that is, its vector space V carries a non-zero $\overline{\mathbb{Q}}_\ell$ -linear form Λ such that $\Lambda(\pi(u)v) = \psi(u)\Lambda(v)$ for all $u \in N$, $v \in V$. Let

$$(2.1) \quad \mathcal{W}(\pi, \psi) \subseteq \text{Ind}_N^G(\psi)$$

denote its Whittaker model with respect to ψ , where Ind_N^G denotes smooth induction from N to G . Let $\mathcal{K}(\pi, \psi)$ denote the Kirillov model of π , that is, the space of smooth $\overline{\mathbb{Q}}_\ell$ -valued functions

on P which extend to a function in $\mathcal{W}(\pi, \psi)$. By Kirillov's theory, restriction from G to P induces a P -equivariant isomorphism from $\mathcal{W}(\pi, \psi)$ to $\mathcal{K}(\pi, \psi)$, and one has the containments

$$\mathrm{ind}_N^P(\psi) \subseteq \mathcal{K}(\pi, \psi) \subseteq \mathrm{Ind}_N^P(\psi)$$

where ind_N^P denotes compact induction from N to P (see [1]).

2.3. In this paragraph, π_1 and π_2 are integral generic irreducible $\overline{\mathbb{Q}}_\ell$ -representations of G . Let \mathfrak{m}_ℓ denote the maximal ideal of $\overline{\mathbb{Z}}_\ell$.

Proposition 2.1. — *Suppose there are Whittaker functions $W_1 \in \mathcal{W}(\pi_1, \psi)$ and $W_2 \in \mathcal{W}(\pi_2, \psi)$ with values in $\overline{\mathbb{Z}}_\ell$ such that:*

- (1) $W_1(1) = W_2(1) = 1$,
- (2) $W_1(g)$ and $W_2(g)$ are congruent mod \mathfrak{m}_ℓ for all $g \in G$.

Then the reductions mod ℓ of π_1 and π_2 share a generic irreducible component, such a generic irreducible component is unique and it occurs with multiplicity 1.

Proof. — We will need the following result.

Lemma 2.2. — *Let π be an integral generic irreducible $\overline{\mathbb{Q}}_\ell$ -representation of G . Then its reduction mod ℓ contains a unique irreducible generic factor, occurring with multiplicity 1.*

Proof. — The existence of an irreducible generic factor follows from the fact that any non-zero linear form in $\mathrm{Hom}_N(\pi, \psi)$ is non-zero on any G -stable $\overline{\mathbb{Z}}_\ell$ -lattice of π . Its uniqueness follows for instance from [15] Proposition 8.4 applied to the representation parabolically induced from the cuspidal support of π . \square

Let $i \in \{1, 2\}$. By [26] Theorem 2, the $\overline{\mathbb{Z}}_\ell$ -module L_i made of all $\overline{\mathbb{Z}}_\ell$ -valued Whittaker functions in $\mathcal{W}(\pi_i, \psi)$ is a G -stable $\overline{\mathbb{Z}}_\ell$ -lattice. Let Λ_i be the $\overline{\mathbb{Z}}_\ell[G]$ -module generated by W_i in $\mathcal{W}(\pi_i, \psi)$. It contains a $\overline{\mathbb{Q}}_\ell$ -basis of $\mathcal{W}(\pi_i, \psi)$ since π_i is irreducible, and it is contained in L_i . It follows that it is a G -stable $\overline{\mathbb{Z}}_\ell$ -lattice in $\mathcal{W}(\pi_i, \psi)$. Let M_i denote the submodule of L_i made of all \mathfrak{m}_ℓ -valued functions. The containment of $\mathfrak{m}_\ell \Lambda_i$ in $\Lambda_i \cap M_i$ implies that we have morphisms:

$$\Lambda_i \otimes \overline{\mathbb{F}}_\ell \rightarrow \Lambda_i / (\Lambda_i \cap M_i) \simeq (\Lambda_i + M_i) / M_i \subseteq L_i / M_i \rightarrow \mathrm{Ind}_N^G(\vartheta \otimes \overline{\mathbb{F}}_\ell)$$

where the left hand side morphism α_i is surjective, the right hand side morphism β_i is injective, and $\vartheta \otimes \overline{\mathbb{F}}_\ell$ denotes the $\overline{\mathbb{F}}_\ell$ -character of N obtained by reducing ϑ mod \mathfrak{m}_ℓ .

As W_1 and W_2 are congruent mod \mathfrak{m}_ℓ on G and take 1 to 1, the intersection:

$$(2.2) \quad \beta_1((\Lambda_1 + M_1) / M_1) \cap \beta_2((\Lambda_2 + M_2) / M_2)$$

is non-zero in $\mathrm{Ind}_N^G(\vartheta \otimes \overline{\mathbb{F}}_\ell)$ for it contains the function $\beta_1(W_1 \bmod M_1) = \beta_2(W_2 \bmod M_2)$ and the latter is non-zero. The socle of (2.2), denoted Σ , is made of generic irreducible $\overline{\mathbb{F}}_\ell$ -representations appearing in both the reductions mod ℓ of π_1 and π_2 . By Lemma 2.2, the reduction mod ℓ of π_i contains a unique irreducible generic factor ρ_i . The socle Σ is thus irreducible, reduced to ρ_i . It follows that ρ_1 and ρ_2 are isomorphic. \square

2.4. Let K be the maximal compact subgroup $GL_n(\mathcal{O}_F)$. In this paragraph, π is an unramified irreducible $\overline{\mathbb{Q}}_\ell$ -representation of G , that is, π has a non-zero K -fixed vector. It defines a conjugacy class of semisimple elements in $GL_n(\overline{\mathbb{Q}}_\ell)$, called its Satake parameter. The characteristic polynomial of this conjugacy class is denoted $\chi(\pi)$. This is a polynomial of degree n in $\overline{\mathbb{Q}}_\ell[X]$.

Lemma 2.3. — *The unramified representation π is integral if and only if the polynomial $\chi(\pi)$ has all its coefficients in $\overline{\mathbb{Z}}_\ell$.*

Proof. — Since $\overline{\mathbb{Z}}_\ell$ is integrally closed, $\chi(\pi)$ has all its coefficients in $\overline{\mathbb{Z}}_\ell$ if and only if its roots are in $\overline{\mathbb{Z}}_\ell$, that is, if and only if π is parabolically induced from an integral unramified character of the diagonal torus of $GL_n(F)$. The lemma then follows from Paragraph 2.1. \square

2.5. Now assume that π is a generic unramified irreducible $\overline{\mathbb{Q}}_\ell$ -representation of G , and that ψ is trivial on \mathcal{O}_F but not on \mathfrak{p}_F^{-1} . Its Whittaker model $\mathcal{W}(\pi, \psi)$ contains a unique Whittaker function W_π such that:

- (1) one has $W_\pi(gk) = W_\pi(g)$ for all $g \in G$ and $k \in K$,
- (2) and $W_\pi(1) = 1$.

Let us recall the Shintani–Casselman–Shalika formula [21, 3], which gives the values of W_π at diagonal elements in terms of the Satake parameter of π .

Fix a representative $(\mu_1, \dots, \mu_n) \in \overline{\mathbb{Q}}_\ell^{\times n}$ of the Satake parameter of π . If we write

$$\chi(\pi) = X^n + c_1(\pi)X^{n-1} + \dots + c_n(\pi) \in \overline{\mathbb{Q}}_\ell[X],$$

then

$$(-1)^r c_r(\pi) = \sum_{1 \leq i_1 < \dots < i_r \leq n} \mu_{i_1} \dots \mu_{i_r}$$

for all $r \in \{1, \dots, n\}$. Let q be the cardinality of the residue field of F . Fix a uniformizer $\varpi \in F$ and let Δ be the subgroup of G made of all diagonal matrices whose eigenvalues are integral powers of ϖ . The Iwasawa decomposition $G = N\Delta K$ shows that W_π is entirely determined by its restriction to Δ . Given $a \in \mathbb{Z}^n$, write ϖ^a for the diagonal matrix whose i th eigenvalue is ϖ^{a_i} .

One has the formula:

$$(2.3) \quad W_\pi(\varpi^a) = q^{\sum_{j=1}^n a_j(j-(n+1)/2)} \cdot \frac{\det((\mu_j^{a_l+n-l})_{j,l})}{\prod_{j < l} (\mu_j - \mu_l)} \quad \text{if } a_1 \geq \dots \geq a_n$$

and W_π vanishes at ϖ^a otherwise.

2.6. Formula (2.3) has the following application.

Proposition 2.4. — *Let π_1 and π_2 be integral generic unramified irreducible representations of G . Assume that the polynomials $\chi(\pi_1)$ and $\chi(\pi_2)$ have the same reduction mod \mathfrak{m}_ℓ in $\overline{\mathbb{F}}_\ell[X]$. Then the Whittaker functions W_{π_1} and W_{π_2} are $\overline{\mathbb{Z}}_\ell$ -valued on G and one has:*

$$(2.4) \quad W_{\pi_1}(g) \equiv W_{\pi_2}(g) \pmod{\mathfrak{m}_\ell}$$

for all $g \in G$.

Proof. — Notice that $\chi(\pi_1)$ and $\chi(\pi_2)$ have coefficients in $\overline{\mathbb{Z}}_\ell$ by Lemma 2.3, thus the reduction of $\chi(\pi_1)$ and $\chi(\pi_2)$ mod \mathfrak{m}_ℓ is well-defined.

It suffices to prove that W_{π_1} and W_{π_2} are $\overline{\mathbb{Z}}_\ell$ -valued on Δ and that the relation (2.4) is satisfied for all $g \in \Delta$. Fix a representative $\boldsymbol{\mu}_i = (\mu_{i,1}, \dots, \mu_{i,n}) \in \overline{\mathbb{Q}}_\ell^{\times n}$ of the Satake parameter of π_i for $i = 1, 2$. The scalars $\mu_{i,1}, \dots, \mu_{i,n}$ are the roots of $\chi(\pi_i)$. They are thus in $\overline{\mathbb{Z}}_\ell$. The fact that the polynomials $\chi(\pi_1)$ and $\chi(\pi_2)$ are congruent mod \mathfrak{m}_ℓ ensures that, up to reordering, we may assume that $\mu_{1,j}$ and $\mu_{2,j}$ are congruent mod \mathfrak{m}_ℓ for all $j = 1, \dots, n$. The lemma now follows from the Shintani formula (2.3). \square

2.7. In this paragraph, π is a generic irreducible $\overline{\mathbb{Q}}_\ell$ -representation of G . We have the following remarkable integrality criterion, based on [8] and [14].

Proposition 2.5. — *Let π be a generic irreducible $\overline{\mathbb{Q}}_\ell$ -representation of G . The following assertions are equivalent:*

- (1) *The representation π is integral.*
- (2) *Given any $\overline{\mathbb{Z}}_\ell$ -valued function $f \in \text{ind}_N^P(\psi)$, the Whittaker function in $\mathcal{W}(\pi, \psi)$ extending f is $\overline{\mathbb{Z}}_\ell$ -valued.*

Remark 2.6. — Assertion 2 can be restated as follows: given any function $W \in \mathcal{W}(\pi, \psi)$ whose restriction to P is compactly supported mod N , the function W is $\overline{\mathbb{Z}}_\ell$ -valued on G if and only if it is $\overline{\mathbb{Z}}_\ell$ -valued on P .

Proof. — That the first assertion implies the second one follows from [14] Corollary 4.3 (which gives an even stronger result: it says that a Whittaker function $W \in \mathcal{W}(\pi, \psi)$ is $\overline{\mathbb{Z}}_\ell$ -valued on G if and only if it is $\overline{\mathbb{Z}}_\ell$ -valued on P).

Let us prove that the second assertion implies the first one. We will use [8] Theorem 3.2, which is stated for Noetherian algebras over the ring \mathbb{W}_ℓ of Witt vectors of $\overline{\mathbb{F}}_\ell$. Let us explain how it applies to a generic $\overline{\mathbb{Q}}_\ell$ -representation π satisfying Assertion 2.

Let V be the $\overline{\mathbb{Q}}_\ell$ -vector space of π . By [25] II.4.9, there exists a finite extension E of $\mathbb{Q}_\ell^{\text{ur}}$, the maximal unramified extension of \mathbb{Q}_ℓ in $\overline{\mathbb{Q}}_\ell$, such that π is defined over E , that is, V contains a G -stable E -vector space V_E such that $V = V_E \otimes_E \overline{\mathbb{Q}}_\ell$. Let π_E denote the E -representation of G on V_E . If ψ_E denotes the character ψ considered as being valued in E , then π_E is generic with respect to ψ_E . Let K be the completion of E . Then $\pi_K = \pi_E \otimes_E K$ is generic with respect to the character $\psi_K = \psi_E \otimes_E K$. Since the complete discrete valuation ring \mathbb{W}_ℓ is isomorphic to the completion of the ring of integers of $\mathbb{Q}_\ell^{\text{ur}}$, the ring \mathcal{O} of integers of K is a Noetherian \mathbb{W}_ℓ -algebra. Let us show that π_K satisfies the analogue of Assertion 2 for the ring \mathcal{O} .

Lemma 2.7. — *Given any \mathcal{O} -valued function $f \in \text{ind}_N^P(\psi_K)$, there exists an \mathcal{O} -valued function in $\mathcal{W}(\pi_K, \psi_K)$ extending f .*

Proof. — This f can be written $a_1 f_1 + \dots + a_r f_r$ with $a_1, \dots, a_r \in \mathcal{O}$, and where the functions $f_1, \dots, f_r \in \text{ind}_N^G(\psi)$ are $\overline{\mathbb{Z}}_\ell$ -valued. By assumption on π , the function $W_i \in \mathcal{W}(\pi, \psi)$ extending f_i is $\overline{\mathbb{Z}}_\ell$ -valued. Thus $a_1 W_1 + \dots + a_r W_r$ is in $\mathcal{W}(\pi_K, \psi_K)$, it extends f and it is \mathcal{O} -valued. \square

Let us collect some results from [7] about the category $\mathbf{Rep}_{\mathbb{W}_\ell}(G)$ of all smooth \mathbb{W}_ℓ -representations of G . This category decomposes into a product of blocks indexed by inertial classes Ω of supercuspidal $\overline{\mathbb{F}}_\ell$ -representations of G . Associated with each block, there is its centre \mathfrak{z}_Ω , which is a finitely generated commutative \mathbb{W}_ℓ -algebra, and its *universal co-Whittaker* module \mathcal{W}_Ω , which is an admissible $\mathfrak{z}_\Omega[G]$ -module.

The representation π_K is absolutely irreducible and generic. It is thus a co-Whittaker $K[G]$ -module in the sense of [8] Definition 2.1. Also, by Schur's lemma, if Ω is the inertial class associated with it, the action of the centre \mathfrak{z}_Ω on π_K defines a morphism of \mathbb{W}_ℓ -algebras $\chi : \mathfrak{z}_\Omega \rightarrow K$. By [7] Theorem 6.3, the representation π_K is a quotient of $\mathcal{W}_\Omega \otimes_{\mathfrak{z}_\Omega} K$.

We now apply [8] Theorem 3.2 to π_K (with $A = K$ and $A' = \mathcal{O}$), which says that, thanks to Lemma 2.7, χ is valued in \mathcal{O} , which makes \mathcal{O} into a \mathfrak{z}_Ω -algebra. By [7] Lemma 6.4 (or more precisely its proof), the image L of $\mathcal{W}_\Omega \otimes_{\mathfrak{z}_\Omega} \mathcal{O}$ in π_K is an \mathcal{O} -torsion free co-Whittaker $\mathcal{O}[G]$ -module such that $L \otimes_{\mathcal{O}} K = \pi_K$. By [8] Definition 2.1, this $\mathcal{O}[G]$ -module L is admissible. The representation π_K is thus \mathcal{O} -integral.

Fix a parabolic subgroup Q of G with Levi subgroup M , and a cuspidal irreducible representation ρ of M such that π embeds in the parabolic induction $i_Q^G(\rho)$. We may and will choose E so that ρ is also defined over E : we thus have an E -representation ρ_E such that $\rho_E \otimes_E \overline{\mathbb{Q}}_\ell = \rho$ and π_E embeds in $i_Q^G(\rho_E)$. Thus π_K embeds in $i_Q^G(\rho_K)$, where $\rho_K = \rho_E \otimes_E K$ is cuspidal and absolutely irreducible. Note that the central character ω of ρ_K takes values in E .

Since L is admissible, [4] Proposition 6.7 implies that the Jacquet module $r_Q^G(L)$ is admissible, thus ρ_K is \mathcal{O} -integral. Its central character ω thus takes values in \mathcal{O}_E , thus the central character of ρ takes values in $\overline{\mathbb{Z}}_\ell$. It follows (see Paragraph 2.1) that π is integral, as expected. \square

3. Global considerations

3.1. Let k be a global field, that is, either a finite extension of \mathbb{Q} or the field of rational functions over a smooth irreducible projective curve X defined over a finite field of cardinality q . Let \mathbb{A} be the ring of adèles of k .

Given an integer $n \geq 2$, let $N = N_n$ be the subgroup of upper triangular unipotent matrices of GL_n and $P = P_n$ be its mirabolic subgroup, made of all matrices whose last row is $(0 \dots 0 \ 1)$. More generally, for $m \in \{0, \dots, n\}$, let $N_{m, n-m}$ denote the unipotent radical of the parabolic subgroup of GL_n generated by upper triangular matrices and the Levi subgroup $GL_m \times GL_{n-m}$.

Let $\psi : \mathbb{A} \rightarrow \mathbb{C}^\times$ be a non-trivial continuous character trivial on k . It defines in the usual way a non-degenerate character of $N(k) \backslash N(\mathbb{A})$, namely

$$u \mapsto \psi(u_{1,2} + \dots + u_{n-1,n})$$

for all $u \in N(\mathbb{A})$, which we still denote by ψ .

For any place v of k , let k_v denote the completion of k at v . If v is finite, we write \mathcal{O}_v for the ring of integers of k_v and \mathfrak{p}_v for its maximal ideal. The character ψ decomposes as

$$(3.1) \quad \psi = \bigotimes_v \psi_v$$

where ψ_v is a non-trivial continuous character of k_v , trivial on \mathcal{O}_v but not on \mathfrak{p}_v^{-1} for almost all finite v .

3.2. Let us fix a Haar measure du on $N(k)\backslash N(\mathbb{A})$. Given a cuspidal irreducible automorphic representation Π of $\mathrm{GL}_n(\mathbb{A})$, the linear form

$$(3.2) \quad \varphi \mapsto \int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1} \varphi(u) \, du$$

on Π is known to be well-defined and non-zero (see [3] Theorem 1.1 or (3.4) below).

Associated to $\varphi \in \Pi$, there is a Whittaker function W_φ defined by:

$$(3.3) \quad W_\varphi(g) = \int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1} \varphi(ug) \, du$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$. The map $\varphi \mapsto W_\varphi$ is a morphism from Π to its Whittaker model $\mathcal{W}(\Pi, \psi)$.

If we choose for du the Haar measure giving measure 1 to the compact group $N(k)\backslash N(\mathbb{A})$, one also has a converse expansion:

$$(3.4) \quad \varphi(g) = \sum_{\gamma \in N_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_\varphi \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$, with absolute and uniform convergence on compact subsets (see for instance [5] Theorem 13.5.4 or [3] Theorem 1.1).

3.3. From now on, assume that k is a function field of characteristic p . There is thus no Archimedean place. Let I be any sub- $\mathbb{Z}[\mu_p]$ -module of \mathbb{C} , where μ_p denotes the subgroup of p th roots of unity in \mathbb{C} . Note that ψ takes values in μ_p .

Theorem 3.1. — *Let $\phi : P_n(\mathbb{A}) \rightarrow \mathbb{C}$ be a smooth function such that:*

- (1) *one has $\phi(\gamma g) = \phi(g)$ for all $\gamma \in P_n(k)$ and all $g \in P_n(\mathbb{A})$,*
- (2) *the function ϕ is cuspidal in the sense that*

$$\int_{N_{m,n-m}(k)\backslash N_{m,n-m}(\mathbb{A})} \phi(ug) \, du = 0$$

for all $g \in P_n(\mathbb{A})$ and all $m \in \{1, \dots, n-1\}$,

- (3) *one has*

$$\int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1} \phi(ug) \, du \in I$$

for all $g \in P_n(\mathbb{A})$.

Then $\phi(g) \in I$ for all $g \in P_n(\mathbb{A})$.

We will prove this theorem by induction on $n \geq 2$. Given a function ϕ as in Theorem 3.1, it is useful to define a function W_ϕ on $P_n(\mathbb{A})$ by setting

$$W_\phi(g) = \int_{N(k) \backslash N(\mathbb{A})} \psi(u)^{-1} \phi(ug) \, du$$

for all $g \in P_n(\mathbb{A})$. We will often use the fact that, by Assumption 3, it takes values in I .

3.4. We first treat the case where $n = 2$. We will need the following lemma.

Lemma 3.2. — *For any smooth functions $f, g \in \mathcal{C}^\infty(\mathbb{A}/k, \mathbb{C})$, we have*

$$\int_{\mathbb{A}/k} f(x)g(x) \, dx = \sum_{\gamma \in k} \int_{\mathbb{A}/k} \psi^{-1}(\gamma x) f(x) \, dx \cdot \int_{\mathbb{A}/k} \psi(\gamma x) g(x) \, dx.$$

Proof. — Start with the Fourier expansion formula

$$f(y) = \sum_{\gamma \in k} \psi(\gamma y) \cdot \int_{\mathbb{A}/k} \psi^{-1}(\gamma x) f(x) \, dx$$

for $y \in \mathbb{A}/k$. Then multiply by $g(y)$ and integrate over \mathbb{A}/k . □

Let $f \in \mathcal{C}^\infty(\mathbb{A}/k, \mathbb{C})$. For any $g \in P_2(\mathbb{A})$, we thus get

$$(3.5) \quad \int_{\mathbb{A}/k} \phi \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) f(u) \, du = \sum_{\gamma \in k} \Phi(\gamma, g) F(\gamma)$$

where

$$\Phi(\gamma, g) = \int_{\mathbb{A}/k} \psi^{-1}(\gamma u) \phi \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) \, du \quad \text{and} \quad F(\gamma) = \int_{\mathbb{A}/k} \psi(\gamma u) f(u) \, du.$$

Therefore, we have $\Phi(0, g) = 0$ by cuspidality of ϕ and, if $\gamma \neq 0$, we have

$$\begin{aligned} \Phi(\gamma, g) &= \int_{\mathbb{A}/k} \psi^{-1}(u) \phi \left(\begin{pmatrix} 1 & \gamma^{-1}u \\ 0 & 1 \end{pmatrix} g \right) \, du \\ &= \int_{\mathbb{A}/k} \psi^{-1}(u) \phi \left(\begin{pmatrix} \gamma & u \\ 0 & 1 \end{pmatrix} g \right) \, du \\ &= W_\phi \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right) \end{aligned}$$

where the first equality follows from the fact that the module of γ is 1 by the product formula, and the second one follows from the fact that ϕ is $P_2(k)$ -invariant.

Now let $U = U(\phi, g)$ be a compact open subgroup of \mathbb{A}/k such that

$$\phi \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) = \phi(g) \quad \text{for all } u \in U.$$

Let f be the characteristic function of U . Thus

$$F(\gamma) = \int_U \psi(\gamma u) \, du.$$

On the one hand, we have

$$\int_{\mathbb{A}/k} \phi \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) f(u) \, du = \phi(g) \cdot |U|$$

(where $|U|$ is the volume of U with respect to du). On the other hand, we have

$$\int_{\mathbb{A}/k} \phi \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) f(u) \, du = \sum_{\gamma \in k^\times} W_\phi \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right) \cdot \int_U \psi(\gamma u) \, du,$$

which gives the identity

$$\phi(g) = \sum_{\gamma \in k^\times} W_\phi \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right) \cdot \int_U \psi(\gamma u) \frac{du}{|U|}.$$

As $W_\phi \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right) \in I$, for all $\gamma \in k^\times$, and

$$\int_U \psi(\gamma u) \frac{du}{|U|} = \begin{cases} 0 & \text{if } \psi \text{ is non-trivial on } \gamma U, \\ 1 & \text{otherwise,} \end{cases}$$

it only remains to prove that the sum over γ is finite, that is, there are only finitely many $\gamma \in k^\times$ such that $\gamma U \subseteq \text{Ker}(\psi)$. Assume that

$$U = \prod_{v \in S} \mathfrak{p}_v^{m_v} \times \prod_{v \notin S} \mathcal{O}_v$$

for some finite set S of places of k and some integers $m_v \in \mathbb{Z}$. Recall that we have the decomposition (3.1) of ψ . By taking a bigger S if necessary, we may (and will) assume that the character ψ_v is trivial on \mathcal{O}_v but not on \mathfrak{p}_v^{-1} for all $v \notin S$. Thus $\gamma U \subseteq \text{Ker}(\psi)$ if and only if $\gamma \mathcal{O}_v \subseteq \text{Ker}(\psi_v)$ for all $v \notin S$ and $\gamma \mathfrak{p}_v^{m_v} \subseteq \text{Ker}(\psi_v)$ for all $v \in S$. Equivalently, this means that γ belongs to the space of $f \in k$, considered as rational functions on the curve X defining the field k , such that

- f has no pole at $v \notin S$,
- f has a pole of order $\geq -m_v$ at $v \in S$.

The expected finiteness result now follows from the fact that these f form a finite dimensional vector space over \mathbb{F}_q (see for instance [22] Proposition 1.4.9).

3.5. We now assume that $n \geq 3$, and that Theorem 3.1 has been proved for $P_{n-1}(\mathbb{A})$.

We fix an arbitrary $g \in P_n(\mathbb{A})$ and define a function $\phi' = \phi'_g$ on $P_{n-1}(\mathbb{A})$ by setting

$$\phi'(h) = \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \phi \left(\begin{pmatrix} h & u \\ 0 & 1 \end{pmatrix} g \right) \, du$$

for all $h \in P_{n-1}(\mathbb{A})$, where $\eta = (0, \dots, 0, 1) \in k^{n-1}$ and $\alpha \cdot u = \alpha_1 u_1 + \dots + \alpha_{n-1} u_{n-1} \in (\mathbb{A}/k)^{n-1}$ for any $\alpha \in k^{n-1}$ and $u \in (\mathbb{A}/k)^{n-1}$. It has the following properties.

Lemma 3.3. — *The function ϕ' is cuspidal on $P_{n-1}(\mathbb{A})$, that is, one has*

$$\int_{N_{m,n-1-m}(k) \backslash N_{m,n-1-m}(\mathbb{A})} \phi'(vh) \, dv = 0$$

for all $h \in P_{n-1}(\mathbb{A})$ and all $m \in \{1, \dots, n-2\}$.

Proof. — Let us fix an $h \in P_{n-1}(\mathbb{A})$ and an $m \in \{1, \dots, n-2\}$. Then

$$(3.6) \quad \int_{N_{m,n-1-m}(k) \backslash N_{m,n-1-m}(\mathbb{A})} \phi'(vh) \, dv = \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \Omega_{g,h}(u) \, du$$

where

$$\Omega_{g,h}(u) = \int_{N_{m,n-1-m}(k) \backslash N_{m,n-1-m}(\mathbb{A})} \phi \left(\begin{pmatrix} vh & u \\ 0 & 1 \end{pmatrix} g \right) \, dv$$

and the right hand side of (3.6) is equal to

$$\int_{(\mathbb{A}/k)^{n-1-m}} \psi^{-1}(\eta \cdot u_2) \int_{(\mathbb{A}/k)^m} \Omega_{g,h} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \, du_1 \, du_2 = \int_{(\mathbb{A}/k)^{n-1-m}} \psi^{-1}(\eta \cdot u_2) \Lambda_{g,h}(u_2) \, du_2$$

where

$$\Lambda_{g,h}(u_2) = \int_{N_{m,n-m}(k) \backslash N_{m,n-m}(\mathbb{A})} \phi \left(w \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g \right) \, dw$$

and this quantity is equal to 0 thanks to the fact that ϕ is cuspidal. \square

Lemma 3.4. — *One has*

$$\phi'(\alpha h) = \phi'(h)$$

for all $\alpha \in P_{n-1}(k)$ and $h \in P_{n-1}(\mathbb{A})$.

Proof. — Let us fix an $\alpha \in P_{n-1}(k)$. Thanks to the fact that ϕ is $P_n(k)$ -invariant, one has

$$\begin{aligned} \phi'(\alpha h) &= \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \phi \left(\begin{pmatrix} \alpha h & u \\ 0 & 1 \end{pmatrix} g \right) \, du \\ &= \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \alpha \cdot u) \phi \left(\begin{pmatrix} h & u \\ 0 & 1 \end{pmatrix} g \right) \, du. \end{aligned}$$

Since $\alpha \in P_{n-1}(k)$, we get $\eta \alpha = \eta$, thus $\phi'(\alpha h) = \phi'(h)$. \square

Lemma 3.5. — *One has*

$$\int_{N_{n-1}(k) \backslash N_{n-1}(\mathbb{A})} \psi^{-1}(v) \phi'(vh) \, dv \in I$$

for all $h \in P_{n-1}(\mathbb{A})$.

Proof. — It suffices to notice that

$$\begin{aligned} \int_{N_{n-1}(k) \backslash N_{n-1}(\mathbb{A})} \psi^{-1}(v) \phi'(vh) \, dv &= \int_{N_n(k) \backslash N_n(\mathbb{A})} \psi^{-1}(w) \phi \left(w \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g \right) \, dw \\ &= W_\phi \left(\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g \right) \end{aligned}$$

which takes values in I for all $g \in P_n(\mathbb{A})$ and $h \in P_{n-1}(\mathbb{A})$. □

Applying now the inductive hypothesis to the function $\phi' = \phi'_g$, we deduce that

$$(3.7) \quad \phi'_g(h) \in I, \quad \text{for all } g \in P_n(\mathbb{A}) \text{ and all } h \in P_{n-1}(\mathbb{A}).$$

We can do even better.

Lemma 3.6. — *For all $g \in P_n(\mathbb{A})$ and all $g' \in \text{GL}_{n-1}(\mathbb{A})$, we have*

$$(3.8) \quad \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \phi \left(\begin{pmatrix} g' & u \\ 0 & 1 \end{pmatrix} g \right) \, du \in I.$$

Proof. — Indeed, we have

$$\int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \phi \left(\begin{pmatrix} g' & u \\ 0 & 1 \end{pmatrix} g \right) \, du = \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \phi \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix} g \right) \, du$$

which is equal to $\phi'_x(1)$ with

$$x = \begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix} g \in P_n(\mathbb{A}).$$

The lemma thus follows from (3.7) applied to the function ϕ'_x . □

We now extend $\phi' = \phi'_g$ to $\text{GL}_{n-1}(\mathbb{A})$ by setting

$$\phi'(g') = \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u) \phi \left(\begin{pmatrix} g' & u \\ 0 & 1 \end{pmatrix} g \right) \, du$$

for all $g' \in \text{GL}_{n-1}(\mathbb{A})$. By Lemma 3.6, it takes values in I on $\text{GL}_{n-1}(\mathbb{A})$ for all $g \in P_n(\mathbb{A})$.

Now, by Fourier analysis on the compact Abelian group $(\mathbb{A}/k)^{n-1}$, we have

$$\begin{aligned} \phi \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) &= \sum_{\beta \in k^{n-1}} \psi(\beta \cdot u) \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\beta \cdot x) \phi \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \, dx \\ &= \sum_{\rho \in P_{n-1}(k) \backslash \text{GL}_{n-1}(k)} \psi(\eta \rho \cdot u) \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot x) \phi \left(\begin{pmatrix} \rho & x \\ 0 & 1 \end{pmatrix} g \right) \, dx \\ &= \sum_{\rho \in P_{n-1}(k) \backslash \text{GL}_{n-1}(k)} \psi(\eta \rho \cdot u) \phi'(\rho). \end{aligned}$$

Multiplying by $f(u)$ for some function $f \in \mathcal{C}^\infty((\mathbb{A}/k)^{n-1}, \mathbb{C})$ and integrating, we get

$$\int_{(\mathbb{A}/k)^{n-1}} \phi \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) f(u) \, du = \sum_{\rho \in P_{n-1}(k) \backslash \mathrm{GL}_{n-1}(k)} \phi'(\rho) \int_{(\mathbb{A}/k)^{n-1}} \psi(\eta\rho \cdot u) f(u) \, du.$$

Now let $U = U(\phi, g)$ be a compact open subgroup of \mathbb{A}/k such that

$$\phi \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right) = \phi(g) \quad \text{for all } u \in U^{n-1} \subseteq (\mathbb{A}/k)^{n-1}.$$

Now take for f the characteristic function of U^{n-1} . We get

$$(3.9) \quad \phi(g) = \sum_{\rho \in P_{n-1}(k) \backslash \mathrm{GL}_{n-1}(k)} \phi'(\rho) \int_{U^{n-1}} \psi(\eta\rho \cdot u) \frac{du}{|U|^{n-1}}.$$

For all ρ , we have

$$\int_{U^{n-1}} \psi(\eta\rho \cdot u) \frac{du}{|U|^{n-1}} = \begin{cases} 0 & \text{if } \psi \text{ is non-trivial on } \eta\rho \cdot U^{n-1}, \\ 1 & \text{otherwise.} \end{cases}$$

A coset $\rho \in P_{n-1}(k) \backslash \mathrm{GL}_{n-1}(k)$ satisfies $\eta\rho \cdot U^{n-1} \subseteq \mathrm{Ker}(\psi)$ if and only if the vector

$$\beta = (\beta_1, \dots, \beta_{n-1}) = \eta\rho \in k^{n-1}$$

satisfies $\beta \cdot U^{n-1} \subseteq \mathrm{Ker}(\psi)$, that is, $\beta_i U \subseteq \mathrm{Ker}(\psi)$ for all i . But it follows from the case where $n = 2$ that there are finitely many $\beta_i \in k$ such that $\beta_i U \subseteq \mathrm{Ker}(\psi)$. There are thus finitely many cosets ρ contributing to the sum (3.9).

Moreover, $\phi'(\rho) \in I$ for all $g \in P_n(\mathbb{A})$ and all $\rho \in P_{n-1}(k) \backslash \mathrm{GL}_{n-1}(k)$. It follows that $\phi(g) \in I$ for all $g \in P_n(\mathbb{A})$. This finishes the proof of Theorem 3.1.

3.6. As in Paragraph 3.3, k is a function field of characteristic p and I is a sub- $\mathbb{Z}[\mu_p]$ -module of \mathbb{C} . Let $Z = Z_n$ denote the centre of GL_n . We will prove the following result.

Theorem 3.7. — *Let $\phi : \mathrm{GL}_n(\mathbb{A}) \rightarrow \mathbb{C}$ be a smooth function such that:*

- (1) *one has $\phi(\gamma g) = \phi(g)$ for all $\gamma \in \mathrm{GL}_n(k)$ and all $g \in \mathrm{GL}_n(\mathbb{A})$,*
- (2) *the function ϕ is cuspidal in the sense that*

$$\int_{N_{m,n-m}(k) \backslash N_{m,n-m}(\mathbb{A})} \phi(ug) \, du = 0$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$ and all $m \in \{1, \dots, n-1\}$,

- (3) *one has $\phi(g) \in I$ for all $g \in Z(\mathbb{A})P(\mathbb{A})$.*

Then

$$\int_{N(k) \backslash N(\mathbb{A})} \psi(u)^{-1} \phi(ug) \, du \in I$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$.

We first prove the following lemma.

Lemma 3.8. — *The image of $\mathrm{GL}_n(k)Z(\mathbb{A})P(\mathbb{A})$ in $\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})$ is dense.*

Proof. — Since the projective space \mathbb{P}^{n-1} is k -isomorphic to GL_n/ZP , it is equivalent to saying that the image of the diagonal embedding of $\mathbb{P}^{n-1}(k)$ in $\mathbb{P}^{n-1}(\mathbb{A})$ is dense, that is, \mathbb{P}^{n-1} satisfies the strong approximation property. We thus have to prove that, given any finite set S of places of k and, for $v \in S$, any non-empty open subset U_v of $\mathbb{P}^{n-1}(k_v)$, the intersection

$$\left(\prod_{v \in S} U_v \times \prod_{v \notin S} \mathbb{P}^{n-1}(\mathcal{O}_v) \right) \cap \mathbb{P}^{n-1}(k)$$

is non-empty. As $\mathbb{P}^{n-1}(\mathcal{O}_v) = \mathbb{P}^{n-1}(k_v)$ for all v (given any point $[x_1 : \dots : x_n] \in \mathbb{P}^{n-1}(k_v)$, one may multiply the x_i by a suitable power of a uniformizer so that the coordinates are in \mathcal{O}_v), this is equivalent to proving the weak approximation property, that is, proving that there is a point $P = [x_1 : \dots : x_n] \in \mathbb{P}^{n-1}(k)$ such that $P \in U_v$ for all $v \in S$. Restricting to the affine open subspace made of all points whose last coordinate is non-zero, we are reduced to proving the weak approximation property for an affine space. This follows from the weak approximation theorem for k (see for instance [22] Theorem 1.3.1). \square

By Assumptions 1 and 3, the function ϕ takes values in I on $\mathrm{GL}_n(k)Z(\mathbb{A})P(\mathbb{A})$. Since ϕ is locally constant on $\mathrm{GL}_n(\mathbb{A})$, it follows from Lemma 3.8 that ϕ takes values in I on $\mathrm{GL}_n(\mathbb{A})$.

We now prove the theorem. Let us apply the map $\phi \mapsto W_\phi$ of Paragraph 3.2 defined by (3.3). This defines a function:

$$W_\phi(g) = \int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1} \phi(ug) \, du, \quad g \in \mathrm{GL}_n(\mathbb{A}).$$

Let us prove that W_ϕ takes values in I on $\mathrm{GL}_n(\mathbb{A})$. Since $N(k)\backslash N(\mathbb{A})$ is compact, there exists a compact open subgroup C of $N(k)\backslash N(\mathbb{A})$ such that $N(k)\backslash N(\mathbb{A})$ is the union of finitely many $u_i C$ and the function $u \mapsto \psi(u)^{-1} \phi(ug)$ is constant on these cosets. This gives us

$$W_\phi(g) = |C| \cdot \sum_{i=1}^r \psi^{-1}(u_i) \phi(u_i g).$$

It thus remains to prove that $|C|$ is in $\mathbb{Z}[\mu_p]$.

The measure du gives measure 1 to the compact group $N(k)\backslash N(\mathbb{A})$. Thus $|C|$ is the index of C in $N(k)\backslash N(\mathbb{A})$. Let us prove that $|C|$ is a p -power. For this, it suffices to prove that $N(k)\backslash N(\mathbb{A})$ is a pro- p -group. For this, by dévissage, it suffices to prove that \mathbb{A}/k is a pro- p -group.

By [19] Theorem 5.8, there are a finite set S of places of k and integers $m_v \geq 0$ for $v \in S$ such that $\mathbb{A} = k + U$ for some compact open subgroup

$$U = \prod_{v \in S} \mathfrak{p}_v^{m_v} \times \prod_{v \notin S} \mathcal{O}_v$$

thus \mathbb{A}/k is a quotient of U . But U is clearly a pro- p -group, as it is a product of pro- p -groups.

4. Modular rigidity

Let k be a global field as in Paragraph 3.1. Fix a prime number ℓ different from the characteristic of k and a field isomorphism $\iota : \mathbb{C} \rightarrow \overline{\mathbb{Q}}_\ell$. Let $\mathbb{A} = \mathbb{A}_k$ be its ring of adèles.

Given any place v of k , write $G_v = GL_n(k_v)$ and $N_v = N(k_v)$, and $P_v = P(k_v)$ for the mirabolic subgroup of G_v .

Recall that \mathfrak{m}_ℓ is the maximal ideal of $\overline{\mathbb{Z}}_\ell$.

4.1. Let us state the following conjecture.

Conjecture 4.1. — *Let Π_1, Π_2 be cuspidal automorphic representations of $GL_n(\mathbb{A})$ with central characters Ω_1, Ω_2 , respectively. Let ι be a field isomorphism from \mathbb{C} to $\overline{\mathbb{Q}}_\ell$ for some prime number ℓ different from the characteristic of k . Suppose that:*

- (1) *the characters $\Omega_1 \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Omega_2 \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are $\overline{\mathbb{Z}}_\ell$ -valued and congruent mod \mathfrak{m}_ℓ ,*
- (2) *there exists a finite set S of places of k , containing all Archimedean places and all finite places above ℓ , such that, for all $v \notin S$, one has:*

- (a) *the local components $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are unramified,*
- (b) *the characteristic polynomials of their Satake parameters belong to $\overline{\mathbb{Z}}_\ell[X]$ and have the same reduction mod \mathfrak{m}_ℓ in $\overline{\mathbb{F}}_\ell[X]$.*

Let w be a finite place not dividing ℓ such that the representations $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are integral. Then the reductions mod ℓ of these representations have a common generic irreducible component, and such a generic component is unique and occurs with multiplicity 1.

Remark 4.2. — Note that the case where $w \notin S$ is easy. Indeed, if $w \notin S$, write

$$\pi_1 = \Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell, \quad \pi_2 = \Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell.$$

These representations are generic (as $\Pi_{1,w}$ and $\Pi_{2,w}$ are local components of cuspidal automorphic representations) and unramified. For each i , there is thus an unramified character ω_i of the diagonal torus T_w of G_w whose parabolic induction is isomorphic to π_i . By Lemma 2.3, these representations are integral, that is, the character ω_i takes values in $\overline{\mathbb{Z}}_\ell^\times$. By [16] Proposition 6.2, the fact that the characteristic polynomials of their Satake parameters are congruent implies that the reductions mod \mathfrak{m}_ℓ of ω_1 and ω_2 are conjugate by the normalizer of T_w in G_w . It follows that $\mathbf{r}_\ell(\pi_1)$ and $\mathbf{r}_\ell(\pi_2)$ are equal, and the conclusion follows from Lemma 2.2. We will thus concentrate on the case where $w \in S$.

Remark 4.3. — The reader should be aware that there are integral unramified irreducible $\overline{\mathbb{Q}}_\ell$ -representations of $GL_n(k_w)$ whose Satake parameters have congruent characteristic polynomials, but whose reductions mod ℓ are unequal. (For instance, this is the case for the trivial $\overline{\mathbb{Q}}_\ell$ -character and any integral unramified principal series $\overline{\mathbb{Q}}_\ell$ -representation whose Satake parameter has a characteristic polynomial congruent to that of the Satake parameter of the trivial $\overline{\mathbb{Q}}_\ell$ -character.) However, this phenomenon does not appear for *generic* unramified representations.

Remark 4.4. — The reductions mod ℓ of $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_{\ell}$ and $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_{\ell}$ won't be equal in general for $w \in S$. Here is an example. Start with a unitary group \mathbf{G} of rank 2 with respect to a totally imaginary quadratic extension l of a totally real number field k . Suppose that:

- the group $\mathbf{G}(k_v)$ is compact for all Archimedean places v ,
- there is a finite place w of k above a prime number $p \neq \ell$ such that $\mathbf{G}(k_w) \simeq \mathrm{GL}_2(k_w)$ and q , the cardinality of the residue field of \mathcal{O}_w , has order 2 mod ℓ .

Thanks to our assumption on q , the $\overline{\mathbb{F}}_{\ell}$ -representation induced from the trivial $\overline{\mathbb{F}}_{\ell}$ -character of a Borel subgroup of $\mathrm{GL}_2(k_w)$ has length 3: its irreducible subquotients are the trivial character, the unramified character of order 2 and a cuspidal subquotient denoted ρ (see [24] Théorème 3, Corollaire 5). Let π be a cuspidal lift of ρ to $\overline{\mathbb{Q}}_{\ell}$, that is, π is an integral cuspidal $\overline{\mathbb{Q}}_{\ell}$ -representation of $\mathrm{GL}_2(k_w)$ such that $\mathbf{r}_{\ell}(\pi) = \rho$. (The existence of such a π is granted by [25] III.5.10.)

Now realize π as the local component at w of some automorphic representation Π_1 of $\mathbf{G}(\mathbb{A}_k)$ which is trivial at infinity, and whose local component at another place $u \neq w$ where \mathbf{G} splits is a given cuspidal representation η of $\mathrm{GL}_2(k_u)$.

We now follow [16] Section 3. Let K_w be the maximal compact subgroup $\mathrm{GL}_2(\mathcal{O}_w)$ and \mathbb{F}_q be the residue field of k_w . Let:

- κ_1 be the inflation to K_w of the cuspidal irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q)$ occurring in the parabolic induction of the trivial $\overline{\mathbb{F}}_{\ell}$ -character of a Borel subgroup (thus the restriction of π to K_w contains κ_1),
- κ_2 be the inflation to K_w of the Steinberg representation of $\mathrm{GL}_2(\mathbb{F}_q)$.

Since the reduction mod ℓ of κ_2 contains that of κ_1 , we get an automorphic representation Π_2 of $\mathbf{G}(\mathbb{A}_k)$ such that:

- the representation Π_2 is trivial at infinity,
- the representation $\Pi_{2,u}$ is isomorphic to η ,
- the restriction of $\Pi_{2,w}$ to K_w contains κ_2 (thus $\Pi_{2,w}$ has non-zero Iwahori fixed vectors),
- there is a finite set of places S of k , containing all Archimedean places and u, w , such that, for all $v \notin S$, the representations $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_{\ell}$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_{\ell}$ are unramified and the characteristic polynomials of their Satake parameters have coefficients in $\overline{\mathbb{Z}}_{\ell}$ and have the same reduction.

Using [12], we transfer Π_1 and Π_2 to algebraic regular, conjugate-selfdual, cuspidal automorphic representations $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ of $\mathrm{GL}_2(\mathbb{A}_l)$. Applying [16] Theorem 8.2, and as the local transfer at w is the identity since the group \mathbf{G} splits at w , we deduce that the representations $\mathbf{r}_{\ell}(\pi) = \rho$ and $\mathbf{r}_{\ell}(\Pi_{2,w})$ share a generic irreducible component. Since \mathbf{r}_{ℓ} commutes to parabolic restriction (by [4] Proposition 6.7), proving that $\mathbf{r}_{\ell}(\Pi_{2,w}) \neq \rho$ reduces to proving that $\Pi_{2,w}$ is not cuspidal. But this follows from the fact that $\Pi_{2,w}$ has non-zero Iwahori fixed vectors.

4.2. An instance of Conjecture 4.1 is provided by [16] Theorem 8.2. More generally, the results of [6, 20, 23] imply the conjecture in the case when k is a totally real or imaginary CM number field and Π_1, Π_2 are algebraic regular, by passing to the Galois side and using a density argument. In that case, note that:

- Assumption 1 on central characters is unnecessary,
- the representations $\Pi_{i,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_{\ell}$ are automatically integral for all finite v not dividing ℓ .

More precisely, assume that k is a totally real or imaginary CM number field and let Π_1, Π_2 be algebraic regular cuspidal automorphic representation of $GL_n(\mathbb{A})$. Assume that there exists a finite set S of places of k , containing all Archimedean places and all finite places dividing ℓ , such that, for all $v \notin S$, one has:

- (1) the local components $\Pi_{1,v}, \Pi_{2,v}$ are unramified,
- (2) the characteristic polynomials of the conjugacy classes of semisimple elements in $GL_n(\overline{\mathbb{Q}}_\ell)$ associated with $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ have coefficients in $\overline{\mathbb{Z}}_\ell$ and are congruent mod \mathfrak{m}_ℓ .

Associated with Π_i in [6] and [20], there is a continuous ℓ -adic Galois representation

$$\Sigma_i : \text{Gal}(\overline{\mathbb{Q}}/k) \rightarrow GL_n(\overline{\mathbb{Q}}_\ell)$$

(depending on $\iota : \mathbb{C} \rightarrow \overline{\mathbb{Q}}_\ell$) for $i = 1, 2$. For any finite place v of k not dividing ℓ , fix a decomposition subgroup Γ_v of $\text{Gal}(\overline{\mathbb{Q}}/k)$. The Weil-Deligne representation associated with $\Sigma_i|_{\Gamma_v}$ is made of a smooth ℓ -adic representation $\rho_{i,v}$ together with a nilpotent operator on the space of $\rho_{i,v}$. On the other hand, the Weil-Deligne representation associated with $\Pi_{i,v} \otimes |\det|_v^{(1-n)/2}$ by the local Langlands correspondence is made of a semisimple smooth complex representation $\sigma_{i,v}$ together with a nilpotent operator on the space of $\sigma_{i,v}$. By [23], for any finite place v of k not dividing ℓ , one has

$$\rho_{i,v}^{\text{ss}} \simeq \sigma_{i,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$$

(where $\rho_{i,v}^{\text{ss}}$ stand for the semisimplification of $\rho_{i,v}$). Arguing as in [16] 8.2, we deduce that, for any finite place v of k not dividing ℓ , the representations $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are integral, their reductions mod ℓ share a generic irreducible component, which occurs with multiplicity 1.

4.3. *From now on, and until the end of this section, k is a function field of characteristic p . We are going to prove Conjecture 4.1 in this case. We will actually prove a stronger result.*

Theorem 4.5. — *Let Π_1, Π_2 be cuspidal automorphic representations of $GL_n(\mathbb{A})$. Let ι be a field isomorphism from \mathbb{C} to $\overline{\mathbb{Q}}_\ell$ for some prime number ℓ different from p . Suppose that there is a finite set S of places of k such that, for all $v \notin S$, one has:*

- (1) *the local components $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are unramified,*
- (2) *the characteristic polynomials of their Satake parameters belong to $\overline{\mathbb{Z}}_\ell[X]$ and have the same reduction mod \mathfrak{m}_ℓ in $\overline{\mathbb{F}}_\ell[X]$.*

Let w be a finite place. Then:

- *the representations $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are integral*
- *the reductions mod ℓ of these representations have a common generic irreducible component,*
- *and such a generic component is unique and occurs with multiplicity 1.*

Remark 4.6. — Note that, by taking a bigger S , we may (and will) assume that the character ψ_v is trivial on \mathcal{O}_v but not on \mathfrak{p}_v^{-1} for all $v \notin S$.

First, let us prove that, under the assumptions of Theorem 4.5, for all v , the central characters of $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ take values in $\overline{\mathbb{Z}}_\ell^\times$ and are congruent mod \mathfrak{m}_ℓ .

Lemma 4.7. — *Let χ be an automorphic character of $\mathbb{A}^\times/k^\times$ and U be a subgroup of \mathbb{C}^\times . Assume that there is a finite set S of places of k such that, for all $v \notin S$, the local component χ_v is unramified and takes values in U . Then, for all v , the character χ_v takes values in U .*

Proof. — If S is empty, there is nothing to prove. We thus assume that there is a place $w \in S$. Let $x \in k_w^\times$. Define an idèle $x' \in \mathbb{A}^\times$ by

$$x'_v = \begin{cases} x & \text{if } v = w, \\ 1 & \text{otherwise.} \end{cases}$$

The weak approximation theorem implies that there is a $y \in k^\times$ such that $y \in \text{Ker}(\chi_v)$ if $v \in S$ and $v \neq w$, and $yx \in \text{Ker}(\chi_w)$. We have

$$\chi_w(x) = \chi(x') = \chi(yx') = \chi_w(yx) \cdot \prod_{\substack{v \in S \\ v \neq w}} \chi_v(y) \cdot \prod_{v \notin S} \chi_v(y).$$

Thanks to the conditions given by the weak approximation theorem, this is equal to the product of $\chi_v(y)$ for all $v \notin S$. (Note that this is a product of finitely many terms, since y is a unit in the ring of integers of k_v for almost all $v \notin S$.) The result follows from the fact that, for such v , one has $\chi_v(y) \in U$. \square

Proposition 4.8. — *Let χ_1 and χ_2 be automorphic characters of $\mathbb{A}^\times/k^\times$, and fix a field isomorphism $\iota : \mathbb{C} \rightarrow \overline{\mathbb{Q}_\ell}$. Assume there is a finite set S of places of k such that, for all $v \notin S$:*

- (1) *the characters $\chi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}_\ell}$ and $\chi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}_\ell}$ are unramified and take values in $\overline{\mathbb{Z}_\ell}^\times$,*
- (2) *the reductions mod ℓ of these characters are equal.*

Then, for all places v , the characters $\chi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}_\ell}$ and $\chi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}_\ell}$ take values in $\overline{\mathbb{Z}_\ell}^\times$ and are congruent mod \mathfrak{m}_ℓ .

Proof. — For Assertion 1 of the proposition, apply Lemma 4.7 to χ_i and $U = \iota^{-1}(A^\times)$. For Assertion 2, apply Lemma 4.7 to $\chi = \chi_1\chi_2^{-1}$ and $U = 1 + \iota^{-1}(\mathfrak{m}_\ell)$. \square

Remark 4.9. — Note that Theorem 4.5 follows from [13] Théorème VI.9 by a global argument in the spirit of §4.2 (see also [9] IV.1.6).

4.4. The remainder of this section is devoted to the proof of Theorem 4.5. By Remark 4.2, we may and will assume that $w \in S$.

Let A denote the image of $\overline{\mathbb{Z}_\ell}$ by ι^{-1} and \mathfrak{m} denote the image of \mathfrak{m}_ℓ by ι^{-1} . Thus A contains the complex p th roots of unity and the character ψ of Paragraph 3.1 takes values in A^\times . Notice that A and \mathfrak{m} are sub- $\mathbb{Z}[\mu_p]$ -modules of \mathbb{C} .

For any place v of k and $i \in \{1, 2\}$, let $W_{i,v}$ be a function in the Whittaker model $\mathcal{W}(\Pi_{i,v}, \psi_v)$ satisfying the conditions:

- if $v \notin S$, then $W_{i,v}$ is the unique $\text{GL}_n(\mathcal{O}_v)$ -invariant function such that $W_{i,v}(1) = 1$ (see Paragraph 2.5),
- if $v \in S$, we fix an arbitrary A -valued function $f_v \in \text{ind}_{N_v}^P(\psi_v)$ and let $W_{i,v} \in \mathcal{W}(\Pi_{i,v}, \psi_v)$ be the unique function extending f_v to G_v (see Paragraph 2.2),
- for all $v \in S$ such that $v \neq w$, we further assume that $f_v(1) = 1$.

For $i \in \{1, 2\}$, we consider the global Whittaker function

$$W_i = \bigotimes_v W_{i,v} \in \mathcal{W}(\Pi_i, \psi).$$

For $x \in P(\mathbb{A})$, we thus have

$$W_i(x) = \prod_{v \in S} f_v(x_v) \cdot \prod_{v \notin S} W_{i,v}(x_v).$$

It follows from Proposition 2.4 that W_1 and W_2 take values in A and $W_1 - W_2$ takes values in \mathfrak{m} on $P(\mathbb{A})$. Let $\varphi_i \in \Pi_i$ be the automorphic form corresponding to W_i via (3.4), that is:

$$\varphi_i(g) = \sum_{\gamma \in N_{n-1}(k) \backslash GL_{n-1}(k)} W_i \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

for all $g \in GL_n(\mathbb{A})$. By Theorem 3.1, the functions φ_1 and φ_2 take values in A and $\varphi_1 - \varphi_2$ takes values in \mathfrak{m} on $P(\mathbb{A})$.

Thanks to Proposition 4.8, the central characters of $\Pi_{1,v}$ and $\Pi_{2,v}$ take values in A^\times and are congruent mod \mathfrak{m} for all v . It follows that φ_1 and φ_2 take values in A and $\varphi_1 - \varphi_2$ takes values in \mathfrak{m} on $Z(\mathbb{A})P(\mathbb{A})$, where Z is the centre of GL_n . Applying Theorem 3.7, we deduce that W_1 and W_2 take values in A and $W_1 - W_2$ takes values in \mathfrak{m} on $GL_n(\mathbb{A})$.

Now let us consider the place w . For $i = 1, 2$ and $g \in G_w \subseteq G(\mathbb{A})$, one has:

$$\begin{aligned} W_i(g) &= \prod_v W_{i,v}(g_v) \\ &= W_{i,w}(g) \cdot \prod_{v \neq w} W_{i,v}(1) \\ &= W_{i,w}(g). \end{aligned}$$

It follows that $W_{1,w}$ and $W_{2,w}$ take values in A , and that $W_{1,w} - W_{2,w}$ takes values in \mathfrak{m} on G_w . We thus proved that, given any A -valued function $f_w \in \text{ind}_{N_w}^{P_w}(\psi_w)$, the functions $W_{1,w}$ and $W_{2,w}$ extending f_w are A -valued. Proposition 2.5 thus implies that $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are integral. Now assume further that $f_w(1) = 1$. Then Theorem 4.5 follows from Proposition 2.1.

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