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► **To cite this version:**

Thierry Coulbois, Anahí Gajardo, Pierre Guillon, Victor Lutfalla. Aperiodic monotiles: from geometry to groups. 2024. hal-04706651

HAL Id: hal-04706651

<https://hal.science/hal-04706651v1>

Preprint submitted on 23 Sep 2024

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Aperiodic monotiles: from geometry to groups

Thierry Coulbois Anahí Gajardo Pierre Guillon Victor Lutfalla

Sept. 2024

Abstract

In 2023, two striking, nearly simultaneous, mathematical discoveries have excited their respective communities, one by Greenfeld and Tao, the other (the *Hat* tile) by Smith, Myers, Kaplan and Goodman-Strauss, which can both be summed up as the following: there exists a single tile that tiles, but not periodically (sometimes dubbed the *einstein* problem). The two settings and the tools are quite different (as emphasized by their almost disjoint bibliographies): one in euclidean geometry, the other in group theory. Both are highly nontrivial: in the first case, one allows complex shapes; in the second one, also the space to tile may be complex.

We propose here a framework that embeds both of these problems. We illustrate our setting by transforming the Hat tile into a new aperiodic group monotile, and we describe its symmetries.

1 Introduction

Tilings were originally defined as coverings of the euclidean plane by compact tiles without overlap. They have been studied since the ancient Greeks, with a special emphasis on periodic tilings in which a finite domain is repeated periodically on the whole plane.

Seminal results on tilings include the classification of Archimedean tilings (tilings by regular polygons where all vertices are the same up to isometry) by Kepler [Kep1619] and the study of the symmetry groups of periodic tilings (crystallographic space groups) by Bieberbach [Bie1911].

In the 60s emerged the question of *aperiodic* tilesets [Wan61]. That is: sets of tile that can tile the whole plane but only in a non-periodic way. Though Wang initially conjectured such tilesets did not exist, Berger [Ber66] soon provided a first aperiodic tileset of 20426 tiles up to translation.

The first simple geometric aperiodic tileset was found by Robinson in 1971 [Rob71], who defined a set of 6 geometric tiles up to isometry with the shape of squares with bumps and dents, that can tile the 2-dimensional plane, but without any translational symmetry. This started the challenge that was taken two years later by Penrose [Pen74], who defined an aperiodic set of only *two* geometric tiles. Penrose's tiling has fivefold rotational symmetry, and cannot be arranged in a square grid as Robinson's. In 1992, Ammann, Grünbaum and Shephard [AGS92] restricted the problem to polyominoes and they found an aperiodic set of *three* polyominoes. The tiling problem with polyominoes is more limited, and in 1991, Beauquier and Nivat [BN91] showed that, if no rotation of the tiles is allowed, then no aperiodic connected polyomino monotile exists.

The problem of tiling with polyominoes can be expressed in terms of *group tiling* of \mathbb{Z}^2 , since a polyomino is a subset of \mathbb{Z}^2 , thus tiling with polyominoes is the same as covering the group \mathbb{Z}^2 with translations of finite connected subsets of \mathbb{Z}^2 , taken from a given set.

While we usually consider geometric tilings in \mathbb{R}^2 , group tilings may be defined in any group, which fully changes the techniques that can be applied to tackle the problem. The problem of tiling \mathbb{Z}^2 with translations of only one tile was definitely solved in 2020 by Bhattacharya [Bha20], who proved that no finite set of \mathbb{Z}^2 (possibly disconnected in the usual Cayley graph) is an aperiodic monotile. Is this a limitation of \mathbb{Z}^2 ? Does it come from the fact of having only *one* tile? An answer was found in 2023 by Greenfeld and Tao [GT24], who proved that an aperiodic monotile exists in \mathbb{Z}^d when d is big enough, and also such a monotile exists for a group of the form $\mathbb{Z}^2 \times H$, for a well-chosen finite abelian group H .

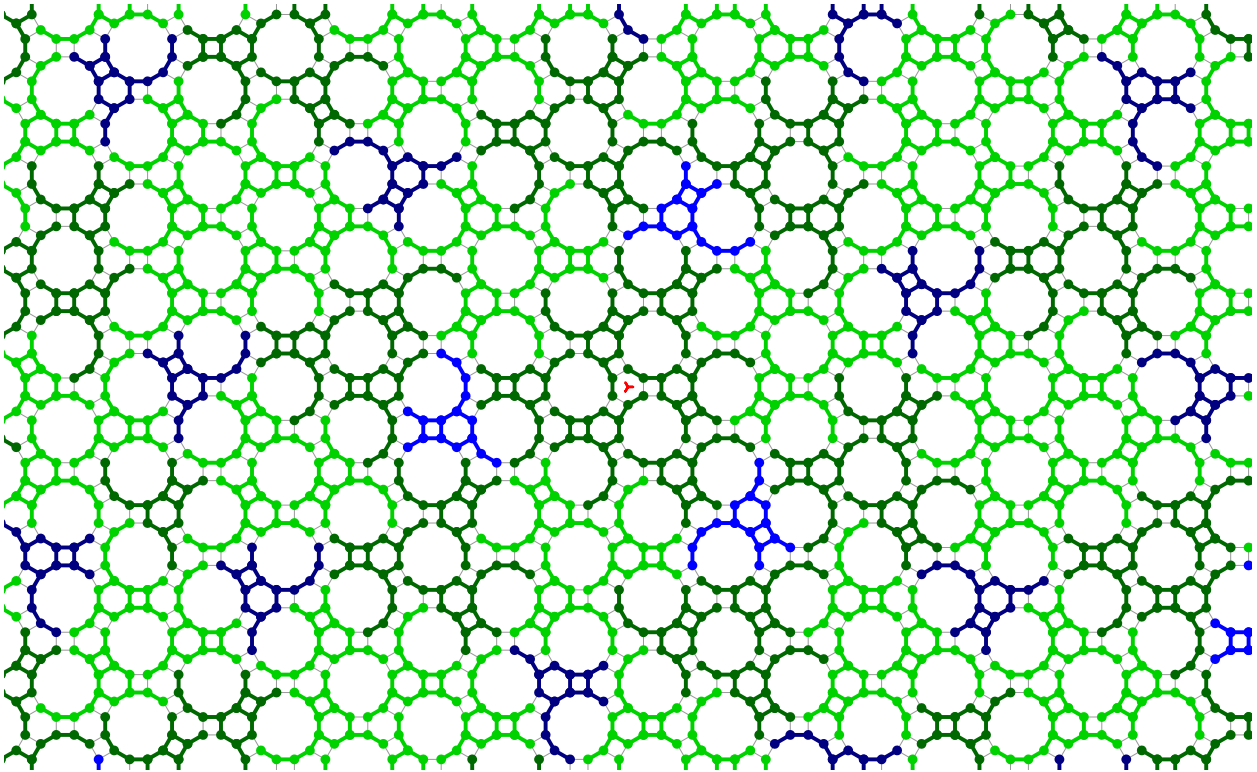
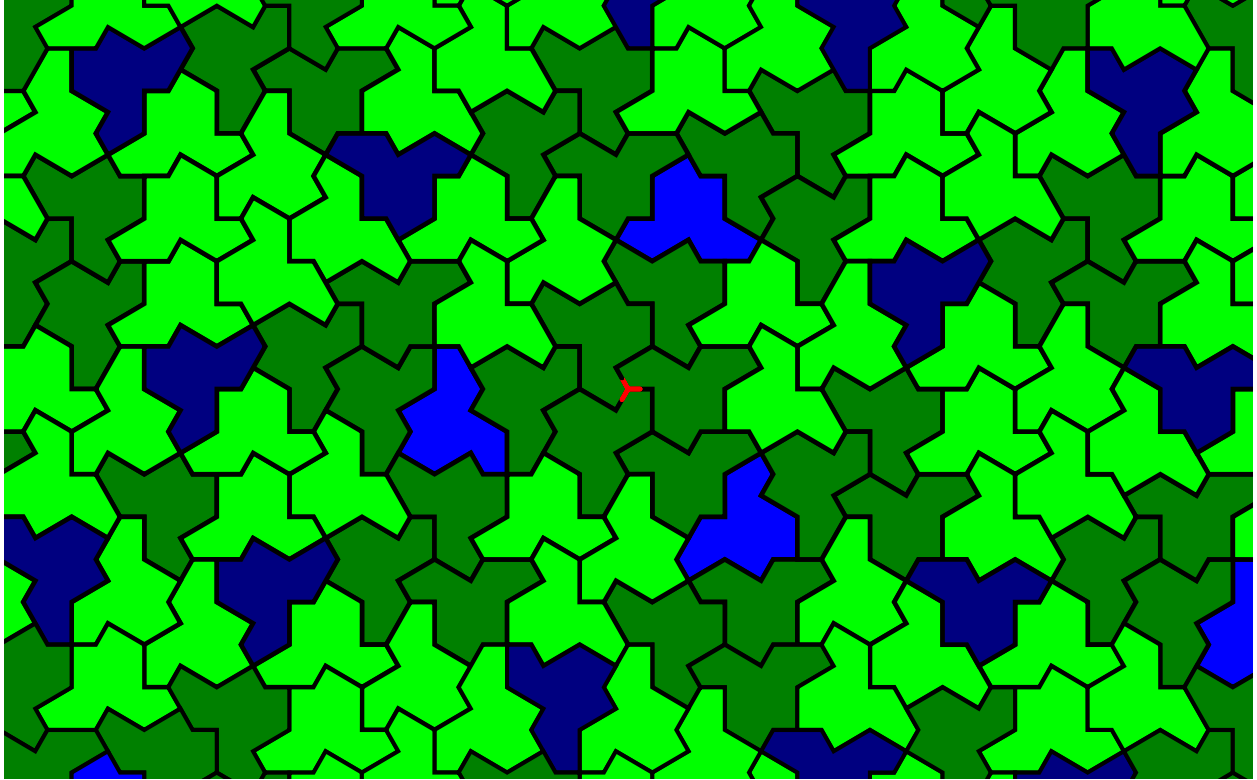


Figure 1: A Hat tiling of \mathbb{R}^2 and a corresponding Roach tiling on group Γ . The red tricross is a center of 3-fold rotational symmetry.

Coming back to geometric tilings, the tiling by Penrose kept the record for tile number for 50 years, unless one authorizes tiles to be disconnected or to communicate through their vertices, see [ST11]. In 2023, Smith, Myers, Kaplan and Goodman-Strauss [SMKGS24] described new aperiodic tilings with only *one* tile. The work is very interesting, not only because a whole family of aperiodic monotiles was discovered, but also because two of these tiles (the Hat and the Turtle) are polykites and can be embedded in a *Kitegrid*, which is the dual of an Archimedean tiling (which can be seen as the Cayley graph of a group), suggesting that the Hat and the Turtle tilings can also be seen as tilings of a group.

Given a compact K which tiles the plane periodically, we call poly- K a tile which is a union of copies of K . Classical examples include polyominoes (K is the unit square), polyamonds (K is the unit equilateral triangle) and polykites (K is the Kite).

In this paper, we provide a toolkit to translate geometric tilings of a poly- K tile to tilings of the symmetry group a K -grid. In particular, we apply this toolkit to the Hat polykite [SMKGS24] to obtain a new aperiodic monotile (which we call Roach) in an explicit finitely presented group Γ , see Figure 1. It can be compared to [GT24, Theorem 1.4]: our result is somehow weaker in the sense that the extension of \mathbb{Z}^2 is not direct. On the other hand, it is more explicit, and enjoys a nice Coxeter definition, as well as nice pictures.

Section 2 provides terminology and definitions for geometric and group tilings. Section 3 presents our toolkit on translation from poly- K tiles to group tilings. Section 4 applies this toolkit to the Hat tile. Section 5 presents more general results on links between geometric and group tilings outside of the poly- K case.

The readers interested in abstract tilings and the link between geometric and group tilings may skip Section 4, while readers only interested in the Roach tile can use Corollary 7 as a blackbox and focus on Section 4.

2 Tilings

Let \mathcal{W} be a topological space endowed with a nontrivial Borel measure λ . Let G be a group which acts by self-homeomorphisms g of \mathcal{W} that we assume **negligibility-preserving** (that is, $\lambda(A) > 0 \implies \lambda(g \cdot A) > 0$). A **tile** is a positive-measure subset of \mathcal{W} (often considered up to negligible sets). A **cotiler** for a set \mathcal{T} of tiles is a countable subset $C \subset G \times \mathcal{T}$ such that:

$$\bigsqcup_{(g,T) \in C} g \cdot T \equiv_{\lambda} \mathcal{W},$$

where \equiv_{λ} is an equality and \sqcup a disjoint union, both up to λ -negligible sets. If \mathcal{T} admits a cotiler, we say that \mathcal{T} **tiles** \mathcal{W} with respect to G .

The terminology is inspired from the case when \mathcal{T} is a singleton $\{T\}$; in that case we will say that T tiles and confuse the cotiler with a subset of $G \sim G \times \{T\}$.

The cotiler should not be confused with the **tiling** $\{g \cdot T \mid (g, T) \in C\}$.

Remark 1. *We keep a very large setting because very few assumptions are needed for our results (some more will be added to the tiles and the group in some results). Nevertheless, one can keep in mind that natural examples arise when \mathcal{W} is a second-countable locally compact space, λ has full support (is positive for every nonempty open subset), and the tiles T are compact, the closure of their interior, have finite positive measure and λ -negligible boundaries. In this setting, $T \equiv_{\lambda} T'$ if and only if $T = T'$.*

Here are some classical examples of settings.

Example 2. *Relevant examples of spaces \mathcal{W} include the following:*

- $\mathcal{W} = \mathbb{R}^d$ endowed with canonical topology, Lebesgue measure, $G = \mathbb{R}^d$ acting by translations or $G = \text{Isom}^+(\mathbb{R}^d) \simeq \mathbb{R}^d \rtimes \text{SO}_d(\mathbb{R})$ acting by direct isometries or $G = \text{Isom}(\mathbb{R}^d) \simeq \mathbb{R}^d \rtimes \text{O}_d(\mathbb{R})$ acting by isometries. We are also interested in tilings with respect to some countable subgroups of these. It is also natural to require additional constraints over the tiles, like polygonality. . .

- Consider a countable group $\mathcal{W} = G$ endowed with discrete topology, the counting measure, self-acting by left multiplication. The notion of *monotileable group* [CC17] and its generalization into *polytileable group* fit in this framework.
- More generally, $\mathcal{W} = G$ is a locally compact group endowed with the Haar measure, self-acting by left multiplication.
- $\mathcal{W} = \mathbb{H}^2$, the hyperbolic plane and $G = \text{Isom}(\mathbb{H}^2) \simeq \text{PGL}_2(\mathbb{R})$ or $G = \text{Isom}^+(\mathbb{H}^2) \simeq \text{PSL}_2(\mathbb{R})$.
- Tilings of half-planes like $\mathbb{N} \times \mathbb{Z}$ or $\mathbb{R}_+ \times \mathbb{R}$ could be handled via a suitable generalization of our theory to monoids, which would nevertheless have the price of a complexification of many notions.

The simplest examples of tilings are periodic tilings, introduced through the following formalism.

Definition 3. A finite tileset \mathcal{K} tiles with cotiler $C = G \times \mathcal{K}$ if, and only if, G is countable and $\bigsqcup_{K \in \mathcal{K}} K$ is, up to λ -negligible sets, a fundamental domain for $G \curvearrowright \mathcal{W}$. We then say that \mathcal{K} **yields a grid**, in the sense that:

$$\mathcal{W} = \bigsqcup_{(h,K) \in G \times \mathcal{K}} h \cdot K.$$

This includes regular tilings like the square lattice, but also some with different tiles like hexagonal-square-triangle lattice (note that \mathcal{K} can include several tiles from the same translation class).

Remark 4.

1. Any tiling with respect to some G can always be considered as a tiling with respect to the group of all negligibility-preserving self-homeomorphisms. Nevertheless, our goal is to reduce G as much as possible (while still allowing tilings).
2. If \mathcal{T} tiles with some cotiler C , then a relevant group of self-homeomorphisms of \mathcal{W} is the (countable) **cotiler group** $G_C := \langle gh^{-1} \mid (g, T), (h, T) \in C, T \in \mathcal{T} \rangle$. Indeed, for every $(g_T)_{T \in \mathcal{T}}$ such that $(g_T, T) \in C$, one can see that the tileset $\{g_T \cdot T \mid T \in \mathcal{T}\}$ (essentially equal to \mathcal{T} , up to translating each tile) tiles with cotiler $\{(gg_T^{-1}, g_T \cdot T) \mid (g, T) \in C\} \subseteq G_C \times \{g_T \cdot T \mid T \in \mathcal{T}\}$.

2.1 Periods

In general, if \mathcal{T} tiles, then it admits many cotilers. Let us denote $\mathcal{C}_G(\mathcal{T})$ the set of its cotilers.

G acts naturally on $\mathcal{C}_G(\mathcal{T})$ by $g \cdot C := \{(gh, T) \mid (h, T) \in C\}$. $\mathcal{C}_G(\mathcal{T})$ can be seen as some kind of SFT (or as a Delone set with local rules), that is, one can check that something is indeed a cotiler by looking at bounded windows: one can encode it as a configuration $x \in (\mathbf{2}^{\mathcal{T}})^G$ (where $\mathbf{2}^{\mathcal{T}}$ consists of the subsets of \mathcal{T}), defined by $x_g = \{T \in \mathcal{T} \mid (g, T) \in C\}$.

A **period** for a cotiler C is some $h \in G$ such that $h \cdot C = C$. The **stabilizer** of C is the subgroup $\text{Stab}(C) \leq G$ of its periods. A cotiler C is:

- **weakly periodic** if $\text{Stab}(C)$ is nontrivial.
- **mildly periodic** if $\text{Stab}(C)$ is infinite.
- **strongly periodic** if $\text{Stab}(C) \curvearrowright \mathcal{W}$ cocompactly (in the case when $\mathcal{W} = G$ is a discrete group, this is equivalent to the classical definition that $\text{Stab}(C)$ has finite index in G).

A tileset which tiles is **strongly** (resp. **mildly**, **weakly**) **aperiodic** if no cotiler is weakly (resp. mildly, strongly) periodic. Of course, strong periodicity implies mild periodicity, which implies weak periodicity, except in the case of finite groups (where all cotilers are strongly, but not mildly, periodic), which does not interest us.

Note also that in the case when G contains other isometries than translations, (a)periodicity differs from the usual notion of (a)periodicity with respect to the subgroup of translations.

A classical metaquestion, in each world \mathcal{W} (and with some constraints on G and on the possible tiles), is whether there exists a finite aperiodic tileset. In \mathbb{Z} and \mathbb{R} , the answer is “no” (folklore), and the proof carries to virtually- \mathbb{Z} groups. In \mathbb{R}^2 and \mathbb{Z}^2 , the answer is “yes”. The technique of turning colored tiles into shapes could be extended to prove that the answer is still “yes” in \mathbb{R}^d and \mathbb{Z}^d , for $d \geq 2$. We are not aware of other groups for which this question has been studied (though extensive literature exists for sets of “colored tiles”; see [Bit24] for a pleasant survey).

In cases where aperiodic tilesets do exist, it is then natural to look for the simplest ones; in particular, in the recent decades, the community has asked about the existence of an aperiodic monotile, that is, an aperiodic tile singleton. In some settings no weakly aperiodic monotile exists, in particular:

- in \mathbb{Z}^2 [Bha20, GT21];
- in \mathbb{R}^2 when G is the group of translations and the tile is required to be a topological disk [BN91, GBN91];
- in \mathbb{R}^n when G is the group of translations and the tile is required to be convex [Ven54, McM80];
- in \mathbb{R}^2 when G is $\text{Isom}(\mathbb{R}^2)$ the group of isometries of the plane and the tile is required to be a convex polygon [Rao17].

Nevertheless, there exist weakly aperiodic monotiles in some other settings:

- in \mathbb{R}^2 when G is the group of isometries and the tile is connected [SMKGS24];
- in \mathbb{R}^2 when G is the group of direct isometries ([ST11] with a disconnected tile, or more recently [SMKGS23] with a connected tile);
- in $\mathbb{Z}^2 \times H$ for some finite abelian group H [GT24, Theorem 1.3];
- in some \mathbb{Z}^d for some large $d \in \mathbb{N}$ [GT24, Corollary 1.5].

There is a very tight boundary between group \mathbb{Z}^2 which does not admit aperiodic monotile, and some commensurable groups which do. It is interesting to understand how small such a group can be.

3 Discretization of poly- \mathcal{K} tiles

This section is devoted to the translation between two worlds: a geometric, continuous world, and a combinatorial, countable world.

Remark 5. *If G is a group and \mathcal{K} is any finite set, G naturally acts on $G \curvearrowright G \times \mathcal{K}$ by $g \cdot (h, K) \mapsto (gh, K)$. Tiling $G \times \mathcal{K}$ is equivalent to tiling G with \mathcal{K} layers that are synchronized (such considerations are used in [GT21, Section 3]): If $\mathcal{T} \subset G \times \mathcal{K}$ is a tileset, and $K \in \mathcal{K}$, one can define the set \mathcal{T}_K of tiles $\{g \mid (g, K) \in \mathcal{T}\}$, for $T \in \mathcal{T}$. Then the cotiler sets are related by $\mathcal{C}_G(\mathcal{T}) = \bigcap_{K \in \mathcal{K}} \mathcal{C}_G(\mathcal{T}_K)$.*

Let \mathcal{K} be a finite set of tiles $K \subset \mathcal{W}$ that yields a grid with respect to G in the sense of Definition 3, that is: \mathcal{K} tiles \mathcal{W} with G as a cotiler for each tile.

One says that a tile T is a **poly- \mathcal{K}** if it is (up to λ -negligible sets) the disjoint union of finitely many copies of tiles from \mathcal{K} , that is,

$$T \equiv_\lambda \bigsqcup_{(h,K) \in \Psi_{\mathcal{K}}(T)} h \cdot K,$$

for some finite $\Psi_{\mathcal{K}}(T) \subset G \times \mathcal{K}$. For $K \in \mathcal{K}$, let us also denote $\Psi_K(T) := \Psi_{\mathcal{K}}(T) \cap G \times \{K\}$.

Theorem 6 (Main result; poly- \mathcal{K} version). *Let G be countable and \mathcal{K} a finite set which yields a grid for $G \curvearrowright \mathcal{W}$. Then $\Psi_{\mathcal{K}}$ is a bijection from the set of (\equiv_λ -classes of) poly- \mathcal{K} sets \mathcal{T} of \mathcal{W} onto the set of finite tilesets of $G \times \mathcal{K}$, such that the cotiler sets $\mathcal{C}_G(\Psi_{\mathcal{K}}(\mathcal{T}))$ and $\mathcal{C}_G(\mathcal{T})$ are equal, up to syntactically renaming each tile T into $\Psi_{\mathcal{K}}(T)$.*

Formally, the last equality means that $\mathcal{C}_G(\Psi_{\mathcal{K}}(\mathcal{T})) = \{\{(g, \Psi_{\mathcal{K}}(T)) \mid (g, T) \in C\} \mid C \in \mathcal{C}_G(\mathcal{T})\}$. Consequently, \mathcal{W} admits (weakly, mildly, strongly) aperiodic poly- \mathcal{K} sets (for the action of G) if, and only if, $G \times \mathcal{K}$ admits (weakly, mildly, strongly) aperiodic sets of finite tiles, with the same number of tiles.

Let us restate the particular case when \mathcal{K} is a singleton $\{K\}$ (in that case we talk about poly- K) and \mathcal{T} is a singleton $\{T\}$.

Corollary 7. *Let G be countable and $K \subset \mathcal{W}$ yield a grid for $G \curvearrowright \mathcal{W}$. Then Ψ_K is a bijection from the set of (\equiv_{λ} -classes of) poly- K T of \mathcal{W} onto the set of finite tilesets of G , such that $\mathcal{C}_G(\Psi_K(T)) = \mathcal{C}_G(T)$, up to syntactically renaming T into $\Psi_K(T)$.*

Proof of Theorem 6. The fact that $\Psi_{\mathcal{K}}$ is a bijection comes directly from the definition and that $h \cdot K$ and $h' \cdot K'$ do not intersect if $(h, K) \neq (h', K')$.

C is cotiler for \mathcal{T} if and only if:

$$\begin{aligned}
&\iff \mathcal{W} \equiv_{\lambda} \bigsqcup_{(g,T) \in C} g \cdot T && \text{(definition of cotiler)} \\
&\iff \mathcal{W} \equiv_{\lambda} \bigsqcup_{(g,T) \in C} \bigsqcup_{(h,K) \in \Psi_{\mathcal{K}}(T)} g \cdot h \cdot K && (T \text{ is poly-}\mathcal{K}) \\
&\iff \bigsqcup_{(h,K) \in G \times \mathcal{K}} h \cdot K \equiv_{\lambda} \bigsqcup_{(g,T) \in C} \bigsqcup_{(h,K) \in \Psi_{\mathcal{K}}(T)} g \cdot h \cdot K && (\mathcal{K} \text{ yields a grid)} \\
&\iff \bigsqcup_{(h,K) \in G \times \mathcal{K}} \{(h, K)\} = \bigsqcup_{(g,T) \in C} \bigsqcup_{(h,K) \in \Psi_{\mathcal{K}}(T)} \{g \cdot h, K\} && (h \cdot K \cap h' \cdot K' \equiv_{\lambda} \emptyset \text{ if } (h, K) \neq (h', K')) \\
&\iff G \times \mathcal{K} = \bigsqcup_{(g,T) \in C} g \cdot \Psi_{\mathcal{K}}(T),
\end{aligned}$$

which means that $\{(g, \Psi_{\mathcal{K}}(T)) \mid (g, T) \in C\}$ is a cotiler for tile $\Psi_{\mathcal{K}}(T)$. □

4 Application to the Hat

4.1 The Hat

The Hat is a polykite tile, hence we first define the kite and Kitegrid.

Definition 8 (Kite and Kitegrid). We call **Kite** the quadrilateral from Figure 2, which we consider as a compact subset of \mathbb{R}^2 . We call **Kitegrid** the periodic Kite tiling which consists of the superposition of a hexagonal grid (with sides 2) and a triangular grid (with sides $2\sqrt{3}$), superimposed in such a way that the center of the hexagons are the vertices of the triangle, see Figure 2.

We also call **Semikite** the right triangle obtained when bisecting the Kite along its long diagonal, and **Semikitegrid** the periodic Semikite tiling obtained from bisecting the Kites of the Kitegrid into Semikites, see Figure 3 (right).

We call **symmetry** of a tiling \mathbf{T} an isometry of the plane that preserves the tiling, that is, $h \in \text{Isom}(\mathbb{R}^2)$ is a symmetry of $\mathbf{T} = \{g \cdot T \mid (g, T) \in C\}$ when $h\mathbf{T} = \mathbf{T}$. Denote $\text{Sym}(\mathbf{T})$ the group of the symmetries of \mathbf{T} . Similarly, we call **symmetry** of a tileset \mathcal{T} an isometry of the plane that sends a tile $T \in \mathcal{T}$ to some tile $T' \in \mathcal{T}$ (possibly the same tile).

Let us emphasize that h is a symmetry of a tiling \mathbf{T} with cotiler C when for any $(g, T) \in C$ there exists $(g', T) \in C$ such that $h \cdot g \cdot T = g' \cdot T$ (here the equality is as subsets of \mathcal{W}), whereas h is a period of C when for any $(g, T) \in C$ there exists $(g', T) \in C$ such that $(h \cdot g, T) = (g', T)$, that is, $h \cdot g = g'$. In particular, the symmetry group $\text{Sym}(\mathbf{T})$ includes the stabilizer $\text{Stab}(C)$ of the cotiler as a subgroup. These two groups are not always equal: in particular for the Kitegrid, the stabilizer contains only direct isometries of the plane whereas the symmetry group also contains reflections.

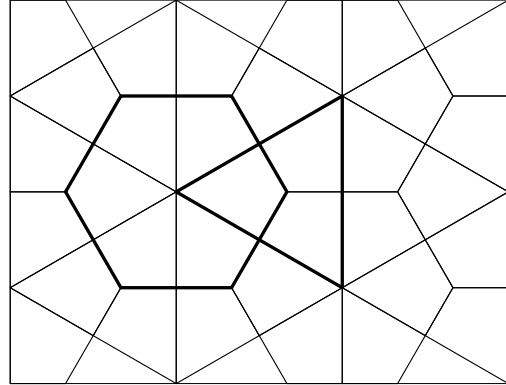
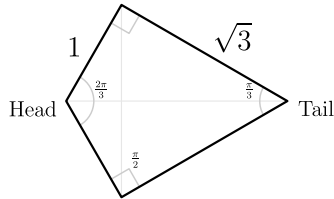


Figure 2: The Kite (left) and Kitegrid (right).

However, if the tileset has no non-trivial symmetry itself, then $\text{Stab}(C) = \text{Sym}(\mathbf{T})$.

An **Archimedean tiling** is a tiling of \mathbb{R}^2 by regular polygons such that every vertex has the same configuration (the same sequence of polygons when reading around the vertex), see Figure 3 in black. An Archimedean tiling is characterized by this vertex configuration; for example the 4.3.4.6 Archimedean tiling is a tiling by regular triangles, squares and hexagons such that around each vertex there is a square, a triangle, another square and a hexagon, see Figure 3 (left).

The dual tilings of Archimedean tilings are called **Laves tiling**, see Figure 3 in grey. Laves tilings are characterized by the vertex configuration of their dual Archimedean tiling, and have only one tile up to isometry. The Kitegrid is the 4.3.4.6 Laves tiling, and the Semikitegrid is the 4.6.12 Laves tiling.

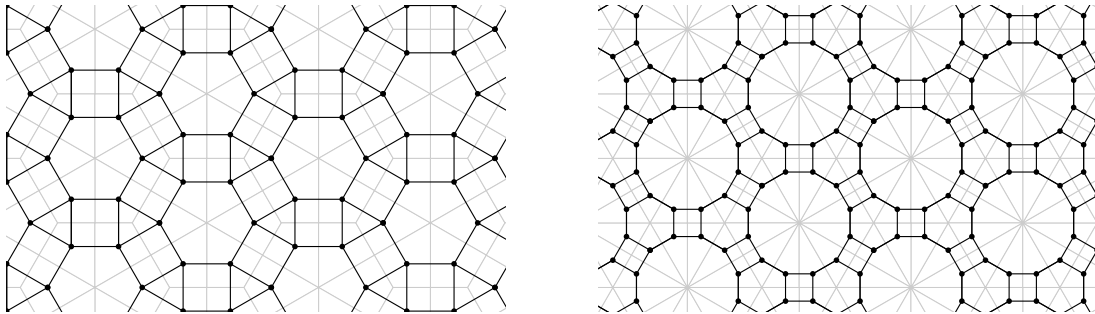


Figure 3: The Kitegrid (left) and Semikitegrid (right) in light grey, with their dual graph. The dual graphs induce the Archimedean tilings of vertex configurations 3.4.6.4 (left) and 4.6.12 (right).

Definition 9 (Hat). We call **Hat** the 13-gon tile represented in Figure 4 as a union of 8 kites.

Note that we can either see the Hat as a 13-gon (6 edges of length $\sqrt{3}$, 6 edges of length 1 and 1 edge of length 2) or as a 14-gon with two adjacent colinear edges (6 edges of length $\sqrt{3}$ and 8 edges of length 1).

We now consider tilings of $\mathcal{W} = \mathbb{R}^2$ with respect to the group G of isometries of \mathbb{R}^2 .

Among the main known results on the Hat tilings, we now present two that are useful in our construction.

Theorem 10 (Hat tilings [SMKGS24]).

1. The Hat is an aperiodic geometric monotile [SMKGS24, Theorem 1.1]; in our terminology: the Hat tiles \mathbb{R}^2 with respect to $G = \text{Isom}(\mathbb{R}^2)$; and no stabilizer of a cotiler for the Hat contains a translation, in particular it is weakly aperiodic.

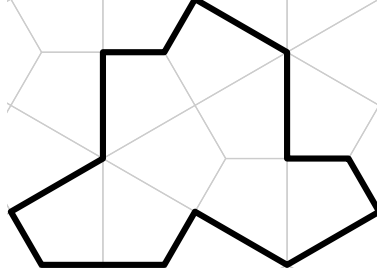


Figure 4: The Hat tile as a polykite.

2. *Decomposing any Hat tiling in kites yields the Kitegrid [SMKGS24, Lemma A.6]; in our terminology: any cotiler group of the Hat is included in the symmetry group Γ of the Kitegrid.*

Remark 11. *Though it is not formally stated in the original paper, the stabilizer of the Hat is finite, so that it is mildly aperiodic. This comes from the fact that Γ is virtually \mathbb{Z}^2 (see Section 4.2, thus any infinite subgroup contains a translation. This remark also holds for the weakly aperiodic monotile from [GT24, Theorem 1.3].*

Section 4.2 presents the symmetry group Γ of the Kitegrid; Section 4.3 describes the possible stabilizers of Hat tilings; and Section 4.4 introduces the Roach, a mildly aperiodic monotile in Γ derived from the Hat. Sections 4.2 and 4.3 are mostly independent.

4.2 The symmetry group of the Kitegrid

In this Section, we present the symmetry group of the Kitegrid, which we denote Γ . The readers who want to avoid some group formalism can refer to the Cayley graph from Figure 5.

Let R_6 be the rotation of angle $\pi/3$ centered at the tail of (any fixed translate of) the Kite and R_3 be the rotation of angle $2\pi/3$ centered at the head of the Kite. Let $\Gamma^+ := \langle R_6, R_3 \rangle$ be the subgroup of $\text{Isom}^+(\mathbb{R}^2)$ (the group of orientation-preserving isometries of \mathbb{R}^2) generated by these two rotations. Γ^+ is the cotiler of the Kite K yielding the Kitegrid, or equivalently, K is a fundamental domain for the action of Γ^+ (up to λ -negligible sets):

$$\mathcal{W} = \mathbb{R}^2 = \bigsqcup_{g \in \Gamma^+} g \cdot K, \text{ up to } \lambda\text{-negligible sets.}$$

We remark that $t_1 = R_6^{-1}R_3R_6^{-1}$ and $t_2 = R_6^{-2}R_3$ are two translations of the plane that generate a lattice $L = \langle t_1, t_2 \rangle \simeq \mathbb{Z}^2$. The regular hexagon $\bigsqcup_{i=0}^5 R_6^i \cdot K$ is a fundamental domain for the group L acting on the plane.

We also remark that the equilateral triangle $\bigsqcup_{i=0}^3 R_3^i \cdot K$ is a fundamental domain for the action of $\Gamma' := L \times \langle R_6^3 \rangle = \langle t_1, t_2, R_6^3 \rangle \simeq \mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z}$.

For the sake of completeness, we remark that $\Gamma^+ = L \times \langle R_6 \rangle = \Gamma' \times \langle R_6^2 \rangle = \Gamma' \times \langle R_3 \rangle \simeq \mathbb{Z}^2 \times \mathbb{Z}/6\mathbb{Z}$.

We now add the reflection β along the axis of the Kite going from the tail to the head. We get a group $\Gamma := \Gamma^+ \rtimes \langle \beta \rangle = \langle R_6, R_3, \beta \rangle$ which is the cotiler of the Semikite K_+ yielding the Semikitegrid; equivalently, K_+ is a fundamental domain for the action of Γ (up to λ -negligible sets). The reflections $\alpha = R_6^{-1} \circ \beta$ and $\gamma = R_3 \circ \beta$ along the other two sides of the Semikite (see Figure 6) are also in the group Γ .

Remark 12. *The group Γ^+ of the Kitegrid acts freely and transitively on the vertices of the Archimedean tiling 4.3.4.6; it acts freely on edges with two orbits: one orbit for the sides of the hexagons, one orbit for the sides of the triangles. Labelling the edges around the hexagons by R_6 and edges around triangles by R_3 , we get that the 1-skeleton of the 4.3.4.6 Archimedean tiling is the Cayley graph of Γ^+ with respect to generators $\{R_6, R_3\}$.*

The group Γ of the Semikitegrid acts freely and transitively on the vertices of the Archimedean tiling 4.6.12. Labelling the edges of the Archimedean tiling 4.6.12 by α , β and γ according to the edge of the Semikitegrid that they cross, we get that the 1-skeleton of the Archimedean tiling 4.6.12 is the Cayley graph of Γ with respect to generators $\{\alpha, \beta, \gamma\}$ (with non-oriented edges, because α , β and γ are involutions).

The symmetry group Γ of the Semikitegrid has presentation:

$$\Gamma = \langle \alpha, \beta, \gamma \mid \alpha^2, \beta^2, \gamma^2, (\alpha\beta)^6, (\beta\gamma)^3, (\alpha\gamma)^2 \rangle,$$

which is the presentation of a Coxeter group.

Remark that Γ is also the symmetry group of the Kitegrid.

Its index-2 subgroup Γ^+ has presentation $\langle R_3, R_6 \mid R_3^3, R_6^6, (R_3 R_6)^2 \rangle$ with $R_3 = \gamma\beta$ and $R_6 = \beta\alpha$.

Figure 5 shows the Cayley graph of the group Γ .

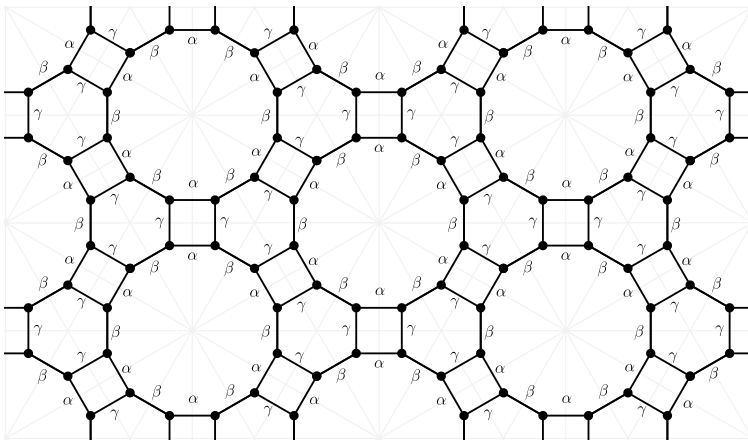


Figure 5: The Cayley graph of Γ . As the generators α , β and γ are involutions, the edges are not directed.

In two senses, these groups are not far from \mathbb{Z}^2 .

Remark 13. Γ^+ and Γ are virtually \mathbb{Z}^2 and both admit planar Cayley graphs (see Figures 3 and 5).

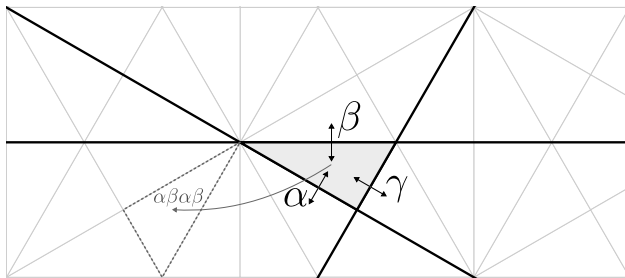


Figure 6: The symmetries of the Semikitegrid.

Any semikite in the tiling is the image of the grey semikite by a composition of the reflections α , β and γ . In particular the semikite in dotted lines is the image of the grey semikite by $\alpha\beta\alpha\beta$.

4.3 Stabilizers of Hat tilings

In this section, we study symmetries and periods of the Hat tilings. We chose to make these statements and proofs as independent as possible from the formalism of Section 3 so that a reader might choose to avoid most of the group-action formalism.

Recall that since the Hat tiles has no symmetries, for any Hat tiling \mathbf{T} with cotiler $C \subset \text{Isom}(\mathbb{R}^2)$ we have $\text{Stab}(C) = \text{Sym}(\mathbf{T})$.

Recall that R_3 is the rotation of angle $\frac{2\pi}{3}$ around the origin head of the Kite, that we assume to be on the origin $0 \in \mathbb{R}^2$.

Proposition 14. *The symmetry group of any Hat tiling is either $\{\text{id}\}$ or conjugated to $\{\text{id}, R_3, R_3^2\}$.*

Proof. From Item 2 of Theorem 10, any Hat tiling yields the Kitegrid when decomposing the tiles in kites, so that its symmetries must be included in the symmetry group Γ of the Kitegrid.

Let us analyse the subgroups of Γ which could be candidate for stabilizers of Hat tilings:

- From Item 1 of Theorem 10, no Hat tiling admits a translation as a symmetry.
- no reflection is possible: indeed the axis of a reflection symmetry of a Hat tiling cannot intersect the interior of any tile. As any Hat tiling induces a Kitegrid when decomposing the tiles, such an axis would be a line in the Kitegrid. There are only two such lines on the boundary of a Hat (which do not intersect its interior), both of which are obstructed by a neighbouring tile, see Figure 7. Hence a Hat tiling does not admit reflection symmetries.

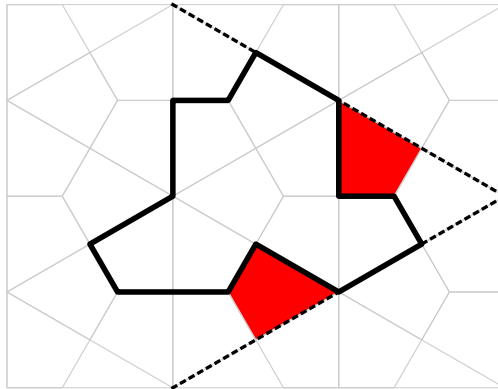


Figure 7: A Hat tile and its two non-intersecting adjacent KiteGrid axes (dashed), in red the obstructing kites: any neighbour Hat tile containing a red kite would intersect the corresponding axis.

- no nontrivial sliding reflection is possible: otherwise iterating it twice would give a translation symmetry.
- rotations of order 6 are not possible either: indeed the Hat tile does not have a symmetry of order 6, so a center of symmetry would have to be a vertex; this it is impossible because the Hat has only angles of $\frac{\pi}{2}$ and $\frac{2\pi}{3}$.

This leaves only the rotations of order 3 as possible symmetries for Hat tilings. Every rotation of order 3 in Γ is conjugated to R_3 by some translation of Γ . \square

This results gives us an upper bound on the stabilizers and symmetry group of Hat tilings.

We now prove that this bound is tight, that is: there exists a Hat tiling $\mathbf{T} = (\text{Hat}, C)$ such that $\text{Sym}(\mathbf{T}) = \text{Stab}(C) = \{\text{id}, R_3, R_3^2\}$.

As a tool to generate tilings, we introduce an auxiliary tileset of 4 polysemikites called $HTPF$, which was introduced in [SMKGS24], see Figure 9 in dotted lines, and a substitution on this tileset, see Figure 8. Note that the H , T , P and F tiles have an arrow as label, this arrow is an important part of the tiles; in particular adjacent tiles cannot have arrows facing each other.

We call **substitution** a function that maps each tile to a patch of tiles, see Figure 8. Note that in Figure 8, only the tiles in full lines and with labels are in the image, and the shapes in dotted lines are called

forced-neighbour tiles. The substitution is then extended to patches of tiles by applying it separately to each tile and gluing together the obtained patches (on the forced-neighbour tiles in our case). For more about geometric and combinatorial substitutions see for example [Jol13].

A substitution σ is called **well-defined** when for any single tile, the substitution can be iterated infinitely many times and in iterating the substitution the whole plane is covered. In this case, we call **admissible** any pattern that is a subpattern of $\sigma^k(t)$ for some integer k and tile t . An infinite tiling \mathbf{T} is called **valid** for the substitution if all of its patterns $M \sqsubset \mathbf{T}$ is admissible.

We call **scaling** a substitution σ for which there exists a scalar $\lambda > 1$ such that for any admissible pattern M , we have $\lambda \text{supp}(M) \subset \text{supp}(\sigma(M))$, where $\text{supp}(M)$ is the support of the pattern (the union of its tiles).

Theorem 15 (HTPF substitution [SMKGS24]).

- The HTPF substitution from Figure 8 is well-defined and scaling;
- Any HTPF tiling which is valid for the substitution can be decomposed into a Hat tiling (see Figure 9) with the same symmetries [SMKGS24, proof of Theorem 2.1].

The original proof that the Hat can tile the whole plane is based on the HTPF substitution. Note that in [SMKGS24], a second substitution (with clusters H_7 and H_8) is proposed, but it is less suited to our needs as it is not scaling.

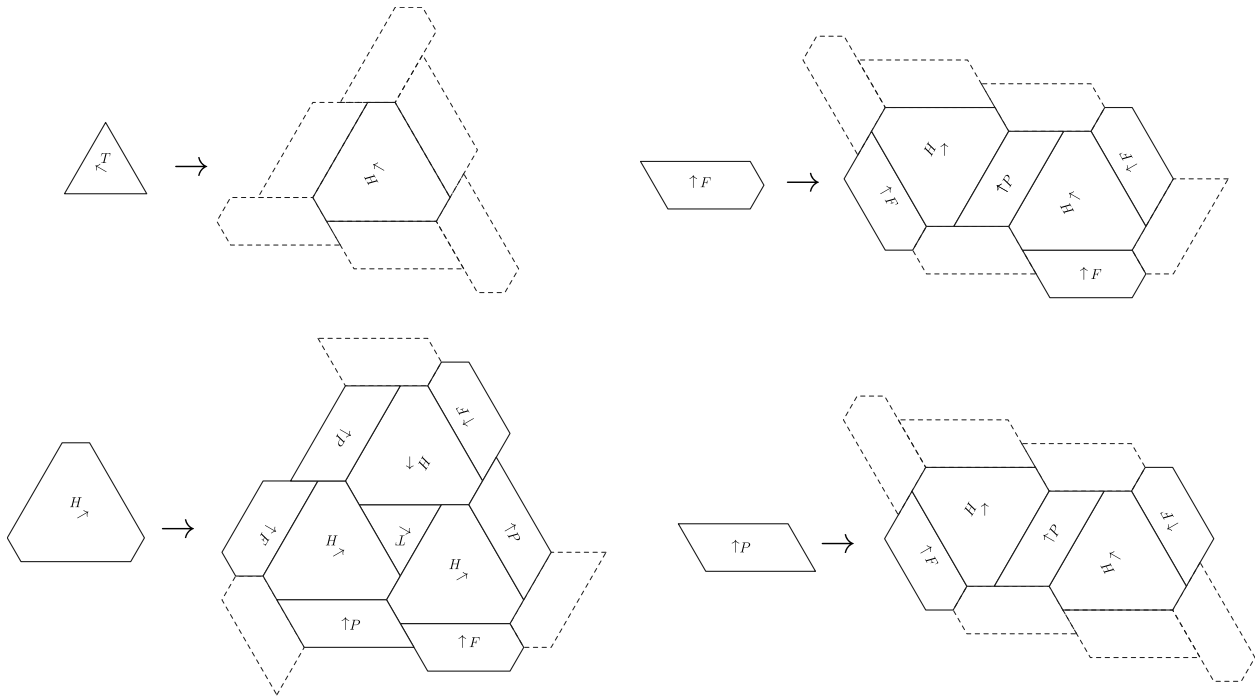


Figure 8: The HTPF substitution. In the image of the tiles, shapes drawn in dotted lines are forced-neighbour tiles.

Proposition 16. *There exists a Hat tiling \mathbf{T} with cotiler C whose stabilizer is exactly $\{\text{id}, R_3, R_3^2\}$, that is: $\text{Stab}(C) = \text{Sym}(\mathbf{T}) = \{\text{id}, R_3, R_3^2\}$.*

Proof. We use the HTPF substitution to construct a fixpoint (limit) tiling with symmetries $\{\text{id}, R_3, R_3^2\}$. We denote by σ the HTPF substitution.

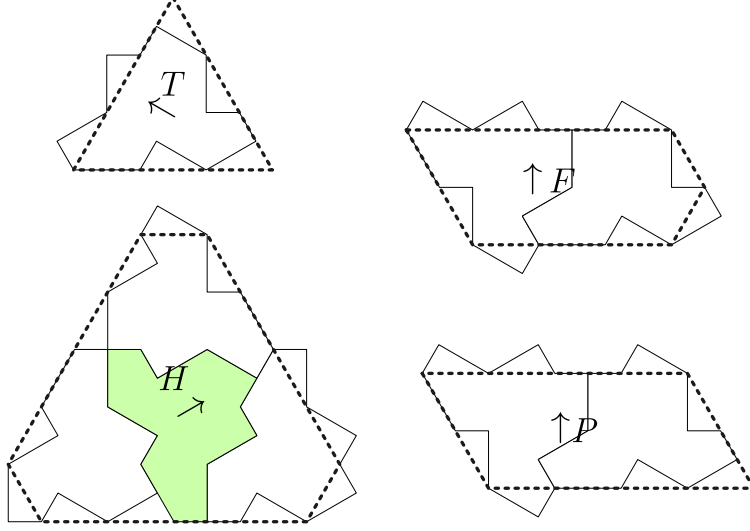


Figure 9: *HTPF* tiles as clusters of Hats, note that these clusters can also be defined as polysemikites.

We use the **fylfot** or F_3 pattern consisting of 3 F tiles (see Figure 10 left) with the central vertex on the origin $0 \in \mathbb{R}^2$. First note that this pattern is admissible for the substitution as it appears in $\sigma^2(H)$. When iterating σ on the fylfot, we see that F_3 is at the center of $\sigma^2(F_3)$ up to a rotation of angle π , so that F_3 appears exactly at the center of $\sigma^4(F_3)$. From this we deduce that the sequence $(\sigma^{4n}(F_3))_{n \in \mathbb{N}}$ is an increasing (for inclusion) sequence of patches. This, together with the fact that the substitution is well-defined and scaling, gives us the existence of a limit tiling $\mathbf{T}_{R_3}^{HTPF}$ to this sequence. $\mathbf{T}_{R_3}^{HTPF}$ is a fixpoint of σ^4 and is invariant under rotations R_3 and R_3^2 . Decomposing the *HTPF* clusters into Hat tiles in $\mathbf{T}_{R_3}^{HTPF}$, we obtain a Hat tiling $\mathbf{T}_{R_3}^{Hat}$ which also has symmetries $\{\text{id}, R_3, R_3^2\}$. \square

Remark 17. *As the *HTPF* substitution σ is sufficiently well-behaved, we can also construct a limit tiling without any non-trivial symmetry.*

*Indeed, a variation around classical recognizability arguments [Sol98] gives that every valid *HTPF* tiling has a unique preimage by the substitution. Also, no Hat decomposition of *HTPF* tile t has a 3-fold symmetry, and this is inherited to supertiles $\sigma^k(t)$, where $k \in \mathbb{N}$. The latter property forces any center of 3-fold symmetry in a tiling to be at the boundary of an infinite-level supertile, and the former states that the boundary of infinite-level supertiles is uniquely defined.*

As the triangle T is at the center of $\sigma^2(T)$ up to a rotation of angle $\frac{\pi}{3}$, we construct the fixpoint (limit) tiling $\mathbf{T}_{\text{id}}^{HTPF}$ by iterating σ^{12} from T . This tiling has no non-trivial symmetry because no point is at the boundary of an infinite-level supertile. We can then decompose $\mathbf{T}_{\text{id}}^{HTPF}$ in $\mathbf{T}_{\text{id}}^{Hat}$ without any non-trivial symmetry.

4.4 The Roach

Recall that $\Gamma^+ \curvearrowright \mathcal{W} = \mathbb{R}^2$ and $\Gamma \curvearrowright \mathcal{W}$ properly discontinuously, and their fundamental domains (up to λ -negligible sets) are the Kite K and the Semikite K_+ . The Hat is a polykite and a polysemikite, which allow to apply the work of Section 3.

We get

$$\Psi_{\Gamma^+, K}(\text{Hat}) = \{1, R_6, R_6^{-1}, R_6^{-2}, R_3, R_3^2, R_3 R_6^{-1}, R_3^2 R_6^{-1}\} \subset \Gamma^+$$

and

$$\Psi_{\Gamma, K_+}(\text{Hat}) = \{1, \alpha, \beta, \gamma, \alpha\beta, \beta\alpha, \beta\gamma, \gamma\beta, \alpha\beta\alpha, \beta\alpha\beta, \beta\alpha\gamma, \beta\gamma\beta, \gamma\beta\alpha, \alpha\beta\alpha\beta, \beta\alpha\gamma\beta, \gamma\beta\alpha\beta\} \subset \Gamma;$$

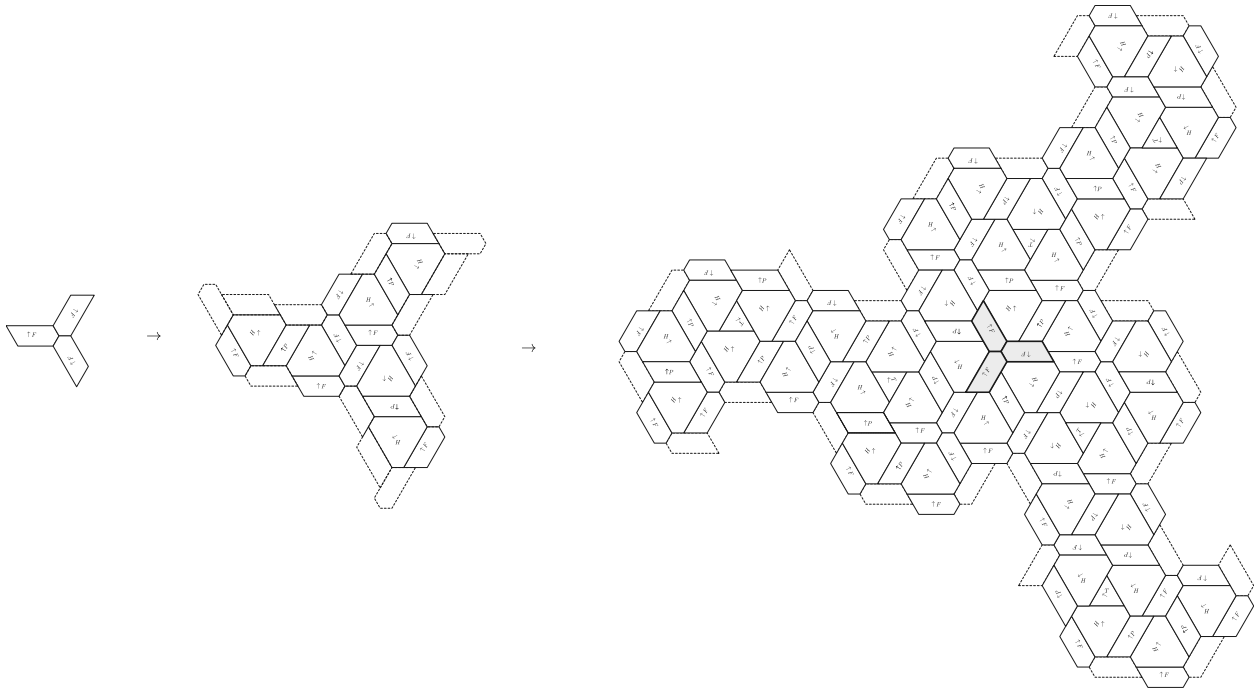


Figure 10: Iterating the *HTPF* substitution on the fylfot (left) pattern. We observe the fylfot at the center of the second iteration.

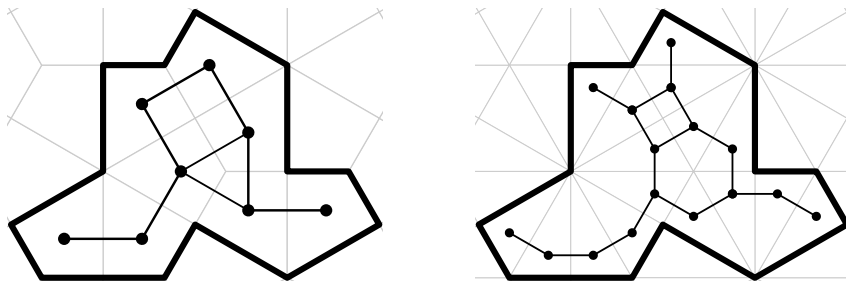


Figure 11: Group tiles induced in Γ^+ (left) and Γ (right) by the Hat.

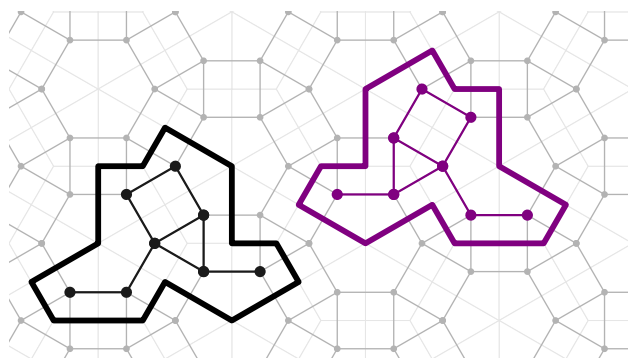


Figure 13: The 2-tile mildly aperiodic tileset induced by the Hat and its symmetric in Γ^+ (up to translation); the two tiles are not Γ^+ -translations of each other.

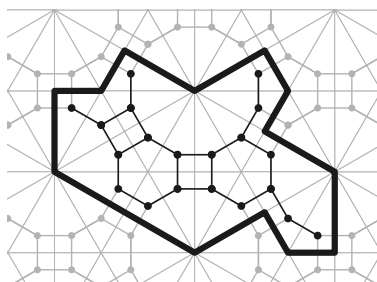


Figure 14: The Turtle tile [SMKGS24] induces a second mildly aperiodic monotile in Γ .

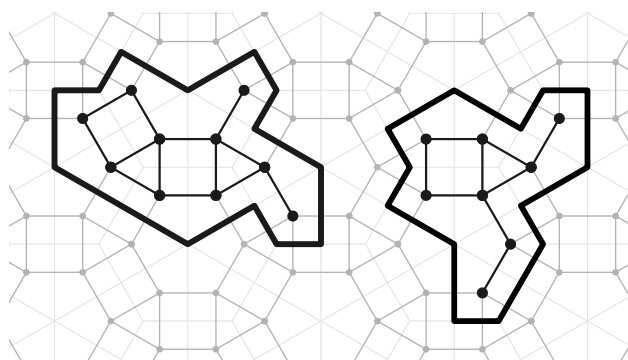


Figure 15: The Hat-Turtle system [SMKGS23] induces a 2-tile mildly aperiodic tileset in Γ^+ (without the reflections).

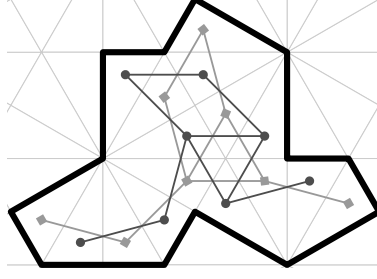


Figure 16: We can partition the vertices of the Roach into two copies of the Γ^+ -discrete Hat (from Figure 11, left), each of which corresponds to one orientation (even / odd signature of the transformation).

5 General version

In this section, we consider a topological space \mathcal{W} endowed with a nontrivial Borel measure λ , a group G acting on it by negligibility-preserving self-homeomorphisms, and a finite set \mathcal{T} of positive-measure tiles $T \subset \mathcal{W}$.

When is the discretization as performed in Section 3 possible? We need a transformation map turning geometric tiles into group tiles. The first candidate is the following. Given a base point $0 \in \mathcal{W}$, one can define its **orbit map** from G to \mathcal{W} : $\omega_0 : g \mapsto g \cdot 0$. One can build, from any set $T \subset \mathcal{W}$, the set $\Psi_0(T) := \omega_0^{-1}(T) = \{g \in G \mid g \cdot 0 \in T\} \subset G$.

In this section, we give two general statements: one for transitive actions (Subsection 5.1), and one for discrete actions (Subsection 5.2, somewhat generalizing Section 3).

5.1 Building tiles in continuous groups

Here is a continuous variation on our theorem. We include it for completeness and the parallelism with the discrete version.

We assume in this subsection that G is a topological group endowed with some nontrivial Borel measure. A map between measure spaces is **nonsingular** if the preimage of any negligible subset is negligible. It is **finite-measure-preserving** if the image of any finite-measure subset has finite measure. An action of a topological group G on \mathcal{W} is **proper** if for every compact $K, K' \subset \mathcal{W}$, $\{g \in G \mid g \cdot K \cap K'\}$ is compact.

The tile obtained in the group enjoys nice properties.

Remark 20.

1. If ω_0 is negligibility-preserving and T has positive measure, then $\Psi_0(T)$ has positive measure.
2. If ω_0 is finite-measure-preserving and T has finite measure, then $\Psi_0(T)$ has finite measure.
3. If $G \curvearrowright \mathcal{W}$ is proper and T is relatively compact, then $\Psi_0(T)$ is compact.

Proposition 21. Endow G with a measure and chose 0 such that ω_0 is nonsingular.

If C is a cotiler of \mathcal{T} in \mathcal{W} with respect to G , then C is a cotiler of $\Psi_0(\mathcal{T})$ in G , up to syntactically renaming each T into $\Psi_0(T)$.

Proof. By nonsingularity and the covering condition for tilings, for λ -almost all $g \in G$, there exists a unique $(g, T) \in C$ such that $g \cdot 0 \in h \cdot T$, which is equivalent to $h^{-1} \cdot g \in \Psi_0(T)$, and to $g \in h \cdot \Psi_0(T)$. This proves that

$$G \equiv_{\lambda} \bigsqcup_{(g,T) \in C} g \cdot \Psi_0(T). \quad \square$$

Theorem 22 (Main result; continuous setting). *If $G \curvearrowright \mathcal{W}$ is transitive and T is any tile, then $\omega_0(\Psi_0(T)) = T$. Moreover, if \mathcal{T} is a finite tiling, μ is a measure on G which makes ω_0 nonsingular and negligibility-preserving, then \mathcal{T} and $\Psi_0(\mathcal{T})$ have the same cotilers (up to syntactically renaming each T into $\Psi_0(T)$).*

Proof. $\omega_0(\Psi_0(T)) = G \cdot 0 \cap T$. If the action is transitive, then this is equal to T . By Proposition 21, the cotilers of \mathcal{T} are (up to name change) cotilers of $\Psi_0(\mathcal{T})$. Conversely, if $G \equiv_\mu \bigsqcup_{(g, \Psi_0(T)) \in C} g \cdot \Psi_0(T)$, then $G \cdot 0 \equiv_\lambda \bigsqcup_{(g, \Psi_0(T)) \in C} g \cdot \omega_0(\Psi_0(T))$. This gives $\mathcal{W} \equiv_\lambda \bigsqcup_{(g, \Psi_0(T)) \in C} g \cdot T$. By the assumption on the tiles, $\mathcal{W} \equiv_\lambda \bigsqcup_{(g, \Psi_0(T)) \in C} g \cdot T$. \square

5.2 Building tiles in discrete groups

Rather than tiling with respect to continuous groups (often Lie groups) such as $\text{Isom}(\mathbb{R}^d)$ like in the previous subsection, we are now interested in countable subgroups (like lattices over Lie groups), for which the orbit map is generally singular; hence Theorem 22 does not apply.

We propose two constructions for carrying a tiling inside the group.

5.2.1 Using the orbit map

In this subsection, G is a countable group, endowed with discrete topology and counting measure. Although Theorem 22 does not apply, we can compensate by a suitable choice of base point.

An immediate remark is that our discretization map is relevant for \equiv_λ -classes of subsets.

Remark 23. *If $0 \notin G \cdot (T \Delta T')$ (where Δ stands for the symmetric difference), then $\Psi_0(T) = \Psi_0(T')$. In particular, if $T \equiv_\lambda T'$, then for λ -almost every 0 , $\Psi_0(T) = \Psi_0(T')$.*

One says that the action $G \curvearrowright \mathcal{W}$ is λ -**properly discontinuous** if for every compact $K \subset \mathcal{W}$, only finitely many $g \cdot K$ intersect K λ -non-trivially. Note that this property is equivalent to: for every relatively compact (that is, included in a compact) $K, K' \subset \mathcal{W}$, only finitely many $g \cdot K$ intersect K' λ -non-trivially. Any properly discontinuous (in the sense that for every compact $K \subset \mathcal{W}$, only finitely many $g \cdot K$ intersect K) action is λ -properly discontinuous for every measure λ .

Our transformation map is nice in the sense that the tiles enjoy simpleness properties inherited from the action.

Proposition 24. *Let T be relatively compact and $G \curvearrowright \mathcal{W}$ be properly discontinuous (resp, λ -properly discontinuous), then $\Psi_0(T)$ is finite for every (resp, λ -almost every) $0 \in \mathcal{W}$.*

Proof. $\Psi_0(T)$ can be written as $\{g \in G \mid T \cap g \cdot \{0\} \neq \emptyset\}$, which is finite if the action is properly discontinuous. If the action is λ -properly discontinuous, this implies that $\mathcal{U} := \{g \in G \mid \lambda(T \cap g \cdot T) > 0\}$ is finite. By definition and countable additivity, $R := G \cdot \bigcup_{g \notin \mathcal{U}} T \cap g \cdot T$ has measure 0. For every $0 \notin R$, $\Psi_0(T) \subset \mathcal{U}$, which is finite. \square

Proposition 25. *Let G be a countable group, endowed with the counting measure. If C is a cotiler of \mathcal{T} in \mathcal{W} , then for almost every $0 \in \mathcal{W}$, C is a cotiler of $\Psi_0(\mathcal{T})$ (with respect to G) in G (up to syntactically renaming each T into $\Psi_0(T)$).*

Proof. Let us show the result for 0 in the set $\bigcap_{h \in G} h \cdot (\bigcup_{(g, T) \in C} g \cdot \Psi_0(T) \setminus \bigcup_{(g', T') \in C} g' \cdot \Psi_0(T) \cap g' \cdot \Psi_0(T'))$. By definition of tilings and countable additivity of the measure, this set has full measure.

By construction, for every $(g, T), (g', T') \in C$, $g \cdot \Psi_0(T) \cap g' \cdot \Psi_0(T') = \emptyset$, unless $(g, T) = (g', T')$; and for every $h \in G$, there exists $(g, T) \in C$ such that $h \cdot 0 \in g \cdot \Psi_0(T)$. \square

5.2.2 Decomposing tiles

We now propose an alternative approach to the orbit map.

We assume that G is a countable group acting cocompactly λ -properly discontinuously on \mathcal{W} and that \mathcal{T} is a finite set of relatively compact tiles.

Let F be a relatively compact fundamental domain. For $x \in F$, consider the set $\chi(x) := \{(g, T) \in G \times \mathcal{T} \mid x \in g \cdot T\}$ of tiles that could cover x according to some transformation. We also define the **cell** $K_x := \chi^{-1}(\{\chi(x)\}) \subset F$ of x .

Let $\mathcal{K} := \{K_x \mid x \in F, \lambda(K_x) > 0\}$.

Lemma 26.

1. Every $K \in \mathcal{K}$ has the form $K = \chi^{-1}(\{\chi(x)\})$ for some finite (possibly empty) $\chi(x)$.
2. \mathcal{K} is finite.
3. $\bigsqcup_{K \in \mathcal{K}} K \equiv_\lambda F$.

Proof. Since the action is λ -properly discontinuous and the tiles are relatively compact, the set S_T of $g \in G$ such that $g \cdot T$ intersects F λ -nontrivially is finite, for each $T \in \mathcal{T}$. The set $S := \bigsqcup_{T \in \mathcal{T}} S_T \times \{T\}$ is still finite, and by construction, $\forall (g, T) \notin S, \lambda(F \cap g \cdot T) = 0$. Let $F' := \{x \in F \mid \chi(x) \subset S\}$. We have:

$$\begin{aligned} F \setminus F' &= \bigcup_{(g, T) \notin S} \{x \in F \mid (g, T) \in \chi(x)\} \\ &= \bigcup_{(g, T) \notin S} F \cap g \cdot T. \end{aligned}$$

By definition of S (and countable additivity), $\lambda(F \setminus F') = 0$. We get that if $\lambda(K_x) > 0$, then $\chi(x) \subset S$. This gives Item 1, and since each $K \in \mathcal{K}$ has the form $\chi^{-1}(\chi(x))$ for some $\chi(x) \in S$, we get Item 2.

By definition of K_x as preimages, we know that they are all disjoint. Now

$$F \setminus \bigsqcup_{K \in \mathcal{K}} K \subset (F \setminus F') \cup \bigcup_{S' \subset S, \lambda(\chi^{-1}(S'))=0} \chi^{-1}(S').$$

By finite additivity, this set has measure 0. This gives Item 3. □

Lemma 27. *Every tile $T \in \mathcal{T}$ is poly- \mathcal{K} .*

Proof. By Item 3 of Lemma 26, $T \subset \mathcal{W} \equiv_\lambda \bigsqcup_{(g, K) \in G \times \mathcal{K}} g \cdot K$. Moreover, if $g \cdot K$ nontrivially intersects T , this means that K nontrivially intersects $g^{-1} \cdot T$: $(g^{-1}, T) \in \chi(K)$. Hence $K \subset g^{-1} \cdot T$, and $g \cdot K \subset T$. Finally, λ -proper discontinuity gives that the set $\Psi_{\mathcal{K}}(T)$ of (g, K) such that $g \cdot K$ intersects T nontrivially is finite. We obtain $T \equiv_\lambda \bigsqcup_{(g, K) \in \Psi_{\mathcal{K}}(T)} g \cdot K$. □

Theorem 28 (Main result; discrete setting). *Assume that G is countable and acts on \mathcal{W} cocompactly λ -properly discontinuously, and that \mathcal{T} is a finite set of relatively compact subsets of \mathcal{W} . Then there exists a finite set \mathcal{K} of relatively compact tiles yielding a grid on \mathcal{W} such that the tiles in \mathcal{T} are poly- \mathcal{K} s, and \mathcal{T} and $\Psi_0(\mathcal{T})$ have the same cotilers (up to syntactically renaming each T into $\Psi_0(T)$).*

Consequently, if G is countable and $G \curvearrowright \mathcal{W}$ is λ -properly discontinuous and admits some (weakly, mildly, strongly) aperiodic set of relatively compact tiles in \mathcal{W} , then one can find some (weakly, mildly, strongly) aperiodic set of finite tiles in G .

Proof. This comes directly from Lemma 26, Lemma 27, and Theorem 6. □

The following proposition links the discretization as poly- \mathcal{K} with the discretization Ψ_0 , which is always possible but allows in general only one inclusion between the cotiler sets. It can be applied for instance just after Theorem 28, to cocompact λ -properly discontinuous actions.

Proposition 29. *If \mathcal{K} yields a grid, then for λ -almost every $0 \in \mathcal{W}$, there exists a unique $(g, K) \in G \times \mathcal{K}$ such that $0 \in g \cdot K$. Then if T is poly- \mathcal{K} , then $\Psi_0(T) = \Psi_{g \cdot K}(T)$. In particular, if $0_K \in \mathcal{K}$ is such a generic point for each $K \in \mathcal{K}$, then*

$$T \equiv_\lambda \bigsqcup_{K \in \mathcal{K}} \Psi_{0_K}(T) \cdot K.$$

In particular, if \mathcal{T} is a set of poly- \mathcal{K} , then (up to syntactically renaming each T into $\Psi_0(T)$):

$$\mathcal{C}_G(\mathcal{T}) = \bigcap_{K \in \mathcal{K}} \mathcal{C}_G(\Psi_{0_K}(\mathcal{T})) = \bigcap_{0 \in \mathcal{W}'} \mathcal{C}_G(\Psi_0(\mathcal{T})),$$

for some full-measure subset $\mathcal{W}' \subset \mathcal{W}$.

Proof. Let $R := G \cdot \bigcup_{\lambda(g \cdot K \cap T)=0} (g \cdot K \cap T)$. By countable additivity, $\lambda(R) = 0$. By construction, for every $0 \in g \cdot K \setminus R$, $h \cdot 0 \in T \iff \lambda(hg \cdot K \cap T) > 0$, which means that $\Psi_0(T) = \Psi_{g \cdot K}(T)$. We conclude by Theorem 6. \square

5.3 Crystallographic groups

From the Hat and our constructions above, we get a finite monotile that tiles mildly aperiodically the group Γ . Moreover, the dual tiling of the Semikitegrid is an Archimedean tiling and its 1-skeleton can be labelled to become a Cayley graph of Γ . Thus, the geometric properties of the Hat carry towards the monotile of Γ : it is finite and connected.

This highly favorable context is that of crystallographic groups which we explore in this subsection.

A **crystallographic group** G is a group acting properly discontinuously and cocompactly on a euclidian space $\mathcal{W} = E$.

From the classical work of Bieberbach, the subgroup L of the translations in G is generated by n linearly independant translations, is isomorphic to \mathbb{Z}^n and has finite index in G . For such a crystallographic group G , there exists a compact convex polytope K that yields a grid. From Corollary 7 a poly- K monotile T is strongly (resp. mildly) aperiodic with respect to G if, and only if, the monotile $\Psi_K(T)$ is strongly (resp. mildly) aperiodic in G .

In dimension $n = 2$, following Grünbaum and Shephard [GS87, Statement 4.3.1], the crystallographic grid is the Laves tiling of an Archimedean tiling. The finite set $S = \{s \in G \mid s \cdot K \text{ and } K \text{ share a hyperface}\}$ generates G . Labelling its edges, the 1-skeleton of the dual Archimedean tiling is the Cayley graph of (G, S) (there is a subtlety regarding orientation of edges: S contains s^{-1} for each of its element s ; if such an s is a reflection, then one needs not choose an orientation). The geometric properties of T translate to geometric properties of $\Psi_K(T)$; for instance, if T is connected, then $\Psi_K(T)$ is connected.

We warn that we only consider tilings by T with respect to G , that is to say the cotilers are inside G . It may happen that using more isometries $G \leq G'$ allows new tilings by T . This situation happened for the Hat: adding a reflection α to $G = \Gamma^+$, we studied the group $G' = \Gamma$ and the Semikitegrid. The Hat does not tile with respect to Γ^+ , but tiles (aperiodically) with respect to Γ . For the Hat, no other isometry can be added: two adjacent copies of the Hat have to share edges and thus have to be drawn on the same Semikitegrid.

In this context, we raise the following two questions:

Question 30. *For which crystallographic groups does there exist a mildly aperiodic monotile?*

For the Laves tiling of which Archimedean tiling does there exist a mildly aperiodic poly- K monotile?

6 Conclusion

We are interested in the following metaquestion:

Question 31. *Which settings $G \curvearrowright \mathcal{W}$ admit aperiodic monotiles (or aperiodic finite tilesets)? Which countable groups G (with translation action)?*

After the big 2023 breakthroughs on monotileability, our work brings: a general framework to compare the results, a tool to transfer geometric tiles into group tiles, and an explicit construction for an original aperiodic monotile in a very small group (in the sense that it is close to a group where such monotiles do not exist). The result is comparable to [GT24, Theorem 1.3], yet more explicit. On the other hand, the direct product is replaced in our construction by some semi-direct product. This prevents to use the lifting operation from [MSS22] in order to recover (an explicit version for) [GT24, Corollary 1.5], building a weakly aperiodic monotile for some \mathbb{Z}^d (and some \mathbb{R}^d). It is not clear how to do a similar operation for semi-direct products.

Expliciting the stabilizers also quantifies somehow how far we are from strong aperiodicity. A great improvement would be to obtain strong aperiodicity (in the sense that the tilings have no symmetry at all, instead of the order-3 R_3 for our tile). One strategy could be to start modify our existing Roach tile into something that breaks the rotation (while still tiling), up to finitely extending the group Γ .

Another strategy would be to use our translation result to groupify a geometric strongly aperiodic monotile. But, though this is not emphasized in the geometric literature (where aperiodicity simply involves translations), it is surprising to notice that strongly aperiodic monotiles are unknown in all settings, including both the geometric and the group settings.

In particular, the Spectre from [SMKGS23] also admits tilings with stabilizer $\{\text{id}, R_3, R_3^2\}$ (which are combinatorially equivalent to tilings by the Hat). Anyway, though it is tempting to think of the Spectre as having fewer symmetries, because it tiles with respect to $\text{Isom}^+(\mathbb{R}^2)$ (unlike the Hat), on the contrary, the cotiler group of any tiling is much bigger for the Spectre than for the Hat, in the sense that the action is minimal (the orbits are dense), as seen by the fact that the tiling does not lie over an Archimedean tiling. This is also true for other famous examples of tilesets, like Penrose's.

Question 32. *Does there exist a geometric monotile whose cotilers have trivial stabilizers? Does there exist one which is union of tiles from an Archimedean tiling?*

Another remark is that our Theorem 6 is an equivalence, which could in theory lift some results from the group world to the geometric world: for instance, if one knows that G admits no aperiodic monotile, then no space \mathcal{W} over which it acts properly discontinuously admits an aperiodic monotile with respect to G . An example is that \mathbb{R}^2 cannot have aperiodic monotiles with respect to \mathbb{Z}^2 , thanks to the discrete result from [Bha20], but this is already folklore.

The link formalized in this paper could help formalize the parallelism between the theory of FLC tilings and symbolic dynamics that can be read in many results with similar flavor.

Finally, let us note that aperiodic tilings are an important ingredient for undecidability results (since settings where aperiodic tile sets do not exist imply decidability of tileability and of many properties), both in logical or in algorithmic terms. The first undecidability results about monotileability were achieved by [GT23] in some virtually- \mathbb{Z}^2 group. Along history, each work establishing the possibility of aperiodicity in some setting was simultaneous to or soon followed by some work establishing undecidability in this setting (and usually based on similar constructions). We are thus expecting more of them.

Question 33. *Is it undecidable whether a polygonal monotile tiles \mathbb{R}^2 ? Is it undecidable whether a finite monotile tiles the symmetry group Γ of the Semikite grid? Does there exist a dimension $d \in \mathbb{N}$ such that it is undecidable whether a finite monotile tiles \mathbb{Z}^d ?*

Acknowledgements: We thank Ch. Goodman-Strauss and T. Meyerovitch for discussions related to some points of this article, and N. Pytheas Fogg for raising the discussion that brought the original idea.

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