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REGULARITY AND TREND TO EQUILIBRIUM FOR A NON-LOCAL ADVECTION-DIFFUSION MODEL OF ACTIVE PARTICLES

LUCA ALASIO, JESSICA GUERAND, AND SIMON SCHULZ

ABSTRACT. We establish regularity and, under suitable assumptions, convergence to stationary states for weak solutions of a parabolic equation with a non-linear non-local drift term. This equation was derived from a model of active Brownian particles with repulsive interactions in the previous work [8], and incorporates advection-diffusion processes both in particle position and orientation. We apply De Giorgi's method and differentiate the equation with respect to the time variable iteratively to show that weak solutions become smooth away from the initial time. This strategy requires that we obtain improved integrability estimates in order to cater for the presence of the non-local drift. The instantaneous smoothing effect observed for weak solutions is shown to also hold for very weak solutions arising from distributional initial data; the proof of this result relies on a uniqueness theorem in the style of M. Pierre for low-regularity solutions. The convergence to stationary states is proved under a smallness assumption on the drift term.

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1. INTRODUCTION

This work is concerned with the study of the regularity and convergence to stationary states for the non-local advection-diffusion equation

$$(1.1) \quad \partial_t f + \text{Pe} \operatorname{div}((1 - \varrho) f \mathbf{e}(\theta)) = D_e \Delta f + \partial_\theta^2 f,$$

where $\varrho(t, x) = \int_0^{2\pi} f(t, x, \theta) d\theta$ is the *angle-independent density* and $\mathbf{e}(\theta) = (\cos \theta, \sin \theta)$, with periodic boundary conditions both in the space variable $x \in \Omega = (0, 2\pi)^2$ and the angle variable $\theta \in (0, 2\pi)$; we use the notation $\Upsilon = \Omega \times (0, 2\pi) = (0, 2\pi)^3$. The main unknown f represents the density of particles at time t and position x aligned in the direction θ . The operators div and Δ are taken with respect to x only. The constant parameters $\text{Pe} \in \mathbb{R}$ and $D_e > 0$ are called the *Péclet number* and *spatial diffusion coefficient*, respectively. This equation was formally derived from a many-particle system in [8]; the particles are said to be *active* in the sense that they are self-propelled, with velocity pointing in the direction of the vector $\mathbf{e}(\theta)$. For details concerning the derivation and well-posedness of (1.1) and the related models presented in [8], we refer the reader to the works [8, 9, 10].

Context. The regularity of elliptic and parabolic equations has been a topic of primordial importance within analysis and partial differential equations since the advent of Hilbert's 19th problem; we mention in particular the seminal contributions of De Giorgi [16, 17], Nash [42, 43], and Moser [40, 41]. The strategy in the aforementioned works is common: one estimates the L^p norms of the weak solution *locally* for arbitrary large p by an iteration procedure so as to obtain local L^∞ bounds. This interior boundedness is then used to obtain local oscillation estimates from which one infers Hölder continuity of

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the solution; either by contradiction arguments (*e.g.* [47]) or by quantitative methods (*e.g.* [29]). These approaches were subsequently refined and generalised to cover certain classes of degenerate elliptic equations by Ladyzhenskaya *et al.* [38], and similar degenerate parabolic equations by DiBenedetto *et al.* [20, 22, 23, 24, 21]. In recent years, the method has been employed to obtain analogous regularity results for the Navier–Stokes system [46], systems of reaction-diffusion equations [13, 28], equations incorporating fractional diffusion [11, 12, 25, 37], as well as for kinetic Fokker–Planck equations [31, 6, 30, 44, 48, 49]; also known as ultraparabolic or Kolmogorov-type equations. Some aspects of regularity for a class of Fokker–Planck equations without advection were discussed in [34], while a more general account of the underlying physics encapsulated by such equations may be found in the monograph [26].

The long-time behaviour of solutions to semi-linear advection-diffusion equations of the form

$$\partial_t v(t, \mathbf{x}) = \Delta_{\mathbf{x}} v + \operatorname{div}_{\mathbf{x}}(v \mathbf{b}) \quad (\mathbf{x} \in \mathbb{R}^d),$$

which includes (1.1), is also a classical area of interest. Solutions of such equations exhibit a rich panorama of possible properties, depending on the structure of the term \mathbf{b} ; some typical examples are as follows.

- (1) *Linear equation* $\mathbf{b} = G(\mathbf{x})$: the asymptotic behaviour depends on the existence of a suitable Lyapunov functional. A necessary condition for the existence of stationary states is (*cf.* [32]):

$$\lim_{\mathbf{x} \rightarrow \infty} G(\mathbf{x}) \cdot \mathbf{x} > 0.$$

This is guaranteed, for example, if $G = \nabla V$, with $V : \mathbb{R}^d \rightarrow \mathbb{R}$ being a convex function. Indeed exponential convergence to the stationary state may be obtained in this case thanks to the gradient flow approach, the entropy/entropy-dissipation method, or the Bakry–Emery Theory; see *e.g.* [2, Ch. 10], [35, Ch. 2], and [14].

- (2) *Aggregation-diffusion equation* $\mathbf{b} = \nabla W * v$: for sufficiently smooth kernels, the solution approaches the heat kernel in $L^1(\mathbb{R}^d)$ with a polynomial rate (see [15]). For singular interaction kernels blow-up may occur as in the Keller–Segel case.
- (3) *Blow-up and non-existence of stationary states*: examples exist even in the case of one spatial dimension. In the super-linear case we mention the case of $\mathbf{b} = v^q$, for $q > 1$, for which the critical mass was determined in [1]. In the non-local setting we mention the case of $\mathbf{b} = x + \delta + \int yv \, dy$ for which there are no stationary states on \mathbb{R} , see [5].
- (4) *Space-time periodic solutions*: we refer, for example, to [33, 4]. A very interesting example of advection-diffusion equation displaying “anomalous diffusion” was recently obtained in [3], where the vector field \mathbf{b} is constructed to be space-time periodic, divergence-free, Hölder-continuous and “fractal”.

Furthermore, the structure of the drift term \mathbf{b} has a profound impact on the regularity of the stationary solutions, and it is *a priori* not clear if such stationary states are smooth with respect to the space variable.

Motivation and originality. Equation (1.1), derived in [8] and analysed in [9], is different to the ones studied in the aforementioned works. It incorporates a drift-diffusion mechanism similar to kinetic Fokker–Planck equations, yet the drift term comprises the angle-independent density and is thereby non-local in addition to not being divergence-free. To the authors’ knowledge, the regularity and long-time behaviour of equations involving such drift terms has not been addressed in the existing literature, except under restrictive assumptions (see, *e.g.*, [5]); however, concerning existence and uniqueness for similar models, we highlight the recent work [7]. In particular, it is *a priori* not clear whether solutions are merely Hölder continuous away from the initial time or if they become infinitely-differentiable, nor is it obvious that they converge to stationary states. While some continuity results had been obtained in [9, §4] by employing the Duhamel principle and assuming more regularity on the initial data, a thorough local regularity analysis had yet to be performed, which motivates the need for the current paper. Additionally, the aforementioned examples of long-time behaviour illustrate that, to the best of our knowledge, there is no unified approach to the study of convergence to equilibrium for generic advection-diffusion equations. We thereby propose an approach based on *a priori* estimates tailored to the problem at hand and, as a result, obtain strong regularity properties.

The present contribution provides a regularity analysis *à la* De Giorgi for (1.1). We first show interior local boundedness of weak subsolutions of (1.1) by an iteration procedure, which then enables

us to show smoothness of weak solutions away from the initial time by means of a bootstrapping argument; the periodicity in space-angle means that we do not need to restrict ourselves to small subcylinders with respect to these variables, whence our result is global in space-angle. We then extend this higher-regularity result to very weak solutions, which arise from merely distributional initial data; the proof relies on a uniqueness theorem for such low-regularity solutions (see Theorem 2). Throughout the paper, we shall make use of the fact that the angle-independent density ρ (and its derivatives) admits higher integrability than the original unknown f ; reminiscent of velocity averaging lemmas in kinetic theory (see, *e.g.*, [45]). Finally, by adapting the aforementioned bootstrapping approach, we show the smoothness of stationary solutions. We also show exponential convergence to these stationary states with respect to the L^2 norm under a smallness assumption on the Péclet number.

Plan of the paper. The rest of the paper is organised as follows. In §1.1 we recall the original notions of weak and very weak solutions for the equation (1.1) introduced in the previous work [9], and provide the statements of our main theorems in §1.2. In §2 we provide an alternative notion of weak solution which is better suited to our regularity analysis as well as the main rescaling lemma used for the De Giorgi method. §3 is concerned with the boundedness of weak solutions; we show the local-in-time boundedness away from the initial time in §3.1 for generic admissible initial data, and also provide a global-in-time estimate in §3.2 for more regular initial data. We study the higher regularity of weak solutions in §4 by means of a bootstrapping argument, which yields smoothness of weak solutions away from the initial time. §5 generalises the higher regularity results obtained for weak solutions in §4 to the very weak solutions, where we also prove a uniqueness result for very weak solutions using an argument *à la* Michel Pierre. In §6 we prove the smoothness of non-negative stationary solutions, and also show the convergence to stationary states under the assumption of small Péclet number. Appendix A is devoted to the proofs of some technical lemmas and is divided into two parts: Appendix A.1 contains the proof of a version of a Calderón–Zygmund Theorem applicable to our periodic setting; Appendix A.2 contains the proof of the alternative weak formulation introduced in §2.

Notations and functional setting. Throughout this work, we use the shorthand $\boldsymbol{\xi} = (x, \theta)$ for the concatenated space-angle variable. The letter C will always denote a positive constant independent of $t, \boldsymbol{\xi}$, unless explicitly stated otherwise, and may change from line to line. For a point $(t_0, \boldsymbol{\xi}_0)$, we define the *open parabolic cylinder* of radius r by

$$(1.2) \quad Q_r(t_0, \boldsymbol{\xi}_0) := \{(t, \boldsymbol{\xi}) : -r^2 < t - t_0 \leq 0, |\boldsymbol{\xi} - \boldsymbol{\xi}_0| < r\},$$

and we shall write $Q_r(0) = Q_r$ when $(t_0, \boldsymbol{\xi}_0) = 0$. The domain on which the equation is posed is denoted by $\Upsilon_T := (0, T) \times \Upsilon$, and similarly we write $\Omega_T = (0, T) \times \Omega$. We also define the usual parabolic norm

$$(1.3) \quad \|f\|_{\mathcal{P}}^2 := \|f\|_{L^\infty(0, T; L^2(\Upsilon))}^2 + \|\nabla_{\boldsymbol{\xi}} f\|_{L^2(\Upsilon_T)}^2,$$

and the corresponding space \mathcal{P} of functions $f : \Upsilon_T \rightarrow \mathbb{R}$ with $\|f\|_{\mathcal{P}}$ finite. Unless stated explicitly otherwise, the symbols $\operatorname{div}, \nabla, \Delta$ denote divergences, gradients, and Laplacians taken with respect to the space variable x , while the symbols $\operatorname{div}_{\boldsymbol{\xi}}, \nabla_{\boldsymbol{\xi}}, \Delta_{\boldsymbol{\xi}}$ denote operators with respect to the concatenated space-angle variable $\boldsymbol{\xi} = (x, \theta)$.

The topological dual of a function space E is denoted by E' , and the bracket $\langle \cdot, \cdot \rangle$ denotes the dual pairing between elements of a space and its dual. In what follows, the functions spaces denoted by $Z_{\text{per}}(S)$ for $S \in \{(0, 2\pi)^d\}_{d=1}^3$, *i.e.* $d = 2$ corresponding to Ω and $d = 3$ to Υ , with $Z \in \{L^p, W^{k,p}, C^k\}$, are understood to mean

$$Z_{\text{per}}(S) := \{g : \mathbb{R}^d \rightarrow \mathbb{R} : \|g\|_{Z(S)} < \infty, \text{ and } g(y + 2\pi \mathbf{e}_i) = g(y) \ \forall y \in \mathbb{R}^d, i \in \{1, \dots, d\}\},$$

where $\{\mathbf{e}_i\}_{i=1}^d$ is the standard basis of \mathbb{R}^d . We also refer to such functions as being *S-periodic*; by which we mean that such functions are periodic with periodic cell S . We denote the spaces

$$Y := H_{\text{per}}^1(\Upsilon) \cap L_{\text{per}}^2(\Omega; H_{\text{per}}^2(0, 2\pi)), \quad X := L^2(0, T; Y).$$

We denote the space-angle average of a function f by

$$\langle f \rangle := \int_{\Upsilon} f \, d\boldsymbol{\xi} = \frac{1}{|\Upsilon|} \int_{\Upsilon} f \, d\boldsymbol{\xi}.$$

1.1. Definitions and problem set-up. For the purposes of the regularity analysis, the coefficients Pe and D_e appearing in (1.1) do not matter; the following paragraph is concerned with rescaling the variables so as to make them both vanish from the equation. By defining $(a, b, c) := (D_e \text{Pe}^{-2}, D_e \text{Pe}^{-1}, \sqrt{D_e} \text{Pe}^{-1})$, we see that the rescaled functions

$$\tilde{f}(t, x, \theta) := f(at, bx, c\theta), \quad \tilde{\varrho}(t, x) := c \int_0^{\frac{2\pi}{c}} \tilde{f}(t, x, \theta) d\theta = \varrho(at, bx), \quad \tilde{\mathbf{e}}(\theta) := \mathbf{e}(c\theta),$$

satisfy, on the rescaled domain $(0, a^{-1}T) \times b^{-1}\Omega \times (0, c^{-1}2\pi)$, the equation

$$\partial_t \tilde{f} + \text{div}((1 - \tilde{\varrho})\tilde{f}\tilde{\mathbf{e}}(\theta)) = \Delta \tilde{f} + \partial_\theta^2 \tilde{f},$$

whence the constants Pe, D_e have vanished. The above rescaling alters the periodicity of the functions $\tilde{f}, \tilde{\varrho}$ and $\tilde{\mathbf{e}}(\theta)$. This is of no importance whatsoever, since the proofs of the well-posedness results of [9] extend easily to any choice of space-angle period. Moreover, the period with respect to the variable x need not be equal to that with respect to θ .

Therefore, for the sake of clarity and concision, the regularity analysis in this work is concerned with the study of the drift-diffusion equation

$$(1.4) \quad \partial_t f + \text{div}_\xi(Uf) = \Delta_\xi f \quad \text{in } \Upsilon_T,$$

where $\xi = (x, \theta)$ is the concatenated space-angle variable and

$$(1.5) \quad U = (1 - \varrho) \begin{pmatrix} \mathbf{e}(\theta) \\ 0 \end{pmatrix}.$$

We emphasise that, if the solution of (1.4) is smooth, then so is the solution of (1.1) by the rescaling above. Likewise, if the solution of (1.4) is bounded in a particular space, then so is the solution of (1.1); modulo the specific bound being influenced by Pe, D_e . For this reason, all estimates in §3–5 are written for equation (1.4). On the other hand, the smallness of Pe is used in §6, and that is the only place in the paper where the values of Pe, D_e influence the results.

For convenience, we recall the definitions of weak and very weak solutions introduced in [9], as well as the well-posedness results proved therein.

Definition 1.1 (Notions of Solution).

We introduce the notions of *weak solution* and *very weak solution*.

(1) A *weak solution* of (1.1) with non-negative initial data $f_0 \in L^2_{\text{per}}(\Upsilon)$ satisfying

$$(1.6) \quad \varrho_0(x) = \int_0^{2\pi} f_0(x, \theta) d\theta \in [0, 1] \text{ a.e. } x \in \Omega$$

is a function $f \in C([0, T]; L^2_{\text{per}}(\Upsilon)) \cap L^2(0, T; H^1_{\text{per}}(\Upsilon))$ with $\partial_t f \in L^2(0, T; (H^1_{\text{per}})'\!(\Upsilon))$ such that, for all $\varphi \in L^2(0, T; H^1_{\text{per}}(\Upsilon))$, there holds

$$(1.7) \quad \begin{cases} \langle \partial_t f, \varphi \rangle = \text{Pe} \int_{\Upsilon_T} (1 - \varrho) f \mathbf{e}(\theta) \cdot \nabla \varphi d\xi dt - D_e \int_{\Upsilon_T} \nabla f \cdot \nabla \varphi d\xi dt - \int_{\Upsilon_T} \partial_\theta f \cdot \partial_\theta \varphi d\xi dt, \\ \lim_{t \rightarrow 0^+} f(t) = f_0 \text{ in } L^2_{\text{per}}(\Upsilon), \end{cases}$$

where $\varrho(t, x) = \int_0^{2\pi} f(t, x, \theta) d\theta$ satisfies

$$(1.8) \quad 0 \leq \varrho(t, x) \leq 1 \text{ a.e. } (t, x) \in \Omega_T.$$

(2) A *very weak solution* of (1.1) with non-negative initial data $f_0 \in L^2_{\text{per}}(\Omega; (H^1_{\text{per}})'\!(0, 2\pi))$ satisfying

$$(1.9) \quad \varrho_0(x) = \langle f_0(x, \cdot), 1 \rangle \in [0, 1] \text{ a.e. } x \in \Omega$$

is a function

$$f \in L^2(0, T; H^1_{\text{per}}(\Omega; (H^1_{\text{per}})'\!(0, 2\pi))) \cap L^\infty(0, T; L^2_{\text{per}}(\Omega; (H^1_{\text{per}})'\!(0, 2\pi))) \cap L^2(\Upsilon_T)$$

with $\partial_t f \in X'$ such that, for all $\varphi \in X$, there holds

$$(1.10) \quad \begin{cases} \langle \partial_t f, \varphi \rangle = \text{Pe} \int_{\Upsilon_T} (1 - \varrho) f \mathbf{e}(\theta) \cdot \nabla \varphi d\xi dt - D_e \int_{\Upsilon_T} \nabla f \cdot \nabla \varphi d\xi dt + \int_{\Upsilon_T} f \partial_\theta^2 \varphi d\xi dt, \\ \lim_{t \rightarrow 0^+} f(t) = f_0 \text{ in } Y', \end{cases}$$

where $\varrho(t, x) = \int_0^{2\pi} f(t, x, \theta) d\theta$ satisfies the estimate (1.8).

We recall that it was proved in [9] that, if the initial data $f_0 \in L^2_{\text{per}}(\Upsilon)$ is non-negative and satisfies (1.6), then there exists a unique weak solution of (1.1). Similarly, if the initial data $f_0 \in L^2_{\text{per}}(\Omega; (H^1_{\text{per}})'(0, 2\pi))$ is non-negative and satisfies (1.9), then there exists a very weak solution of (1.1). Furthermore, these solutions are *global-in-time* in the sense that they exist on the time interval $(0, T)$ for all $T > 0$.

We also recall that for f a weak (or very weak) solution of (1.4), the density ϱ satisfies

$$(1.11) \quad \partial_t \varrho + \operatorname{div}((1 - \varrho)\mathbf{p}) = \Delta \varrho$$

in the weak sense, with $\varrho \in L^\infty(\Upsilon_T) \cap L^2(0, T; H^1_{\text{per}}(\Omega))$. The vector quantity \mathbf{p} is the *polarization*, defined by

$$(1.12) \quad \mathbf{p}(t, x) := \int_0^{2\pi} f(t, x, \theta) \mathbf{e}(\theta) d\theta,$$

which, by virtue of $0 \leq \varrho \leq 1$ a.e. and the non-negativity of f , satisfies

$$(1.13) \quad |\mathbf{p}(t, x)| \leq 1.$$

Before proceeding to the main results, we introduce some technical results that will be used recurrently. In this work, we shall frequently employ a well-known interpolation inequality; we refer to it throughout the paper as the *Interpolation Lemma*.

Lemma 1.2 (DiBenedetto, Proposition 3.2 of §1, [19]). *Let $d \in \mathbb{N}$ and $\omega \subset \mathbb{R}^d$ have piecewise smooth boundary, and let $p, m \geq 1$. There exists a positive constant C depending only on d, p, m and the structure of $\partial\omega$ such that, for all*

$$v \in L^\infty(0, T; L^m(\omega)) \cap L^p(0, T; W^{1,p}(\omega)) =: V^{m,p},$$

there holds

$$\|v\|_{L^q((0,T) \times \omega)} \leq C \left(1 + \frac{T}{|\omega|^{\frac{d(p-m)+mp}{md}}} \right)^{\frac{1}{q}} \|v\|_{V^{m,p}}, \quad q = p \frac{d+m}{d}.$$

We shall also employ the following version of the Calderón–Zygmund Theorem applicable to our periodic setting. The proof is delayed to Appendix A.1. We do not claim that this result is sharp nor that it is novel; it is merely sufficient for our purposes.

Lemma 1.3 (Periodic Calderón–Zygmund Inequality). *Let $p \in (1, \infty)$. There exists a positive constant $C = C(p, \Upsilon)$ such that for all $v \in W^{1,p}_{\text{per}}(\Upsilon)$ with $\Delta_{\xi} v \in L^p(\Upsilon)$, there holds*

$$\|\nabla_{\xi}^2 v\|_{L^p(\Upsilon)} \leq C \left(\|\Delta_{\xi} v\|_{L^p(\Upsilon)} + \|v\|_{W^{1,p}(\Upsilon)} \right).$$

In the case $p = 2$, there holds

$$\|\nabla_{\xi}^2 v\|_{L^2(\Upsilon)} = \|\Delta_{\xi} v\|_{L^2(\Upsilon)}.$$

1.2. Main theorems. We state our main results, which are essentially partitioned into three: regularity of the time-dependent solutions, regularity of stationary solutions, and convergence to stationary states. For the regularity, as already mentioned, we first prove the smoothness of weak solutions away from the initial time, and then extend this result to the very weak solutions.

Our first main theorem is the following.

Theorem 1 (Smoothness away from Initial Time). *Assume $f_0 \in L^2_{\text{per}}(\Upsilon)$ is non-negative and satisfies (1.6), $T > 0$, and let f be the unique weak solution of (1.1) with initial data f_0 . Then, for a.e. $t \in (0, T)$, there holds $f \in C^\infty((t, T) \times \mathbb{R}^3)$.*

We then establish that very weak solutions coincide with weak solutions away from the initial time.

Theorem 2 (Uniqueness for Very Weak Solutions). *For any $T > 0$, let f be a very weak solution of (1.1). Then, for a.e. $t \in (0, T)$, f coincides with the unique weak solution of (1.1) on the interval (t, T) with initial data $f(t, \cdot)$.*

From Theorems 1 and 2 we obtain the regularity result for very weak solutions.

Theorem 3 (Regularity for Very Weak Solutions). *Assume $f_0 \in L^2_{\text{per}}(\Omega; (H^1_{\text{per}})'(0, 2\pi))$ is non-negative and satisfies (1.9), $T > 0$, and let f be a very weak solution of (1.1) with initial data f_0 . Then, for a.e. $t \in (0, T)$, there holds $f \in C^\infty((t, T) \times \mathbb{R}^3)$.*

We also record the following global-in-time boundedness for initial data in L^∞ , based on an iterative argument which had been used in the context of degenerate diffusion equations with drift (cf. [36]); this result is proved in §3.2.

Theorem 4 (Global-in-time Boundedness for Bounded Initial Data). *Assume $f_0 \in L^\infty(\Upsilon)$ is non-negative and satisfies (1.6), and let f be the unique weak solution of (1.1) with initial data f_0 . Then there holds the global-in-time estimate*

$$\|f\|_{L^\infty((0, \infty) \times \Upsilon)} \leq C(\text{Pe}, D_e, \|f_0\|_{L^\infty(\Upsilon)}).$$

The final section of this manuscript is concerned with the long-time behaviour of solutions of equation (1.1). Our main results are concerned with the regularity of solutions of the stationary elliptic problem

$$(1.14) \quad \text{Pe} \operatorname{div}((1 - \varrho_\infty)f_\infty \mathbf{e}(\theta)) = D_e \Delta f_\infty + \partial_\theta^2 f_\infty,$$

and the convergence of the time-dependent solutions to such stationary solutions.

Theorem 5 (Smoothness of Stationary States). *Let $f_\infty \in H^1_{\text{per}}(\Upsilon)$ with $\varrho_\infty = \int_0^{2\pi} f_\infty \, d\theta \in [0, 1]$ be a non-negative periodic weak solution of (1.14), i.e. for all $\phi \in H^1_{\text{per}}(\Upsilon)$ there holds*

$$\text{Pe} \int_\Upsilon (1 - \varrho_\infty) f_\infty \mathbf{e}(\theta) \cdot \nabla \phi \, d\xi = D_e \int_\Upsilon \nabla_\xi f_\infty \cdot \nabla_\xi \phi \, d\xi.$$

Then, f_∞ is a smooth periodic function on \mathbb{R}^3 .

Theorem 6 (Convergence to Constant Stationary States for Small Péclet Number). *Assume $f_0 \in L^2_{\text{per}}(\Upsilon)$ is non-negative and satisfies (1.6), and let f be the unique solution of (1.1) with initial data f_0 . Assume that there holds*

$$|\text{Pe}| < \frac{\min\{D_e, 1\}}{2\sqrt{2}\pi C_P(1 + \langle f_0 \rangle)},$$

where C_P is the Poincaré constant associated to Υ , and define

$$(1.15) \quad \kappa := \frac{1}{2} \left(\frac{1}{2} C_P^{-2} \min\{D_e, 1\} - \frac{(2\pi)^2 \text{Pe}^2 (1 + \langle f_0 \rangle)^2}{\min\{D_e, 1\}} \right) > 0.$$

Then,

$$\|f(t, \cdot) - \langle f_0 \rangle\|_{L^2(\Upsilon)} \leq e^{-\kappa t} \|f_0 - \langle f_0 \rangle\|_{L^2(\Upsilon)} \quad \text{for all } t \geq 0.$$

Remark 1.4 (Stationary States). We remark that Theorem 6 shows that, under the aforementioned assumptions on the initial data and on the Péclet number, all weak solutions of (1.1) converge to a constant stationary state in the limit of infinite time; by Theorem 2 this is also the case for all very weak solutions. We note that this result is in accordance with the linear stability analysis performed in [8, §3]. For large Péclet number, the simulations in [8, §4] suggest that phase separation occurs, and we do not expect convergence to a constant stationary state. The study of the long-time behaviour of the solutions for large Péclet number will be the subject of future investigations.

To conclude this subsection, we briefly outline the mechanism by which the equation improves the regularity of its solution.

Remark 1.5 (Regularity Bootstrap). One begins by applying De Giorgi's method/Moser's iteration technique to (1.4) and obtain that $f \in L^\infty((t, T) \times \Upsilon)$ for a.e. $t > 0$. This boundedness is then sufficient to perform the classical H^2 -type estimate on (1.4); yielding boundedness in $L^2((t, T) \times \Upsilon)$ both for $\Delta_\xi f$ and $\partial_t f =: \dot{f}$. Further work shows that $\dot{f} \in L^\infty(t, T; L^2(\Upsilon)) \cap L^2(t, T; H^1(\Upsilon))$ satisfies

$$(1.16) \quad \partial_t \dot{f} + \operatorname{div}((\dot{f}(1 - \varrho) - f \dot{\varrho}) \mathbf{e}(\theta)) = \Delta_\xi \dot{f};$$

and, *a priori*, the boundedness of $\dot{\varrho}$ in $L^2((t, T) \times \Upsilon)$ is insufficient to apply De Giorgi's method to the above and obtain boundedness of \dot{f} in $L^\infty((t, T) \times \Upsilon)$. Indeed, it appears at first glance from the formula $\dot{\varrho} = \int_0^{2\pi} \dot{f} \, d\theta$ that $\dot{\varrho}$ inherits the same boundedness properties as \dot{f} and none more. However, the dimensionality reduction in the angle-independent density plays a crucial role. Using the Interpolation

Lemma 1.2, one obtains $\dot{f} \in L^{10/3}(\Upsilon_T)$, as $\Upsilon \subset \mathbb{R}^3$; see Proof of Lemma 4.5. Meanwhile, using this very same interpolation result, since $\Omega \subset \mathbb{R}^2$, we obtain that $\dot{\varrho} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ in fact belongs to $L^4(\Omega_T)$. This improved integrability in $\dot{\varrho}$ leads to classical H^2 -type estimate on the equation

$$\partial_t \dot{\varrho} + \operatorname{div}(-\mathbf{p}\dot{\varrho} + (1 - \varrho)\dot{\mathbf{p}}) = \Delta \dot{\varrho}, \quad \text{where} \quad \mathbf{p}(t, x) = \int_0^{2\pi} f(t, x, \theta) \mathbf{e}(\theta) d\theta,$$

which corresponds to equation (1.11) differentiated in time, which yields higher integrability of $\dot{\varrho}$ again by interpolation. We deduce a sufficient gain of integrability in the non-local drift term of (1.16) to then apply De Giorgi's method again and obtain boundedness in L^∞ of the time derivative \dot{f} . This procedure is then performed iteratively for all time-derivatives $\partial_t^n f$, from which we deduce smoothness of the solution away from the initial time.

2. PRELIMINARY NOTIONS

In this section, we present the main rescaling lemma for De Giorgi's method, for which we introduce an alternative weak formulation of the equation; we explain our reasons for doing so in the paragraphs that follow.

The strategy of our regularity analysis is to use the method of De Giorgi to obtain interior regularity, which involves "zooming in" on subcylinders to obtain local boundedness. Central to this strategy is the appropriate parabolic rescaling of the functions at hand to deduce the required *localised* estimates by first obtaining analogous bounds on the unit cylinder Q_1 .

It is apparent that the aforementioned rescaling affects the periodicity of the test functions that can be inserted into the weak formulation of Definition 1.1. In turn, it is more convenient to employ an alternative weak formulation, for which we do not require the test functions to be periodic; this is encapsulated in the following lemma, the proof of which is delayed to Appendix A.2.

Lemma 2.1 (Alternative Weak Formulation). *Assume $f_0 \in L^2_{\text{per}}(\Upsilon)$ is non-negative and satisfies (1.6), and let $f \in C([0, T]; L^2_{\text{per}}(\Upsilon)) \cap L^2(0, T; H^1_{\text{per}}(\Upsilon))$ with $\partial_t f \in L^2(0, T; (H^1_{\text{per}})'(\Upsilon))$ be the unique weak solution of (1.1) with initial data f_0 . Then, for all $\varphi \in C^\infty([0, T] \times \mathbb{R}^3)$ with $\varphi(t, \cdot) \in C_c^\infty(\mathbb{R}^3)$ for all t and for a.e. $t_1, t_2 \in [0, T]$, there holds*

$$(2.1) \quad \int_{t_1}^{t_2} \int_{\mathbb{R}^3} f \partial_t \varphi d\xi dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (1 - \varrho) f \mathbf{e}(\theta) \cdot \nabla \varphi d\xi dt - \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \nabla_\xi f \cdot \nabla_\xi \varphi d\xi dt \\ = \int_{\mathbb{R}^3} f \varphi d\xi \Big|_{t_2} - \int_{\mathbb{R}^3} f \varphi d\xi \Big|_{t_1}.$$

With this alternative weak formulation at our disposal, we define our notion of weak subsolution for the generalisation

$$(2.2) \quad \partial_t f + \operatorname{div}_\xi(Uf + V) = \Delta_\xi f$$

of the equation (1.4), where $V \in L^q(\Upsilon_T)$ for $q > 5$, which is the form which we shall use when employing De Giorgi's method. The reason as to why we generalise the analysis to include the term V will be made clear in §4.

Definition 2.2 (Weak Subsolution). We say that $f \in C([0, T]; L^2_{\text{loc}}(\mathbb{R}^3)) \cap L^2(0, T; H^1_{\text{loc}}(\mathbb{R}^3))$ with $\partial_t f \in L^2(0, T; (H^1_{\text{loc}})'(\mathbb{R}^3))$ is a *weak subsolution* of (2.2) if, for all non-negative $\varphi \in C^\infty(0, T; C_c^\infty(\mathbb{R}^3))$ and for a.e. $t_1, t_2 \in [0, T]$, there holds

$$(2.3) \quad - \int_{t_1}^{t_2} \int_{\mathbb{R}^3} f \partial_t \varphi d\xi dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \nabla_\xi f \cdot \nabla_\xi \varphi d\xi dt \\ \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (fU + V) \cdot \nabla_\xi \varphi d\xi dt - \int_{\mathbb{R}^3} f \varphi d\xi \Big|_{t_2} + \int_{\mathbb{R}^3} f \varphi d\xi \Big|_{t_1}.$$

Remark 2.3 (Parabolic Norm). We recall that, given admissible initial data f_0 , it was shown in [9, Proof of Theorem 3.1, (3.15)] that the weak solution f admits, for some positive constant C independent of T , the estimate

$$(2.4) \quad \|f\|_{\mathcal{P}}^2 \leq \frac{C}{\pi \min\{1, D_e\}} e^{2CT \frac{Pe^2}{D_e}} \|f_0\|_{L^2(\Upsilon)}^2,$$

where the parabolic norm was defined in (1.3).

We now provide the main rescaling lemma used in the proof of the first De Giorgi lemma, using the notion of subsolution given in the previous definition.

Lemma 2.4 (Rescaling Lemma). *Let f be a weak subsolution of (1.4). Let $\delta \in (0, 1)$, $(t_0, \boldsymbol{\xi}_0) \in (0, T) \times \mathbb{R}^3$, and r satisfy the constraint:*

$$0 < r < \min \{1, \sqrt{t_0/2}\}.$$

Let $(t, \boldsymbol{\xi}) \in Q_r(t_0, \boldsymbol{\xi}_0)$ and define

$$\ell(r, \delta) := \delta^{\frac{1}{2}} \frac{r^{\frac{3}{2}}}{\|f\|_{\mathcal{D}} + \|V\|_{L^q(\Upsilon_T)}},$$

as well as the rescaled functions $f_r, U_r, V_r : Q_1 \rightarrow \mathbb{R}$ by

$$(2.5) \quad \begin{aligned} f_r(\tau, \boldsymbol{\zeta}) &:= \ell f(t + r^2\tau, \boldsymbol{\xi} + r\boldsymbol{\zeta}), \\ U_r(\tau, \boldsymbol{\zeta}) &:= rU(t + r^2\tau, \boldsymbol{\xi} + r\boldsymbol{\zeta}), \\ V_r(\tau, \boldsymbol{\zeta}) &:= r\ell V(t + r^2\tau, \boldsymbol{\xi} + r\boldsymbol{\zeta}). \end{aligned}$$

Then, $f_r \in C([-1, 0]; L^2(B_1)) \cap L^2(-1, 0; H^1(B_1))$, $\partial_t f_r \in L^2(-1, 0; (H^1)'(B_1))$,

$$|U_r| \leq 1 \text{ a.e. } Q_1, \quad \|V_r\|_{L^q(Q_1)} \leq 1, \quad \text{ess sup}_{\tau \in [-1, 0]} \int_{B_1} |f_r(\tau)|^2 d\boldsymbol{\zeta} + \int_{Q_1} |\nabla_{\boldsymbol{\zeta}} f_r|^2 d\boldsymbol{\zeta} d\tau \leq \delta,$$

and f_r is a weak subsolution of

$$(2.6) \quad \partial_{\tau} f_r + \text{div}_{\boldsymbol{\zeta}}(U_r f_r + V_r) = \Delta_{\boldsymbol{\zeta}} f_r \quad \text{in } Q_1,$$

i.e., for all non-negative $\varphi \in C^\infty(-1, 0; C_c^\infty(B_1))$ and for a.e. $\tau_1, \tau_2 \in [-1, 0]$, there holds

$$(2.7) \quad \begin{aligned} & - \int_{\tau_1}^{\tau_2} \int_{B_1} f_r \partial_t \varphi d\boldsymbol{\zeta} d\tau + \int_{\tau_1}^{\tau_2} \int_{B_1} \nabla_{\boldsymbol{\zeta}} f_r \cdot \nabla_{\boldsymbol{\zeta}} \varphi d\boldsymbol{\zeta} d\tau \\ & \leq \int_{\tau_1}^{\tau_2} \int_{B_1} (f_r U_r + V_r) \cdot \nabla_{\boldsymbol{\zeta}} \varphi d\boldsymbol{\zeta} d\tau - \int_{B_1} f_r \varphi d\boldsymbol{\xi} \Big|_{\tau_2} + \int_{B_1} f_r \varphi d\boldsymbol{\xi} \Big|_{\tau_1}. \end{aligned}$$

Proof. The smallness of r and the boundedness of ϱ immediately yield the pointwise estimate on U_r . Next, observe that

$$(2.8) \quad \int_{B_1} |f_r(\tau)|^2 d\boldsymbol{\zeta} \leq \frac{\delta}{\|f\|_{\mathcal{D}}^2} \int_{B_r} |f(t + r^2\tau, \boldsymbol{\xi} + \boldsymbol{\zeta}')|^2 d\boldsymbol{\zeta}'.$$

Note from the definition of the cylinders that

$$-2r^2 \leq -r^2 + (t - t_0) \leq t + r^2\tau - t_0 \leq 0,$$

whence the conditions on r imply

$$(2.9) \quad 0 < t + r^2\tau < T.$$

Furthermore, the smallness of r implies

$$(2.10) \quad \{\boldsymbol{\xi} + \boldsymbol{\zeta}' : \boldsymbol{\xi} \in B_r(\boldsymbol{\xi}_0), \boldsymbol{\zeta}' \in B_r\} \subset \{\boldsymbol{\zeta}_0 + \boldsymbol{\zeta} : \boldsymbol{\zeta} \in (-\pi, \pi)^3\}.$$

It follows from (2.9) and (2.10) that there is no overlap in the integration in $\boldsymbol{\zeta}'$ from one periodic cell to another when performing the integration in (2.8), and thus

$$\int_{B_r} |f(t + r^2\tau, \boldsymbol{\xi} + \boldsymbol{\zeta}')|^2 d\boldsymbol{\zeta}' \leq \|f\|_{L^\infty(0, T; L^2(\Upsilon))}^2.$$

Similarly,

$$\int_{Q_1} |\nabla_{\boldsymbol{\zeta}} f_r|^2 d\boldsymbol{\zeta} d\tau \leq \frac{\delta}{\|f\|_{\mathcal{D}}^2} \int_{Q_r} |\nabla_{\boldsymbol{\xi}} f(t + \tau', \boldsymbol{\xi} + \boldsymbol{\zeta}')|^2 d\boldsymbol{\zeta}' d\tau',$$

and by applying the same reasoning as before, we obtain

$$\int_{Q_r} |\nabla_{\boldsymbol{\xi}} f(t + \tau', \boldsymbol{\xi} + \boldsymbol{\zeta}')|^2 d\boldsymbol{\zeta}' d\tau' \leq \|\nabla_{\boldsymbol{\xi}} f\|_{L^2(\Upsilon_T)}^2,$$

and we deduce

$$\operatorname{ess\,sup}_{\tau \in [-1,0]} \int_{B_1} |f_r(\tau)|^2 \, d\zeta + \int_{Q_1} |\nabla_{\zeta} f_r|^2 \, d\zeta \, d\tau \leq \delta,$$

as required. Furthermore,

$$\|V_r\|_{L^q(Q_1)}^q \leq \frac{r^{5(\frac{q}{2}-1)} \delta^{\frac{q}{2}}}{\|V\|_{L^q(\Upsilon_T)}^q} \int_{Q_r} |V(t', \xi')|^q \, d\xi' \, dt' \leq 1.$$

The weak subsolution formulation (2.7) of the drift-diffusion equation (2.6) is easily verified by direct computation. \square

It will therefore suffice to study the following equation:

$$(2.11) \quad \partial_t f + \operatorname{div}_{\xi}(Uf + V) = \Delta_{\xi} f \quad \text{in } Q_1,$$

with $f \in C([-1, 0]; L^2(B_1)) \cap L^2(-1, 0; H^1(B_1))$, with $\partial_t f \in L^2(-1, 0; (H^1)')(B_1)$, $|U| \leq 1$ a.e. in Q_1 , and $\|V\|_{L^q(Q_1)} \leq 1$ where $q > 5$.

3. BOUNDEDNESS OF WEAK SOLUTIONS

3.1. Boundedness away from initial time. The goal of this section is to prove the following proposition, which will subsequently be used to prove Theorem 1 in §4.

Proposition 3.1 (Boundedness away from Initial Time). *Assume $f_0 \in L^2_{\text{per}}(\Upsilon)$ is non-negative and satisfies (1.6), $T > 0$, and let f be the unique weak solution of (1.1) with initial data f_0 . There exists a positive constant C depending only on $\Upsilon, T, \text{Pe}, D_e$ such that, for all $t \in (0, T)$,*

$$(3.1) \quad \|f\|_{L^\infty((t,T) \times \Upsilon)} \leq C(1 + t^{-\frac{13}{4}}) \|f\|_{\mathcal{D}}.$$

Notice that the right-hand side of inequality (3.1) is bounded thanks to (2.4). The proof is broken down into several steps, which constitute the subsections that follow. The local boundedness for solutions of equations of the form (2.11) by means of De Giorgi's method is classical (*cf. e.g.* [39, Ch. VI §5]). We nevertheless highlight that a novel aspect of our approach is that we only need to consider the solution away from the initial time, and not on a more restrictive subcylinder; this is a consequence of the choice of periodic boundary conditions in space-angle. Details of the iterative procedure for more general systems may also be found in, *e.g.*, [47, §3.2]. We include these details in the present section so as to make the paper self-contained, and to make the proof of the higher-regularity result of §4 easier to follow; this latter proof uses De Giorgi's method inductively on repeated time-derivatives of the equation.

3.1.1. Caccioppoli inequality.

Lemma 3.2 (Caccioppoli Inequality). *Let $f \in C([-1, 0]; L^2(B_1)) \cap L^2(-1, 0; H^1(B_1))$, with $\partial_t f \in L^2(-1, 0; (H^1)')(B_1)$, $|U(t, \xi)| \leq 1$ a.e. $(t, \xi) \in Q_1$, and $\|V\|_{L^q(Q_1)} \leq 1$ for $q > 5$, be a weak subsolution of (2.11). Let $\eta \in C_c^\infty(B_1)$ be any compactly supported function independent of t , and $K \geq 0$. Define $v = (f - K)_+$. Then, there exists a positive constant C , independent of η, K, f, U, V , such that there holds, for all $-1 < s < t < 0$,*

$$\begin{aligned} & \left(\int_{B_1} |\eta v|^2 \, d\xi \right)(t) - \left(\int_{B_1} |\eta v|^2 \, d\xi \right)(s) + \int_s^t \int_{B_1} |\nabla_{\xi}(\eta v)|^2 \, d\xi \, d\tau \\ & \leq C(1 + K^2) \int_s^t \int_{B_1} (\eta + |\nabla_{\xi} \eta|)^2 (1 + |V|^2) (v^2 + \mathbf{1}_{\{v>0\}}) \, d\xi \, d\tau. \end{aligned}$$

Proof. A standard argument shows (*e.g.* [50, Theorem 2.1.11]), using the compact support of η with respect to the space-angle variable ξ , that $\eta^2 v \in C([-1, 0]; L^2(B_1)) \cap L^2(-1, 0; H^1(B_1))$ may be approximated by elements of $C_c^\infty(Q_1)$. In turn, we may insert $\eta^2 v$ into the weak subsolution formulation (2.3). Using also the relations $v f = v^2 + K v$, $\nabla_{\xi} v = \mathbf{1}_{v \geq 0} \nabla_{\xi} v$, $f \mathbf{1}_{v \geq 0} = (v + K) \mathbf{1}_{v \geq 0}$, and

$\nabla_{\xi}(\eta^2 v) \cdot \nabla_{\xi} v = |\nabla_{\xi}(\eta v)|^2 - v^2 |\nabla_{\xi} \eta|^2$, we obtain

$$(3.2) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{B_1} \eta^2 v^2 d\xi + \int_{B_1} |\nabla_{\xi}(\eta v)|^2 d\xi \\ & \leq \int_{B_1} v^2 |\nabla_{\xi} \eta|^2 d\xi + \int_{B_1} \eta^2 v \nabla_{\xi} v \cdot U d\xi + K \int_{B_1} \eta^2 \nabla_{\xi} v \cdot U d\xi + \int_{B_1} \nabla_{\xi}(\eta v) \cdot V \eta d\xi \\ & \quad + 2 \int_{B_1} v^2 \eta \nabla_{\xi} \eta \cdot U d\xi + 2K \int_{B_1} v \eta \nabla_{\xi} \eta \cdot U d\xi + \int_{B_1} \nabla_{\xi} \eta \cdot V v \eta d\xi. \end{aligned}$$

Using the bound on U and the Cauchy–Young inequality, also writing $v \leq \frac{1}{2}(\mathbf{1}_{\{v>0\}} + v^2)$, it follows that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{B_1} \eta^2 v^2 d\xi + \frac{1}{2} \int_{B_1} |\nabla_{\xi}(\eta v)|^2 d\xi & \leq \int_{B_1} \eta^2 v |\nabla_{\xi} v| d\xi + K \int_{B_1} \eta^2 |\nabla_{\xi} v| d\xi \\ & \quad + C(1+K) \int_{B_1} (\eta + |\nabla_{\xi} \eta|)^2 (v^2 + \mathbf{1}_{\{v>0\}}) d\xi \\ & \quad + C \int_{B_1} \eta |\nabla_{\xi} \eta| |V| v d\xi + C \int_{B_1} \eta^2 |V|^2 \mathbf{1}_{\{v>0\}} d\xi, \end{aligned}$$

for some universal constant C , whence, using the relation $\eta \nabla_{\xi} v = \nabla_{\xi}(\eta v) - v \nabla_{\xi} \eta$ and the Cauchy–Young inequality to estimate the first term on the right-hand side of the previous inequality, there holds

$$\frac{d}{dt} \int_{B_1} \eta^2 v^2 d\xi + \int_{B_1} |\nabla_{\xi}(\eta v)|^2 d\xi \leq C(1+K^2) \int_{B_1} (\eta + |\nabla_{\xi} \eta|)^2 (1+|V|^2)(v^2 + \mathbf{1}_{\{v>0\}}) d\xi,$$

where we also used $|V| \leq \frac{1}{2}(1+|V|^2)$. Integrating the final inequality with respect to the time variable gives the result. \square

3.1.2. Interior local boundedness on subcylinders. The goal of this subsection is to prove the following result.

Proposition 3.3 (Interior Local Boundedness). *Let f be a weak subsolution of (2.11). Let $(t, \xi) \in (0, T) \times \mathbb{R}^3$, and r satisfy the constraint:*

$$(3.3) \quad 0 < r < \min \{1, \sqrt{t/2}\}.$$

Then, there exists a positive constant C , independent of $r, (t, \xi)$, such that there holds

$$(3.4) \quad \|f\|_{L^\infty(Q_r(t, \xi))} \leq C(1+r^{-\frac{3}{2}})(\|f\|_{\mathcal{D}} + \|V\|_{L^q(\Upsilon_T)}).$$

We begin by proving the following lemma.

Lemma 3.4. *Let $q > 5$ be fixed. There exists $\delta_* > 0$, depending only on T, Υ, q , such that: for all weak subsolutions f of (2.11), where $f \in C([-1, 0]; L^2(B_1)) \cap L^2(-1, 0; H^1(B_1))$, with $\partial_t f \in L^2(-1, 0; (H^1)'(B_1))$, $|U| \leq 1$ a.e. in Q_1 , and $\|V\|_{L^q(Q_1)} \leq 1$, if*

$$\text{ess sup}_{t \in [-1, 0]} \int_{B_1} |f(t)|^2 d\xi + \int_{Q_1} |\nabla_{\xi} f|^2 d\xi dt \leq \delta_*,$$

then

$$f_+ \leq \frac{1}{2} \quad \text{in } Q_{\frac{1}{2}}.$$

Proof. The proof is divided in several steps.

1. *Iterative set-up:* Consider the sequence of times $T_k = -\frac{1}{2}(1+2^{-k})$ as well as the sequence of cylinders $\tilde{Q}_k = (T_k, 0) \times \tilde{B}_k$, where $\tilde{B}_k = \{\xi : |\xi| < \frac{1}{2}(1+2^{-k})\}$, and define the truncations $\mathcal{T}_k f = (f - C_k)_+$ with $C_k = \frac{1}{2}(1-2^{-k})$. We consider a family of non-negative cut-off functions $\{\eta_k\}_{k \in \mathbb{N}}$, compactly supported in \tilde{B}_{k+1} , identically equal to 1 in \tilde{B}_k , and such that $|\nabla_{\xi} \eta_k| \leq C2^k$ for some positive universal constant C . Correspondingly, we define, for all $k \in \mathbb{N}$,

$$\begin{aligned} \mathcal{E}_k & := \text{ess sup}_{t \in [T_k, 0]} \left(\int_{B_1} |\eta_k \mathcal{T}_k f|^2 d\xi \right) (t) + \int_{T_k}^0 \int_{B_1} |\nabla_{\xi}(\eta_k \mathcal{T}_k f)|^2 d\xi dt, \\ \mathcal{E}_0 & := \text{ess sup}_{t \in [-1, 0]} \int_{B_1} |f_+(t)|^2 d\xi + \int_{Q_1} |\nabla_{\xi} f_+|^2 dt. \end{aligned}$$

2. *Non-linear recursive estimate:* Our goal is to prove the non-linear recursive estimate:

$$(3.5) \quad \mathcal{E}_{k+1} \leq C_*^k \mathcal{E}_k^{1+\epsilon},$$

for some positive universal constant C_* depending only on Υ, T , where

$$\epsilon := 1 - \frac{2}{q} - \frac{3}{5} \in (0, 1)$$

depends only on $q > 5$. By substituting $\eta = \eta_{k+1}$ and $K = C_k$ into the inequality of Lemma 3.2, and constraining $T_k \leq s \leq T_{k+1} \leq t \leq 0$, we get

$$\begin{aligned} & \left(\int_{B_1} |\eta_{k+1} \mathcal{T}_{k+1} f|^2 d\xi \right) (t) + \int_{T_{k+1}}^t \int_{B_1} |\nabla_{\xi}(\eta_{k+1} \mathcal{T}_{k+1} f)|^2 d\xi d\tau \\ & \leq \left(\int_{B_1} |\eta_{k+1} \mathcal{T}_{k+1} f|^2 d\xi \right) (s) \\ & \quad + C(1 + C_k^2) \int_{T_k}^0 \int_{B_1} (\eta_{k+1} + |\nabla_{\xi} \eta_{k+1}|)^2 (1 + |V|^2) (\mathcal{T}_{k+1} f^2 + \mathbf{1}_{\{\mathcal{T}_{k+1} f > 0\}}) d\xi d\tau, \end{aligned}$$

which, by integrating the entire inequality in s over the interval $[T_k, T_{k+1}]$ and noting $T_{k+1} - T_k = 2^{-(k+2)}$, yields

$$\begin{aligned} & \left(\int_{B_1} |\eta_{k+1} \mathcal{T}_{k+1} f|^2 d\xi \right) (t) + \int_{T_{k+1}}^t \int_{B_1} |\nabla_{\xi}(\eta_{k+1} \mathcal{T}_{k+1} f)|^2 d\xi d\tau \\ & \leq 2^{k+2} \int_{T_k}^{T_{k+1}} \int_{B_1} |\eta_{k+1} \mathcal{T}_{k+1} f|^2 d\xi d\tau \\ (3.6) \quad & + C \int_{T_k}^0 \int_{B_1} \underbrace{(1 + 2^{k+1})^2}_{\leq 2 \cdot 2^{k+1}} \mathbf{1}_{\tilde{Q}_k} (1 + |V|^2) ((\mathcal{T}_{k+1} f)^2 + \mathbf{1}_{\{\mathcal{T}_{k+1} f > 0\}}) d\xi d\tau \\ & \leq C^k \int_{T_k}^{T_{k+1}} \int_{B_1} |\eta_{k+1} \mathcal{T}_{k+1} f|^2 d\xi d\tau + \underbrace{C^k \int_{T_k}^0 \int_{B_1} \mathbf{1}_{\tilde{Q}_k \cap \{\mathcal{T}_{k+1} f > 0\}} (1 + |V|^2) ((\mathcal{T}_{k+1} f)^2 + 1) d\xi d\tau}_{=: I}, \end{aligned}$$

where the value of C has changed from line to line. We proceed to estimating the term I . Notice that, provided $(t, \xi) \in \tilde{Q}_k \cap \{\mathcal{T}_{k+1} f > 0\}$, there holds

$$(3.7) \quad \mathcal{T}_k f(t, \xi) = f(t, \xi) - C_k = \mathcal{T}_{k+1} f(t, \xi) + 2^{-(k+2)} > 2^{-(k+2)},$$

whence, by squaring the above inequality (noting that all quantities are non-negative), we get

$$\mathbf{1}_{\{\mathcal{T}_{k+1} f > 0\}} \leq 2^{2(k+2)} (\mathcal{T}_k f)^2 \mathbf{1}_{\{\mathcal{T}_{k+1} f > 0\}}.$$

In view of $\mathcal{T}_{k+1} f \leq \mathcal{T}_k f$, we therefore estimate the final term of (3.6) as

$$\begin{aligned} I & \leq \int_{T_k}^0 \int_{B_1} \mathbf{1}_{\tilde{Q}_k \cap \{\mathcal{T}_{k+1} f > 0\}} (1 + |V|^2) (1 + 2^{2(k+2)}) (\mathcal{T}_k f)^2 d\xi d\tau \\ & \leq C^k \int_{T_k}^0 \int_{B_1} \mathbf{1}_{\tilde{Q}_k \cap \{\mathcal{T}_{k+1} f > 0\}} (1 + |V|^2) (\mathcal{T}_k f)^2 d\xi d\tau, \end{aligned}$$

and there holds

$$\mathcal{E}_{k+1} \leq C^k \int_{T_k}^{T_{k+1}} \int_{B_1} |\eta_{k+1} \mathcal{T}_{k+1} f|^2 d\xi d\tau + C^k \int_{T_k}^0 \int_{B_1} \mathbf{1}_{\tilde{Q}_k \cap \{\mathcal{T}_{k+1} f > 0\}} (1 + |V|^2) |\mathcal{T}_k f|^2 d\xi d\tau.$$

Similarly, using the boundedness of the cut-off as well as $\mathcal{T}_{k+1} f \leq \mathcal{T}_k f$, the first term on the right-hand side of the previous estimate may be rewritten as

$$\int_{T_k}^{T_{k+1}} \int_{B_1} |\eta_{k+1} \mathcal{T}_{k+1} f|^2 d\xi d\tau \leq \int_{T_k}^0 \int_{B_1} \mathbf{1}_{\tilde{Q}_k \cap \{\mathcal{T}_{k+1} f > 0\}} |\mathcal{T}_{k+1} f|^2 d\xi d\tau \leq I,$$

and thus

$$(3.8) \quad \mathcal{E}_{k+1} \leq C^k \int_{T_k}^0 \int_{B_1} \mathbf{1}_{\tilde{Q}_k \cap \{\mathcal{T}_{k+1} f > 0\}} (1 + |V|^2) |\mathcal{T}_k f|^2 d\xi d\tau.$$

Using the Sobolev inequality, there holds, for a positive constant C independent of t, k ,

$$\|\eta_k \mathcal{T}_k f(t, \cdot)\|_{L^6(B_1)}^2 \leq C \|\eta_k \mathcal{T}_k f(t, \cdot)\|_{H^1(B_1)}^2,$$

whence $\|\eta_k \mathcal{T}_k f\|_{L^2(T_k, 0; L^6(B_1))} \leq C \mathcal{E}_k^{\frac{1}{2}}$, and

$$\begin{aligned} \|\mathcal{T}_k f\|_{L^2(\tilde{Q}_k)}^2 &= \|\eta_k \mathcal{T}_k f\|_{L^2(\tilde{Q}_k)}^2 \leq \int_{T_k}^0 \left(\int_{B_1} |\eta_k \mathcal{T}_k f(t, \xi)|^2 d\xi \right) dt \\ (3.9) \qquad \qquad \qquad &\leq \int_{T_k}^0 |B_1|^{\frac{2}{3}} \left(\int_{B_1} |\eta_k \mathcal{T}_k f(t, \xi)|^6 d\xi \right)^{\frac{1}{3}} dt \\ &= C \|\eta_k \mathcal{T}_k f\|_{L^2(T_k, 0; L^6(B_1))}^2 \\ &\leq C \mathcal{E}_k. \end{aligned}$$

We remark that this estimate alone would be insufficient for bounding the first term on the right-hand side of (3.6); indeed, we must have a *non-linear* estimate in order for the iteration procedure to succeed.

Meanwhile, $\|\eta_k \mathcal{T}_k f\|_{L^\infty(T_k, 0; L^2(B_1))} \leq \mathcal{E}_k^{\frac{1}{2}}$. We interpolate between these two norms. More precisely, using the Interpolation Lemma 1.2, there exists a positive constant C , independent of k , such that

$$\|\eta_k \mathcal{T}_k f\|_{L^p(\tilde{Q}_k)} \leq C(1 + |T_k|) \left(\|\eta_k \mathcal{T}_k f\|_{L^2(T_k, 0; H^1(B_1))} + \|\eta_k \mathcal{T}_k f\|_{L^\infty(T_k, 0; L^2(B_1))} \right) \leq C \mathcal{E}_k^{\frac{1}{2}},$$

where $p = \frac{2}{3}(2 + 3) = \frac{10}{3}$, *i.e.*, $\|\eta_k \mathcal{T}_k f\|_{L^{\frac{10}{3}}(\tilde{Q}_k)}^2 \leq C \mathcal{E}_k$, from which we deduce

$$\|(\mathcal{T}_k f)^2\|_{L^{\frac{5}{3}}(\tilde{Q}_k)} \leq \|\eta_k \mathcal{T}_k f\|_{L^{\frac{10}{3}}(\tilde{Q}_k)}^2 \leq C \mathcal{E}_k.$$

In turn, returning to (3.8) and using the Hölder, Jensen, and Minkowski inequalities with the assumption $\|V\|_{L^q(Q_1)} \leq 1$, there holds

$$\begin{aligned} \mathcal{E}_{k+1} &\leq C^k |\tilde{Q}_k \cap \{\mathcal{T}_{k+1} f > 0\}|^{1 - \frac{2}{q} - \frac{3}{5}} \underbrace{\|1 + |V|^2\|_{L^{\frac{q}{2}}(Q_1)}}_{\leq C_q(1 + \|V\|_{L^q(Q_1)}^2)} \|(\mathcal{T}_k f)^2\|_{L^{\frac{5}{3}}(\tilde{Q}_k)} \\ &\leq C^k |\tilde{Q}_k \cap \{\mathcal{T}_{k+1} f > 0\}|^{1 - \frac{2}{q} - \frac{3}{5}} \mathcal{E}_k; \end{aligned}$$

note that the application of Hölder's inequality is justified due to the condition $V \in L^q(Q_1)$ for $q > 5$, *i.e.*, $\epsilon = 1 - 2/q - 3/5 \in (0, 1)$. By applying the Markov inequality, using (3.7) to write $\tilde{Q}_k \cap \{\mathcal{T}_{k+1} f > 0\} \subset \{|\eta_k \mathcal{T}_k f|^2 > 2^{-2(k+2)}\}$, we estimate, using also the bound (3.9),

$$|\tilde{Q}_k \cap \{\mathcal{T}_{k+1} f > 0\}| \leq 2^{2(k+2)} \|\eta_k \mathcal{T}_k f\|_{L^2(\tilde{Q}_k)}^2 \leq C^k \mathcal{E}_k,$$

whence we get the desired non-linear estimate (3.5).

3. Initialisation and iterative procedure: Applying the standard iteration lemma [19, §1, Lemma 4.1] to the recursive estimate relation (3.5), we deduce that there exists $\delta_* = \delta(C_*, \epsilon) > 0$ sufficiently small such that if $\mathcal{E}_0 \leq \delta_*$, then $\lim_{k \rightarrow \infty} \mathcal{E}_k = 0$. The Monotone Convergence Theorem and (3.9) then imply

$$\int_{Q_{\frac{1}{2}}} (f - \frac{1}{2})_+^2 d\xi dt = \lim_{k \rightarrow \infty} \|\mathcal{T}_k f\|_{L^2(\tilde{Q}_k)}^2 \leq \lim_{k \rightarrow \infty} \mathcal{E}_k = 0,$$

which yields the conclusion of the lemma. \square

Proposition 3.3 now follows as a simple corollary of Lemma 3.4 by a standard scaling argument using Lemma 2.4.

Proof of Proposition 3.3. Let f_r, U_r be defined from f as per equation (2.5) in the proof of Lemma 2.4, with δ chosen to be the specific value δ_* . We then apply Lemma 3.4 to f_r , from which we obtain

$$(3.10) \qquad \qquad \qquad (f_r)_+ \leq \frac{1}{2} \quad \text{in } Q_{\frac{1}{2}}.$$

Then, using the non-negativity of f and the rescaling (2.5) to transfer the bound (3.10), we obtain the result. \square

3.1.3. *Boundedness away from initial time.* The smallness of the radius r of the subcylinder $Q_r(t, \boldsymbol{\xi})$ constrains the result of Proposition 3.3 to being local in the interior. However, the periodicity of the problem actually means that the result is global-in-space, while being local-in-time; this is manifestly clear from the fact that the constraint on the size of r in $Q_r(t, \boldsymbol{\xi})$ depends only on the coordinate t , and not on $\boldsymbol{\xi}$, as is shown in the criterion (3.3).

Our objective is therefore to extend the result of Proposition 3.3 from subcylinders to infinite strips away from the initial time, *i.e.* Theorem 3.1, which is proved by an exhaustion argument.

Proof of Proposition 3.1. Fix $t \in (0, T)$. Define $r_t := \frac{1}{2} \min\{1, \sqrt{t/2}\}$. Given this radius r_t , select $\{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_N\}$ to be any finite collection of points in Υ chosen such that

$$\Upsilon \subset \bigcup_{j=1}^N B_{r_t}(\boldsymbol{\xi}_j);$$

it is clear that such a collection exists, and an easy argument shows that we may take $N = \lceil Cr_t^{-3} \rceil$ for some positive constant C depending only on Υ . Similarly, let $\{t_0, \dots, t_M\}$ be any collection of times such that $t = t_0 < t_1 < \dots < t_M = T$ such that $|t_j - t_{j-1}| < r_t^2$ for all $j \in \{1, \dots, M\}$; as before, one may take $M = \lceil Cr_t^{-2} \rceil$ for a suitable constant C depending only on T . It follows that

$$(t, T) \times \Upsilon \subset \bigcup_{i=0}^M \bigcup_{j=1}^N Q_{r_t}(t_i, \boldsymbol{\xi}_j),$$

and thus

$$\|f\|_{L^\infty((t, T) \times \Upsilon)} \leq \sum_{i=0}^M \sum_{j=1}^N \|f\|_{L^\infty(Q_{r_t}(t_i, \boldsymbol{\xi}_j))}.$$

Observe that, for each subcylinder $Q_{r_t}(t_i, \boldsymbol{\xi}_j)$ there holds $r_t < \min\{1, \sqrt{t_i/2}\}$, whence we may apply Proposition 3.3 to get

$$\|f\|_{L^\infty((t, T) \times \Upsilon)} \leq \underbrace{MN}_{\leq Cr_t^{-5}} (1 + r_t^{-\frac{3}{2}}) \|f\|_{\mathcal{D}}.$$

The result then follows from the definition of r_t . \square

3.2. **Global-in-time boundedness for L^∞ initial data.** In this section we prove Theorem 4, concerning the global-in-time boundedness of weak solutions, assuming initial data in L^∞ . Our strategy is based on [36, §3], and was used for a similar model in [18, Proposition 3.5]. We note that this section is separate from the rest of the regularity analysis.

Before proceeding to the proof of Theorem 4, we recall a version of a technical lemma from [36]; we omit the proof, which can be found in [36, Appendix A].

Lemma 3.5 (Lemma 3.2 of [36]). *Let $A_k : [0, \infty) \rightarrow [0, \infty)$ be a sequence of functions satisfying the differential inequality*

$$\frac{d}{dt} A_k + C_0 A_k \leq C_1^k (A_{k-1})^2 \quad \text{for all } t, k,$$

for some positive constants C_0, C_1 , and assume that $A_0(t)$ is uniformly bounded in time and $\{A_k(0)\}_k$ is uniformly bounded in k . Then, with $n_k = 2^k$, the sequence $\{A_k^{1/n_k}(t)\}_k$ is uniformly bounded in time.

Proof of Theorem 4. Let $n \geq 2$. We test the equation (1.1) against nf^{n-1} and, using the boundedness of ϱ , obtain

$$\frac{d}{dt} \int_{\Upsilon} f^n d\boldsymbol{\xi} + \frac{4(n-1)}{n} \min\{D_e, 1\} \int_{\Upsilon} |\nabla_{\boldsymbol{\xi}} f^{\frac{n}{2}}|^2 d\boldsymbol{\xi} \leq 2(n-1) \text{Pe} \int_{\Upsilon} f^{\frac{n}{2}} |\nabla f^{\frac{n}{2}}| d\boldsymbol{\xi},$$

which, by applying Young's inequality and using the lower bound $4(n-1)/n \geq 2$, implies

$$\frac{d}{dt} \int_{\Upsilon} f^n d\boldsymbol{\xi} + \frac{3}{2} \min\{D_e, 1\} \int_{\Upsilon} |\nabla_{\boldsymbol{\xi}} f^{\frac{n}{2}}|^2 d\boldsymbol{\xi} \leq \frac{2(n-1)^2 \text{Pe}^2}{\min\{D_e, 1\}} \|f^{\frac{n}{2}}\|_{L^2(\Upsilon)}^2.$$

By applying the Gagliardo–Nirenberg inequality, we get

$$\|f^{\frac{n}{2}}\|_{L^2(\Upsilon)} \leq C_{GN} \left(\|\nabla_{\boldsymbol{\xi}} f^{\frac{n}{2}}\|_{L^2(\Upsilon)}^{\frac{3}{5}} \|f^{\frac{n}{2}}\|_{L^1(\Upsilon)}^{\frac{2}{5}} + \|f^{\frac{n}{2}}\|_{L^1(\Upsilon)} \right),$$

from which, by using Young's inequality, we deduce

$$(3.11) \quad \|\nabla_{\xi} f^{\frac{n}{2}}\|_{L^2(\Upsilon)} \geq \frac{1}{C'_{GN}} \|f^{\frac{n}{2}}\|_{L^2(\Upsilon)} - \|f^{\frac{n}{2}}\|_{L^1(\Upsilon)},$$

with $C'_{GN} = \frac{7}{5}C_{GN}$, as well as

$$\|f^{\frac{n}{2}}\|_{L^2(\Upsilon)}^2 \leq \epsilon^{\frac{5}{3}} \frac{6C_{GN}^2}{5} \|\nabla_{\xi} f^{\frac{n}{2}}\|_{L^2(\Upsilon)} + 2C_{GN}^2 \left(1 + \frac{2}{5\epsilon^{\frac{5}{2}}}\right) \|f^{\frac{n}{2}}\|_{L^1(\Upsilon)}^2,$$

for all $\epsilon > 0$. Consequently, by setting

$$\epsilon = \left(\frac{5 \min\{D_e, 1\}^2}{24(n-1)^2 \text{Pe}^2 C_{GN}^2} \right)^{3/5},$$

we get

$$\frac{d}{dt} \int_{\Upsilon} f^n d\xi + \min\{D_e, 1\} \|\nabla_{\xi} f^{\frac{n}{2}}\|_{L^2(\Upsilon)}^2 \leq (n-1)^2 \frac{4\text{Pe}^2 C_{GN}^2}{\min\{D_e, 1\}} \left(1 + \frac{2}{5\epsilon^{\frac{5}{2}}}\right) \|f^{\frac{n}{2}}\|_{L^1(\Upsilon)}^2.$$

Thus, by defining

$$c_0 := \frac{\min\{D_e, 1\}}{2(C'_{GN})^2}, \quad c_1 := \min\{D_e, 1\} + \frac{4(n-1)^2 \text{Pe}^2 C_{GN}^2}{\min\{D_e, 1\}} \left(1 + \frac{2}{5\epsilon^{\frac{5}{2}}}\right),$$

using the lower bound (3.11), we obtain

$$\frac{d}{dt} \|f^n\|_{L^1(\Upsilon)} + c_0 \|f^n\|_{L^1(\Upsilon)} \leq c_1 n^2 \|f^{\frac{n}{2}}\|_{L^1(\Upsilon)}^2,$$

whence, by setting $n = 2^k$ and

$$A_k(t) := \|f^{n_k}(t, \cdot)\|_{L^1(\Upsilon)},$$

we get

$$\frac{d}{dt} A_k + c_0 A_k \leq c_1 4^k (A_{k-1})^2.$$

Using the boundedness of the initial data $f_0 \in L^\infty(\Upsilon)$ and applying Lemma 3.5, we deduce global-in-time estimate

$$\|f(t, \cdot)\|_{L^{2^k}(\Upsilon)} \leq C(c_0, c_1, \|f_0\|_{L^\infty}) \quad \text{for all } t, k,$$

whence the result follows. \square

4. HIGHER REGULARITY OF WEAK SOLUTIONS

We obtain smoothness of all weak solutions f away from the initial time. Our strategy is to take derivatives in time in the equation (1.1), and then to apply the iteration procedure used to prove Proposition 3.1 to the resulting equation. For clarity of exposition, in §4.1 we first show how to do this for the first derivative in time, denoted by $\dot{f} := \partial_t f$ with $\dot{\varrho}$ defined analogously, before moving on to general higher derivatives in §4.2, which are denoted by $f^{(n)} := \partial_t^n f$ and $\varrho^{(n)} := \partial_t^n \varrho$.

The main results of this subsection are as follows, and are used to prove Theorem 1.

Proposition 4.1 (Boundedness of Time Derivatives away from Initial Time). *For all integer n , there exists a decreasing positive function $\psi^n : (0, T) \rightarrow (0, \infty)$ such that*

$$\limsup_{t \rightarrow 0^+} \psi^n(t) = +\infty$$

and, for a.e. $t \in (0, T)$, there holds

$$(4.1) \quad \|f^{(n)}\|_{L^\infty((t, T) \times \Upsilon)} \leq \psi^n(t).$$

We will then use the previous proposition and the computations developed in its proof to prove the next result:

Proposition 4.2 (Sobolev Estimates for Time Derivatives away from Initial Time). *For all integer n , there exists a decreasing positive function $\Psi^n : (0, T) \rightarrow (0, \infty)$ such that*

$$\limsup_{t \rightarrow 0^+} \Psi^n(t) = +\infty$$

and, for a.e. $t \in (0, T)$, there holds

$$(4.2) \quad \|f^{(n)}\|_{W^{2, \frac{10}{3}}((t, T) \times \Upsilon)} \leq \Psi^n(t).$$

Theorem 1 then follows as an immediate corollary of Proposition 4.2, as shown below.

Proof of Theorem 1. Morrey's inequality and Proposition 4.2 implies that $f^{(n)} \in C^{\frac{4}{5}}((t, T) \times \Upsilon)$, where we used that the dimension of $(t, T) \times \Upsilon$ is $4 < 2 \cdot 10/3$. It then follows that, for all integers n , the function $f^{(n)}$ is continuous on the subset $(t, T) \times \Upsilon$. By returning to the equation and differentiating, a straightforward argument shows that the continuity of $\{f^{(n)}\}_n$ implies continuity of the derivatives with respect to ξ of all orders, and analogously for all mixed-derivatives in t and ξ . The proof is complete. \square

4.1. Boundedness of \dot{f} away from initial time. The goal of this subsection is to prove the case $n = 1$ in Proposition 4.1. Our underlying strategy is to derive an equation for \dot{f} to which the De Giorgi method can be applied. First, we must derive a H^2 -type bound on the solution away from the initial time, which is the content of the next lemma.

Lemma 4.3 (H^2 -type Estimate for f away from Initial Time). *There exists a positive constant C , independent of t , such that for a.e. $t \in (0, T)$ there holds*

$$\|\nabla_{\xi} f\|_{L^{\infty}(t, T; L^2(\Upsilon))}^2 + \|\Delta_{\xi} f\|_{L^2((t, T) \times \Upsilon)}^2 \leq C \|\nabla_{\xi} f(t, \cdot)\|_{L^2(\Upsilon)}^2 \exp\left(T(1 + \|f\|_{L^{\infty}((t, T) \times \Upsilon)}^2)\right),$$

and

$$\|\dot{f}\|_{L^2((t, T) \times \Upsilon)} \leq C \left(\|\Delta_{\xi} f\|_{L^2((t, T) \times \Upsilon)} + (1 + \|f\|_{L^{\infty}((t, T) \times \Upsilon)}) \|\nabla_{\xi} f\|_{L^2(\Upsilon_T)} \right).$$

Remark that $\nabla_{\xi} f \in L^2(\Upsilon_T)$ implies, using Markov's inequality, that $\|\nabla_{\xi} f(t, \cdot)\|_{L^2(\Upsilon)}^2$ is finite a.e. in $(0, T)$; using also Proposition 3.1, the right-hand sides of the previous estimates are therefore finite. Furthermore the boundedness of $\|\Delta_{\xi} f\|_{L^2((t, T) \times \Upsilon)}$ yields an identical estimate for the full Hessian $\|\nabla_{\xi}^2 f\|_{L^2((t, T) \times \Upsilon)}$ by virtue of Lemma 1.3.

Proof. Let $\eta \in C_c^{\infty}(B_1)$ be the usual non-negative bump function with $\int_{\mathbb{R}^3} \eta(\xi) d\xi = 1$, and define the sequence of Friedrichs mollifiers $\eta_{\varepsilon}(\xi) := \varepsilon^{-3} \eta(\xi/\varepsilon)$. Correspondingly, define $f_{\varepsilon} := f(t, \cdot) * \eta_{\varepsilon} = \int_{\mathbb{R}^3} \eta_{\varepsilon}(\xi - \zeta) f(t, \zeta) d\zeta$ and $\varrho_{\varepsilon} := \int_0^{2\pi} f_{\varepsilon} d\theta$. Note that this operation preserves the periodicity and that the convolution is well-defined as f, ϱ extend periodically to the full space. It is straightforward to verify that there holds

$$(4.3) \quad \partial_t f_{\varepsilon} + \operatorname{div}_{\xi}(U f_{\varepsilon} + E_{\varepsilon}) = \Delta_{\xi} f_{\varepsilon},$$

in the weak sense, where

$$\begin{aligned} U &= (1 - \varrho) \begin{pmatrix} \mathbf{e}(\theta) \\ 0 \end{pmatrix}, & E_{\varepsilon}(t, \xi) &:= (U f) * \eta_{\varepsilon} - U f_{\varepsilon} \\ & & &= \int_{\mathbb{R}^3} \eta_{\varepsilon}(\xi - \zeta) f(t, \zeta) (U(t, \zeta) - U(t, \xi)) d\zeta, \end{aligned}$$

where the convolution is taken only with respect to the space-angle variable ξ . Using also the positivity of f, η_{ε} , there holds

$$(4.4) \quad |U| \leq 1, \quad |E_{\varepsilon}(t, \xi)| \leq 2f_{\varepsilon}(t, \xi) \quad \text{a.e. in } \Upsilon_T.$$

Observe that, for all fixed ε , we have $\Delta_{\xi} f_{\varepsilon} \in L^2((t, T) \times \Upsilon)$ for a.e. $t \in (0, T)$, whence (4.3) holds in the strong sense and we may test with this quantity to obtain

$$(4.5) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Upsilon} |\nabla_{\xi} f_{\varepsilon}|^2 d\xi + \int_{\Upsilon} |\Delta_{\xi} f_{\varepsilon}|^2 d\xi &= \int_{\Upsilon} \operatorname{div}_{\xi} (U f_{\varepsilon} + E_{\varepsilon}) \Delta_{\xi} f_{\varepsilon} d\xi \\ &\leq \int_{\Upsilon} |\nabla_{\xi} f_{\varepsilon}| |\Delta_{\xi} f_{\varepsilon}| d\xi + \int_{\Upsilon} |\operatorname{div}_{\xi} U| |f_{\varepsilon}| |\Delta_{\xi} f_{\varepsilon}| d\xi \\ &\quad + \int_{\Upsilon} |\operatorname{div}_{\xi} E_{\varepsilon}| |\Delta_{\xi} f_{\varepsilon}| d\xi; \end{aligned}$$

we bound each of the three terms on the right-hand side of the above. The first is dealt with using the Young inequality, while for the second term we have

$$\begin{aligned} \int_{\Upsilon} |\operatorname{div}_{\xi} U| |f_{\varepsilon}| |\Delta_{\xi} f_{\varepsilon}| d\xi &\leq \|f_{\varepsilon}\|_{L^{\infty}((t, T) \times \Upsilon)} \int_{\Upsilon} |\nabla \varrho| |\Delta_{\xi} f_{\varepsilon}| d\xi \\ &\leq \frac{1}{4} \|\Delta_{\xi} f_{\varepsilon}\|_{L^2(\Upsilon)}^2 + \|f_{\varepsilon}\|_{L^{\infty}((t, T) \times \Upsilon)}^2 \|\nabla \varrho\|_{L^2(\Upsilon)}^2. \end{aligned}$$

For the final term, we write

$$\operatorname{div}_{\xi} E_{\varepsilon} = (\operatorname{div}_{\xi} (U f)) * \eta_{\varepsilon} - \underbrace{(\operatorname{div}_{\xi} U) f_{\varepsilon} - U \cdot \nabla_{\xi} f_{\varepsilon}}_{= -\operatorname{div}(U f_{\varepsilon})},$$

from which we estimate, using standard properties of mollifiers,

$$\|\operatorname{div}_{\xi} E_{\varepsilon}\|_{L^2(\Upsilon)} \leq C \left(\|f\|_{L^{\infty}((t, T) \times \Upsilon)} \|\nabla \varrho\|_{L^2(\Upsilon)} + \|\nabla_{\xi} f\|_{L^2(\Upsilon)} \right),$$

and hence the final term on the right-hand side of (4.5) is dealt with again using the Cauchy–Young inequality.

Thus, returning to (4.5), and using the Cauchy–Young inequality and elementary properties of the Friedrichs mollifier, we find that there exists a constant C , independent of ε , such that

$$\frac{d}{dt} \int_{\Upsilon} |\nabla_{\xi} f_{\varepsilon}|^2 d\xi + \int_{\Upsilon} |\Delta_{\xi} f_{\varepsilon}|^2 d\xi \leq C(1 + \|f\|_{L^{\infty}((t, T) \times \Upsilon)}^2) \|\nabla_{\xi} f\|_{L^2(\Upsilon)}^2.$$

We remind the reader that, since $\nabla_{\xi} f \in L^2(\Upsilon_T)$, it follows from Markov's inequality that $\|\nabla_{\xi} f(t, \cdot)\|_{L^2(\Upsilon)}^2$ is finite a.e. in $(0, T)$. In turn, for a.e. $t \in (0, T)$, we deduce from Grönwall's Lemma that there holds

$$\|\nabla_{\xi} f_{\varepsilon}\|_{L^{\infty}(t, T; L^2(\Upsilon))}^2 \leq C \|\nabla_{\xi} f_{\varepsilon}(t, \cdot)\|_{L^2(\Upsilon)}^2 \exp \left(T(1 + \|f\|_{L^{\infty}((t, T) \times \Upsilon)}^2) \right),$$

whence

$$\|\nabla_{\xi} f_{\varepsilon}\|_{L^{\infty}(t, T; L^2(\Upsilon))}^2 + \|\Delta_{\xi} f_{\varepsilon}\|_{L^2((t, T) \times \Upsilon)}^2 \leq C \|\nabla_{\xi} f_{\varepsilon}(t, \cdot)\|_{L^2(\Upsilon)}^2 \exp \left(T(1 + \|f\|_{L^{\infty}((t, T) \times \Upsilon)}^2) \right),$$

and we let $\varepsilon \rightarrow 0$ to deduce the estimate for f . Then, using the equation, there holds

$$(4.6) \quad \|\partial_t f\|_{L^2((t, T) \times \Upsilon)} \leq \|\Delta_{\xi} f\|_{L^2((t, T) \times \Upsilon)} + \|f\|_{L^{\infty}((t, T) \times \Upsilon)} \|\nabla \varrho\|_{L^2(\Omega_T)} + \|\nabla f\|_{L^2(\Upsilon_T)}.$$

We conclude the proof by bounding $\|\nabla \varrho\|_{L^2(\Omega_T)}$ by $\|\nabla f\|_{L^2(\Upsilon_T)}$ using Jensen's inequality. \square

The following corollary of Lemma 4.3 is immediate.

Corollary 4.4. *For a.e. $t \in (0, T)$, there holds*

$$\|\nabla \varrho\|_{L^4((t, T) \times \Omega)} + \|\nabla \mathbf{p}\|_{L^4((t, T) \times \Omega)} \leq C \left(\|\nabla_{\xi} f\|_{L^{\infty}(t, T; L^2(\Upsilon))} + \|\Delta_{\xi} f\|_{L^2(\Upsilon_T)} \right).$$

Proof. Using Jensen's inequality and the relations between f and ϱ, \mathbf{p} , we obtain for a.e. $t \in (0, T)$

$$\begin{aligned} (\|\nabla \varrho\|_{L^{\infty}(t, T; L^2(\Omega))} + \|\nabla^2 \varrho\|_{L^2((t, T) \times \Omega)}) &+ (\|\nabla \mathbf{p}\|_{L^{\infty}(t, T; L^2(\Omega))} + \|\nabla^2 \mathbf{p}\|_{L^2((t, T) \times \Omega)}) \\ &\leq C \left(\|\nabla_{\xi} f\|_{L^{\infty}(t, T; L^2(\Upsilon))} + \|\Delta_{\xi} f\|_{L^2(\Upsilon_T)} \right), \end{aligned}$$

where we again used Lemma 1.3. The result then follows from the conclusion of Lemma 4.3 and the Interpolation Lemma 1.2, where we used that $\Omega \subset \mathbb{R}^2$ has lower dimension than $\Upsilon \subset \mathbb{R}^3$. \square

In the next lemma we derive the equation for \dot{f} .

Lemma 4.5 (Equation for \dot{f}). *For a.e. $t \in (0, T)$, there holds the estimate*

$$(4.7) \quad \|\dot{f}\|_{L^\infty(t, T; L^2(\Upsilon))}^2 + \|\dot{f}\|_{L^2(t, T; H^1(\Upsilon))}^2 \leq \exp(T(1 + \|f\|_{L^\infty((t, T) \times \Upsilon)}^2)) \|\dot{f}(t, \cdot)\|_{L^2(\Upsilon)}^2.$$

Furthermore, for a.e. $t \in (0, T)$, there holds in the weak sense

$$(4.8) \quad \partial_t \dot{f} + \operatorname{div}((\dot{f}(1 - \varrho) - f\dot{\varrho})\mathbf{e}(\theta)) = \Delta_\xi \dot{f}.$$

We emphasise that the right-hand side of the estimate (4.7) is finite by virtue of $\dot{f} \in L^2((t, T) \times \Upsilon)$ for a.e. $t \in (0, T)$, by Lemma 4.3, whence $\|\dot{f}(t, \cdot)\|_{L^2(\Upsilon)}$ is finite for a.e. $t \in (0, T)$.

Proof. Fix $\delta > 0$ arbitrarily and extend f by zero outside of $[0, T]$ to the larger time-interval $(-\delta, T + \delta)$; we note that this extends the weak formulation to the larger time-interval and preserves

$$\|f\|_{L^2(-\delta, T + \delta; H^1(\Upsilon))} = \|f\|_{L^2(0, T; H^1(\Upsilon))}.$$

We define, for $0 < |h| < \delta$ and a.e. $(t, \xi) \in (0, T) \times \Upsilon$, the difference quotients in time:

$$D_h f(t, \xi) := \frac{f(t + h, \xi) - f(t, \xi)}{h},$$

and, correspondingly,

$$D_h \varrho(t, x) := \frac{\varrho(t + h, x) - \varrho(t, x)}{h} = \int_0^{2\pi} D_h f(t, x, \theta) d\theta,$$

whence $|D_h \varrho(t, x)| \leq \int_0^{2\pi} |D_h f(t, x, \theta)| d\theta$ and Jensen's inequality implies

$$(4.9) \quad \|D_h \varrho(t, \cdot)\|_{L^2(\Omega)} \leq 2\pi \|D_h f(t, \cdot)\|_{L^2(\Upsilon)}.$$

In what follows, we use the notation $\tau_h \varrho = \varrho(t + h)$; the estimate on ϱ implies $0 \leq \tau_h \varrho \leq 1$ a.e. Direct computation show that the equation for $D_h f$ reads, for a.e. $t \in (0, T)$,

$$(4.10) \quad \partial_t D_h f + \operatorname{div}((D_h f(1 - \tau_h \varrho) - f D_h \varrho)\mathbf{e}(\theta)) = \Delta_\xi D_h f,$$

in the weak sense. Note that there holds $D_h f \in L^2(0, T; H^1(\Upsilon))$, whence it is an admissible test function to insert into the weak formulation. It follows that, for a.e. $0 < s < t < T$ away from the initial time,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Upsilon |D_h f|^2 d\xi + \int_\Upsilon |\nabla_\xi D_h f|^2 d\xi \\ &= \int_\Upsilon (D_h f(1 - \tau_h \varrho) - f D_h \varrho)\mathbf{e}(\theta) \cdot \nabla D_h f d\xi \\ &\leq \frac{1}{2} \int_\Upsilon |\nabla_\xi D_h f|^2 d\xi + \frac{1}{2} (1 + \|f\|_{L^\infty((s, T) \times \Upsilon)}^2) \int_\Upsilon |D_h f|^2 d\xi, \end{aligned}$$

where we used the Young inequality and the estimate (4.9); we control $D_h \varrho$ with $D_h f$ using Jensen's inequality and the Fubini–Tonelli Theorem. It therefore follows that

$$(4.11) \quad \frac{d}{dt} \int_\Upsilon |D_h f|^2 d\xi + \int_\Upsilon |\nabla_\xi D_h f|^2 d\xi \leq (1 + \|f\|_{L^\infty((s, T) \times \Upsilon)}^2) \int_\Upsilon |D_h f|^2 d\xi,$$

whence the Grönwall Lemma implies

$$\operatorname{ess\,sup}_{[s, T]} \|D_h f\|_{L^2(\Upsilon)}^2 \leq \exp(T(1 + \|f\|_{L^\infty((s, T) \times \Upsilon)}^2)) \|D_h f(s)\|_{L^2(\Upsilon)}^2.$$

By letting $h \rightarrow 0$, it follows that $\dot{f} := \partial_t f \in L^\infty(s, T; L^2(\Upsilon))$ for a.e. $s \in (0, T)$ and moreover

$$(4.12) \quad \|\dot{f}\|_{L^\infty(s, T; L^2(\Upsilon))}^2 \leq \exp(T(1 + \|f\|_{L^\infty((s, T) \times \Upsilon)}^2)) \|\dot{f}(s)\|_{L^2(\Upsilon)}^2.$$

Similarly, we obtain the boundedness of $\|\nabla_\xi \dot{f}\|_{L^2((t, T) \times \Upsilon)}$ away from the initial time using (4.11), from which, using also (4.12), we deduce, for a.e. $t \in (0, T)$,

$$\|\dot{f}\|_{L^2(t, T; H^1(\Upsilon))}^2 \leq \exp(T(1 + \|f\|_{L^\infty((t, T) \times \Upsilon)}^2)) \|\dot{f}(t, \cdot)\|_{L^2(\Upsilon)}^2.$$

Returning to the weak formulation of (4.10), the improved regularity $\dot{f} \in L^2(t, T; H^1(\Upsilon))$ for a.e. $t \in (0, T)$ implies that we may rigorously take the limit as $h \rightarrow 0$ therein. We therefore rigorously differentiate the weak formulation with respect to the time variable and obtain (4.8) in the weak sense over the interval (t, T) for a.e. $t \in (0, T)$. \square

Next, we need to upgrade the integrability of $\dot{\varrho}$; this is essential, as the term $f\dot{\varrho}\mathbf{e}(\theta)$ forms the second drift term in (4.8) which, as per the statement of Lemma 3.4, is required to belong to L^q for $q > 5$. To this end, we record the following lemma.

Lemma 4.6 (Improved Integrability for $\dot{\varrho}$). *There exists a decreasing positive function $\phi^1 : (0, T) \rightarrow (0, \infty)$ such that $\limsup_{t \rightarrow 0^+} \phi^1(t) = +\infty$ and, for a.e. $t \in (0, T)$, there holds*

$$\|\dot{\varrho}\|_{L^8((t, T) \times \Omega)} \leq \phi^1(t).$$

Proof. The proof is divided into several steps.

1. *Equation for $\dot{\varrho}$ and interpolated integrability:* By integrating the equation (4.8) with respect to the angle variable, we obtain that there holds in the weak sense

$$(4.13) \quad \partial_t \dot{\varrho} + \operatorname{div}((1 - \varrho)\dot{\mathbf{p}} - \mathbf{p}\dot{\varrho}) = \Delta \dot{\varrho}.$$

Furthermore, the estimate (4.7) implies

$$(4.14) \quad (\|\dot{\varrho}\|_{L^\infty(t, T; L^2(\Upsilon))}^2 + \|\dot{\varrho}\|_{L^2(t, T; H^1(\Omega))}^2) + (\|\dot{\mathbf{p}}\|_{L^\infty(t, T; L^2(\Omega))}^2 + \|\dot{\mathbf{p}}\|_{L^2(t, T; H^1(\Omega))}^2) \\ \leq C \exp(T(1 + \|f\|_{L^\infty((t, T) \times \Upsilon)}^2)) \|\dot{f}(t, \cdot)\|_{L^2(\Upsilon)},$$

for some positive constant C independent of t . Note that, as per the proof of Corollary 4.4, the Interpolation Lemma 1.2 yields that, for a.e. $t \in (0, T)$,

$$(4.15) \quad \|\dot{\varrho}\|_{L^4((t, T) \times \Omega)} + \|\dot{\mathbf{p}}\|_{L^4((t, T) \times \Omega)} \\ \leq (\|\dot{\varrho}\|_{L^\infty(t, T; L^2(\Upsilon))} + \|\dot{\varrho}\|_{L^2(t, T; H^1(\Omega))}) + (\|\dot{\mathbf{p}}\|_{L^\infty(t, T; L^2(\Omega))} + \|\dot{\mathbf{p}}\|_{L^2(t, T; H^1(\Omega))}).$$

2. *H^2 -bound on $\dot{\varrho}$:* The computation that follows is formal, as we do not know *a priori* that $\Delta \dot{\varrho}$ is square-integrable, however it is easily made rigorous by either the difference quotient technique used in the proof of Lemma 4.5 or the mollification method from the proof of Lemma 4.3. By testing equation (4.13) against $\Delta \dot{\varrho}$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \dot{\varrho}|^2 dx + \int_{\Omega} |\Delta \dot{\varrho}|^2 dx \leq \int_{\Omega} |\Delta \dot{\varrho}|(1 - \varrho)|\nabla \dot{\mathbf{p}}| dx + \int_{\Omega} |\Delta \dot{\varrho}||\dot{\mathbf{p}}||\nabla \varrho| dx \\ + \int_{\Omega} |\Delta \dot{\varrho}||\dot{\varrho}||\nabla \mathbf{p}| dx + \int_{\Omega} |\Delta \dot{\varrho}||\mathbf{p}||\nabla \dot{\varrho}| dx$$

Using Lemma 1.3 and the Hölder inequality, as well as (4.15) and Corollary 4.4, we obtain, for a.e. $t \in (0, T)$,

$$\|\nabla \dot{\varrho}\|_{L^\infty(t, T; L^2(\Omega))}^2 + \|\nabla \dot{\varrho}\|_{L^2(t, T; H^1(\Omega))}^2 \leq C \left(\|\nabla \dot{\varrho}(t, \cdot)\|_{L^2(\Omega)}^2 + \|\nabla \dot{\varrho}\|_{L^2((t, T) \times \Omega)} + \|\nabla \dot{\mathbf{p}}\|_{L^2((t, T) \times \Omega)}^2 \right. \\ \left. + \|\nabla \varrho\|_{L^4((t, T) \times \Omega)}^4 + \|\dot{\varrho}\|_{L^4((t, T) \times \Omega)}^4 \right. \\ \left. + \|\nabla \mathbf{p}\|_{L^4((t, T) \times \Omega)}^4 + \|\dot{\mathbf{p}}\|_{L^4((t, T) \times \Omega)}^4 \right) \\ =: (\phi^1(t))^2,$$

where the right-hand side is finite a.e. and explodes as $t \rightarrow 0^+$. The Interpolation Lemma 1.2 therefore yields $\|\nabla \dot{\varrho}\|_{L^4((t, T) \times \Upsilon)} \leq C \phi^1(t)$ for a.e. $t \in (0, T)$. Using also (4.15), we deduce that, for a.e. $t \in (0, T)$,

$$\|\dot{\varrho}\|_{L^\infty(t, T; L^2(\Omega))} + \|\dot{\varrho}\|_{L^4(t, T; W^{1,4}(\Omega))} \leq C \phi^1(t),$$

whence the Interpolation Lemma 1.2 yields the result; up to a constant which we do not relabel. \square

We are now in a position to prove the first step of the regularity bootstrap.

Proposition 4.7 (Boundedness of \dot{f} away from Initial Time). *There exists a decreasing positive function $\psi^1 : (0, T) \rightarrow (0, \infty)$ such that $\limsup_{t \rightarrow 0^+} \psi^1(t) = +\infty$ and, for a.e. $t \in (0, T)$, there holds*

$$\|\dot{f}\|_{L^\infty((t, T) \times \Upsilon)} \leq \psi^1(t).$$

Proof. We apply the De Giorgi method to (4.8), which we rewrite as:

$$(4.16) \quad \partial_t \dot{f} + \operatorname{div}_{\xi}(U^1 \dot{f} + V^1) = \Delta_{\xi} \dot{f},$$

where

$$U^1 := (1 - \varrho) \begin{pmatrix} \mathbf{e}(\theta) \\ 0 \end{pmatrix}, \quad V^1 := f \dot{\varrho} \begin{pmatrix} \mathbf{e}(\theta) \\ 0 \end{pmatrix}.$$

As $f \in L^\infty((t, T) \times \Upsilon)$ for a.e. $t \in (0, T)$, we conclude from Lemma 4.6 that the drift terms satisfy:

$$|U^1| \leq 1, \quad V^1 \in L^8((t, T) \times \Upsilon) \quad \text{for a.e. } t \in (0, T).$$

1. *Rescaling to the unit subcylinder:* Let $\delta \in (0, 1)$, $(t_0, \xi_0) \in (0, T) \times \mathbb{R}^3$, and r satisfy the constraint:

$$(4.17) \quad 0 < r < \min \{1, \sqrt{t_0/4}\}.$$

For the purposes of what follows, we write

$$(4.18) \quad \|\dot{f}\|_{t_0}^2 := \|\dot{f}\|_{L^\infty(t_0/2, T; L^2(\Upsilon))}^2 + \|\nabla_\xi \dot{f}\|_{L^2((t_0/2, T) \times \Upsilon)}^2.$$

Let $(t, \xi) \in Q_r(t_0, \xi_0)$ and define

$$\ell(r, \delta) := \delta^{\frac{1}{2}} \frac{r^{\frac{3}{2}}}{1 + \|\dot{f}\|_{t_0} + \|V^1\|_{L^8((t_0/2, T) \times \Upsilon)}},$$

as well as the rescaled functions $\dot{f}_r, U_r^1, V_r^1 : Q_1 \rightarrow \mathbb{R}$ by

$$(4.19) \quad \begin{aligned} \dot{f}_r(\tau, \zeta) &:= \ell \dot{f}(t + r^2\tau, \xi + r\zeta), \\ U_r^1(\tau, \zeta) &:= rU^1(t + r^2\tau, \xi + r\zeta), \\ V_r^1(\tau, \zeta) &:= r\ell V^1(t + r^2\tau, \xi + r\zeta). \end{aligned}$$

Then, by arguing as per the proof of Lemma 2.4, we have that $\dot{f}_r \in C([-1, 0]; L^2(B_1)) \cap L^2(-1, 0; H^1(B_1))$ is non-negative, $\partial_\tau \dot{f}_r \in L^2(-1, 0; (H^1)')(B_1)$,

$$|U_r^1| \leq 1 \quad \text{a.e. in } Q_1, \quad \text{ess sup}_{\tau \in [-1, 0]} \int_{B_1} |\dot{f}_r(\tau)|^2 d\zeta + \int_{Q_1} |\nabla_\zeta \dot{f}_r|^2 d\zeta d\tau \leq \delta$$

and as per (2.3) there holds, in the sense of distributions,

$$(4.20) \quad \partial_\tau \dot{f}_r + \text{div}_\zeta (U_r^1 \dot{f}_r + V_r^1) = \Delta_\zeta \dot{f}_r \quad \text{in } Q_1.$$

Similarly, arguing as per the proof of Lemma 2.4 and using the restrictions on δ, r , we obtain

$$\begin{aligned} \|V_r^1\|_{L^8(Q_1)}^8 &\leq \frac{\delta^4 r^7}{\|V^1\|_{L^8((t_0/2, T) \times \Upsilon)}^8} \int_{Q_r} |V^1(t + \tau', \xi + \zeta')|^8 d\zeta' d\tau' \\ &\leq \frac{1}{\|V^1\|_{L^8((t_0/2, T) \times \Upsilon)}^8} \int_{t_0/2}^T \int_\Upsilon |V^1(\xi, t)|^8 d\xi dt, \end{aligned}$$

whence there holds

$$(4.21) \quad \|V_r^1\|_{L^8(Q_1)} \leq 1.$$

2. *Boundedness and compact exhaustion:* We are in a position to apply Lemmas 3.2 and 3.4 to the equation (4.19). In turn, we find that $(\dot{f}_r)_+ \leq 1/2$ inside the subcylinder $Q_{1/2}$, from which we deduce: there exists a positive constant C , independent of $r, (t, \xi)$, such that

$$(4.22) \quad \|\dot{f}\|_{L^\infty(Q_r(t, \xi))} \leq C \delta_*^{-\frac{1}{2}} (1 + r^{-\frac{3}{2}}) (1 + \|\dot{f}\|_{t_0} + \|V_1\|_{L^8((t_0/2, T) \times \Upsilon)}).$$

We notice once again that the smallness requirement on r in (4.17) only depends on t , whence the exhaustion argument in the proof of Proposition 3.1 may be repeated. We deduce that there exists a constant C independent of t such that, for a.e. $t \in (0, T)$, there holds

$$\|\dot{f}\|_{L^\infty((t, T) \times \Upsilon)} \leq C(1 + t^{-\frac{13}{4}}) (1 + \|\dot{f}\|_t + \|V_1\|_{L^8((t/2, T) \times \Upsilon)}) =: \psi^1(t),$$

where the exponent $-13/4$ is obtained as in the proof of Proposition 3.1. The proof is complete. \square

We conclude this subsection by noting that, by replicating the proofs of Lemmas 4.3 and 4.5, it is easy to see that, for a.e. $t \in (0, T)$, there holds $f^{(2)} \in L^\infty(t, T; L^2(\Upsilon)) \cap L^2(t, T; H^1(\Upsilon))$ with $\partial_t f^{(2)} \in L^2(t, T; (H^1)')(\Upsilon)$, which satisfies in the weak sense

$$\partial_t f^{(2)} + \text{div}((1 - \varrho)f^{(2)} - 2f^{(1)}\varrho^{(1)} - f\varrho^{(2)})\mathbf{e}(\theta) = \Delta_\xi f^{(2)}.$$

4.2. Proofs of Propositions 4.1 and 4.2. Proposition 4.1 follows immediately by applying the next lemma inductively.

Lemma 4.8. *Let f be a non-negative weak solution of (1.1). Let $n \geq 2$ be an integer. Assume that for $j \in \{0, \dots, n-1\}$ there exist decreasing positive functions $\psi^j : (0, T) \rightarrow (0, \infty)$ such that $\limsup_{t \rightarrow 0^+} \psi^j(t) = +\infty$ and, for a.e. $t \in (0, T)$, there holds*

$$(4.23) \quad \|f^{(j)}\|_{L^\infty((t, T) \times \Upsilon)} \leq \psi^j(t),$$

and there exists positive decreasing functions $\Phi^j : (0, T) \rightarrow (0, \infty)$ such that

$$(4.24) \quad \|f^{(j)}\|_{L^\infty(t, T; H^1(\Upsilon))} + \|f^{(j)}\|_{L^2(t, T; H^2(\Upsilon))} \leq \Phi^j(t) \quad \text{a.e. } t \in (0, T).$$

Assume also that there exists a positive decreasing function $\varphi^n : (0, T) \rightarrow (0, \infty)$ such that

$$(4.25) \quad \|f^{(n)}\|_{L^\infty(t, T; L^2(\Upsilon))} + \|f^{(n)}\|_{L^2(t, T; H^1(\Upsilon))} \leq \varphi^n(t) \quad \text{a.e. } t \in (0, T),$$

and that there holds in the weak sense, with $\partial_t f^{(n)} \in L^2(0, T; (H^1)'(\Upsilon))$,

$$(4.26) \quad \partial_t f^{(n)} + \operatorname{div} \left[\left((1 - \varrho) f^{(n)} + \sum_{k=1}^{n-1} \binom{n}{k} \varrho^{(n-k)} f^{(k)} - \varrho^{(n)} f \right) \mathbf{e}(\theta) \right] = \Delta_{\xi} f^{(n)}.$$

Then, there exists a decreasing positive function $\psi^n : (0, T) \rightarrow (0, \infty)$ with $\limsup_{t \rightarrow 0^+} \psi^n(t) = +\infty$ and, for a.e. $t \in (0, T)$, there holds

$$(4.27) \quad \|f^{(n)}\|_{L^\infty((t, T) \times \Upsilon)} \leq \psi^n(t).$$

Furthermore, $f^{(n+1)} \in L^\infty(t, T; L^2(\Upsilon)) \cap L^2(t, T; H^1(\Upsilon))$ and $\partial_t f^{(n+1)} \in L^2(0, T; (H^1)'(\Upsilon))$ satisfies

$$(4.28) \quad \partial_t f^{(n+1)} + \operatorname{div} \left[\left((1 - \varrho) f^{(n+1)} + \sum_{k=1}^n \binom{n+1}{k} \varrho^{(n+1-k)} f^{(k)} - \varrho^{(n+1)} f \right) \mathbf{e}(\theta) \right] = \Delta_{\xi} f^{(n+1)},$$

and there exists a positive decreasing function $\varphi^{n+1} : (0, T) \rightarrow (0, \infty)$ such that

$$(4.29) \quad \|f^{(n+1)}\|_{L^\infty(t, T; L^2(\Upsilon))} + \|f^{(n+1)}\|_{L^2(t, T; H^1(\Upsilon))} \leq \varphi^{n+1}(t) \quad \text{a.e. } t \in (0, T),$$

as well as a positive decreasing function $\Phi_n : (0, T) \rightarrow (0, \infty)$ such that

$$(4.30) \quad \|f^{(n)}\|_{L^\infty(t, T; H^1(\Upsilon))} + \|f^{(n)}\|_{L^2(t, T; H^2(\Upsilon))} \leq \Phi^n(t) \quad \text{a.e. } t \in (0, T).$$

Proof. Throughout this proof, we denote the binomial constants by $\binom{n}{k} =: C_k^n$.

1. *Improved integrability for $\varrho^{(n)}$:* By integrating (4.26) with respect to the angle variable, we see that there holds in the weak sense

$$\partial_t \varrho^{(n)} + \operatorname{div} \left((1 - \varrho) \mathbf{p}^{(n)} + \sum_{k=1}^{n-1} C_k^n \varrho^{(n-k)} \mathbf{p}^{(k)} - \mathbf{p} \varrho^{(n)} \right) = \Delta \varrho^{(n)}.$$

The Interpolation Lemma 1.2 and estimates (4.24) imply that, for all $j \in \{0, \dots, n-1\}$, there holds

$$(4.31) \quad \begin{aligned} & \|\varrho^{(j)}\|_{L^4((t, T) \times \Omega)} + \|\mathbf{p}^{(j)}\|_{L^4((t, T) \times \Omega)} \\ & + \|\nabla \varrho^{(j)}\|_{L^4((t, T) \times \Omega)} + \|\nabla \mathbf{p}^{(j)}\|_{L^4((t, T) \times \Omega)} \leq C \Phi^j(t) \quad \text{a.e. } t \in (0, T), \end{aligned}$$

as well as $\|\varrho^{(n)}\|_{L^4((t, T) \times \Omega)} + \|\mathbf{p}^{(n)}\|_{L^4((t, T) \times \Omega)} \leq C \varphi^n(t)$.

We replicate the second derivative estimate from the proof of Lemma 4.6; we write down formal estimates for clarity of presentation, which can easily be made rigorous by means of difference quotients or mollification. We test (4.26) against $\Delta_{\xi} \varrho^{(n)}$ and obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \varrho^{(n)}|^2 dx + \int_{\Omega} |\Delta \varrho^{(n)}|^2 dx \\ & \leq \int_{\Omega} |\Delta \varrho^{(n)}| |\nabla \mathbf{p}^{(n)}| dx + \int_{\Omega} |\Delta \varrho^{(n)}| |\mathbf{p}^{(n)}| |\nabla \varrho| dx + C \sum_{k=1}^{n-1} \int_{\Omega} |\Delta \varrho^{(n)}| |\varrho^{(n-k)}| |\nabla \mathbf{p}^{(k)}| dx \\ & + C \sum_{k=1}^{n-1} \int_{\Omega} |\Delta \varrho^{(n)}| |\mathbf{p}^{(k)}| |\nabla \varrho^{(n-k)}| dx + \int_{\Omega} |\Delta \varrho^{(n)}| |\nabla \varrho^{(n)}| dx + \int_{\Omega} |\Delta \varrho^{(n)}| |\nabla \mathbf{p}| |\varrho^{(n)}| dx. \end{aligned}$$

By applying the Hölder and Young inequalities, we get

$$\begin{aligned}
& \|\nabla \varrho^{(n)}\|_{L^\infty(t,T;L^2(\Omega))}^2 + \|\Delta \varrho^{(n)}\|_{L^2((t,T)\times\Omega)}^2 \\
& \leq C \left(\|\nabla \varrho^{(n)}(t, \cdot)\|_{L^2(\Omega)}^2 + \|\nabla \varrho^{(n)}\|_{L^2((t,T)\times\Omega)}^2 + \|\nabla \mathbf{p}^{(n)}\|_{L^2((t,T)\times\Omega)}^2 + \|\varrho^{(n)}\|_{L^4((t,T)\times\Omega)}^4 \right. \\
& \quad \left. + \|\mathbf{p}^{(n)}\|_{L^4((t,T)\times\Omega)}^4 + \sum_{k=0}^{n-1} \left(\|\varrho^{(k)}\|_{L^4(t,T;W^{1,4}(\Omega))}^4 + \|\mathbf{p}^{(k)}\|_{L^4(t,T;W^{1,4}(\Omega))}^4 \right) \right) \\
& \leq C \left(\|\nabla \varrho^{(n)}(t, \cdot)\|_{L^2(\Omega)}^2 + (\varphi^n(t))^2 + \sum_{k=0}^{n-1} (\Phi^k(t))^4 \right) \\
& =: \hat{\Psi}^n(t);
\end{aligned}$$

the above is bounded by virtue of the boundedness of φ^n (see (4.25)), $\{\Phi^k\}_{k=0}^{n-1}$ and $\|\nabla \varrho^{(n)}(t, \cdot)\|_{L^2(\Omega)}$ being finite for a.e. $t \in (0, T)$ using Markov's inequality and (4.25).

It follows from Lemma 1.3 and the Interpolation Lemma 1.2 that $\|\nabla \varrho^{(n)}\|_{L^4((t,T)\times\Omega)} \leq C \hat{\Psi}^n(t)$. Arguing as per the proof of Lemma 4.6, we deduce that there exists a positive decreasing function ϕ^n such that

$$\|\varrho^{(n)}\|_{L^\infty(t,T;L^2(\Omega))} + \|\varrho^{(n)}\|_{L^4(t,T;W^{1,4}(\Omega))} \leq \phi^n(t),$$

whence the Interpolation Lemma yields

$$(4.32) \quad \|\varrho^{(n)}\|_{L^8((t,T)\times\Omega)} \leq C \phi^n(t).$$

2. *De Giorgi method for $f^{(n+1)}$* : We define the drift term

$$V^n := \sum_{k=1}^{n-1} C_k^n \varrho^{(n-k)} \mathbf{p}^{(k)} - \mathbf{p} \varrho^{(n)}.$$

We deduce directly from (4.23), (4.31), and (4.32) that there holds

$$V^n \in L^8((t, T) \times \Omega) \quad \text{a.e. } t \in (0, T).$$

By following the proof of Proposition 4.7 to the letter, we apply the De Giorgi method to the equation (4.26) and obtain the estimate (4.27).

3. *Second derivative estimate*: We replicate the argument of Lemma 4.3 for the equation (4.26); again the estimates are formal for clarity of presentation, and can easily be made rigorous by means of difference quotients or mollification. We test (4.26) against $\Delta_\xi f^{(n)}$ and obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_\Upsilon |\nabla_\xi f^{(n)}|^2 d\xi + \int_\Upsilon |\Delta_\xi f^{(n)}|^2 d\xi \\
& \leq \int_\Upsilon |\Delta_\xi f^{(n)}| |\nabla_\xi f^{(n)}| d\xi + \int_\Upsilon |\Delta_\xi f^{(n)}| |f^{(n)}| |\nabla \varrho| d\xi \\
& \quad + C \sum_{k=1}^{n-1} \int_\Upsilon |\Delta_\xi f^{(n)}| |\varrho^{(n-k)}| |\nabla f^{(k)}| d\xi + C \sum_{k=1}^{n-1} \int_\Upsilon |\Delta_\xi f^{(n)}| |\nabla \varrho^{(n-k)}| |f^{(k)}| d\xi \\
& \quad + \int_\Upsilon |\Delta_\xi f^{(n)}| |\varrho^{(n)}| |\nabla f| d\xi + \int_\Upsilon |\Delta_\xi f^{(n)}| |\nabla \varrho^{(n)}| |f| d\xi.
\end{aligned}$$

Using the Young and Jensen inequalities as well as the assumption (4.23), the above implies, for a.e. $t \in (0, T)$,

$$\begin{aligned}
& \frac{d}{dt} \|\nabla_\xi f^{(n)}(t)\|_{L^2(\Upsilon)}^2 + \|\Delta_\xi f^{(n)}(t)\|_{L^2(\Upsilon)}^2 \\
& \leq C \left(\|\nabla_\xi f^{(n)}\|_{L^2(\Upsilon)}^2 + \|f^{(n)}\|_{L^\infty((t,T)\times\Upsilon)}^2 \|\nabla f\|_{L^2(\Upsilon)}^2 + \sum_{k=1}^{n-1} \|f^{(k)}\|_{L^\infty((t,T)\times\Upsilon)}^2 \|\nabla f^{(n-k)}\|_{L^2(\Upsilon)}^2 \right) \\
& \leq C \left(\|\nabla_\xi f^{(n)}(t)\|_{L^2(\Upsilon)}^2 + (\psi^n(t))^2 (\Phi^0(t))^2 + \sum_{k=1}^n (\psi^k(t))^2 (\Phi^{n-k}(t))^2 \right),
\end{aligned}$$

and an application of Grönwall's Lemma yields the estimate (4.30). Returning to the equation, we deduce, as per the estimate (4.6) that $\|\partial_t f^{(n)}\|_{L^2((t,T)\times\Upsilon)}$ is finite for a.e. $t \in (0, T)$.

4. *Equation differentiated in time:* We now differentiate (4.26) in time; the new coefficients $\{C_k^{n+1}\}_{k=1}^n$ are determined by the product rule. For clarity of presentation, we justify this step by performing formally the classical parabolic estimate on (4.28), which can be done rigorously by means of difference quotients as per the proof of Lemma 4.5. We test (4.28) with $f^{(n+1)}$ and obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Upsilon} |f^{(n+1)}|^2 d\xi + \int_{\Upsilon} |\nabla_{\xi} f^{(n+1)}|^2 d\xi \\ & \leq \int_{\Upsilon} |\nabla f^{(n+1)}| |f^{(n+1)}| d\xi + C \sum_{k=1}^{\infty} \|\nabla f^{(n+1)}\|_{\varrho^{(n+1-k)}} \|f^{(k)}\| d\xi + \int_{\Upsilon} |\nabla f^{(n+1)}|_{\varrho^{(n+1)}} |f| d\xi, \end{aligned}$$

whence the Young and Jensen inequalities yield, using also (4.23),

$$\begin{aligned} & \frac{d}{dt} \|f^{(n+1)}(t)\|_{L^2(\Upsilon)}^2 \|\nabla_{\xi} f^{(n+1)}(t)\|_{L^2(\Upsilon)}^2 \\ & \leq C \left((1 + \|f\|_{L^\infty((t,T)\times\Upsilon)}) \|f^{(n+1)}(t)\|_{L^2(\Upsilon)}^2 + \sum_{k=1}^n \|f^{(n+1-k)}\|_{L^\infty((t,T)\times\Upsilon)} \|f^{(k)}\|_{L^\infty((t,T)\times\Upsilon)} \right) \\ & \leq C \left((1 + \psi^0(t)) \|f^{(n+1)}(t)\|_{L^2(\Upsilon)}^2 + \sum_{k=1}^n \psi^{n+1-k}(t) \psi^k(t) \right), \end{aligned}$$

whence the estimate (4.29) follows from Grönwall's Lemma and the weak formulation (4.28) is justified. \square

Proof of Proposition 4.1. As previously mentioned, Proposition 4.1 follows from the previous lemma by applying it inductively. \square

We shall now employ Proposition 4.1 and the time-differentiated equations to prove Proposition 4.2.

Proof of Proposition 4.2. Estimate (4.30), Jensen's inequality, and the Interpolation Lemma imply that for a.e. $t \in (0, T)$

$$\|\nabla_{\xi} f^{(n)}\|_{L^{\frac{10}{3}}((t,T)\times\Upsilon)} + \|\nabla \varrho^{(n)}\|_{L^4((t,T)\times\Omega)} \leq \Phi^n(t),$$

and the above holds for all n . It then follows from the equation (4.26) and Jensen's inequality that

$$\|\Delta_{\xi} f^{(n)}\|_{L^{\frac{10}{3}}((t,T)\times\Upsilon)} \leq C \left(\|f^{(n+1)}\|_{L^\infty((t,T)\times\Upsilon)} + \sum_{k=0}^n \|f^{(k)}\|_{L^\infty((t,T)\times\Upsilon)} \|\nabla f^{(n-k)}\|_{L^{\frac{10}{3}}((t,T)\times\Upsilon)} \right).$$

The above and Lemma 1.3 imply an equivalent bound on the full Hessian $\|\nabla_{\xi}^2 f^{(n)}\|_{L^{10/3}((t,T)\times\Upsilon)}$. Meanwhile, $\partial_t^2 f^{(n)} = f^{(n+2)} \in L^\infty((t, T) \times \Upsilon)$, and the result follows from Minkowski's inequality. \square

5. UNIQUENESS AND REGULARITY OF VERY WEAK SOLUTIONS

We begin this section with a uniqueness result *à la* Michel Pierre for very weak solutions of (1.1); this result will then be used to show that very weak solutions coincide with weak solutions away from the initial time, whence they are endowed with the same regularity properties derived in §4.

Lemma 5.1 (Uniqueness of Very Weak Solutions away from Initial Time). *Let f and g be very weak solutions of (1.1). Suppose that, for some $t_0 \in (0, T)$, there holds $f(t_0, \cdot) = g(t_0, \cdot)$ in $L^2(\Upsilon)$. Then, $f = g$ in $L^2((t_0, T) \times \Upsilon)$.*

Before we proceed to the proof of this result, we remark that a standard argument shows that the weak and very weak formulations of Definition 1.1 can be rewritten without the duality product of the time-derivative, provided we employ test functions that vanish along $\{t = 0\}$ and $\{t = T\}$. Indeed, provided $\varphi \in C^\infty(\Upsilon_T)$ is such that $\varphi(0, \cdot) = \varphi(T, \cdot) = 0$, then there holds

$$\langle \partial_t f, \varphi \rangle = - \int_{\Upsilon_T} f \partial_t \varphi d\xi dt.$$

We use this fact in the proof that follows.

Proof. Recall from [9, Proof of Theorem 2.6] that $f, g \in L^2(\Upsilon_T)$ and the integrals

$$\varrho_f(t, x) = \int_0^{2\pi} f(t, x, \theta) d\theta, \quad \varrho_g(t, x) = \int_0^{2\pi} g(t, x, \theta) d\theta$$

are well-defined and $0 \leq \varrho_f, \varrho_g \leq 1$ a.e. in Ω_T . Without loss of generality, we translate the problem in time $t \mapsto t - t_0$ such that we may assume $t_0 = 0$.

Let $w := f - g \in L^2(\Upsilon_T)$ and note that the assumptions of the lemma imply $w(0, \cdot) = 0$. The equation for w reads (in the weak sense)

$$\partial_t w + \operatorname{div} \left[\left((1 - \varrho_f) f - (1 - \varrho_g) g \right) \mathbf{e}(\theta) \right] = \Delta w + \partial_\theta^2 w.$$

Hence, for any smooth function ψ periodic in x, θ satisfying $\psi(T, \cdot) = 0$, there holds

$$(5.1) \quad \int_{\Upsilon_T} w \partial_t \psi d\xi dt = \int_{\Upsilon_T} \nabla w \cdot \nabla \psi d\xi dt - \int_{\Upsilon_T} w \partial_\theta^2 \psi d\xi dt - \int_{\Upsilon_T} w \mathbf{e}(\theta) \cdot \nabla \psi d\xi dt \\ + \int_{\Upsilon_T} \varrho_f w \mathbf{e}(\theta) \cdot \nabla \psi d\xi dt + \int_{\Upsilon_T} w \left(\int_0^{2\pi} g \mathbf{e}(\theta') \cdot \nabla \psi d\theta' \right) d\xi dt,$$

where we used the Fubini–Tonelli Theorem to rewrite the final term, *i.e.*,

$$\int_{\Upsilon_T} \left(\int_0^{2\pi} w(t, x, \theta') d\theta' \right) g(t, x, \theta) \mathbf{e}(\theta) \cdot \nabla \psi(t, x, \theta) d\xi dt \\ = \int_{\Upsilon_T} w \left(\int_0^{2\pi} g(t, x, \theta') \mathbf{e}(\theta') \cdot \nabla \psi(t, x, \theta') d\theta' \right) d\xi dt.$$

A standard density argument implies that (5.1) holds for all $\psi \in L^2(0, T; H^2(\Upsilon))$ periodic in x, θ satisfying the final-time condition $\psi(T, \cdot) = 0$.

In turn, for $\zeta \in C_c^\infty(\Upsilon_T)$ an arbitrary test function periodic in x, θ , we define ϕ to be a strong solution of the following strongly parabolic linear (formal) dual equation:

$$(5.2) \quad \begin{cases} \partial_t \phi - \Delta_\xi \phi - (1 - \varrho_f) \mathbf{e}(\theta) \cdot \nabla \phi + \int_0^{2\pi} g \mathbf{e}(\theta') \cdot \nabla \phi d\theta' = -\zeta, \\ \phi(0, \cdot) = 0, \end{cases}$$

with periodic boundary conditions in x, θ , and consider $\psi(t, \xi) := \phi(T - t, \xi)$, which satisfies

$$(5.3) \quad \begin{cases} \partial_t \psi + \Delta_\xi \psi + (1 - \varrho_f) \mathbf{e}(\theta) \cdot \nabla \psi - \int_0^{2\pi} g \mathbf{e}(\theta') \cdot \nabla \psi d\theta' = \zeta, \\ \psi(T, \cdot) = 0. \end{cases}$$

The standard linear theory implies the existence and uniqueness of $\phi \in C([0, T]; L_{\text{per}}^2(\Upsilon)) \cap L^2(0, T; H_{\text{per}}^1(\Upsilon))$, and regularity arguments akin to those used in the proof of Lemma 4.3 imply $\phi \in L^2(0, T; H_{\text{per}}^2(\Upsilon))$.

Hence ψ is an admissible test function to insert into (5.1), whence integrating (5.3) against w and substituting for the term $\int_{\Upsilon_T} w \partial_t \psi d\xi dt$ using (5.1) yields

$$\int_{\Upsilon_T} w \zeta d\xi dt = 0.$$

This procedure may be repeated for all $\zeta \in C_c^\infty(\Upsilon_T)$, whence the previous equality holds for all such ζ . It follows that $w \equiv 0$ a.e. in Υ_T , which concludes the proof of the lemma. \square

In turn, we are ready to give the proofs of the main results for very weak solutions.

Proof of Theorems 2 and 3. Let f be a very weak solution. There holds $f \in L^2(\Upsilon_T)$, whence, by Markov's inequality, for a.e. $t_0 \in (0, T)$ there holds $f(t_0, \cdot) \in L^2(\Upsilon)$. It follows that $f(t_0, \cdot)$ may be used as initial data to produce a weak solution for a.e. $t \in (t_0, T)$, which we denote by g . It follows from Lemma 5.1 that $f \equiv g$ in (t_0, T) in the sense of $L^2((t_0, T) \times \Upsilon)$, for a.e. $t_0 \in (0, T)$; in other words, we may select g as the *precise representative* of f in $(t_0, T) \times \Upsilon$. In turn, since f is regular enough to satisfy the weak formulation of (1.1), the conclusion of Theorem 2 follows. Furthermore, by Theorem 1, there holds $g \in C^\infty((t, T) \times \mathbb{R}^3)$ a.e. $t \in (0, T)$, whence f satisfies the assertion of Theorem 3. \square

6. STATIONARY STATES

Our objective in this final section is to prove regularity results for the stationary equation (1.14) associated to (1.1) and the convergence of the time-dependent solutions to stationary solutions. We begin with the former, in §6.1, and then consider the latter in §6.2.

6.1. Regularity of Stationary Solutions. Note that the existence of stationary solutions is trivial; by formally rewriting the drift term $\operatorname{div}((1 - \varrho)f\mathbf{e}(\theta)) = \mathbf{e}(\theta) \cdot \nabla((1 - \varrho)f)$, all constant solutions $f_\infty \in \mathbb{R}$ satisfy the stationary equation (1.14), *i.e.*,

$$\operatorname{Pe} \operatorname{div}((1 - \varrho_\infty)f_\infty\mathbf{e}(\theta)) = D_e \Delta f_\infty + \partial_\theta^2 f_\infty$$

with periodic boundary conditions.

We now prove Theorem 5. To do so, we apply a bootstrapping argument similar to the one in §4 and thereby show smoothness of the stationary states. Without loss of generality we restrict this part of our analysis to the case $\operatorname{Pe} = D_e = 1$.

Proof of Theorem 5. The proof is divided into several steps.

1. *H^2 -type bound on f_∞ :* Using the classical Moser's iteration method in the elliptic context (*cf.* §3.1 or [47, §2]), it holds that any weak solution $f_\infty \in H_{\text{per}}^1(\Upsilon)$ of (1.14) with $0 \leq \varrho_\infty \leq 1$ belongs to L^∞ ; we omit the details for concision. Then, by testing the equation against $\Delta_\xi f_\infty$ (which may be performed rigorously by means of difference quotients as in §4), we obtain using Young's inequality the second derivative estimate

$$\int_\Upsilon |\Delta_\xi f_\infty|^2 d\xi \leq C(1 + \|f_\infty\|_{L^\infty(\Upsilon)}^2) \int_\Upsilon |\nabla f_\infty|^2 d\xi,$$

for some positive constant $C = C(\Upsilon)$. It follows from Lemma 1.3 that $\Delta_\xi f_\infty \in L^2(\Upsilon)$ implies $f_\infty \in W^{2,2}(\Upsilon)$ and thus $\varrho_\infty \in W^{2,2}(\Omega)$. The Sobolev Embedding Theorem therefore implies $\nabla f_\infty \in L^6(\Upsilon)$ and $\nabla \varrho_\infty \in \operatorname{BMO}(\Omega)$, whence $\nabla \varrho_\infty \in L^p(\Omega)$ for all $p \in [1, \infty)$.

2. *Equation for ϱ_∞ :* By integrating the equation with respect to the angle variable, we recover the equation for ϱ_∞ , which reads

$$\operatorname{div}((1 - \varrho_\infty)\mathbf{p}_\infty) = \Delta \varrho_\infty,$$

where $\mathbf{p}_\infty = \int_0^{2\pi} f_\infty \mathbf{e}(\theta) d\theta \in L^\infty(\Omega)$. By virtue of $0 \leq \varrho_\infty \leq 1$, we obtain

$$(6.1) \quad \Delta \varrho_\infty = (1 - \varrho_\infty) \underbrace{\operatorname{div} \mathbf{p}_\infty}_{\in L^6} - \nabla \varrho_\infty \cdot \mathbf{p}_\infty,$$

and we deduce from Lemma 1.3 that $\varrho_\infty \in W^{2,6}(\Omega)$ and Morrey's Embedding implies $\nabla \varrho_\infty \in W^{1,6}(\Omega) \subset C^{0,2/3}(\Omega)$; in particular, we have $\varrho_\infty, \nabla \varrho_\infty \in L^\infty(\Omega)$. Returning to (1.14), we get

$$(6.2) \quad \Delta_\xi f_\infty = (1 - \varrho_\infty) \underbrace{\nabla f_\infty \cdot \mathbf{e}(\theta)}_{\in L^6} - \underbrace{f_\infty \nabla \varrho_\infty \cdot \mathbf{e}(\theta)}_{\in L^\infty},$$

whence, again by Lemma 1.3 and Morrey's Embedding, we get $\nabla f_\infty \in W^{1,6}(\Upsilon) \subset C^{0,1/2}(\Upsilon)$; thus $f_\infty, \nabla f_\infty \in L^\infty(\Upsilon)$. By returning to (6.1)–(6.2), we get $\Delta_\xi f_\infty \in L^\infty(\Upsilon)$ and $\Delta \varrho_\infty \in L^\infty(\Omega)$; in fact, both of these quantities are Hölder continuous. By Lemma 1.3, we obtain

$$(6.3) \quad f_\infty \in W^{2,p}(\Upsilon) \quad \text{for all } p \in [1, \infty).$$

3. *Higher derivatives:* Next, one may use difference quotients with respect to the variable ξ to make rigorous the following formal computation: differentiating (1.14) with respect to x and θ , respectively, gives

$$\begin{aligned} \operatorname{div} \left(((1 - \varrho_\infty)\nabla f_\infty - f_\infty \nabla \varrho_\infty) \otimes \mathbf{e}(\theta) \right) &= \Delta_\xi \nabla f_\infty \\ \operatorname{div} \left((1 - \varrho_\infty)\partial_\theta f_\infty \mathbf{e}(\theta) + (1 - \varrho_\infty)f_\infty \mathbf{e}'(\theta) \right) &= \Delta_\xi \partial_\theta f_\infty. \end{aligned}$$

In turn, taking the L^p norm of both sides and estimating directly using (6.3) yields $\Delta_\xi \nabla_\xi f_\infty \in L^p(\Upsilon)$ for all $p \in [1, \infty)$. It follows from Lemma 1.3 that $\nabla_\xi^3 f_\infty \in L^p(\Upsilon)$, *i.e.*,

$$\nabla_\xi^2 f_\infty \in W^{1,p}(\Upsilon) \subset C^{0,\gamma}(\Upsilon)$$

for some $\gamma \in (0, 1)$, by Morrey's Embedding. One may iterate this procedure indefinitely to deduce that $\nabla_\xi^k f_\infty$ is continuous for all $k \in \mathbb{N}$, whence we deduce f_∞ is smooth; we skip the details as they are analogous to those in §4. Note that the aforementioned embeddings may be taken with respect to the

larger set $\Upsilon' = (-2\pi, 4\pi)^3$ using the periodicity of f_∞ , whence Υ is comprised as an interior set and thus all embeddings also hold on the closure $\bar{\Upsilon}$, *i.e.*, there exist $\gamma_k \in (0, 1)$ such that $\nabla_{\xi}^k f_\infty \in C^{0, \gamma_k}(\bar{\Upsilon})$ for all $k \in \mathbb{N}$. \square

Remark 6.1 (Uniqueness for the stationary problem). The uniqueness of stationary solutions is a more delicate issue. Let f_∞ be a solution of (1.14), satisfying the mass constraint $\int_{\Upsilon} f_\infty d\xi = m$ for some $m > 0$ and the periodic boundary condition; we emphasise that in this remark f_∞ is unrelated to the evolution problem (1.1). Then, all the translates of f_∞ , *i.e.* $f_\infty(x + x_0, \theta + \theta_0)$ for arbitrary x_0, θ_0 , are also periodic stationary solutions satisfying the mass constraint. Furthermore, a numerical example of a non-constant stationary state co-existing with (unstable) constant stationary states is shown in [8, §4].

For small Péclet number, we have the following uniqueness result regarding a natural linearisation of (1.14). Note that the stationary function ϱ_∞ is known *a priori*, since the uniform-in-time estimate $0 \leq \varrho(t, \cdot) \leq 1$ implies the weak-* subsequential convergence of the sequence $\{\varrho(t, \cdot)\}_{t>0}$ to ϱ_∞ also satisfying $0 \leq \varrho_\infty \leq 1$ a.e.

Lemma 6.2. *Let $m \geq 0$ and $\varrho_\infty \in L_{\text{per}}^\infty(\Omega)$ be given, and assume that $0 \leq \varrho_\infty \leq 1$ a.e. Provided there holds*

$$|\text{Pe}| < \frac{\min\{D_e, 1\}}{C_P},$$

where C_P is the Poincaré constant on Υ , then the solution of the linear elliptic equation

$$(6.4) \quad \text{Pe} \operatorname{div}((1 - \varrho_\infty)f_\infty \mathbf{e}(\theta)) = D_e \Delta f_\infty + \partial_\theta^2 f_\infty$$

is unique within the class

$$\mathcal{C} = \left\{ f_\infty \in H_{\text{per}}^1(\Upsilon) : f_\infty \geq 0 \text{ a.e., } \int_{\Upsilon} f_\infty d\xi = m \right\}.$$

Proof. Let f_1 and f_2 belong to the aforementioned class of solutions and set $\bar{f} := f_1 - f_2$. Then, $\int_{\Upsilon} \bar{f} d\xi = 0$, whence it follows that there holds the Poincaré inequality

$$\|\bar{f}\|_{L^2(\Upsilon)} \leq C_P \|\nabla_{\xi} \bar{f}\|_{L^2(\Upsilon)}.$$

Furthermore, \bar{f} solves

$$\text{Pe} \operatorname{div}((1 - \varrho_\infty)\bar{f} \mathbf{e}(\theta)) = D_e \Delta \bar{f} + \partial_\theta^2 \bar{f},$$

whence testing the equation against \bar{f} yields

$$D_e \int_{\Upsilon} |\nabla \bar{f}|^2 d\xi + \int_{\Upsilon} |\partial_\theta \bar{f}|^2 d\xi = \text{Pe} \int_{\Upsilon} (1 - \varrho_\infty) \bar{f} \mathbf{e}(\theta) \cdot \nabla \bar{f} d\xi.$$

The Poincaré inequality yields

$$\min\{D_e, 1\} \|\nabla_{\xi} \bar{f}\|_{L^2(\Upsilon)}^2 \leq \text{Pe} \|\bar{f}\|_{L^2(\Upsilon)} \|\nabla \bar{f}\|_{L^2(\Upsilon)} \leq \text{Pe} C_P \|\nabla_{\xi} \bar{f}\|_{L^2(\Upsilon)}^2.$$

We deduce that $\nabla_{\xi} \bar{f} \equiv 0$ a.e., whence f_1 and f_2 differ by a constant; the mass constraint $\int_{\Upsilon} f_1 d\xi = \int_{\Upsilon} f_2 d\xi = m$ implies that this constant is null, which concludes the proof. \square

6.2. Convergence to Equilibrium. We henceforth concentrate entirely on the convergence to the stationary states in the limit of infinite time. Our first result shows the convergence as $t \rightarrow \infty$ of the spatial averages $\int_{\Omega} f(t, x, \theta) dx$ to the constant stationary state $\frac{1}{2\pi} \int_{\Upsilon} f_0 dx d\theta = \frac{1}{2\pi} \int_{\Omega} \varrho_0 dx$, regardless of the Péclet number.

Lemma 6.3 (Convergence of spatial averages). *Let $f_0 \in L_{\text{per}}^2(\Upsilon)$ be admissible non-negative initial data and let f be the unique solution of (1.1). Then, for all $t \geq 0$,*

$$\left\| \int_{\Omega} \left(f(t) - \frac{1}{2\pi} \int_0^{2\pi} f_0 d\theta \right) dx \right\|_{L^2([0, 2\pi])} \leq e^{-t} \left\| \int_{\Omega} \left(f_0 - \frac{1}{2\pi} \int_0^{2\pi} f_0 d\theta \right) dx \right\|_{L^2([0, 2\pi])}.$$

Proof. By integrating the equation (1.1) with respect to the space variable, we see that $h(t, \theta) := \int_{\Omega} f(t, x, \theta) dx$ satisfies the heat equation

$$\begin{cases} \partial_t h = \partial_\theta^2 h, \\ h|_{t=0} = h_0, \end{cases}$$

where we denote $h_0 := \int_{\Omega} f_0(x, \theta) dx$. Setting

$$\bar{h} := h - \frac{1}{2\pi} \int_0^{2\pi} h_0 d\theta,$$

the conservation of the initial mass implies

$$(6.5) \quad \int_0^{2\pi} \bar{h} d\theta = 0.$$

Performing the usual parabolic estimate for the equation satisfied by \bar{h} and using Wirtinger's inequality (*i.e.* Poincaré's inequality in one dimension, for which the constant is 1), we get

$$\frac{1}{2} \frac{d}{dt} \int_0^{2\pi} |\bar{h}|^2 d\theta = - \int_0^{2\pi} |\partial_{\theta} \bar{h}|^2 d\theta \leq - \int_0^{2\pi} |\bar{h}|^2 d\theta.$$

Grönwall's Lemma then implies

$$\int_0^{2\pi} |\bar{h}(t)|^2 d\theta \leq \left(\int_0^{2\pi} |\bar{h}(0)|^2 d\theta \right) e^{-2t},$$

and the result follows. \square

We conclude this section with the proof of Theorem 6, which shows that, under the constraint of small Péclet number, weak solutions converge to a constant stationary state prescribed by the initial data.

Proof of Theorem 6. Define $w := f - \langle f_0 \rangle$, where $\langle f_0 \rangle = \int_{\Gamma} f_0 d\xi$. Observe that w solves

$$(6.6) \quad \begin{cases} \partial_t w + \text{Pe} \operatorname{div}((1 - \varrho)w\mathbf{e}(\theta)) = D_e \Delta w + \partial_{\theta}^2 w - \text{Pe} \langle f_0 \rangle \operatorname{div}((1 - \varrho)\mathbf{e}(\theta)), \\ w(0, \cdot) = f_0 - \langle f_0 \rangle. \end{cases}$$

Testing the above with w and using Young's inequality gives

$$\begin{aligned} \frac{d}{dt} \int_{\Gamma} w^2 d\xi + \min\{D_e, 1\} \int_{\Gamma} |\nabla_{\xi} w|^2 d\xi &\leq \frac{\text{Pe}^2}{\min\{D_e, 1\}} \int_{\Gamma} w^2 d\xi \\ &\quad + 2 \int_{\Gamma} \text{Pe} \langle f_0 \rangle \operatorname{div}((1 - \varrho)\mathbf{e}(\theta)) w d\xi, \end{aligned}$$

and the final term on the right-hand side may be controlled as

$$\begin{aligned} \int_{\Gamma} \operatorname{div}((1 - \varrho)\mathbf{e}(\theta)) w d\xi &= - \int_{\Gamma} \nabla_{\varrho} \cdot \mathbf{e}(\theta) w d\xi = - \int_{\Gamma} \nabla(\varrho - \langle \varrho_0 \rangle) \cdot \mathbf{e}(\theta) w d\xi \\ &= \int_{\Gamma} (\varrho - \langle \varrho_0 \rangle) \mathbf{e}(\theta) \cdot \nabla w d\xi \\ &\leq 2\pi \|w\|_{L^2(\Gamma)} \|\nabla w\|_{L^2(\Gamma)}, \end{aligned}$$

where we used Jensen's inequality to get

$$\begin{aligned} \int_{\Gamma} |\varrho - \langle \varrho_0 \rangle|^2 d\xi &= \int_{\Gamma} \left| \int_0^{2\pi} (f(t, x, \theta') - \langle f_0 \rangle) d\theta' \right|^2 d\xi \\ &\leq 2\pi \int_{\Gamma} \int_0^{2\pi} |f(t, x, \theta') - \langle f_0 \rangle|^2 d\theta' d\xi \\ &= (2\pi)^2 \|w\|_{L^2(\Gamma)}^2. \end{aligned}$$

It follows that

$$\frac{d}{dt} \|w(t, \cdot)\|_{L^2(\Gamma)}^2 d\xi + \frac{1}{2} \min\{D_e, 1\} \|\nabla_{\xi} w(t, \cdot)\|_{L^2(\Gamma)}^2 \leq \frac{(2\pi)^2 \text{Pe}^2 (1 + \langle f_0 \rangle)^2}{\min\{D_e, 1\}} \|w(t, \cdot)\|_{L^2(\Gamma)}^2,$$

and, since $\int_{\Gamma} w(t, \cdot) d\xi = \int_{\Gamma} w(0, \cdot) d\xi = 0$ for all $t \geq 0$, an application of Poincaré's inequality yields

$$\frac{d}{dt} \|w(t, \cdot)\|_{L^2(\Gamma)}^2 d\xi \leq - \underbrace{\left(\frac{1}{2} C_P^{-2} \min\{D_e, 1\} - \frac{(2\pi)^2 \text{Pe}^2 (1 + \langle f_0 \rangle)^2}{\min\{D_e, 1\}} \right)}_{=2\kappa} \|w(t, \cdot)\|_{L^2(\Gamma)}^2,$$

where C_P is the Poincaré constant on Υ . Provided κ is positive, which imposes the smallness condition on Pe , Grönwall's Lemma implies

$$\|w(t, \cdot)\|_{L^2(\Upsilon)}^2 \leq \|w(0, \cdot)\|_{L^2(\Upsilon)}^2 e^{-2\kappa t},$$

and the proof is complete. \square

APPENDIX A. PROOFS OF TECHNICAL LEMMAS

A.1. Proof of Lemma 1.3. We explain the simple idea underlying the proof of Lemma 1.3. To begin with, the classical Calderón–Zygmund Theorem requires the function under consideration to be compactly supported. In order to bypass this, we use the fact that the periodic cell $\Upsilon = (0, 2\pi)^3$ may be interpreted as an interior set of the larger set $\Upsilon' := (-2\pi, 4\pi)^3$, say. Periodicity then implies that, for all $p \in [1, \infty]$ and $v \in W_{\text{per}}^{2,p}(\Upsilon)$, there holds

$$(A.1) \quad \|\nabla_{\xi}^j v\|_{L^p(\Upsilon')} = 3^3 \|\nabla_{\xi}^j v\|_{L^p(\Upsilon)} \quad \text{for } j = 0, 1, 2,$$

so that one may recover the desired estimate after having localised.

Proof of Lemma 1.3. Let $\eta \in C_c^\infty(\Upsilon')$ be a positive bump function such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on Υ . Note that there exists a positive constant $C = C(\Upsilon)$ such that

$$(A.2) \quad \|\nabla_{\xi}^j \eta\|_{L^\infty(\Upsilon')} \leq C \quad \text{for } j = 0, 1, 2.$$

1. *For smooth functions:* Fix v smooth and Υ -periodic, and observe that $\eta v \in W_0^{2,p}(\Upsilon')$. It then follows from the classical Calderón–Zygmund Inequality (cf. [27, Corollary 9.10 §9.4]) that

$$\|\nabla_{\xi}^2(\eta v)\|_{L^p(\Upsilon')} \leq C \|\Delta_{\xi}(\eta v)\|_{L^p(\Upsilon')}.$$

Using the product rule and the triangle inequality, we also have

$$\|\eta \nabla_{\xi}^2 v\|_{L^p(\Upsilon')} \leq \|\nabla_{\xi}^2(\eta v)\|_{L^p(\Upsilon')} + \|\nabla_{\xi} v \otimes \nabla_{\xi} \eta + \nabla_{\xi} \eta \otimes \nabla_{\xi} v + v \nabla_{\xi}^2 \eta\|_{L^p(\Upsilon')},$$

whence combining with the previous estimate and (A.2) yields

$$(A.3) \quad \|\eta \nabla_{\xi}^2 v\|_{L^p(\Upsilon')} \leq C \left(\|\Delta_{\xi}(\eta v)\|_{L^p(\Upsilon')} + \|\nabla_{\xi} v\|_{L^p(\Upsilon')} + \|v\|_{L^p(\Upsilon')} \right).$$

The inclusion $\Upsilon \subset \Upsilon'$ and the condition $\eta \equiv 1$ on Υ implies that the left-hand side of the previous estimate is bounded from below by $\|\nabla_{\xi}^2 v\|_{L^p(\Upsilon)}$. In turn, using (A.1) and the periodicity of v to control the final two terms on the right-hand side of (A.3), we obtain

$$\|\nabla_{\xi}^2 v\|_{L^p(\Upsilon)} \leq C \left(\|\Delta_{\xi}(\eta v)\|_{L^p(\Upsilon')} + \|v\|_{W^{1,p}(\Upsilon)} \right).$$

By expanding

$$\Delta_{\xi}(\eta v) = v \Delta_{\xi} \eta + 2 \nabla_{\xi} \eta \cdot \nabla_{\xi} v + \eta \Delta_{\xi} v,$$

and using again (A.1) and (A.2), we estimate the first term on the right-hand side of (A.3) to deduce

$$(A.4) \quad \|\nabla_{\xi}^2 v\|_{L^p(\Upsilon)} \leq C \left(\|\Delta_{\xi} v\|_{L^p(\Upsilon)} + \|v\|_{W^{1,p}(\Upsilon)} \right)$$

for all smooth Υ -periodic v .

2. *Density argument:* Now fix $v \in W_{\text{per}}^{1,p}(\Upsilon)$ satisfying $\Delta_{\xi} v \in L^p(\Upsilon)$ as per the statement. Let $\{\eta_{\varepsilon}\}_{\varepsilon}$ be the usual sequence of Friedrichs mollifiers on \mathbb{R}^3 ; note that the convolutions

$$v * \eta_{\varepsilon}(\xi) = \int_{\mathbb{R}^3} \eta_{\varepsilon}(\xi - \zeta) v(\zeta) \, d\zeta$$

are well-defined by virtue of v being locally integrable on any subset of \mathbb{R}^3 due to its periodicity. It follows from (A.4) that, since $v * \eta_{\varepsilon}$ is smooth and periodic, there holds

$$\begin{aligned} \|\nabla_{\xi}^2 v * \eta_{\varepsilon}\|_{L^p(\Upsilon)} &\leq C \left(\|\Delta_{\xi} v * \eta_{\varepsilon}\|_{L^p(\Upsilon)} + \|v * \eta_{\varepsilon}\|_{W^{1,p}(\Upsilon)} \right) \\ &\leq C \left(\|\Delta_{\xi} v\|_{L^p(\Upsilon)} + \|v\|_{W^{1,p}(\Upsilon)} \right), \end{aligned}$$

where the second line follows from elementary results on mollifiers. The conclusion of the lemma for general $p \in (1, \infty)$ now follows from letting $\varepsilon \rightarrow 0$.

3. *Case $p = 2$:* In the specific case $p = 2$, we integrate by parts twice, noting that the periodicity of v implies that all boundary terms cancel, to obtain

$$\|\nabla_{\xi}^2 v\|_{L^2(\Upsilon)}^2 = \sum_{ij} \int_{\Upsilon} \partial_{ij} v \partial_{ij} v \, d\xi = \sum_{ij} \int_{\Upsilon} \partial_{ii} v \partial_{jj} v \, d\xi = \int_{\Upsilon} \left(\sum_i \partial_{ii} v \right) \left(\sum_j \partial_{jj} v \right) \, d\xi,$$

and hence $\|\nabla_{\xi}^2 v\|_{L^2(\Upsilon)}^2 = \|\Delta_{\xi} v\|_{L^2(\Upsilon)}^2$, which concludes the proof. \square

A.2. Proof of Lemma 2.1.

Proof of Lemma 2.1. Let f be the unique weak solution of (1.1). Recall from [9, §3] that f was constructed via a Galerkin approximation, and may be written as a subsequential limit of the sequence of smooth functions $\{f_n\}_n$ which solve, as a pointwise equality between continuous functions,

$$(A.5) \quad \partial_t f_n + \operatorname{div}((1 - (\varrho_n)_+) + f_n \mathbf{e}(\theta)) = \Delta_{\xi} f_n,$$

where $\varrho_n = \int_0^{2\pi} f_n \, d\theta$ is also smooth, with initial data $f_n(0, \cdot) = f_{0,n} \in L^2_{\text{per}}(\Upsilon)$ a smooth approximation of f_0 in terms of the Galerkin basis functions; *i.e.* $\lim_{n \rightarrow \infty} \|f_0 - f_{0,n}\|_{L^2(\Upsilon)} = 0$. More precisely, with f_n as above, there holds

$$(A.6) \quad \lim_{n \rightarrow \infty} \|f_n - f\|_{L^2(0,T;H^1(\Upsilon))} = 0,$$

and, for each n , the function $f_n : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is such that, for each fixed $t \in [0, T]$, the function $f_n(t, \cdot)$ is triply 2π -periodic and C^1 . The strong convergence in $L^2(0, T; H^1(\Upsilon))$ and the periodicity implies, using a straightforward covering argument, that for any bounded domain $\omega \subset \mathbb{R}^3$, there holds

$$(A.7) \quad \lim_{n \rightarrow \infty} \|f_n - f\|_{L^2(0,T;H^1(\omega))} = 0.$$

Similarly, $\varrho_n(t, \cdot)$ is doubly 2π -periodic for each fixed $t \in [0, T]$, and

$$(A.8) \quad \lim_{n \rightarrow \infty} \|\varrho_n - \varrho\|_{L^2(0,T;H^1(\omega'))} = 0$$

for all bounded domains $\omega' \subset \mathbb{R}^2$. Using (A.7), a standard argument using Minkowski's inequality implies (up to a subsequence which we do not relabel)

$$(A.9) \quad \lim_{n \rightarrow \infty} \|f_n(t, \cdot) - f(t, \cdot)\|_{H^1(\omega)} = 0 \quad \text{a.e. } t \in [0, T].$$

Let φ be as in the statement of Lemma 2.1. By testing against the pointwise equality (A.5) and integrating by parts, using the compact support of $\varphi(t, \cdot)$ for all t , we obtain

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\mathbb{R}^3} f_n \partial_t \varphi \, d\xi \, dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (1 - (\varrho_n)_+) + f_n \mathbf{e}(\theta) \cdot \nabla \varphi \, d\xi \, dt - \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \nabla_{\xi} f_n \cdot \nabla_{\xi} \varphi \, d\xi \, dt \\ = \int_{\mathbb{R}^3} f_n \varphi \, d\xi \Big|_{t_2} - \int_{\mathbb{R}^3} f_n \varphi \, d\xi \Big|_{t_1} \end{aligned}$$

for all n . By passing to the limit in n , using the strong local convergences of (A.7)-(A.9) and the compact support of the test function in the variable ξ , we obtain

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\mathbb{R}^3} f \partial_t \varphi \, d\xi \, dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (1 - (\varrho)_+) + f \mathbf{e}(\theta) \cdot \nabla \varphi \, d\xi \, dt - \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \nabla_{\xi} f \cdot \nabla_{\xi} \varphi \, d\xi \, dt \\ = \int_{\mathbb{R}^3} f \varphi \, d\xi \Big|_{t_2} - \int_{\mathbb{R}^3} f \varphi \, d\xi \Big|_{t_1}. \end{aligned}$$

We recall from [9, §3.3] that ϱ satisfies the estimates (1.8), whence the relation (2.1) follows immediately. \square

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