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Absence of percolation for infinite Poissonian systems of stopped paths

David Coupier ^{*}, David Dereudre[†], Jean-Baptiste Gouéré [‡]

September 23, 2024

Abstract

The state space of our model is the Euclidean space in dimension $d = 2$. Simultaneously, from all points of a homogeneous Poisson point process, we let grow independent and identically distributed random continuum paths. Each path stops growing at time $t > 0$ if it hits the trace of the other curves realized up until time t . Such dynamic is well-defined as long as the distribution of paths has a finite second moment at each time $t > 0$. Letting the time runs until infinity so that each path reaches its stopping curve, we study the connected property of the graph formed by all stopped curves. Our main result states the absence of percolation in this graph, meaning that each cluster consists of a finite number of curves. The assumptions on the distribution of paths are very mild, with the main one being the so-called 'loop assumption' which ensures that finite clusters (necessarily containing a loop) occur with positive probability. The main issue in this model comes from the long-range dependence arising from long sequences of causalities in the hitting/stopping procedure. Most methods based on block approaches fail to effectively address the question of percolation in this setting.

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1 Introduction

Our model belongs to the family of stopped germ-grain models in \mathbb{R}^2 . In this context, a point (germ) from a point process in \mathbb{R}^2 initiates the growth of a random set (grain), which stops growing when it encounters another grain. These models naturally arise in material science and biology, for example, to model impurities or cracks in materials or the propagation of infection in a cellular tissue. They are also a significant class of models in stochastic geometry, presenting interesting and challenging percolation issues. The first example of such a model, introduced by Häggström and Meester in 1996, is the lilypond model [11], where the germs are given by an homogeneous Poisson point process and the grains are discs with radii that increase over time with constant velocity. They proved that the clusters of stopped discs do not percolate. The main conjecture, which emerged from Question 6.1 in [11], can be rephrased as follows: for a large class of distributions of grains, the associated stopped germ-grain model does not percolate. Since then, the absence of percolation has been investigated for many germ-grain models in which grains are mainly convex bodies [8, 12] or line-segments that can be unilateral or bilateral, with constant or i.i.d. (even unbounded) velocities, according to isotropic directions or not [3, 4, 5, 13].

We address this problem in the present article. The primary difficulties encountered in this class of models arise from the long-range dependence of the connection graph and the lack of monotony or usefull correlation inequalities (such as FKG inequalities).

Our model

In our case, the point process ξ is an homogeneous Poisson point process in \mathbb{R}^2 with intensity one (other intensities could be considered by a simple rescaling procedure). The space of grains \mathcal{C} is the space of continuous functions $h : \mathbb{R}_+ \rightarrow \mathbb{R}^2$ such that $h(0) = 0$, and we denote by μ a probability measure on \mathcal{C} . Each point x in ξ is then equipped with a random path h_x with distribution μ . We assume that the paths $(h_x)_{x \in \xi}$ are independent of each other and independent of ξ . This description is rigorously defined by a marked Poisson point processes $\bar{\xi}$ on $\mathbb{R}^2 \times \mathcal{C}$ with intensity $dx \otimes \mu$.

Now, from each point $x \in \xi$, we let the path $x + h_x$ starts to grow, and it stops growing at time $t > 0$ when it hits the curves of other grains produced until time t . If a grain does not meet any other grain, it continues to grow indefinitely. The existence of this infinite-dimensional dynamics is not obvious, but it is possible to prove its existence as soon as $\mathbb{E}(\sup_{0 \leq s \leq t} \|h_x(s)\|^2) < +\infty$ for all positive time t (see Proposition 4).

Denote by $\tau(x)$ the stopping time of the grain $x \in \xi$ (which is infinite if x never stops growing). The main question is whether the random set

$$\bigcup_{x \in \xi} \{x + h_x(t), 0 \leq t \leq \tau(x)\}$$

percolates; i.e., whether some of the connected components of this random set consist of an infinite number of curves. The main conjecture since the paper by Häggström and Meester mentioned above is that, under reasonable assumptions, **percolation should not occur**. Let us be more specific about those reasonable assumptions. Consider the graph of connections $\overrightarrow{\text{Graph}}(\bar{\xi})$ with vertex set ξ where we put an edge $x \rightarrow y$ from x to y if the path from x hits the curve from y . If loops are not possible, then the connected components of this graph are infinite. Therefore the possibility of existence of loops is a necessary condition for non percolation. Because of the continuous nature of our model, if one seeks for a simple general sufficient condition, it is natural to assume that loops arise at arbitrary small space and time scales. We thus formulate the following question.

Question 1. *Do the following assumptions imply absence of percolation?*

- *The model is well defined. (We give a precise meaning to this assumption in Section 2.2 with the notion of tempered measure).*
- *The "loop-assumption" holds at any small spatial and temporal scale $\alpha > 0$. It means that for any $\alpha > 0$, with positive probability, the points (and the paths) in the ball $B(0, \alpha)$ produce before time $\alpha > 0$ a loop inside the ball that disconnects the center 0 from the outside of the ball. See Figure 1 (We refer to Section 2.3 for a precise definition.)*

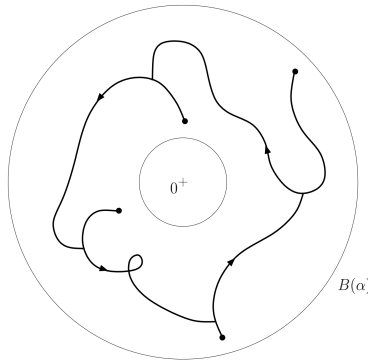


Figure 1: In this picture, the loop is made up of $n = 4$ points (the black points). The union of their curves separates 0 from the outside of $B(\alpha)$. The black arrows indicate the direction in which the grains grow.

Let us mention that it is easy to see that some models lack the loop condition and therefore exhibit percolation. For instance, if the paths go in a straight line to the right or to the top with equal probability 1/2, the loops are not possible. See also the navigation models studied in [1, 2] which are directed according to a given deterministic vector.

Results

Our main result is a positive answer to Question 1 under the following mild extra assumption on the distribution of paths μ : there exists $t > 0$ such that $\mathbb{E}(\sup_{0 \leq s \leq t} \|h_x(s)\|^2) < +\infty$. See Theorem 5.

Comparison with models in the literature.

As mentioned above, the first stopped germ-grain model in the literature is the lilypond model [6, 7, 11], where the grains are balls. In this case, the grains are thick, and achieving the locality of the interaction

is easier, as the probability of a distant grain hitting another grain decreases exponentially with the distance. See also generalizations to convex grains [8, 12]. Studying thin grains, like curves, poses more challenges. The question has been raised in [5] where line-segment models are introduced. The first non percolation result for this kind of model has been obtained in [13] in a non isotropic setting. Here, paths move with constant velocity in straight lines to the right, left, top, or down with equal probability $1/4$. Similar to the ball model, percolation does not occur, but proving this is significantly more challenging. The proof relies on a block method and a sprinkling procedure, and it heavily uses the fact that directions are limited to the four cardinal points and that the velocity is bounded. The absence of percolation for the general isotropic line-segment model with unbounded velocity V have been proved later in [3, 4] with a super exponential moment assumption $\mathbb{E}(e^{V^s}) < +\infty$, $s > 1$. Note that, as a corollary of our main theorem, we relax the super exponential moment in the line-segment model since we require only a finite second moment $\mathbb{E}(V^2) < +\infty$ (Proposition 7).

All strategies developed prior to this paper relied on paths moving in straight lines with constant or strongly integrable velocity. These settings enable the proof of good upper bounds on the probability of crossing a large box by introducing several obstacles to block any line. This leads to useful localization properties. Dealing with less stringent assumptions (whether on speed or on the paths of the grains) is more challenging and requires new approaches. Regarding paths, it appears that regularity beyond mere continuity is irrelevant. For example, rough curves (as Brownian curves, see Section 2.4.1) or space-filling curves (as Peano curves) are permitted. This allows for unusual hitting behavior, including simultaneous or coincident hits.

To sum up, in the present paper we greatly generalize the non percolation results for germ-grain models with thin grains in two main directions: on the growth of grains since a second moment condition is assumed instead of a super exponential one, and on the shape of grains since only the continuity of curves is assumed instead of line-segments.

This is achieved through two main ingredients.

- We develop a quantitative multi-scale analysis to control the dependencies in a related technical model.
- As it not possible to localize in a useful way the properties of interest, we develop a notion of scenarios in which what happens in some bounded window is only known up to a finite number of locally defined scenarios.

We give a more detailed sketch in the next part.

Ingredients of the proof.

Putting an edge between x and y when x hits y or y hits x defines an undirected graph with vertex set ξ , denoted by $\text{Graph}(\bar{\xi})$, for which we aim to state the absence of percolation (Theorem 5). Putting an arrow $x \rightarrow y$ from x to y when x hits y defines an out-degree one graph (up to adequat and irrelevant definitions about vertices hitting zero or more than two vertices) denoted by $\overrightarrow{\text{Graph}}(\bar{\xi})$. The forward set $\text{For}(x)$ of a vertex x is defined as the set of all vertices y such that there exists an oriented path from x to y in $\overrightarrow{\text{Graph}}(\bar{\xi})$. Soft arguments relying only on stationarity and the out-degree one property allows to reduce the proof of Theorem 5 to that of the absence of infinite forward sets in $\overrightarrow{\text{Graph}}(\bar{\xi})$. Such arguments, sometimes quoted as *mass transport principle*, were already present in [5, 6].

Our global strategy to prove the absence of infinite forward sets comes from [3, 4] and can be explained in rough terms as follows. Consider by absurd an infinite forward set. When one travels along this forward set, one should encounter opportunities to add to the current configuration a suitable (small) loop that would break the considered forward set, making it finite and contradicting our assumption. The crux of the matter consists in proving that the number of these opportunities is infinite but the non-local character of the model makes this task tricky. Indeed, when one travels along a forward set, one gathers more and more information and it is difficult to rule out the fact that such opportunities could become increasingly rare. Moreover, this difficulty is compounded by the long range dependence in the model: modifying a grain arbitrarily far away from a given grain $x \in \xi$ can change the lifetime of x by a domino effect. In particular, these strong dependencies forbid any useful stabilizing definition for $\tau(x)$ (or other relevant quantities) and therefore prevent the use of the associated tools from stochastic geometry.

Let us be more concrete about opportunities to add loops. We refer to Figure 2 for an illustration. Consider a point x_0 . One looks for some $\alpha > 0$ such that we can place a ball $B(v, 2\alpha)$ *just before* the impact of x_0 on its stopping grain $s(x_0)$ and overlapping no other grains. Henceforth, a loop suitably placed inside $B(v, 2\alpha)$ will stop the grain x_0 without reducing its backward set $\text{Back}(x)$ (which is the set of vertices y whose the forward set contains x). This idea was already present in [3, 4]. Note that it is important not to alter the backward set: imagine that we are exploring a forward set and that, at some point, we arrive at x_0 where we want to create (say by a modification argument) a loop after x_0 in order to break the forward set. However if this creation modifies the backward of x_0 then the forward set we were exploring could by-pass x_0 and continues forever.

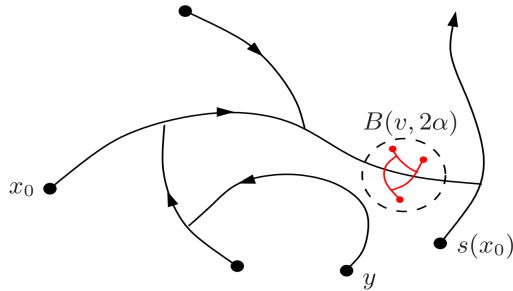


Figure 2: Grains of $\bar{\xi}$, especially x_0 and its stopping grain $s(x_0)$, are represented in black. Black arrows indicate the sense in which these grains grow. A loop built with three new grains (in red) and located inside the small ball $B(v, 2\alpha)$ allows to stop the grain x_0 without reducing its backward set. Note that if $B(v, 2\alpha)$ overlapped the grain y , the red loop could stop it and then would reduce the backward set of x_0 (to which y belongs).

Say that a vertex $x_0 \in \xi$ is α -good if we can place a ball $B(v, 2\alpha)$ as described above. Otherwise say that x_0 is α -bad. Still by soft arguments, we can reduce our task to proving that, for small enough $\alpha > 0$, there exists no infinite forward sets composed only with α -bad points— and we now have a parameter $\alpha > 0$ to play with. Besides, determining a suitable location for the small ball $B(v, 2\alpha)$ just before the impact of x_0 on its successor $s(x_0)$ requires to know the local picture around the grain created by x_0 , especially who is its successor and the lifetime at which x_0 ceases growing, which are clearly non-local informations. This is a crucial obstacle in establishing the non existence of infinite forward sets of α -bad points. This difficulty were overcome in [4] by the introduction of an additional technical assumption called "shield" whose fulfillnes was only proved under strong integrability assumptions. As we aim at reducing such extra technical conditions, we need a new approach.

To overcome this difficulty, we develop in this article a way to witness α -goodness without revealing too much of $\bar{\xi}$. There are two layers in our approach.

- First, we introduce an augmented model, called the \sharp -model, in which the augmented grain $G^\sharp(x) := x + h_x([0, \tau^\sharp(x)])$ contains the grain $x + h_x([0, \tau(x)])$ of the original model. One crucial point is that the dependencies are less strong in this model. Under our second moment assumption, the \sharp -model is amenable to some **quantitative multi-scale analysis** which enables us to prove that the $G^\sharp(x)$'s are not too large. See Proposition 18 for a formal statement. The aforementioned multi-scale analysis provides a lower bound in the passage time in some kind of first-percolation problem associated with a subcritical Boolean model (see Proposition 25) which could be of independent interest.
- There is not enough information in the \sharp -neighbourhood of x to determine exactly what happens around the grain x (two vertices x and y are \sharp -neighbours when, roughly, $G^\sharp(x)$ and $G^\sharp(y)$ overlap). Therefore, in order to gain locality we have no choice but to lose information. **We achieve to determine what really happens around x up to a finite number of scenarios** and we declare x as (α, \sharp) -good if it is α -good in each of these scenarios. See Proposition 19 for a formal statement. An advantage of the concept of (α, \sharp) -goodness is locality: the fact that a vertex x is (α, \sharp) -good only depends on its \sharp -neighbours, and their \sharp -neighbours, and so on, up to four levels

deep. Above all, (α, \sharp) -goodness witnesses α -goodness, i.e. (α, \sharp) -goodness of a vertex x implies α -goodness of x .

Once the key Propositions 18 and 19 are proved, a multi-scale approach (initially introduced in [9]) can be used to prove that for small enough $\alpha > 0$ there exists no infinite forward sets of (α, \sharp) -bad points. Therefore, there exists no infinite forward sets of α -bad points. As mentioned above, this is sufficient to ensure absence of percolation in the model.

Plan of the paper.

The paper is organized as follows. In the next Section 2, we introduce the model and present our main result, Theorem 5. Additionally, two main examples are provided: the line-segment model and the Brownian model. The general scheme of the proof of the theorem is given in Section 3, which relies on two major Propositions, 18 and 19, whose proofs are postponed to Sections 4 and 5, respectively.

2 Model and main results

2.1 Deterministic model

2.1.1 The space of configurations

Let \mathcal{C} be the space of continuous functions $h : \mathbb{R}_+ \rightarrow \mathbb{R}^2$ such that $h(0) = 0$. We equip \mathcal{C} with the product σ -algebra $\mathcal{F}^{\mathcal{C}}$. Let $\bar{\pi} : \mathbb{R}^2 \times \mathcal{C} \rightarrow \mathbb{R}^2$ denotes the projection onto \mathbb{R}^2 . The state space is

$$\bar{\mathcal{S}} = \{\bar{S} \subset \mathbb{R}^2 \times \mathcal{C} : \bar{S} \text{ is finite or countable and the restriction of } \bar{\pi} \text{ to } \bar{S} \text{ is one-to-one}\}.$$

Let $\bar{S} \in \bar{\mathcal{S}}$. We write $S = \bar{\pi}(\bar{S})$. As $\bar{\pi}|_{\bar{S}}$ is one-to-one we can write

$$\bar{S} = \{(x, h_x), x \in S\}$$

where $(h_x)_{x \in S}$ is a family of continuous functions from \mathbb{R}_+ to \mathbb{R}^2 which vanish at 0. Let $x \in S$. We define a continuous function $g_x : \mathbb{R}_+ \rightarrow \mathbb{R}^2$ by

$$g_x(t) = x + h_x(t).$$

For all $t \geq 0$ we also set

$$G_x(t) = g_x([0, t]).$$

When considering some configuration \bar{S} in the sequel, we freely use the notations S, h_x, g_x and G_x without introducing them. For a configuration \bar{S} and a set $B \in \mathcal{B}(\mathbb{R}^2)$, we use the following short notation for the restriction of \bar{S} to $B \times \mathcal{C}$:

$$\bar{S}|_B := \bar{S} \cap (B \times \mathcal{C}).$$

Write $\#X$ for the cardinality of a set X . For any A in $\mathcal{B}(\mathbb{R}^2) \otimes \mathcal{F}^{\mathcal{C}}$, define a map N_A from $\bar{\mathcal{S}}$ to $\mathbb{N} \cup \{\infty\}$ by

$$N_A(\bar{S}) = \#(\bar{S} \cap A).$$

We equip $\bar{\mathcal{S}}$ with the σ -field $\mathcal{F}^{\bar{\mathcal{S}}}$ generated by the family of maps $N_A, A \in \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{F}^{\mathcal{C}}$.

2.1.2 The static point of view

Fix $\bar{S} \in \bar{\mathcal{S}}$. In the introduction we gave an informal dynamical description of the model which will be formalized in Section 2.1.3. It appears that a static point of view – which focuses on the properties of the family of times at which each grain stops growing – is more appropriate to formulate a robust definition. This approach was already used to define the Lilypond model in [6]. Here we follow the definition of [4] with some adaptations.

Let $\bar{S} \in \bar{\mathcal{S}}$ be a configuration. Let $\tau : S \rightarrow (0, +\infty]$. For any $x \in S$, we think about $\tau(x)$ as the time at which the grain x stops growing (if $\tau(x) = \infty$, then the growth never stops). Before stating the

properties we require about τ , let us define a useful notion. For $x \neq y \in S$ and $t_x \geq 0$, we say that x **τ -hits y at time t_x** if there exists $t_y \geq 0$ such that

$$t_x \leq \tau(x) \text{ and } t_y \leq \tau(y) \text{ and } g_x(t_x) = g_y(t_y) \text{ and } t_y \leq t_x .$$

The property in display can be rephrased informally as follows.

- The grain y is still growing at time t_y (as $t_y \leq \tau(y)$) and thus the point $g_y(t_y)$ belongs to the grain y at time t_y .
- The point $g_y(t_y)$ thus also belongs to the grain y at time t_x (as $t_x \geq t_y$).
- But the grain x is still growing at time t_x (as $t_x \leq \tau(x)$).
- Therefore the grain x hits the grain y at time t_x (as moreover $g_x(t_x) = g_y(t_y)$).

Note that when x **τ -hits y at time t_x** , it does not mean that it is for the first time. We are now ready to give the property we require regarding τ , meaning the first time of hitting. A **\bar{S} -lifetime function** is a function $\tau : S \rightarrow (0, +\infty]$ such that:

1. **Stopping property.** For all $x \neq y \in S$ and $t_x \geq 0$, if x τ -hits y at time t_x , then $\tau(x) = t_x$.
2. **Hitting property.** For all $x \in S$ such that $\tau(x) < \infty$, there exists $y \in S \setminus \{x\}$ such that x τ -hits y at time $\tau(x)$.

Very roughly speaking, the Stopping and Hitting properties can be respectively understood as follows: “a hit implies a stop” and “a stop requires a hit”. From now, a function $\tau : S \rightarrow (0, +\infty]$ will always denotes a **\bar{S} -lifetime function** with the stopping and hitting properties.

As a first consequence of the Stopping property, the Hard-core property asserts that only the extremity $g_x(\tau(x))$ of a grain may hit another grain:

Lemma 2 (Hard-core property). *For all $x \neq y \in S$, $g_x([0, \tau(x))) \cap g_y([0, \tau(y))) = \emptyset$.*

Proof. Let $x \neq y \in S$ and $t_x, t_y \geq 0$ be such that $t_x \leq \tau(x), t_y \leq \tau(y)$ and $g_x(t_x) = g_y(t_y)$. If $t_y \leq t_x$ then x τ -hits y at time t_x and thus the Stopping property implies $t_x = \tau(x) \notin [0, \tau(x))$. Otherwise y τ -hits x at time t_y and the Stopping property implies $t_y = \tau(y) \notin [0, \tau(y))$. To sum up, we got $t_x \notin [0, \tau(x))$ or $t_y \notin [0, \tau(y))$ which is the searched result. \square

Note that if x τ -hits $y \in S \setminus \{x\}$ it necessarily happens at time $\tau(x)$ by the Stopping property. Therefore, x τ -hits y is equivalent to

$$\tau(x) < \infty \text{ and there exists } t_y \geq 0 \text{ such that } t_y \leq \tau(y) \text{ and } g_x(\tau(x)) = g_y(t_y) \text{ and } t_y \leq \tau(x) . \quad (1)$$

In our model, the hits on itself are ignored so that a grain may overlap itself infinitely many times without being stopped (think of a Brownian trajectory). We say that x **is stopped at time $\tau(x)$** if $\tau(x) < \infty$ is finite. In that case, the Hitting property means that x τ -hits at least one $y \in S \setminus \{x\}$, but nothing prevents $g_x(\tau(x))$ to belong to several grains.

We say that x **τ -hits y in a regular way** if

$$\tau(x) < \infty \text{ and there exists } t_y \geq 0 \text{ such that } t_y < \tau(y) \text{ and } g_x(\tau(x)) = g_y(t_y) \text{ and } t_y \leq \tau(x) . \quad (2)$$

This is a stronger than (1) as we require $t_y < \tau(y)$. It is important to note that, by the Hard-core property (Lemma 2), if x τ -hits y in a regular way then there exists a unique such y .

A configuration \bar{S} is said **tempered** if there exists a unique \bar{S} -lifetime function τ . In that case, the model is well defined (from the configuration \bar{S}). In the sequel, we will work with tempered configurations and shorten “ x τ -hits y ” to “ x hits y ”.

2.1.3 The dynamic point of view

Let us present a procedure, called the **Dynamical algorithm**, corresponding to the dynamical description of the model given in introduction. The intuition is the following: for any $x \in S$ and any time $t \geq 0$, $G_x(t)$ is the state of the grain x at time t if its growth has not been stopped yet; its growth stops as soon as it hits another grain. From a *finite* configuration \bar{S} , the Dynamical algorithm returns a function $\tau_{\text{dyn}} : S \rightarrow (0, +\infty]$ such that $\tau_{\text{dyn}}(x)$ is the time at which the grain x is stopped and $\tau_{\text{dyn}}(x) = \infty$ if it is not stopped. The Dynamical algorithm consists of a While loop which involves the set $S' \subset S$ of grains still alive (or growing) at each step of the loop.

For any $x \in S$, let $\tau_{\text{dyn}}(x) = +\infty$ and $g'_x([0, t]) = g_x([0, t])$ for any $t \geq 0$.

Let $S' = S$.

While $S' \neq \emptyset$:

For any $x \neq y$, $x \in S'$ and $y \in S$, let $t(x \rightarrow y) = \inf\{t \geq 0 : g_x(t) \in g'_y([0, t])\}$.

Let $t^* = \inf\{t(x \rightarrow y) : x \neq y, x \in S' \text{ and } y \in S\}$.

If $t^* = +\infty$ **Then** $S' = \emptyset$. **Else**:

Let $S'' = \{x \in S' : \exists y \in S \setminus \{x\}, t(x \rightarrow y) = t^*\}$.

For any $x \in S''$, let $\tau_{\text{dyn}}(x) = t^*$ and $g'_x([0, t]) = g'_x([0, t^*])$ for any $t \geq t^*$.

Let $S' = S' \setminus S''$.

End If

End While

Let us specify that the infimum $t(x \rightarrow y)$, for $x \neq y$, is reached since the g_x 's are continuous functions. The same holds for the infimum t^* because the configuration \bar{S} is finite. Also the set S'' is allowed, at a given step of the While loop, to contain several elements: this corresponds to simultaneous hits between different grains. Finally, each auxiliary function g'_x is possibly *frozen* during the algorithm. This ensures that the piece of trajectory of $(g_x(t))_{t \geq 0}$ after its stopping time will play no role in the sequel.

Let us prove that τ_{dyn} is a \bar{S} -lifetime function and this is the only one. This means that any finite configuration \bar{S} is tempered and both definitions of the model– the static one with lifetime functions and the dynamical one –coincide on each finite \bar{S} .

Lemma 3 (Reconciliation Lemma). *Let \bar{S} be a finite configuration. The function τ_{dyn} generated by the dynamical algorithm from \bar{S} is the only \bar{S} -lifetime function.*

Proof. Let \bar{S} be a finite configuration. Let us prove that τ_{dyn} is a \bar{S} -lifetime function and first focus on the Stopping property. For all grains $x \neq y$, one has

$$\tau_{\text{dyn}}(x) \leq t(x \rightarrow y) \in [0, +\infty]$$

where $t(x \rightarrow y) = \inf\{t \geq 0 : g_x(t) \in g'_y([0, t])\}$ and $g'_y([0, t]) = g_y([0, t \wedge \tau_{\text{dyn}}(y)])$. Assume that x τ_{dyn} -hits y at time t_x for $x \neq y$ in S and $t_x \geq 0$, i.e. there exists t_y such that $g_x(t_x) = g_y(t_y)$ with $t_x \leq \tau_{\text{dyn}}(x)$, $t_y \leq \tau_{\text{dyn}}(y)$ and $t_y \leq t_x$. Then

$$g_x(t_x) = g_y(t_y) \subset g_y([0, t_x \wedge \tau_{\text{dyn}}(y)]) = g'_y([0, t_x])$$

which means $t_x \geq t(x \rightarrow y)$. Hence we get $\tau_{\text{dyn}}(x) \leq t(x \rightarrow y) \leq t_x \leq \tau_{\text{dyn}}(x)$, i.e. $t_x = \tau_{\text{dyn}}(x)$. The Stopping property holds.

In order to check the Hitting property, let us consider a grain $x \in S$ with $\tau_{\text{dyn}}(x) < \infty$. By the Dynamical algorithm, there exists a grain $y \in S \setminus \{x\}$ such that $\tau_{\text{dyn}}(x) = t(x \rightarrow y) < \infty$. We then have to prove that x τ_{dyn} -hits y at time $\tau_{\text{dyn}}(x)$. Since the g_x 's are continuous, the infimum $t(x \rightarrow y)$ is reached :

$$g_x(\tau_{\text{dyn}}(x)) \in g'_y([0, \tau_{\text{dyn}}(x)]) = g_y([0, \tau_{\text{dyn}}(x) \wedge \tau_{\text{dyn}}(y)]) .$$

So there exists $t_y \leq \tau_{\text{dyn}}(x) \wedge \tau_{\text{dyn}}(y)$ satisfying $t_y \leq \tau_{\text{dyn}}(y)$, $g_x(\tau_{\text{dyn}}(x)) = g_y(t_y)$ and $t_y \leq \tau_{\text{dyn}}(x)$. In other words, x τ_{dyn} -hits y at time $\tau_{\text{dyn}}(x)$.

It remains to prove that τ_{dyn} is the only \bar{S} -lifetime function. By absurd, assume that there exist two \bar{S} -lifetime function, say τ_1 and τ_2 . Thus let us consider the grain x minimizing $\tau_1(x) \wedge \tau_2(x)$ among grains

satisfying $\tau_1(x) \neq \tau_2(x)$. This minimal element is well defined since $\tau_1 \neq \tau_2$ and \bar{S} is finite. Without loss of generality, we can assume $\tau_1(x) < \tau_2(x)$ which in particular implies that $\tau_1(x) < \infty$. By the Hitting property (for τ_1), there exists a grain $y \neq x$ such that x τ_1 -hits y at time $\tau_1(x)$. Roughly speaking, the grain x is stopped by y for τ_1 and it lives a little longer for τ_2 than for τ_1 . Then the Hitting property (for τ_2) forces the grain y to live a little shorter for τ_2 than for τ_1 . This contradicts the minimality of x . Precisely, x τ_1 -hits y at time $\tau_1(x)$ means $g_x(\tau_1(x)) = g_y(t_y)$ for some time t_y such that $t_y \leq \tau_1(y)$ and $t_y \leq \tau_1(x)$. The hypothesis $t_y \leq \tau_2(y)$ together with $g_x(\tau_1(x)) = g_y(t_y)$, $\tau_1(x) \leq \tau_2(x)$ and $t_y \leq \tau_1(x)$ would mean that x τ_2 -hits y at time $\tau_1(x)$ and then $\tau_1(x) = \tau_2(x)$ by the Hitting property (for τ_2). As this conclusion fails, we necessarily have $t_y > \tau_2(y)$. To sum up,

$$\tau_2(y) < t_y \leq \tau_1(x) \wedge \tau_1(y) < \tau_2(x) .$$

The grain y then satisfies $\tau_1(y) \neq \tau_2(y)$ and $\tau_1(y) \wedge \tau_2(y) < \tau_1(x) \wedge \tau_2(x)$ which contradicts the minimality of x . \square

2.2 Model driven by a probability measure μ

We need to introduce a few objects to handle some measurability issues about the families of stopping times $\tau(x)$. Consider the set

$$\widehat{S} = \{\widehat{S} \subset \mathbb{R}^2 \times [0, +\infty] : \widehat{S} \text{ is finite or countable and the restriction of } \widehat{\pi} \text{ to } \widehat{S} \text{ is one-to-one}\}$$

where $\widehat{\pi}$ denotes the projection from $\mathbb{R}^2 \times [0, +\infty]$ onto \mathbb{R}^2 . As before, if $\widehat{S} \in \widehat{S}$, we can write

$$\widehat{S} = \{(x, \tau(x)), x \in S\}$$

where $S = \widehat{\pi}(\widehat{S})$. For any A in $\mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}([0, +\infty])$, define a map \widehat{N}_A from \widehat{S} to $\mathbb{N} \cup \{\infty\}$ by

$$\widehat{N}_A(\widehat{S}) = \#(\widehat{S} \cap A).$$

We equip \widehat{S} with the σ -field $\mathcal{F}^{\widehat{S}}$ generated by the family of maps $\widehat{N}_A, A \in \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}([0, +\infty])$.

Let $\bar{\mathcal{A}} \subset \bar{S}$. We say that $\bar{\mathcal{A}}$ is a **tempered set of configurations** if the following conditions hold:

1. $\bar{\mathcal{A}}$ is measurable, that is $\bar{\mathcal{A}} \in \mathcal{F}^{\bar{S}}$.
2. For all $\bar{S} \in \bar{\mathcal{A}}$, \bar{S} is a tempered configuration.
3. The set $\bar{\mathcal{A}}$ is invariant under the action of spatial translations. In other words, for all $\bar{S} \in \bar{\mathcal{A}}$ and all $u \in \mathbb{R}^2$,

$$\bar{S} - u \in \bar{\mathcal{A}}$$

where $\bar{S} - u = \{(x - u, h), (x, h) \in \bar{S}\}$.

4. The map $\varphi : \bar{\mathcal{A}} \rightarrow \widehat{S}$ defined by

$$\varphi(\bar{S}) = \{(x, \tau(x; \bar{S})), x \in S\}$$

is measurable where $\bar{\mathcal{A}}$ is equipped with the σ -field induced by $\mathcal{F}^{\bar{S}}$ and \widehat{S} is equipped with the σ -field $\mathcal{F}^{\widehat{S}}$. Here $\tau(\cdot, \bar{S})$ denotes the unique \bar{S} -lifetime.

Note that translating by a common vector the starting points of the grains does not modify the lifetime of the grains. More formally, if $\bar{\mathcal{A}}$ is a tempered set of configuration, then for any $\bar{S} \in \bar{\mathcal{A}}$ and any $u \in \mathbb{R}^2$ we have

$$\varphi(\bar{S} - u) = \varphi(\bar{S}) - u$$

where $\varphi(\bar{S}) - u = \{(x - u, \tau), (x, \tau) \in \varphi(\bar{S})\}$.

Let μ be a probability measure on $(\mathcal{C}, \mathcal{F}^{\mathcal{C}})$. Let $\bar{\xi}$ be a Poisson point process on $\mathbb{R}^2 \times \mathcal{C}$ with intensity measure $dx \otimes \mu$ where dx denotes the Lebesgue measure on \mathbb{R}^2 . Equivalently, $\bar{\xi}$ is a marked Poisson point process on \mathbb{R}^2 with intensity dx and independent marks in \mathcal{C} with distribution μ . We refer to the

book by Last and Penrose [14] for background on Poisson point processes. The projection $\bar{\pi}$ from $\mathbb{R}^2 \times \mathcal{C}$ onto \mathbb{R}^2 is almost surely one-to-one on $\bar{\xi}$. In the remaining of this article we assume that this condition holds. We can thus write, as before,

$$\bar{\xi} = \{(x, h_x), x \in \xi\}$$

where $\xi = \pi(\bar{\xi})$ and $(h_x)_x$ is a family of elements of \mathcal{C} . As before, we use the associated notations g_x and G_x . The process ξ is a Poisson point process on \mathbb{R}^2 with intensity dx . Conditioned on ξ , $(h_x)_x$ is a family of i.i.d.r.v. with distribution μ .

We say that **the measure μ is tempered** if there exists a tempered set of configurations $\bar{\mathcal{A}}$ such that $\bar{\xi} \in \bar{\mathcal{A}}$ with probability 1. In this case, we always work on a full probability event such that $\bar{\xi} \in \bar{\mathcal{A}}$. Moreover we use as before the notation $\tau(x; \bar{\xi})$ – or simply $\tau(x)$ – for $x \in \xi$.

In the article we focus on percolation questions. We however provide a simple sufficient condition (3) which ensures that a given measure μ is tempered. In other words, under (3), the model built from the marked Poisson point process $\bar{\xi}$ (and driven by the probability measure μ) is well defined.

Proposition 4. *A probability measure μ on $(\mathcal{C}, \mathcal{F}^{\mathcal{C}})$ is tempered as soon as, for all $t \geq 0$,*

$$\mathbb{E} \left[\sup_{s \in [0, t]} \|h(s)\|^2 \right] < \infty \quad (3)$$

where h is a random variable with distribution μ .

The proof of Proposition 4 essentially follows the same strategy as the one of Theorem 3.1 of [4] but with some differences about the hypotheses on which these results are based. For the convenience of the reader, we provide a sketched proof in Appendix 6.1.

2.3 Absence of percolation

Let $\bar{\xi}$ be a Poisson point process on $\mathbb{R}^2 \times \mathcal{C}$ with intensity measure $dx \otimes \mu$ where μ is a tempered probability measure. We put an edge between $x, y \in \xi$ if x hits y or if y hits x . We write in this case $x \sim y$. We thus get an undirected graph structure $\text{Graph}(\bar{\xi})$ whose set of vertices is ξ . We say that $\text{Graph}(\bar{\xi})$ **percolates** if one of its connected component is infinite.

The **loop property** expresses the possibility for the model to contain loops as small as we want. Further, this property will be used to locally modify the current configuration by adding a loop at the right scale and at the right place so as to block or stop a connected component. Precisely, we say that the loop property at scale $\alpha > 0$ holds if there exists $\mathcal{L}(\alpha) \in \mathcal{F}^{\bar{\mathcal{S}}}$ and $r(\alpha) \in (0, \alpha)$ such that:

1. The event $\mathcal{L}(\alpha)$ is possible, that is $\mathbb{P}[\bar{\xi} \in \mathcal{L}(\alpha)] > 0$.
2. The event is local. More precisely $\mathcal{L}(\alpha)$ only depends on the grains starting in $B(\alpha)$. In other words,

$$\forall \bar{S} \in \bar{\mathcal{S}}, \bar{S} \in \mathcal{L}(\alpha) \iff \bar{S}|_{B(\alpha)} \in \mathcal{L}(\alpha).$$

3. On the event $\{\bar{\xi} \in \mathcal{L}(\alpha)\}$, the set $\xi \cap B(\alpha)$ can be written as $\{x_1, \dots, x_n\}$ for some integer $n \geq 2$. Furthermore, in the model associated to $\bar{\xi}|_{B(\alpha)}$ (whose existence is ensured by the Reconciliation lemma), the following properties hold:

- (a) For any $1 \leq i \leq n - 1$, x_i is stopped by x_{i+1} and x_n is stopped by x_1 .
- (b) For any $1 \leq i \leq n$, $\tau(x_i; \bar{\xi}|_{B(\alpha)}) \leq \alpha$.
- (c) The set

$$\bigcup_{1 \leq i \leq n} G_{x_i}(\tau(x_i; \bar{\xi}|_{B(\alpha)}))$$

is included in the open annulus $\dot{B}(\alpha) \setminus B(r(\alpha))$ and it separates $B(r(\alpha))$ from $\mathbb{R}^2 \setminus \dot{B}(\alpha)$.

Let us remark that in the loop property, the parameter α is used as a spatial parameter and also as a time parameter (Item 3.(b)). Note that the the radius $r(\alpha)$ could be chosen equals to 0 in this definition without changing anything. Indeed, since the union of grains $\bigcup_{1 \leq i \leq n} G_{x_i}(\tau(x_i; \bar{\xi}|_{B(\alpha)}))$ is a compact set,

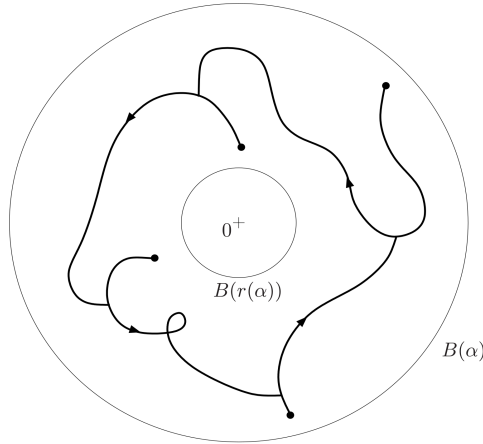


Figure 3: On this picture, $\xi \cap B(\alpha)$ is made up with $n = 4$ points (the black points). The union of their grains separates $B(r(\alpha))$ from the outside of $B(\alpha)$. The black arrows indicate the sense in which grains grow.

if Item 3.(c) holds for $r(\alpha) = 0$ it holds for $r(\alpha) > 0$ as well. By dominated convergence Theorem, if the loop property at scale α holds with $r(\alpha) = 0$, it holds also for $r(\alpha) > 0$. We keep the presence of $r(\alpha) > 0$ in the definition since it plays a crucial role in the following constructions. Note also that the loop property is monotone w.r.t. α : if it holds at scale $\alpha > 0$ then it holds at any scale $\alpha' > \alpha$. Indeed, if $\mathcal{L}(\alpha)$ and $r(\alpha)$ satisfy the above conditions for a given $\alpha > 0$, then $\mathcal{L}(\alpha') = \mathcal{L}(\alpha) \cap \{\xi \cap (B(\alpha') \setminus B(\alpha)) = \emptyset\}$ and $r(\alpha') = r(\alpha)$ satisfy the above conditions for $\alpha' > \alpha$.

The main result of the article is the following result.

Theorem 5. *Let $\bar{\xi}$ be a Poisson point process on $\mathbb{R}^2 \times \mathcal{C}$ with intensity measure $dx \otimes \mu$ where μ is a tempered probability measure. We denote by h a random variable on \mathcal{C} with distribution μ . Assume*

- *The loop property holds at any scale $\alpha > 0$.*
- *There exists $t > 0$ such that*

$$\mathbb{E} \left[\sup_{s \in [0, t]} \|h(s)\|^2 \right] < \infty.$$

Then, almost surely, $\text{Graph}(\bar{\xi})$ does not percolate.

2.4 Examples

2.4.1 The Brownian model

The Brownian model consists in i.i.d. standard Brownian motions (BM) in \mathbb{R}^2 starting from the Poisson points of ξ . Equivalently, the Brownian model corresponds to the PPP $\bar{\xi}$ whose intensity measure $dx \otimes \mu$ is such that μ denotes the probability distribution on $(\mathcal{C}, \mathcal{F}^{\mathcal{C}})$ of the standard BM in \mathbb{R}^2 starting at the origin. Let $B = \{B_t\}_{t \geq 0}$ be a random variable with distribution μ . We also write $B = \{(B_t^{(1)}, B_t^{(2)})\}_{t \geq 0}$ where $\{B_t^{(1)}\}_{t \geq 0}$ and $\{B_t^{(2)}\}_{t \geq 0}$ are two i.i.d. one-dimensional (standard) BM starting at 0. Using the fact that $\sup_{[0, t]} B_s^{(1)}$ is distributed as $|B_t^{(1)}|$, we easily check that

$$\mathbb{E} \left[\sup_{s \in [0, t]} \|h(s)\|^2 \right]$$

is finite for any t . By Proposition 4, this suffices to ensure that μ is tempered, i.e. the Brownian model exists or is well-defined.

The existence of that model was already stated in [4] (Corollary 3.2). But the absence of percolation in the Brownian model is a new result.

Proposition 6. *The Brownian model previously defined does not percolate (in the sense of Section 2.3) with probability one.*

Proof. By Theorem 5, it is enough to prove that the loop property $\mathcal{L}(\alpha)$ holds at any space-time scale $\alpha > 0$ for the Brownian model since the moment condition has been checked just above. We only prove it for $\alpha = 1$ (the same construction works for any $\alpha > 0$, by a standard re-scaling procedure). Let us first introduce some notations. Given $\varepsilon > 0$ (small), let us set

- $R_1^\varepsilon := [-\frac{1}{2} - 3\varepsilon, \frac{1}{2} - \varepsilon] \times [-\varepsilon, \varepsilon]$;
- $R_2^\varepsilon := [0, \frac{1}{2} + 3\varepsilon] \times [-\varepsilon, \varepsilon]$;
- $R_3^\varepsilon := [\frac{1}{2} + \varepsilon, \frac{1}{2} + 3\varepsilon] \times [-\varepsilon, \varepsilon]$.

Let $B = \{B_t\}_{t \geq 0}$ be a standard BM in \mathbb{R}^2 starting at $(-\frac{1}{2} - 2\varepsilon, 0)$. We claim (without proof) that the event \mathcal{A} depicted in the left hand side of Fig. 4 has positive probability:

$$\mathcal{A} := \{\forall t \in [0, 1/2], B_t \in R_1^\varepsilon\} \cap \{\forall t \in [1/2, 1], B_t \in R_2^\varepsilon\} \cap \{B_1 \in R_3^\varepsilon\} .$$

Thus, taking ε small enough and four Poisson points respectively located nearby the four points $(\pm\frac{1}{2}, \pm\frac{1}{2})$ and using the conjunction of four (translated and rotated) copies of the event \mathcal{A} , we can check that the loop property $\mathcal{L}(1)$ holds (with $r(1)$ small enough, say $r(1) = \frac{1}{4}$). See the right hand side of Fig. 4. The details are left to the reader. \square

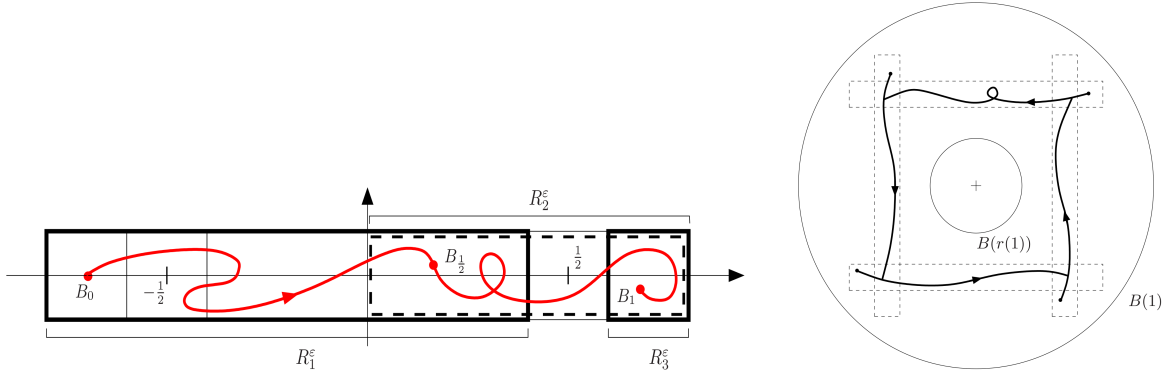


Figure 4: The picture on the left hand side represents the event \mathcal{A} . The Brownian trajectory (in red) starting at $B_0 = (-\frac{1}{2} - 2\varepsilon, 0)$ lies in $R_1^\varepsilon \cap R_2^\varepsilon$ at time $t = 1/2$. So, it horizontally crosses the square $(-\frac{1}{2}, 0) + [-\varepsilon, \varepsilon]^2$ during the time interval $[0, 1/2]$. After that it remains trapped in R_2^ε and finishes in R_3^ε at time $t = 1$. In particular, it horizontally crosses the square $(\frac{1}{2}, 0) + [-\varepsilon, \varepsilon]^2$ during the time interval $[1/2, 1]$. To the right hand side is represented the Brownian model built from four grains whose trajectories satisfy translated and rotated copies of the event \mathcal{A} . The resulting grain configuration forms a loop included in the open annulus $\mathring{B}(1) \setminus B(r(1))$ which separates $B(r(1))$ from $\mathbb{R}^2 \setminus \mathring{B}(1)$.

2.4.2 The line-segment model

The line-segment model corresponds to the PPP $\bar{\xi}$ whose intensity measure $dx \otimes \mu$ is such that μ denotes the probability distribution on $(\mathcal{C}, \mathcal{F}^{\mathcal{C}})$ of the trajectory $Y = \{Y(t)\}_{t \geq 0}$ defined as follows: for any $t \geq 0$,

$$Y(t) := tV(\cos \Theta, \sin \Theta) \in \mathbb{R}^2$$

where V and Θ are two independent random variables such that $V > 0$ a.s. and Θ is uniformly distributed on $[0, 2\pi]$. Hence $Y = \{Y(t)\}_{t \geq 0}$ is a unilateral line-segment growing linearly with velocity V and according to the direction Θ .

The next result states the existence and the absence of percolation in the line-segment model improving moment conditions given in [4] (finite fourth and exponential moments were resp. needed to ensure existence and absence of percolation in Proposition 3.4 and Theorem 3.5 of [4]).

Proposition 7. *Whenever $\mathbb{E}[V^2] < \infty$, the line-segment model exists and does not percolate with probability one.*

Proof. Let us first compute

$$\mathbb{E} \left[\sup_{s \in [0, t]} \|Y(s)\|^2 \right] = \mathbb{E} \left[\sup_{s \in [0, t]} (sV)^2 \right] = t^2 \mathbb{E} [V^2] .$$

So $\mathbb{E}[V^2] < \infty$ implies the moment conditions in Proposition 4– so the existence of the line-segment model –and in Theorem 5. It then remains to prove the loop property $\mathcal{L}(\alpha)$ at any space-time scale $\alpha > 0$. A basic construction involving four grains in the same spirit as the right hand side of Fig. 4 will work. The details are left to the reader. \square

3 Proof of Theorem 5

3.1 Preliminaries on Palm measure and mass transport principle

In this section we summarize some standard results around the stationarity of point processes. In particular we recall the special form of the Palm measure for the Poisson point process with marks and the mass transport principle as well. We refer to [14] for a modern and pedagogical presentation.

Let us recall that $\bar{\xi}$ is a Poisson point process on \bar{S} with intensity measure $dx \otimes \mu$. The "heuristic" definition of the Palm measure of $\bar{\xi}$ at a point $x \in \mathbb{R}^d$ is the conditional distribution of $\bar{\xi}$ given the event "there is a marked point at the location x ". Obviously this definition is senseless since this event has null probability. Nevertheless, using the independence properties of the Poisson point process (a spatial version of absence of memory) it is possible to provide a sens to this heuristic definition and show that the Palm measure of $\bar{\xi}$ at the location x is simply the process $\bar{\xi}$ itself with an extra point at x with a random mark h following the distribution μ and being independent to $\bar{\xi}$. Precisely we define this point process by

$$\bar{\xi}^x = \bar{\xi} \cup \{(x, h)\}$$

and the classical "Slivnyak-Mecke formula" provides a rigorous setting of the heuristic described above: For any measurable and non-negative function $M : \bar{S} \times \bar{S} \rightarrow [0, +\infty)$

$$\mathbb{E} \left[\sum_{y \in \bar{\xi}} M(y, \bar{\xi}) \right] = \mathbb{E} \left[\int_{\mathbb{R}^d} M((x, h), \bar{\xi}^x) dx \right]. \quad (4)$$

By stationarity of the Poisson point process $\bar{\xi}$, it is easy to see that $\bar{\xi}^x$ has the same distribution than $\bar{\xi}^0 + x$ and therefore $\bar{\xi}^0$ is the main object of interest. In particular we have the mass transport principle.

Lemma 8. *Let $M : \mathbb{R}^d \times \mathbb{R}^d \times \bar{S} \rightarrow [0, +\infty)$ be measurable and non-negative. We assume that M is equivariant under the action of translations of \mathbb{R}^d :*

$$\forall x, y, t \in \mathbb{R}^d, \quad \forall \bar{S} \in \bar{S}, \quad M(x, y, \bar{S}) = M(x - t, y - t, \bar{S} - t).$$

Then,

$$\mathbb{E} \left[\sum_{y \in \xi^0} M(0, y, \bar{\xi}^0) \right] = \mathbb{E} \left[\sum_{x \in \xi^0} M(x, 0, \bar{\xi}^0) \right].$$

In words, under the Palm distribution, the mass coming out of the origin is equal to the mass coming into the origin. The lemma admits a proof relying only on stationarity. For completeness, we choose to provide a very short and intuitive proof relying on the fact that $\bar{\xi}$ is a Poisson point process.

Proof. Let h and h' be two independent random variables with distribution μ .

$$\begin{aligned}
\mathbb{E} \left[\sum_{y \in \xi^0} M(0, y, \bar{\xi}^0) \right] &= \mathbb{E} \left[M(0, 0, \{(0, h)\} \cup \bar{\xi}) \right] + \int_{\mathbb{R}^d} \mathbb{E} \left[M(0, y, \{(0, h), (y, h')\} \cup \bar{\xi}) \right] dy \\
&= \mathbb{E} \left[M(0, 0, \{(0, h)\} \cup \bar{\xi}) \right] + \int_{\mathbb{R}^d} \mathbb{E} \left[M(-y, 0, \{(-y, h), (0, h')\} \cup (\bar{\xi} - y)) \right] dy \\
&= \mathbb{E} \left[M(0, 0, \{(0, h)\} \cup \bar{\xi}) \right] + \int_{\mathbb{R}^d} \mathbb{E} \left[M(-y, 0, \{(-y, h), (0, h')\} \cup \bar{\xi}) \right] dy \\
&= \mathbb{E} \left[M(0, 0, \{(0, h)\} \cup \bar{\xi}) \right] + \int_{\mathbb{R}^d} \mathbb{E} \left[M(x, 0, \{(x, h), (0, h')\} \cup \bar{\xi}) \right] dx \\
&= \mathbb{E} \left[\sum_{x \in \xi^0} M(x, 0, \bar{\xi}^0) \right].
\end{aligned}$$

We used the Slivnyak-Mecke formula in Steps 1 and 4, the equivariance of M in step 2 and the stationarity of $\bar{\xi}$ in step 3. \square

3.2 Reduction to an out-degree one graph

In the undirected graph $\text{Graph}(\bar{\xi})$ (see Section 2.3), two distinct vertices $x, y \in \xi$ are linked by an edge if x hits y or y hits x . We write $x \sim y$ when there is such an edge.

For all $x \in \xi$ we now define a successor $s(x) = s(x; \bar{\xi}) \in \xi$ as follows:

1. If $\tau(x) = +\infty$, we set $s(x) = x$.
2. If $\tau(x) < +\infty$, there are two cases:
 - (a) If x is stopped in a regular way by some y (see (2)), then there exists a unique such y because of the hard-core property expressed in Lemma 2. In that case we set $s(x) = y$.
 - (b) If x is not stopped in a regular way by any y , we set $s(x) = x$.

We now define a directed graph $\overrightarrow{\text{Graph}}(\bar{\xi})$ with set of vertices ξ by putting an arrow from x to y when $s(x) = y$. We write $x \rightarrow y$ when there is such an arrow. The graph $\overrightarrow{\text{Graph}}(\bar{\xi})$ satisfies two basic and elementary properties stated in the following lemma.

Lemma 9.

1. For any $x \in \xi$, the outdegree of x in $\overrightarrow{\text{Graph}}(\bar{\xi})$ is 1.
2. The graph $\overrightarrow{\text{Graph}}(\bar{\xi})$ is shift-invariant. This means that for all $u \in \mathbb{R}^d$,

$$\overrightarrow{\text{Graph}}(\bar{\xi} - u) = \overrightarrow{\text{Graph}}(\bar{\xi}) - u.$$

Proof. Recall that we work on a full probability event on which $\bar{\xi} \in \bar{\mathcal{A}}$ where $\bar{\mathcal{A}}$ is a tempered set of configurations. The first item is a straightforward consequence of the definition of the graph. Let us check the second item. As $\bar{\xi}$ belongs to $\bar{\mathcal{A}}$ and as $\bar{\mathcal{A}}$ is invariant under spatial translations, $\bar{\xi} - u \in \bar{\mathcal{A}}$. Moreover, by uniqueness of the lifetime function, for all $x \in \xi$, $\tau(x - u; \bar{\xi} - u) = \tau(x; \bar{\xi})$. The result easily follows. \square

We denote by $\overleftrightarrow{C}(x; \bar{\xi})$ the undirected component of x in $\overrightarrow{\text{Graph}}(\bar{\xi})$. This is the set of vertices we can reach from x by following the arrows forward or backwards. More formally,

$$\begin{aligned}
\overleftrightarrow{C}(x) = \overleftrightarrow{C}(x; \bar{\xi}) &= \{y \in \xi : \text{there exists } n \geq 0 \text{ and a sequence } x = a_0, \dots, a_n = y \\
&\quad \text{such that, for all } i \in \{1, \dots, n\}, a_{i-1} \rightarrow a_i \text{ or } a_i \rightarrow a_{i-1}\}. \tag{5}
\end{aligned}$$

There is a strong link between percolation in the graphs $\text{Graph}(\bar{\xi})$ and $\overrightarrow{\text{Graph}}(\bar{\xi})$ as stated in the next lemma.

Lemma 10. *On a full probability event, the following holds. For any $x \in \xi$, if the connected component of x in $\text{Graph}(\bar{\xi})$ is infinite, then the undirected component $\overleftrightarrow{C}(x; \bar{\xi})$ of x in $\overrightarrow{\text{Graph}}(\bar{\xi})$ is infinite.*

Thanks to Lemma 10, in order to prove Theorem 5, it is sufficient to prove that all the undirected component of $\overrightarrow{\text{Graph}}(\bar{\xi})$ are finite almost surely.

Proof of Lemma 10. There is a simple proof in the case where all the hits are regular (see (2) for the definition). This means that for all $x \neq y \in \xi$ then $x \sim y$ implies that x hits y or y hits x which is equivalent to $x \rightarrow y$ or $y \rightarrow x$. Therefore the connected components of $\text{Graph}(\bar{\xi})$ coincide with the undirected components of $\overrightarrow{\text{Graph}}(\bar{\xi})$.

We now give the proof in the general case where non-regular hits are possible. The basic idea is that, if in a connected component C of $\text{Graph}(\bar{\xi})$ there exists two points $x \sim y$ such that neither $x \rightarrow y$ nor $y \rightarrow x$, then one can associate with the connected component C a unique special point in a translation equivariant invariant way. By a standard application of the mass transport principle, this implies that the connected component C is finite.

We say that $x \in \xi$ stops in a singular way if

1. $\tau(x)$ is finite.
2. There exists no $y \in \xi$ (we do not exclude $y = x$) such that $g_x(\tau(x))$ belongs to $g_y([0, \tau(y)))$.

In this case we say that x stops in a singular way at $g_x(\tau(x))$. Let C be a connected component of $\text{Graph}(\bar{\xi})$. We say that $s \in \mathbb{R}^2$ is a singular point of C if there exists $x \in C$ such that x stops in a singular way at s . The proof relies on the following observation whose proof is postponed.

Claim 11. *The number of singular points of a connected component of $\text{Graph}(\bar{\xi})$ is either 0 or 1.*

The following claim follows from the previous one and stationarity through standard mass transport arguments. The proof is also postponed.

Claim 12. *With probability one, any connected component of $\text{Graph}(\bar{\xi})$ which contains a singular point is finite.*

We first show how to conclude using Claim 12. Fix C an infinite connected component of $\text{Graph}(\bar{\xi})$. By Claim 12, C admits no singular point. Let $x \neq x' \in C$ be such that $x \sim x'$. By symmetry we can assume that x hits x' . There are two cases:

1. If x hits x' in a regular way, then there is an arrow $x \rightarrow x'$ in $\overrightarrow{\text{Graph}}(\bar{\xi})$.
2. Otherwise, $\tau(x)$ and $\tau(x')$ are finite and

$$g_x(\tau(x)) = g_{x'}(\tau(x')) \text{ and } g_x(\tau(x)) \notin g_{x'}([0, \tau(x'))).$$

But C admits no singular point. Therefore there exists $y \in \chi$ such that $g_x(\tau(x)) \in g_y([0, \tau(y)))$. Now there are three subcases:

- (a) Subcase $y \notin \{x, x'\}$. Then x and x' hits y in a regular way: there are two arrows $x \rightarrow y$ and $x' \rightarrow y$ in $\overrightarrow{\text{Graph}}(\bar{\xi})$.
- (b) Subcase $y = x'$. Then x hits x' in a regular way. This subcase is ruled out by Case 1.
- (c) Subcase $y = x$. Then x' hits x in a regular way: there is an arrow $x' \rightarrow x$ in $\overrightarrow{\text{Graph}}(\bar{\xi})$.

In all cases, x and y are in the same undirected cluster \overleftrightarrow{C} of $\overrightarrow{\text{Graph}}(\bar{\xi})$. Therefore C is a subset of such an undirected cluster \overleftrightarrow{C} . As C is infinite, \overleftrightarrow{C} is infinite and the proof of the lemma is over up to the two postponed proofs.

Proof of Claim 12 using Claim 11. This is a standard application of stationarity through mass transport principle. Denote by $C(x; \bar{\xi})$ the connected component of $x \in \xi$ in $\text{Graph}(\bar{\xi})$. By Slivnyak-Mecke formula (4) and Lemma 8, for any bounded set $\Lambda \subset \mathbb{R}^2$

$$\begin{aligned}
& \mathbb{E} \left[\sum_{x,y \in \xi} \mathbb{1}_{\{C(x;\bar{\xi}) \text{ admits the singular point } y \text{ and } y \text{ belongs to } \Lambda\}} \right] \\
&= \int_{\mathbb{R}^2} \mathbb{E} \left[\sum_{x \in \xi^y} \mathbb{1}_{\{C(x;\bar{\xi}^y) \text{ admits the singular point } y \text{ and } y \text{ belongs to } \Lambda\}} \right] dy \\
&= |\Lambda| \mathbb{E} \left[\sum_{x \in \xi^0} \mathbb{1}_{\{C(x;\bar{\xi}^0) \text{ admits the singular point } 0\}} \right] \\
&= |\Lambda| \mathbb{E} \left[\sum_{x \in \xi^0} \mathbb{1}_{\{C(0;\bar{\xi}^0) \text{ admits the singular point } x\}} \right] \\
&\leq |\Lambda|. \tag{6}
\end{aligned}$$

It is enough to show that for any bounded Λ , the variable $\sum_{x,y \in \xi} \mathbb{1}_{\{C(x;\bar{\xi}) \text{ admits the singular point } y \text{ and } y \text{ belongs to } \Lambda\}}$ is finite almost surely. Therefore if it exists a connected component with a singular point, necessary this connected component is bounded. \square

Proof of Claim 11. Let C be a connected component of $\text{Graph}(\bar{\xi})$. Assume, for a contradiction, that C admits two distinct singular points s and s' . Then, there exists $x, x' \in C$ and n such that

$$\begin{aligned}
& \tau(x) < \infty \text{ and } g_x(\tau(x)) = s \\
& \text{and } \tau(x') < \infty \text{ and } g_{x'}(\tau(x')) = s' \\
& \text{and there exists a path } x = x_0, \dots, x_n = x' \text{ such that for all } i \in \{1, \dots, n\}, x_{i-1} \sim x_i.
\end{aligned}$$

Choose x, x' and n as above and such that n is minimal. Necessarily, the x_i are distinct (as the length of the path is minimal) and $n \geq 1$ (as $x \neq x'$ because $s \neq s'$). We now show by induction on i the following property:

$$\forall i \in \{1, \dots, n\}, \tau(x_i) < \infty \text{ and } g_{x_i}(\tau(x_i)) \in g_{x_{i-1}}([0, \tau(x_{i-1}))]. \tag{7}$$

Let $i \in \{1, \dots, n\}$. As $x_i \sim x_{i-1}$, at least one of the following three conditions holds:

1. $\tau(x_{i-1}) < \infty$ and $g_{x_{i-1}}(\tau(x_{i-1})) \in g_{x_i}([0, \tau(x_i)))$.
2. $\tau(x_i) < \infty$ and $g_{x_i}(\tau(x_i)) \in g_{x_{i-1}}([0, \tau(x_{i-1}))$.
3. $\tau(x_{i-1}) < \infty$ and $\tau(x_i) < \infty$ and $g_{x_{i-1}}(\tau(x_{i-1})) = g_{x_i}(\tau(x_i))$.

Consider the case $i = 1$. The first condition above can not hold. Otherwise we would have $s = g_x(\tau(x)) \in g_{x_1}([0, \tau(x_1)))$ and s would not be a singular point. The third condition above can not hold. Otherwise we could consider the path $x = x_1, \dots, x_n = x'$ and n would not be minimal. Therefore the second condition holds and the desired property is proven for $i = 1$.

Consider now the case $i \in \{2, \dots, n\}$ and assume that the desired property is true for $i - 1$. The first condition can not hold. Otherwise $g_{x_{i-1}}(\tau(x_{i-1}))$ would belong to $g_{x_i}([0, \tau(x_i)))$ and to $g_{x_{i-2}}([0, \tau(x_{i-2}))$ (by the induction hypothesis) and therefore the intersection between $g_{x_i}([0, \tau(x_i)))$ and $g_{x_{i-2}}([0, \tau(x_{i-2}))$ would be non empty. But as $x_{i-2} \neq x_i$, this is forbidden by the hard-core property, see Lemma 2. The third condition can not hold. Otherwise, we would have $g_{x_i}(\tau(x_i)) = g_{x_{i-1}}(\tau(x_{i-1})) \in g_{x_{i-2}}([0, \tau(x_{i-2}))$ and thus x_i would hit x_{i-2} . We could then consider the path $x = x_0, \dots, x_{i-2}, x_i, \dots, x_n = x'$ and n would not be minimal. Therefore the second condition holds and the desired property is proven for i .

By induction this establishes (7). In particular $s' = g_{x'}(\tau(x')) = g_{x_n}(\tau(x_n)) \in g_{x_{n-1}}([0, \tau(x_{n-1}))$. But this is not possible as s' is a singular point. This is our contradiction and this completes the proof of Claim 11. \square

The proof of Lemma 10 is complete. \square

3.3 Absence of forward percolation is sufficient

For any $x \in \xi$ we define three components in $\overrightarrow{\text{Graph}}(\bar{\xi})$:

- The forward component of x . This is the set of all vertices we can reach from x by following the arrows. More formally,

$$\text{For}(x) = \text{For}(x; \bar{\xi}) = \{y \in \xi : \text{there exists } n \geq 0 \text{ and a sequence } x = a_0, \dots, a_n = y \text{ such that, for all } i \in \{1, \dots, n\}, a_{i-1} \rightarrow a_i\}.$$

- The backward component of x . This is the set of all vertices we can reach from x by following the arrows in reverse. More formally,

$$\text{Back}(x) = \text{Back}(x; \bar{\xi}) = \{z \in \xi : x \in \text{For}(z)\}.$$

- The undirected connected component of x . This is the set $\overleftrightarrow{C}(x) = \overleftrightarrow{C}(x; \bar{\xi})$ defined by (5).

The following lemma is a known consequence of Lemmas 8 and 9. A proof can find in [3] for instance, but for the convenience of the reader we provide a sketch of the proof below.

Lemma 13. *One a full probability event, the following holds:*

$$\forall x \in \xi, \text{For}(x) \text{ is finite} \iff \overleftrightarrow{C}(x) \text{ is finite}.$$

Combining Lemmas 10 and Lemmas 13 we get that, in order to prove that all the connected component of $\overrightarrow{\text{Graph}}(\bar{\xi})$ are almost surely finite, it is sufficient to prove that all forward components $\text{For}(x)$ of $\overrightarrow{\text{Graph}}(\bar{\xi})$ are finite.

Proof. A loop in $\overrightarrow{\text{Graph}}(\bar{\xi})$ is a finite sequence x_0, \dots, x_n , $n \geq 0$, of points of ξ such that $x_0 = x_n$ and, for all $i \in \{1, \dots, n\}$, $x_{i-1} \rightarrow x_i$. Using only the first property of Lemma 9 we get for each $x \in \xi$ that exactly one of the following properties is satisfied:

1. $\text{For}(x)$ is infinite and $C(x)$ contains no loop. In that case $C(x)$ is infinite.
2. $\text{For}(x)$ is finite and $C(x)$ contains a unique loop. In that case $C(x)$ can be finite or infinite.

This can be proven as follows. First one checks that $\overleftrightarrow{C}(x)$ is the union of $\text{For}(x)$ and of all $\text{Back}(y)$ for $y \in \text{For}(x)$. Then one distinguishes two cases: $\text{For}(x)$ is finite (and thus contains a unique loop) or infinite (and thus contains no loop). One concludes by noticing that adding the backwards components $\text{Back}(y)$ adds no loops.

The proof of Lemma 13 is then complete once we have proven

$$\mathbb{P}[\exists x \in \xi : C(x) \text{ is infinite and contains a unique loop}] = 0. \quad (8)$$

This property can be proved as in the Claim 12. Indeed a cluster with an unique loop contains a special point which is for instance the most right point in the loop. Following similar computation as in (6), we show that almost surely this cluster is finite. \square

3.4 Absence of forward percolation of bad points is sufficient

In the previous sections, we showed that the proof of Theorem 5 is reduced to the proof of the absence of forward percolation in the graph $\overrightarrow{\text{Graph}}(\bar{\xi})$. In this section, we show that we can reduce the setting further since it is only necessary to prove the absence of forward percolation for a class of "bad" points that we define below. We start with the definition of "good" points.

3.4.1 Definition of good points

Loops. We say that $x \in \xi$ belongs to a loop if there exists $n \geq 1$ such that $s^{(n)}(x) = x$ where $s^{(n)}$ denotes the n -th iterate of s . In particular, in the degenerate case where $s(x) = x$, x belongs to a loop. We say that $x \in \xi$ is followed by a loop if x does not belong to a loop and if its successor $s(x)$ belongs to a loop.

Notion of (α, u) -good points: informal definition. Let $\alpha > 0$ and $u \in \mathbb{R}^2$ be such that $0 \notin B(u, \alpha)$. Slightly informally, we say that $x \in \xi$ is an (α, u) -good point if replacing the configuration inside $B(x + u, \alpha)$ by a local configuration belonging to $x + u + \mathcal{L}(\alpha)$ creates a loop after x and does not decrease the cardinality of the backward of x .

Notion of (α, u) -good points: formal definition. Let $\bar{\mathcal{S}}^0$ denote the set of configurations $\bar{S} \in \mathcal{S}$ such that 0 belongs to S . We can focus on such configurations thanks to the fact that everything is well behaved under the action of translations.

For any $\bar{S}, \bar{S}' \in \bar{\mathcal{S}}$ we set

$$\text{resample}(\bar{S}, \bar{S}') = \bar{S}|_{B(u, \alpha)^c} \cup \bar{S}'|_{B(u, \alpha)}. \quad (9)$$

This define a measurable map from $\bar{\mathcal{S}}^2$ to $\bar{\mathcal{S}}$. Let $\bar{\xi}'$ be an independent copy of $\bar{\xi}$. In words, $\text{resample}(\bar{\xi}, \bar{\xi}')$ is obtained from $\bar{\xi}$ by resampling inside $B(u, \alpha)$ using $\bar{\xi}'$. Define $\text{Good}(\alpha, u)$ as the measurable subset of configurations $\bar{S} \in \bar{\mathcal{S}}^0$ such that:

1. Almost surely, $\text{resample}(\bar{S}, \bar{\xi}')$ belongs to $\bar{\mathcal{A}}$.
2. Almost surely, on the event $\{\bar{\xi}' \in u + \mathcal{L}(\alpha)\}$, the following conditions hold:
 - (a) In $\text{resample}(\bar{S}, \bar{\xi}')$, 0 is followed by a loop.
 - (b) $\#\text{Back}(0, \bar{S}) \leq \#\text{Back}(0, \text{resample}(\bar{S}, \bar{\xi}'))$.

We then say that $x \in S$ is an (α, u) -good point in \bar{S} if $\bar{S} - x \in \text{Good}(\alpha, u)$.

The first condition is needed to ensure that the second condition makes sense. It is however harmless as stated in Lemma 40 in Appendix 6.2.

Notion of α -good point. Let $0 < \alpha < 1$. Recall that $r(\alpha)$ is part of the definition of $\mathcal{L}(\alpha)$. We fix a finite set $U(\alpha)$ in \mathbb{R}^2 such that

$$B(0, \alpha^{-1}) \setminus B(0, \alpha) \subset \bigcup_{u \in U(\alpha)} B(u, r(\alpha)) \subset \mathbb{R}^2 \setminus \{0\}.$$

We say that $x \in \bar{S}$ is α -good in the configuration \bar{S} if there exists $u \in U(\alpha)$ such that x is (α, u) -good. In other words, we set

$$\text{Good}(\alpha) = \bigcup_{u \in U(\alpha)} \text{Good}(\alpha, u)$$

and we say that $x \in \bar{S}$ is α -good in the configuration \bar{S} if $\bar{S} - x$ belongs to $\text{Good}(\alpha)$. The definition depends on the choice of $U(\alpha)$ but it will not be an issue. Recall that $U(\alpha)$ is fixed.

3.4.2 The number of good points in a forward is finite

In the following lemma we show that almost surely an infinite forward cluster can not contain an infinite number of good points. The proof is inspired by similar results developed in Section 4.3 in [3].

Lemma 14. *Let $\alpha > 0$. With probability one, for all $x \in \xi$, the set $\{y \in \text{For}(x) : y \text{ is good for } \bar{\xi}\}$ is finite.*

Before proving Lemma 14 let us state an immediate consequence. We say that a point $x \in \bar{\xi}$ is α -bad when it is not α -good.

Lemma 15. *Assume that there exists $\alpha > 0$ such that, almost surely, there is no x in ξ such that*

- $\text{For}(x)$ is infinite.
- $\forall y \in \text{For}(x)$, y is α -bad for $\bar{\xi}$.

Then, almost surely, for all $x \in \xi$, $\text{For}(x)$ is finite.

Combining Lemmas 10, Lemmas 13 and Lemma 15 we get that, in order to prove that all the connected component of $\text{Graph}(\bar{\xi})$ are almost surely finite, it is sufficient to prove that all infinite forward components contains α -good points. In other words, there is no percolation in the original non oriented graph as soon as there is no forward percolation of α -bad points for α small enough. The problem is thus reduced to the proof of the existence of a subcritical regime in some oriented percolation model. This is a huge improvement as it enables us to play with α .

Proof of Lemma 15. Let α be as in the statement. With probability one, by assumption and by Lemma 14, we have

$$\forall x \in \xi, \quad \text{For}(x) \text{ is finite or there exists } y \in \text{For}(x) \text{ such that } y \text{ is } \alpha\text{-good} \quad (10)$$

and

$$\forall x \in \xi, \quad \{y \in \text{For}(x) : y \text{ is good for } \bar{\xi}\} \text{ is finite.} \quad (11)$$

We work on this event. Let $x \in \xi$. Assume that $\text{For}(x)$ is infinite. By (11) there exists $x^* \in \text{For}(x)$ such that all points of $\text{For}(x^*)$ are α -bad. This contradicts (10). This establishes the lemma. \square

Lemma 14 is a consequence of the following result.

Lemma 16. *Let $\alpha > 0$ and $u \in \mathbb{R}^2$ be such that $0 \notin B(u, \alpha)$. Then,*

$$\mathbb{E} \left[\sum_{y \in \text{For}(0, \bar{\xi}^0)} \mathbf{1}_{y \text{ is } (\alpha, u)\text{-good for } \bar{\xi}^0} \right] \leq \frac{1}{\mathbb{P}[\bar{\xi} \in \mathcal{L}(\alpha)]}.$$

Proof. Let $\bar{\xi}'$ be a copy of $\bar{\xi}$ independent of $\bar{\xi}$ and of $\bar{\xi}^0$. By definition of (α, u) -good we have

$$\begin{aligned} & \#\text{Back}(0, \bar{\xi}^0) \mathbf{1}_{0 \text{ is } (\alpha, u)\text{-good for } \bar{\xi}^0} \mathbf{1}_{\bar{\xi}' \in u + \mathcal{L}(\alpha)} \\ & \leq \#\text{Back}(0, \text{resample}(\bar{\xi}^0, \bar{\xi}')) \mathbf{1}_{0 \text{ is followed by a loop in } \text{resample}(\bar{\xi}^0, \bar{\xi}')} \end{aligned}$$

But $\bar{\xi}^0$ is independent of $\bar{\xi}'$, $\text{resample}(\bar{\xi}^0, \bar{\xi}')$ has the same distribution as $\bar{\xi}^0$ (as $0 \notin B(u, \alpha)$) and $\bar{\xi}' - u$ has the same distribution as $\bar{\xi}$. Therefore, taking expectation, we get

$$\mathbb{E} \left[\#\text{Back}(0, \bar{\xi}^0) \mathbf{1}_{0 \text{ est } (\alpha, u)\text{-good for } \bar{\xi}^0} \right] \mathbb{P}[\bar{\xi} \in \mathcal{L}(\alpha)] \leq \mathbb{E} \left[\#\text{Back}(0, \bar{\xi}^0) \mathbf{1}_{0 \text{ is followed by a loop in } \bar{\xi}^0} \right]. \quad (12)$$

We plan to apply Lemma 8 (mass transport principle). Recall $\bar{\mathcal{A}}$ the set of tempered configuration whose existence is due to the fact that μ is a tempered distribution. Recall $\mathbb{P}[\bar{\xi}^0 \in \bar{\mathcal{A}}] = 1$ by Lemma 40. We consider the map M defined for $x, y \in \mathbb{R}^d$ and $\bar{S} \in \bar{\mathcal{S}}$ by

$$M(x, y, \bar{S}) = \mathbf{1}_{\bar{S} \in \bar{\mathcal{A}}} \mathbf{1}_{x \in \text{Back}(y, \bar{S})} \mathbf{1}_{y \text{ is } (\alpha, u)\text{-good for } \bar{S}} = \mathbf{1}_{\bar{S} \in \bar{\mathcal{A}}} \mathbf{1}_{y \in \text{For}(x, \bar{S})} \mathbf{1}_{y \text{ is } (\alpha, u)\text{-good for } \bar{S}}.$$

In particular $M(x, y, \bar{S}) = 0$ if x or y does not belong to S . Now,

$$\begin{aligned} \mathbb{E} \left[\#\text{Back}(0, \bar{\xi}^0) \mathbf{1}_{0 \text{ is } (\alpha, u)\text{-good for } \bar{\xi}^0} \right] &= \mathbb{E} \left[\sum_{x \in \text{Back}(0, \bar{\xi}^0)} \mathbf{1}_{0 \text{ is } (\alpha, u)\text{-good for } \bar{\xi}^0} \right] \\ &= \mathbb{E} \left[\sum_{x \in \bar{\xi}^0} M(x, 0, \bar{\xi}^0) \right] \\ &= \mathbb{E} \left[\sum_{y \in \bar{\xi}^0} M(0, y, \bar{\xi}^0) \right] \quad (\text{by Lemma 8}) \\ &= \mathbb{E} \left[\sum_{y \in \text{For}(0, \bar{\xi}^0)} \mathbf{1}_{y \text{ is } (\alpha, u)\text{-good for } \bar{\xi}^0} \right]. \end{aligned} \quad (13)$$

Similarly,

$$\mathbb{E} \left[\# \text{Back}(0, \bar{\xi}^0) \mathbf{1}_{0 \text{ is followed by a loop in } \bar{\xi}^0} \right] = \mathbb{E} \left[\sum_{y \in \text{For}(0, \bar{\xi}^0)} \mathbf{1}_{y \text{ is followed by a loop in } \bar{\xi}^0} \right]. \quad (14)$$

But there is at most one point followed by a loop in any forward. Combining this fact, (12), (13) and (14) we get the lemma. \square

Lemma 16 has a nice intuitive interpretation. One may think – this is a rough picture – that the forward ends by a loop after each (α, u) -good point with probability at least $\mathbb{P}[\bar{\xi} \in \mathcal{L}(\alpha)]$. One may then hope that the number of (α, u) -good points in a forward is dominated by a geometrical distribution with parameter $\mathbb{P}[\bar{\xi} \in \mathcal{L}(\alpha)]$. The expected value of this number of points would then be at most the inverse of this parameter. While the actual picture is more complex, the lemma states that the latter prediction actually holds true.

Proof of Lemma 14. For any $\bar{S} \in \bar{\mathcal{S}}$ For any $x \in \xi$ write $\text{For}^{\text{Good}}(x; \bar{\xi}) = \{y \in \text{For}(x; \bar{\xi}) : y \text{ is } \alpha\text{-good for } \xi\}$. Define similarly $\text{For}^{\text{Good}}(0; \bar{\xi}^0)$. Set $C = [0, 1]^d$. Thanks to Slivnyak-Mecke Formula (4), the Lemma 16 and the definition of $\#U(\alpha)$, we have

$$\mathbb{E} \left[\sum_{x \in \xi \cap C} \# \text{For}^{\text{Good}}(x; \bar{\xi}) \right] = \mathbb{E} \left[\# \text{For}^{\text{Good}}(0; \bar{\xi}^0) \right] \leq \frac{\#U(\alpha)}{\mathbb{P}[\bar{\xi} \in \mathcal{L}(\alpha)]} < \infty.$$

We get the result. \square

3.5 Absence of forward percolation of \sharp -bad points is sufficient

In this section, we introduce a new model, an augmented model in the sense that the collection of new stopped curves contains the curves of the original models. This model has better stochastic properties and its study will be easier. In particular, the proof of Theorem 5 will be reduced to the absence of forward percolation of bad points in this new model.

Before starting its construction, recall that we work under the framework and assumptions of Theorem 5. In particular, there exists $t > 0$ such that

$$\mathbb{E} \left[\sup_{s \in [0, t]} \|h(s)\|^2 \right] < \infty.$$

In fact we can assume $t = 1$. If $t \geq 1$, this is straightforward. If $t < 1$, it suffices to perform a linear change of time for the probability measure μ : replace μ by its image under the map $h \mapsto (s \mapsto h(st))$. The assumptions are still satisfied for the new probability measure (for the loop property at scale α , one can use the fact that the initial measure satisfies the loop property at scale αt and argue as in the proof of the monotony of the loop property) and the graph obtained does not change.

We therefore assume $t = 1$ to simplify notations,

$$\mathbb{E} \left[\sup_{s \in [0, 1]} \|h(s)\|^2 \right] < \infty.$$

3.5.1 The \sharp -model

Recall $\mathcal{L}(1)$ and $r(1)$ from the definition of the loop property at scale 1. We say that there is an obstacle around $c \in \mathbb{R}^2$ if the following conditions hold:

1. $\bar{\xi} - c \in \mathcal{L}(1)$.
2. For all $x \in \xi \setminus B(c, 1)$, $G_1(x) \cap B(c, 1) = \emptyset$.

We then define $\mathcal{O}_c = \mathcal{O}_c(\bar{\xi})$ as follows.

- If there is an obstacle around $c \in \mathbb{R}^2$ we define \mathcal{O}_c as the union of the terminal state of all grains which started in $B(c, 1)$:

$$\mathcal{O}_c = \bigcup_{x \in \xi \cap B(c, 1)} G_x(\tau(x)).$$

- Otherwise, we set $\mathcal{O}_c = \emptyset$.

We then define the global obstacle by

$$\mathcal{O} = \mathcal{O}(\bar{\xi}) = \bigcup_{c \in \mathbb{R}^2} \mathcal{O}(c).¹$$

For all $x \in \xi$ we define

$$\tau^\sharp(x) = \tau^\sharp(x; \bar{\xi}) = \inf\{t \geq 1 : g_t(x) \in \mathcal{O}\}$$

and

$$G^\sharp(x) = G^\sharp(x; \bar{\xi}) = G_x(\tau^\sharp(x)).$$

As mentioned above, this defines a \sharp -model which is an augmented version of the initial model. More precisely, we have the following lemma.

Lemma 17. *For all $x \in \bar{\xi}$, $\tau(x) \leq \tau^\sharp(x)$.*

Proof. Assume there is an obstacle around some point c . As $G_1(x) \cap B(c, 1) = \emptyset$ for all $x \in \xi \setminus B(c, 1)$, no grain starting outside $B(c, 1)$ reached $B(c, 1)$ at time 1. As $\bar{\xi} - c \in \mathcal{L}(1)$, when considering only the dynamics of the grains inside $B(c, 1)$, those grains stops before time 1 and their union at time 1 is $\mathcal{O}(c)$. Therefore, when considering the whole dynamic, the evolution of the grains starting inside $B(c, 1)$ is not modified by the evolution of the grains starting outside $B(c, 1)$. Thus the set $\mathcal{O}(c)$ is created before time 1 and is, indeed, an obstacle for the other grains after time 1.

Let now $x \in \xi$. There are two cases.

- The grain x is one of the grains involved in the creation of the obstacles. In this case, we know that $\tau(x)$ is at most 1. But by definition $\tau^\sharp(x)$ is at least 1. Therefore $\tau(x) \leq \tau^\sharp(x)$.
- The grain x is not one of the grains involved in the creation of the obstacles. In this case, the grain starting at x is stopped at the latest at the first time after 1 at which it meets an obstacle. We thus also get $\tau(x) \leq \tau^\sharp(x)$ in this case as well.

□

Fix $\rho > 1$. For all $c \in \mathbb{R}^2$ and $n \geq 1$, we define the annulus

$$A(c, n) = B(c, \rho^n) \setminus \mathring{B}(c, \rho^{n-1})$$

and $A(n) = A(0, n)$ when the center c equals to the origin. For $x \in \xi$, we consider the fattened set

$$G^{\sharp\text{-fat}}(x) = G^\sharp(x) + B(3) = \{g + b : g \in G^\sharp(x), b \in B(3)\}.$$

For $n \geq 1$, we say $G^{\sharp\text{-fat}}(x)$ crosses the annulus $A(n)$ if

$$G^{\sharp\text{-fat}}(x) \cap \mathring{B}(\rho^{n-1}) \neq \emptyset \text{ and } G^{\sharp\text{-fat}}(x) \cap B(\rho^n)^c \neq \emptyset.$$

The following result will enable us to control the dependencies in the \sharp -model.

Proposition 18. *There exists $\rho_0 > 1$ such that, for any $\rho \geq \rho_0$, with probability 1, there exists n_0 such that:*

1. For all $n \geq n_0$ and all $x \in \xi$, $G^{\sharp\text{-fat}}(x)$ does not cross $A(n)$.
2. For all $n \geq n_0$ and all $x \in \xi \cap B(\rho^n)$, $G^\sharp(x) = G^\sharp(x; \bar{\xi}_{B(\rho^{n+1})})$.

Let us note that the last object $G^\sharp(x; \bar{\xi}_{B(\rho^{n+1})})$ is well defined since the definition of obstacles can be extended to the case of any finite configuration, in particular $\bar{\xi}_{B(\rho^{n+1})}$. The definitions of $\tau^\sharp(x; \bar{\xi}_{B(\rho^{n+1})})$ and $G^\sharp(x; \bar{\xi}_{B(\rho^{n+1})})$ follow. The proof of the proposition is postponed in Section 4.

¹Note that if there is an obstacle around $c \in \mathbb{R}^2$, then for any $c' \in \mathbb{R}^2$ close enough to c one has $\mathcal{O}'_c = \mathcal{O}_c$. This yields $\mathcal{O} = \bigcup_{c \in \mathbb{Q}^2} \mathcal{O}_c$. This alternative description of \mathcal{O} enables to solve some potential measurability issues.

3.5.2 Definition of \sharp -good points

Let us define a new graph structure on the set of vertices ξ by putting an edge between $x, y \in \xi$ when

$$G^{\sharp\text{-fat}}(x) \cap G^{\sharp\text{-fat}}(y) \neq \emptyset .$$

In this case, we write $x \sim^\sharp y$. A \sharp -path is a (finite or not) path in this graph. In other words, this is a (possibly infinite) sequence (x_i) of points of ξ such that, for any index i , $x_{i-1} \sim^\sharp x_i$. Using the natural distance in this graph, we denote by $B^\sharp(x, k)$ the ball with center $x \in \xi$ and radius $k \geq 0$.

The next proposition which considers the case where 0 belongs to the configuration, introduces the notion of (α, \sharp) -goodness through the event $\text{Good}^\sharp(\alpha)$.

Proposition 19. *There exists a family of events $\text{Good}^\sharp(\alpha) \in \mathcal{F}^{\bar{S}}$ indexed by $\alpha > 0$ such that the following properties hold.*

1. For all $\alpha > 0$, $\text{Good}^\sharp(\alpha) \subset \{\bar{S} \in \bar{\mathcal{S}} : 0 \in S\}$.
2. A.s. for all $\alpha > 0$, any $x \in \xi$ stopped in a regular way (in $\bar{\xi}$) and such that $\bar{\xi} - x \in \text{Good}^\sharp(\alpha)$ satisfies $\bar{\xi} - x \in \text{Good}(\alpha)$.
3. For all $\alpha > 0$, the event $\text{Good}^\sharp(\alpha)$ only depends on $\{(y, g_y), y \in B^\sharp(0, 4)\}$.
4. A.s. for any $x \in \xi$, there exists $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$, $\bar{\xi} - x \in \text{Good}^\sharp(\alpha)$.

An element $x \in \xi$ is an (α, \sharp) -good point if $\bar{\xi} - x$ belongs to $\text{Good}^\sharp(\alpha)$. Otherwise, x is an (α, \sharp) -bad point. The statements of Proposition 19 can thus be paraphrased as follows:

- A.s. any (α, \sharp) -good point stopped in a regular way is an α -good point.
- Whether $x \in \xi$ is (α, \sharp) -good or not only depends on $\{(y, g_y), y \in B^\sharp(x, 4)\}$.
- A.s. any point of ξ is (α, \sharp) -good for α small enough.

The proof of the proposition is postponed to Section 5. While it may be frustrating for the reader to encounter only an abstract definition of $\text{Good}^\sharp(\alpha)$ points at this stage, providing a precise definition now is impractical due to its technical nature, which involves extensive work, such as introducing scenarios, blurred lifetime functions, and more. Therefore, we have chosen to extract and present only the significant properties of $\text{Good}^\sharp(\alpha)$ points necessary to complete the proof of Theorem 5.

3.6 Proof of Theorem 5 using Propositions 18 and 19

In the previous section, we successively reduced the proof of Theorem 5 to the proof of the existence of a subcritical regime for forward percolation of (α, \sharp) -bad points. In other words, it remains to prove the following lemma.

Lemma 20. *There exists $\alpha > 0$ such that, with probability 1, there exists no $x \in \xi$ such that $\text{For}(x)$ is infinite and contains only (α, \sharp) -bad points.*

Proof of Theorem 5 using Lemma 20. Let $\alpha > 0$ be as in the statement of the lemma. First remark that any element y belonging to an infinite forward set is stopped in a regular way. We can then use the second item of Proposition 19 to assert that if such y is an α -bad point, it is also an (α, \sharp) -bad point. Hence, Lemma 20 implies that a.s. there exists no $x \in \xi$ such that $\text{For}(x)$ is infinite and contains only α -bad points.

We conclude by invoking almost sure results stated previously.

- By Lemma 15, all forward clusters $\text{For}(x)$ are finite.
- By Lemma 13, all undirected clusters $\overleftrightarrow{C}(x)$ are finite.
- By Lemma 10, all the clusters $C(x)$ of the original undirected graph $\text{Graph}(\bar{\xi})$ are finite.

Note that in the first two bullet points, forward clusters and undirected clusters of the directed graph $\overrightarrow{\text{Graph}}(\bar{\xi})$ are considered while the third one concerns clusters of the original undirected graph $\text{Graph}(\bar{\xi})$. Theorem 5 is proven. \square

Proof of Lemma 20 using Propositions 18 and 19. Let π be a \sharp -path (recall the definition in Section 3.5.2). We say that π is a \sharp -path of (α, \sharp) -bad points if all its vertices are (α, \sharp) -bad points. We say that π is self-avoiding if all its vertices are distinct.

Besides, two consecutive elements x_i and x_{i+1} in a forward cluster in $\overrightarrow{\text{Graph}}(\bar{\xi})$ satisfies $x_i \rightarrow x_{i+1}$. This means that there are different and $x_i \sim^\sharp x_{i+1}$ by Lemma 17. Hence, any infinite forward set is an infinite self-avoiding \sharp -path. So, in order to get Lemma 20, it is sufficient to prove that:

Claim 21. *There exists $\alpha > 0$ (small) such that a.s. there is no infinite self-avoiding \sharp -path of (α, \sharp) -bad points.*

Given two subsets A and B of \mathbb{R}^2 , we say that there exists a \sharp -path from A to B if there is a \sharp -path (x_0, \dots, x_k) such that $G^{\sharp\text{-fat}}(x_0) \cap A \neq \emptyset$ and $G^{\sharp\text{-fat}}(x_k) \cap B \neq \emptyset$. Let $\rho_0 > 1$ given by Proposition 18 and pick $\rho \geq \max\{\rho_0, 2\}$. For any $\alpha > 0$, $n \geq 1$ and $c \in \mathbb{R}^2$, let us consider the event

$$\text{Cross}_\alpha(c, n) = \{ \exists \text{ a self-avoiding } \sharp\text{-path of } (\alpha, \sharp)\text{-bad points from } B(c, \rho^{10n}) \text{ to } B(c, \rho^{10(n+1)})^c \} .$$

By stationarity, it is sufficient to get Claim 21 to prove that

$$\mathbb{P}[\text{Cross}_\alpha(0, n)] \rightarrow 0 \text{ as } n \rightarrow \infty \tag{15}$$

for $\alpha > 0$ small enough. Indeed, the previous limit asserts that any self-avoiding \sharp -path of (α, \sharp) -bad points starting from $c + [0, 1]^2$ is a.s. finite, for any $c \in \mathbb{R}^2$, and Claim 21 immediately follows.

The proof of (15) relies on a multiscale analysis: see [9] for details. One key point is that the event $\text{Cross}_\alpha(c, n)$ can be localized on the following nice event:

$$\text{Nice}(c, n) = \left\{ \begin{array}{l} \text{for all } k \geq 10n \text{ and all } x \in \xi, G^{\sharp\text{-fat}}(x) \text{ does not cross } A(c, k) \text{ and} \\ \text{for all } k \geq 10n \text{ and all } x \in \xi \cap B(c, \rho^k), G^\sharp(x) = G^\sharp(x; \bar{\xi}_{|B(c, \rho^{k+1})}) \end{array} \right\}$$

recalling that $A(c, k) = B(c, \rho^k) \setminus \mathring{B}(c, \rho^{k-1})$. This is Lemma 22 below. Later, we will use that the event $\text{Nice}(c, n)$ has a probability tending to 1 as n tends to ∞ (uniformly on the center c by stationarity) thanks to Proposition 18.

Lemma 22. *For any $\alpha > 0$, $c \in \mathbb{R}^2$ and $n \geq 1$, there exists an event $\text{CrossLocal}_\alpha(c, n)$, depending only on $\bar{\xi}_{|B(c, \rho^{10n+16})}$, such that*

$$\text{Cross}_\alpha(c, n) \cap \text{Nice}(c, n) = \text{CrossLocal}_\alpha(c, n) \cap \text{Nice}(c, n) .$$

Proof of Lemma 22. Assume that $\text{Nice}(c, n)$ holds. In order to know whether $\text{Cross}_\alpha(c, n)$ holds or not, we proceed in two steps. First, we focus on all possible self-avoiding \sharp -path (x_0, \dots, x_k) from $B(c, \rho^{10n})$ to $B(c, \rho^{10n+10})^c$. W.l.o.g. we can assume that (x_0, \dots, x_k) is minimal in the following sense: $G^{\sharp\text{-fat}}(x_i)$ overlaps $B(c, \rho^{10n+10})^c$ only when $i = k$. Then, the fattened set $G^{\sharp\text{-fat}}(x_k)$ (and in particular the vertex x_k) is included in $B(c, \rho^{10n+11})$. Otherwise, it would touch $B(c, \rho^{10n+11})^c$ and $B(c, \rho^{10n+10})$ since by hypothesis x_k is the unique element among the x_i 's such that $G^{\sharp\text{-fat}}(x_i) \cap B(c, \rho^{10n+10})^c \neq \emptyset$. This is forbidden when $\text{Nice}(c, n)$ holds. Consequently, x_0, \dots, x_k are all points of $\xi \cap B(c, \rho^{10n+11})$. On $\text{Nice}(c, n)$, the \sharp -grains $G^\sharp(x_i)$'s (and also the fattened sets $G^{\sharp\text{-fat}}(x_i)$'s) only depend on $\bar{\xi}_{|B(c, \rho^{10n+12})}$.

In a second step, we have to check whether each x_i is an (α, \sharp) -bad point or not. Thanks to the second item of Proposition 19, this only involves elements in $B^\sharp(x_i, 4)$. Repeating the same argument as in the first step on the event $\text{Nice}(c, n)$, we prove that any vertex belonging to some $B^\sharp(x_i, 4)$, for $0 \leq i \leq k$, has to be inside $B(c, \rho^{10n+15})$. On $\text{Nice}(c, n)$, the \sharp -grains of vertices of $\xi \cap B(c, \rho^{10n+15})$ only depend on $\bar{\xi}_{|B(c, \rho^{10n+16})}$. \square

The next step toward the proof of (15) consists in the following induction relation.

Lemma 23. *There exists a constant $C = C(\rho, d) \geq 1$ such that, for all $n \geq 1$ and all $\alpha > 0$,*

$$\mathbb{P}[\text{Cross}_\alpha(0, n+1)] \leq C\mathbb{P}[\text{Cross}_\alpha(0, n)]^2 + C\mathbb{P}[\text{Nice}(0, n)^c] .$$

Proof of Lemma 23. Let \mathcal{C}_1 be a finite subset of $\mathcal{S}(\rho^{10})$ and \mathcal{C}_2 be a finite subset of $\mathcal{S}(\rho^{20})$ such that

$$\mathcal{S}(\rho^{10}) \subset \bigcup_{c \in \mathcal{C}_1} B(c, 1) \text{ and } \mathcal{S}(\rho^{20}) \subset \bigcup_{c \in \mathcal{C}_2} B(c, 1).$$

Denote by $K = K(\rho, d)$ the maximum of the cardinal of \mathcal{C}_1 and \mathcal{C}_2 . For any $n \geq 1$, we immediately get

$$\mathcal{S}(\rho^{10(n+1)}) \subset \bigcup_{c \in \rho^{10n}\mathcal{C}_1} B(c, \rho^{10n}) \text{ and } \mathcal{S}(\rho^{10(n+2)}) \subset \bigcup_{c \in \rho^{10n}\mathcal{C}_2} B(c, \rho^{10n})$$

where $\rho^{10n}\mathcal{C}_1$ (resp. $\rho^{10n}\mathcal{C}_2$) is a subset of $\mathcal{S}(\rho^{10(n+1)})$ (resp. $\mathcal{S}(\rho^{10(n+2)})$) with cardinality at most K .

Assume that $\text{Cross}_\alpha(0, n+1)$ occurs and let us denote by $\pi = (x_0, \dots, x_k)$ a path realizing this event. W.l.o.g. we can assume that π is minimal in the following sense: $G^\#(x_i)$ touches $B(0, \rho^{10(n+1)})$ (resp. $B(0, \rho^{10(n+2)})^c$) only when $i = 0$ (resp. $i = k$). So, $G^{\#\text{-fat}}(x_0)$ touches $B(0, \rho^{10(n+1)})$ and its complement: it also touches $\mathcal{S}(\rho^{10(n+1)})$. Therefore there exists $c_1 \in \rho^{10n}\mathcal{C}_1$ such that $G^{\#\text{-fat}}(x_0)$ touches $B(c_1, \rho^{10n})$. Similarly there exists $c_2 \in \rho^{10n}\mathcal{C}_2$ such that $G^{\#\text{-fat}}(x_k)$ touches $B(c_2, \rho^{10n})$. As both distances between c_1 and $\mathcal{S}(\rho^{10(n+2)})$ and between c_2 and $\mathcal{S}(\rho^{10(n+1)})$ are at least $\rho^{10(n+2)} - \rho^{10(n+1)} > \rho^{10(n+1)}$ (since $\rho \geq 2$), we get that both events $\text{Cross}_\alpha(c_1, n)$ and $\text{Cross}_\alpha(c_2, n)$ occur. To sum up,

$$\text{Cross}_\alpha(0, n+1) \subset \bigcup_{c_1, c_2} \text{Cross}_\alpha(c_1, n) \cap \text{Cross}_\alpha(c_2, n)$$

where $c_1 \in \rho^{10n}\mathcal{C}_1$ and $c_2 \in \rho^{10n}\mathcal{C}_2$.

Thus Lemma 22 leads to the following inclusion

$$\text{Cross}_\alpha(0, n+1) \subset \left(\bigcup_{c_1, c_2} \text{CrossLocal}_\alpha(c_1, n) \cap \text{CrossLocal}_\alpha(c_2, n) \right) \cup \left(\bigcup_{c_3} \text{Nice}(c_3, n)^c \right)$$

where c_1, c_2 are as before and where $c_3 \in \rho^{10n}\mathcal{C}_1 \cup \rho^{10n}\mathcal{C}_2$. By Lemma 22 and for $i = 1, 2$, the event $\text{CrossLocal}_\alpha(c_i, n)$ is measurable w.r.t. $\bar{\xi}_{B(c_i, \rho^{10n+16})}$. So they are independent since $\|c_2 - c_1\| \geq \rho^{10(n+2)} - \rho^{10(n+1)} > 2\rho^{10n+16}$ as $\rho \geq 2$. Therefore, the previous inclusion yields

$$\mathbb{P}[\text{Cross}_\alpha(0, n+1)] \leq \sum_{c_1, c_2} \mathbb{P}[\text{CrossLocal}_\alpha(c_1, n)] \mathbb{P}[\text{CrossLocal}_\alpha(c_2, n)] + \sum_{c_3} \mathbb{P}[\text{Nice}(c_3, n)^c].$$

By stationarity, we can write

$$\mathbb{P}[\text{Cross}_\alpha(0, n+1)] \leq K^2 \mathbb{P}[\text{CrossLocal}_\alpha(0, n)]^2 + 2K \mathbb{P}[\text{Nice}(0, n)^c].$$

Using Lemma 22 once again, we get $\mathbb{P}[\text{CrossLocal}_\alpha(0, n)] \leq \mathbb{P}[\text{Cross}_\alpha(0, n)] + \mathbb{P}[\text{Nice}(0, n)^c]$ and then

$$\mathbb{P}[\text{Cross}_\alpha(0, n+1)] \leq K^2 \mathbb{P}[\text{Cross}_\alpha(0, n)]^2 + (3K^2 + 2K) \mathbb{P}[\text{Nice}(0, n)^c].$$

The lemma follows with $C = 3K^2 + 2K$. \square

Let us conclude the proof of (15). Let $C \geq 1$ be the constant given in Lemma 23. Since the event $\text{Nice}(0, n)^c$ has a vanishing probability as $n \rightarrow \infty$ by Proposition 18, we can pick n_0 large enough such that, for all $n \geq n_0$,

$$C \mathbb{P}[\text{Nice}(0, n_0)^c] \leq C^2 \mathbb{P}[\text{Nice}(0, n)^c] \leq \frac{1}{4}. \quad (16)$$

As a consequence of the fourth item of Proposition 19, any point in a given bounded set is $(\alpha, \#)$ -good with probprobability tending to 1 as α tends to 0. Hence, pick $\alpha > 0$ small enough such that

$$C \mathbb{P}[\exists x \in \xi \cap B(0, \rho^{10n_0+16}) \text{ such that } x \text{ is } (\alpha, \#)\text{-bad}] \leq \frac{1}{4}.$$

The argument used in the first step of Lemma 22 says that the conjunction of both events $\text{Cross}_\alpha(0, n_0)$ and $\text{Nice}(0, n_0)$ ensures the existence of a $(\alpha, \#)$ -bad point in $B(0, \rho^{10n_0+11})$. Using what precedes, we get:

$$\begin{aligned} C \mathbb{P}[\text{Cross}_\alpha(0, n_0)] &\leq C \mathbb{P}[\exists x \in \xi \cap B(0, \rho^{10n_0+16}) \text{ such that } x \text{ is } (\alpha, \#)\text{-bad}] + C \mathbb{P}[\text{Nice}(0, n_0)^c] \\ &\leq \frac{1}{2}. \end{aligned} \quad (17)$$

Henceforth, combining (16), (17) and the induction relation

$$C\mathbb{P}[\text{Cross}_\alpha(0, n+1)] \leq (C\mathbb{P}[\text{Cross}_\alpha(0, n)])^2 + C^2\mathbb{P}[\text{Nice}(0, n)^c] \quad (18)$$

providing by Lemma 23 (for all $n \geq n_0$), we easily show by induction that $C\mathbb{P}[\text{Cross}_\alpha(0, n)] \leq \frac{1}{2}$ for all $n \geq n_0$. In other words,

$$0 \leq \limsup_n C\mathbb{P}[\text{Cross}_\alpha(0, n)] \leq \frac{1}{2}.$$

Besides, from (16) and (18) we also deduce that

$$\limsup_n C\mathbb{P}[\text{Cross}_\alpha(0, n)] \leq \left(\limsup_n C\mathbb{P}[\text{Cross}_\alpha(0, n)] \right)^2.$$

Putting together the two previous displays, we finally get (15). Lemma 20 is proven. \square

4 Proof of Proposition 18: study of the \sharp -model

4.1 Local obstacles, pollution and strategy

Rough strategy. We mainly have to prove that augmented grains $G^\sharp(x)$ do not travel too far. As augmented grains are stopped when they hit an obstacle, we have to prove a conveniently quantified version of "any long path touches an obstacle". The difficulty lies in the fact that the definition of the event "there is an obstacle around point c " is not local. Indeed, the second item in the definition requires that there is no $x \in \xi \setminus B(c, 1)$ such that $G_1(x)$ touches $B(c, 1)$. This leads to long range dependence. To deal with that difficulty we introduce two objects:

- The event "there is a local obstacle around c ". This is a local event. They are therefore easily dealt with.
- The pollution Σ . This is a random subset of \mathbb{R}^2 such that, for any point c out of Σ , "there is a local obstacle around c " if and only if "there is an obstacle around c ".

The difficulty is thus concentrated in the study of the pollution Σ . We are led to show that on any long path one can find many points outside of Σ . This is related, at least in spirit, to percolation and first passage percolation. We make a few comment on this point in Section 4.3.3. In the remaining of this section, we give precise definitions.

V_0 -local obstacle. We define a notion of local obstacle. This notion depends on a positive real number $V_0 > 0$. We also write $V_0^+ = V_0 + 1$. For any $x \in \xi$ we define

$$V(x) = \sup_{s \in [0,1]} \|g_x(s) - g_x(0)\| = \sup_{s \in [0,1]} \|h_x(s)\| \quad \text{and} \quad V^+(x) = V(x) + 1.$$

In the line-segment model introduced in Section 2.4.2, $V(x)$ denotes the speed of growth of the associated grain. We keep this interpretation as a speed in the general case and speak of low-speed or high-speed grains x depending on whether $V(x) < V_0$ or not. Recall the discussion at the beginning of Section 3.5, the moment assumption of Theorem 5 can be rewritten with $t = 1$. Hence, in the whole of Section 4, we assume

$$\mathbb{E}[V^2] = \mathbb{E}\left[\sup_{s \in [0,1]} \|h(s)\|^2 \right] < \infty$$

or equivalently $\mathbb{E}[(V^+)^2] < \infty$.

Recall also $\mathcal{L}(1)$ and $r(1)$ from the definition of the loop property at scale 1. Let us first reformulate the definition of obstacle as stated in Section 3.5.1. There is an obstacle around $c \in \mathbb{R}^2$ if the following conditions hold:

1. $\bar{\xi} - c \in \mathcal{L}(1)$.
2. For all $x \in \xi \setminus B(c, 1)$, if $V(x) < V_0$, then $G_1(x) \cap B(c, 1) = \emptyset$.

3. For all $x \in \xi \setminus B(c, 1)$, if $V(x) \geq V_0$, then $G_1(x) \cap B(c, 1) = \emptyset$.

We say that there is a local obstacle around $c \in \mathbb{R}^2$ if the first two of the above properties are true. Let us make a few elementary remarks.

- If there is an obstacle around c , then there is a local obstacle around c .
- By definition of the loop property, the probability of having an obstacle around a point is positive. The same therefore holds for the probability of having a local obstacle around a point.
- Property 1 only depends on $\bar{\xi}_{|B(c,1)}$. Property 2 only depends on $\xi_{|B(c, V_0^+)}$. Indeed any grain x such that $G_1(x) \cap B(c, 1) \neq \emptyset$ satisfies $B(x, V(x)) \cap B(c, 1) \neq \emptyset$. If the grain is moreover low-speed, then $B(x, V_0) \cap B(c, 1) \neq \emptyset$ and thus x belongs to $B(c, V_0^+)$. Therefore,

$$\text{The event "there is a local obstacle at } c \text{" is measurable with respect to } \xi_{|B(c, V_0^+)}. \quad (19)$$

- If there is no high-speed grain $x \in \xi$ such that $B(x, V(x))$ touches $B(c, 1)$ – which is equivalent to $c \in B(x, V^+(x))$ –, then there is a local obstacle around c if and only if there is an obstacle around c .

V_0 -pollution and strategy. The latter remark motivates the definition of the following object. We define a Boolean model by

$$\Sigma = \Sigma(V_0) = \bigcup_{x \in \xi: V(x) \geq V_0} B(x, V^+(x)).$$

We can reformulate the remark ending the previous paragraph as follows:

If $c \in \mathbb{R}^2 \setminus \Sigma$, then there is a local obstacle around c if and only if there is an obstacle around c . (20)

Plan of the section. We fix some constants in Section 4.2. The key result on pollution, Proposition 25, is stated and proven in Section 4.3. Its main consequence, which is essentially the first item of Proposition 18, is then stated and proven in Section 4.4. The conclusion of the proof of Proposition 18 is then given in Section 4.5.

4.2 Choice of constants for Sections 4.3 and 4.4

We fix ρ and V_0 large enough for Sections 4.3 and 4.4. In Section 4.5 we do not use V_0 any more and ρ become a free parameter again.

Scale parameter ρ . We fix $\rho = 100$.

Speed parameter V_0 . In order to choose V_0 , we need to introduce a further event. We say that there is a local low-speed obstacle around $c \in \mathbb{R}^2$ if the following conditions hold:

1. $\bar{\xi}^{<V_0} - c \in \mathcal{L}(1)$ where $\bar{\xi}^{<V_0} = \{(x, h_x) \in \bar{\xi} : V(x) < V_0\}$.
2. For all $x \in \xi \setminus B(c, 1)$, if $V(x) < V_0$, then $G_1(x) \cap B(c, 1) = \emptyset$.

The advantage of this new event is that it is measurable with respect to $\bar{\xi}^{<V_0}$. If all the grains of $\xi \cap B(c, 1)$ are low-speed, then there is a local low-speed obstacle around $c \in \mathbb{R}^2$ if and only if there is a local obstacle around $c \in \mathbb{R}^2$. Recall that the probability of having a local obstacle around c is positive. Therefore, for V_0 large enough, the probability of having a local low-speed obstacle around c is positive.

Let now C be the absolute constant define in (25). Fix V_0 large enough to ensure

$$\mathbb{P}[\text{local low-speed obstacle at } 0] > 0 \quad (21)$$

and there exists an integer n_0 such that

$$\rho^{n_0-1} < V_0^+ \text{ and } 6V_0^+ < \rho^{n_0-1}(\rho - 1) \text{ and } C^2 \int_{(\rho^{n_0-2}, +\infty)} (v^+)^2 d\mathbb{P}_{V^+(v^+)} \leq \frac{1}{4}. \quad (22)$$

We can have (21) thanks to the discussion of the previous paragraph. We can also have (22) as $\mathbb{E}[(V^+)^2]$ is finite and as $\rho - 1 = 99 > 6$.

4.3 Pollution control

Recall that $\rho = 100$ and that V_0 has been chosen in Section 4.2.

4.3.1 Notations, result and plan

Let $n \geq 1$. We consider the annulus

$$A_n = B(\rho^n) \setminus \mathring{B}(\rho^{n-1})$$

and the Boolean model

$$\Sigma_n = \bigcup_{x \in \xi: V_0^+ \leq V^+(x) \leq \rho^{n-1}} B(x, V^+(x)).$$

Let us recall the definition of the pollution Σ :

$$\Sigma = \bigcup_{x \in \xi: V_0 \leq V(x)} B(x, V^+(x)) = \bigcup_{x \in \xi: V_0^+ \leq V^+(x)} B(x, V^+(x)).$$

Thus Σ_n is a local version of the pollution Σ .

To any deterministic continuous path π crossing A_n (that is starting from the inner boundary of A_n , ending on the outer boundary of A_n and remaining within the annulus A_n in the meantime) we associate the score

$$S_n(\pi) = \max\{\text{card}(W) : W \subset \pi, W \subset \Sigma^c(V_0), W \text{ is separated}\}$$

and the score

$$S'_n(\pi) = \max\{\text{card}(W) : W \subset \pi, W \subset \Sigma_n^c(\rho, V_0), W \text{ is separated}\}$$

where " W is separated" means that any two distinct points of W are at least at distance $3V_0^+$ from each other and that each point is at least at distance $3V_0^+$ from the boundaries of the annulus.

Finally, we introduce

$$T_n = \inf_{\pi \text{ crosses } A_n} S_n(\pi)$$

and

$$T'_n = \inf_{\pi \text{ crosses } A_n} S'_n(\pi).$$

The latter quantity is local:

$$T'_n \text{ is measurable with respect to } \xi|_{B(2\rho^n)}. \quad (23)$$

Indeed T'_n depends only on $A_n \cap \Sigma_n$, and the radius of balls considered in the definition of Σ_n is bounded from above by ρ^{n-1} (and thus by ρ^n).

Let us collect a few results in the following lemma.

Lemma 24. • *If W is separated, then the events " w is a local obstacle" indexed by $w \in W$ are independent.*

- *If $W \subset \Sigma^c$, then for any $w \in W$, " w is a local obstacle" if and only if " w is an obstacle".*

Proof. The first item is a consequence of (19). The second item is (20). \square

Our aim is to show that any crossing of an annulus touches an obstacle with high probability. Thanks to the previous lemma, the task is mainly reduced to showing that T_n is large. This motivates the following key result about pollution.

Proposition 25. *With probability 1, for any n large enough, $T_n \geq (4/3)^n$.*

The lower bound $(4/3)^n$ is negligible compare with the width of the annulus A_n which is of order 100^n . The lower bound is thus sub-linear. This is however sufficient for our purpose. The proposition is an immediate consequence of the following results and of Borel-Cantelli lemma.

Lemma 26. *There exists $A > 0$ such that $\sum_n \mathbb{P}[T'_n < (3/2)^n A] < \infty$.*

Lemma 27. *With probability 1, for any n large enough, $T'_n = T_n$.*

Proof of Lemma 27. Let $n \geq 0$. If $T'_n \neq T_n$, then there exists $x \in \xi$ such that

$$V^+(x) \geq \rho^{n-1} \text{ and } B(x, V^+(x)) \cap B(\rho^n) \neq \emptyset$$

and therefore such that

$$\|x\| \leq (1 + \rho)V^+(x).$$

But the expected value of the number of $x \in \xi$ fulfilling the previous condition is $\pi(1 + \rho)^2 \mathbb{E}[(V^+)^2]$, using the Campbell's formula, which is finite as $V^+ = V + 1$ and $\mathbb{E}[V^2]$ is finite. Therefore, almost surely, the number of such x is finite. The lemma follows. \square

The proof of Lemma 26 is given in the next section.

4.3.2 Proof of Lemma 26

We need the following lemma, which relates the behavior at scales n and $n - 1$.

Lemma 28. *There exists an absolute constant C - given by (25) - such that, for all $n \geq 1$ and all $A > 0$,*

$$\mathbb{P}[T'_n < (3/2)A] \leq C\mathbb{P}[T'_{n-1} < A]^2 + C \int_{(\rho^{n-2}, \rho^{n-1}]} (v^+)^2 d\mathbb{P}_{V^+}(v^+).$$

Proof. Let $n \geq 1$. Consider the event

$$\mathcal{D}(n) = \{\text{there exists } x \in \xi \text{ such that } V^+(x) \in (\rho^{n-2}, \rho^{n-1}] \text{ and } B(x, V^+(x)) \text{ touches } B(\rho^n)\}.$$

On the complement of this event, we have

$$\Sigma_n \cap B(\rho^n) = \Sigma_{n-1} \cap B(\rho^n). \quad (24)$$

Recall that we set $\rho = 100$. Fix three finite sets L_1, L_2 and L_3 such that, for all $i \in \{1, 2, 3\}$,

$$L_i \subset \partial B(20i) \text{ and } \partial B(20i) \subset \bigcup_{w \in L_i} B(w, \rho^{-1}).$$

We define an absolute constant C by

$$C = \max(\text{card}(L_1)\text{card}(L_2) + \text{card}(L_1)\text{card}(L_3) + \text{card}(L_2)\text{card}(L_3), \pi(1 + \rho^2)^2). \quad (25)$$

Fix $n \geq 1$. For all $i \in \{1, 2, 3\}$ we have

$$\rho^{n-1}L_i \subset \partial B(20i\rho^{n-1}) \text{ and } \partial B(20i\rho^{n-1}) \subset \bigcup_{w \in \rho^{n-1}L_i} B(w, \rho^{n-2}).$$

The key is the inclusion

$$\{T'_n < (3/2)A\} \subset \mathcal{D}(n) \cup \bigcup_{w_1 \in \rho^{n-1}L_1} \bigcup_{w_2 \in \rho^{n-1}L_2} \bigcup_{w_3 \in \rho^{n-1}L_3} \{T'_{n-1}{}^{w_1} + T'_{n-1}{}^{w_2} + T'_{n-1}{}^{w_3} < (3/2)A\} \quad (26)$$

where $T'_{n-1}{}^w$ is the event T'_{n-1} "centered at w " (in other words this is a translate of T_{n-1}). Let us prove (26). We work on $\{T'_n < (3/2)A\} \setminus \mathcal{D}(n)$. Consider a path π which realizes the minimum in the definition of T'_n . Hence,

$$T'_n = S'_n(\pi) \geq \text{card}(W)$$

for any subset W such that $W \subset \pi$, $W \subset \Sigma_n^c(\rho, V_0)$ and W is separated. Let us construct a 'suitable' set W . For each i choose $w_i \in \rho^{n-1}L_i$ such that $B(w_i, \rho^{n-2})$ contains a point of π . Then consider for each i some restrictions π_i of π crossing the corresponding annuli. One can for example define π_i as follows:

$$a_i = \sup\{t : \pi(t_i) \in B(w_i, \rho^{n-2})\} \text{ and } b_i = \inf\{t \geq a_i : \pi(t_i) \notin B(w_i, \rho^{n-1})\}$$

and then set $\pi_i = \pi|_{[a_i, b_i]}$. See Fig. 5. Now define for each i a set W_i which realizes the maximum in the definition of $S'_{n-1}(\pi_i)$ (with a definition for an annulus centered at w_i). So, for any i ,

$$\text{card}(W_i) = S'_{n-1}(\pi_i) \geq T'_{n-1}{}^{w_i}.$$

Finally, set $W = W_1 \cup W_2 \cup W_3$. This set enjoys the following properties.

1. $\text{card}(W) = \text{card}(W_1) + \text{card}(W_2) + \text{card}(W_3)$. Indeed each W_i is included in $B(w_i, \rho^{n-1})$ and these balls are disjoint.
2. $W \subset \pi$.
3. $W \subset \Sigma_n^c(\rho, V_0)$. This is due to (24).
4. W is V_0 -separated. This is a consequence of the fact that each W_i is V_0 -separated (which implies in particular that each point of W_i is at distance at least $3V_0^+$ of the complement of $B(w_i, \rho^{n-1})$), the fact that the W_i are included in the disjoint balls $B(w_i, \rho^{n-1})$ and the fact that these balls are included in the annulus at scale A_n .

Therefore

$$\begin{aligned} T'_n = S'_n(\pi) \geq \text{card}(W) &= \text{card}(W_1) + \text{card}(W_2) + \text{card}(W_3) \\ &= S'_{n-1}(\pi_1) + S'_{n-1}(\pi_2) + S'_{n-1}(\pi_3) \geq T'_{n-1}{}^{w_1} + T'_{n-1}{}^{w_2} + T'_{n-1}{}^{w_3}. \end{aligned}$$

This establishes (26).

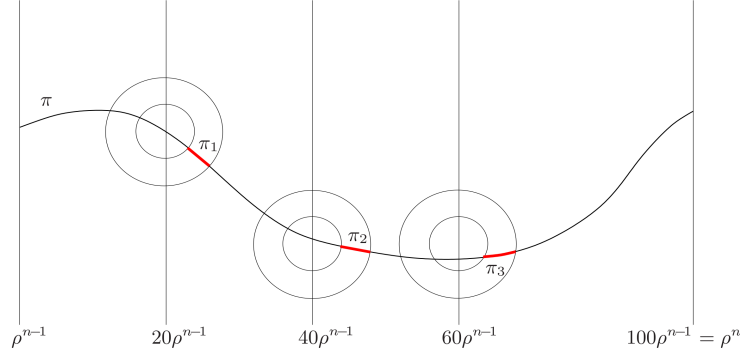


Figure 5: On this picture, circular arcs of $\partial B(\rho^{n-1})$, $\partial B(20\rho^{n-1})$ etc. are represented with vertical lines. The (black) path π crosses the annulus $A_n = B(\rho^n) \setminus \dot{B}(\rho^{n-1})$. When π crosses the smaller annuli $w_i + A_{n-1}$ for $i = 1, 2, 3$, it provides the three (red) paths π_1, π_2, π_3 .

If the sum of three numbers is smaller than $(3/2)A$, then the sum of the two smallest of them is smaller than A . Indeed, if $a \leq b \leq c$ are such that $a + b + c \leq (3/2)A$, then $(a + b)/2 \leq c$ and thus $(a + b) + (a + b)/2 \leq a + b + c \leq (3/2)A$ from which we get $a + b \leq A$. From this simple remark (which is the reason why we need the three layers L_1, L_2 and L_3 in (26)) and (26) we get, by union bound, by (23) and by stationarity

$$\mathbb{P}[T'_n < (3/2)A] \leq \mathbb{P}[\mathcal{D}(n)] + C\mathbb{P}[T'_{n-1} < A]^2$$

where C is defined by (25).

Finally, using the Campbell's formula,

$$\begin{aligned} \mathbb{P}[\mathcal{D}(n)] &\leq \mathbb{E}[\text{card}(x \in \xi : V^+(x) \in (\rho^{n-2}, \rho^{n-1}] \text{ and } B(x, V^+(x)) \text{ touches } B(\rho^n))] \\ &= \mathbb{E}[\text{card}(x \in \xi : V^+(x) \in (\rho^{n-2}, \rho^{n-1}] \text{ and } \|x\| \leq V^+(x) + \rho^n)] \\ &\leq \mathbb{E}[\text{card}(x \in \xi : V^+(x) \in (\rho^{n-2}, \rho^{n-1}] \text{ and } \|x\| \leq (1 + \rho^2)V^+(x))] \\ &= \pi(1 + \rho^2)^2 \int_{(\rho^{n-2}, \rho^{n-1}]} (v^+)^2 d\mathbb{P}_{V^+}(v^+). \end{aligned}$$

Plugging this inequality in the previous one and using the definition of C by (25) ends the proof. \square

Proof of Lemma 26. Let C be the constant given by Lemma 28. Fix n_0 such that (22) holds. As $\rho^{n_0-1} < V_0^+$, $\Sigma_{n_0} = \emptyset$. For any path π crossing A_{n_0} , as $6V_0^+ < \rho^{n_0-1}(\rho - 1)$, we thus get $S'_{n_0}(\pi) \geq 1$

by considering for W the singleton containing some point of π in the middle of A_n (that is at distance $\rho^{n_0-1} + \rho^{n_0-1}(\rho - 1)/2$ of 0). Thus $T'_{n_0} \geq 1$. Set $A = (3/2)^{-n_0}$. We get

$$\mathbb{P}[T'_{n_0} < (3/2)^{n_0} A] = 0. \quad (27)$$

By Lemma 28 we have, for all $n \geq 1$,

$$C\mathbb{P}[T'_n < (3/2)^n A] \leq C^2\mathbb{P}[T'_{n-1} < (3/2)^{n-1} A]^2 + C^2 \int_{(\rho^{n-2}, \rho^{n-1})} (v^+)^2 d\mathbb{P}_{V^+}(v^+). \quad (28)$$

From (22), (27) and (28) we first get, by induction, that for all $n \geq n_0$,

$$C\mathbb{P}[T'_n < (3/2)^n A] \leq \frac{1}{2}.$$

From (28) we then get, for all $n \geq n_0 + 1$,

$$C\mathbb{P}[T'_n < (3/2)^n A] \leq \frac{1}{2}C\mathbb{P}[T'_{n-1} < (3/2)^{n-1} A] + C^2 \int_{(\rho^{n-2}, \rho^{n-1})} (v^+)^2 d\mathbb{P}_{V^+}(v^+).$$

Summing over n we obtain, for all $k \geq n_0$,

$$\sum_{n=n_0+1}^{k+1} C\mathbb{P}[T'_n < (3/2)^n A] \leq \frac{1}{2} \sum_{n=n_0}^k C\mathbb{P}[T'_n < (3/2)^n A] + C \int_{(\rho^{n_0-1}, \rho^k)} (v^+)^2 d\mathbb{P}_{V^+}(v^+)$$

and then (using (27))

$$\sum_{n=n_0+1}^k C\mathbb{P}[T'_n < (3/2)^n A] \leq \frac{1}{2} \sum_{n=n_0+1}^k C\mathbb{P}[T'_n < (3/2)^n A] + C \int_{(\rho^{n_0-1}, +\infty)} (v^+)^2 d\mathbb{P}_{V^+}(v^+)$$

and thus

$$\sum_{n=n_0+1}^k C\mathbb{P}[T'_n < (3/2)^n A] \leq 2C \int_{(\rho^{n_0-1}, +\infty)} (v^+)^2 d\mathbb{P}_{V^+}(v^+) < \infty.$$

The lemma follows by letting k go to ∞ . \square

4.3.3 Comments on some related results

We review here some results which are related to the first part of the plan (Proposition 25) but which are not sufficient to prove it, explaining that we have to develop new strategies here.

We can consider $\Sigma(V_0)$ from the percolation point of view. As $\mathbb{E}[V^2]$ is finite, for V_0 large enough, all the connected components of $\Sigma(V_0)$ are bounded with probability one. Actually, the following stronger result is true. For all V_0 large enough, for all $\rho > 1$,

$$\mathbb{P}[\text{there exists a path in } \Sigma(V_0) \text{ from } B(0, r) \text{ to the complement of } B(0, \rho r)] \rightarrow 0 \text{ as } r \rightarrow \infty. \quad (29)$$

In words, with high probability, we can find at least one point outside $\Sigma(V_0)$ on any path crossing a large annulus. Those results are implicit in [9]. They are not sufficient to establish the first part of the plan which requires many points and qualitative results (on the number of points and on the probability).

We can also consider $\Sigma(V_0)$ from the first passage percolation point of view. In [10], the following model is considered. Let $T(x, y)$ be the minimum time needed to go from $x \in \mathbb{R}^2$ to $y \in \mathbb{R}^2$ by a traveler who walks at speed ∞ inside $\Sigma(V_0)$ and at speed 1 outside $\Sigma(V_0)$. By standard techniques in first passage percolation, one can show

$$\frac{T(0, x)}{\|x\|} \rightarrow \mu \text{ a.s. as } \|x\| \rightarrow \infty$$

where μ is a deterministic constant. As consequence of the main result of [10] and of (29), μ is positive for V_0 large enough under a condition which is slightly stronger than $\mathbb{E}[V^2] < \infty$. This is closer to what we need here but it is not sufficient to prove the first part of the plan as it gives no good probabilistic estimates.

4.4 Augmented grains do not cross large annuli

Recall that $\rho = 100$ and that V_0 has been chosen in Section 4.2. For $n \geq 1$, consider the following bad event:

$$\text{Bad}_n = \left\{ \text{there exists } x \in \xi \text{ such that } G^{\sharp\text{-fat}}(x) \text{ crosses the big annulus } \bigcup_{i=0}^5 A_{n+i} \right\}.$$

Let

$$N = \inf\{n_0 \geq 1 : \forall n \geq n_0, \text{Bad}_n \text{ does not occur}\}.$$

Lemma 29. *With probability one, N is finite.*

Proof of Lemma 29. Let

$$N_1 = \inf\{n_0 \geq 1 : \forall n \geq n_0, T_n \geq (4/3)^n\}.$$

By Proposition 25, N_1 is finite with probability 1. This is our control on the influence of high-speed grains. We say that a ball B crosses an annulus A_n if B touches $\mathring{B}(\rho^{n-1})$ and $B(\rho^n)^c$. Let

$$N_2 = \inf\{n_0 \geq 1 : \forall n \geq n_0, \text{there exists no } x \in \xi \text{ such that } B(x, V^+(x)) \text{ crosses } A_n\}.$$

If n is such that A_n is crossed by $B(x, V^+(x))$ for some $x \in \xi$, then $V^+(x) \geq \rho^{n-1}(\rho - 1)/2 \geq \rho^{n-1}$ and $B(x, V^+(x)) \cap B(\rho^n)$ is non empty. Arguing as in the proof of Lemme 27 we deduce that N_2 is finite with probability 1.

If k is an integer, we consider the event

$$\begin{aligned} \text{SmallBad}_k = \{ \exists x \in \xi \cap B(\rho^{k-2}) \cup A_{k+2} \text{ such that } g_x \text{ crosses } A_k \\ \text{and there is no obstacle at any point of } \text{Crossing}(g_x, A_k) \} \end{aligned}$$

where $\text{Crossing}(g_x, A_k)$ denotes the first crossing of the annulus A_k by the path g_x , that is the path g_x restricted to the time interval (s, t) where

$$t = \inf\{u \geq 0 : \|g_x(u)\| = \rho^k\} \text{ and } s = \sup\{u \leq t : \|g_x(u)\| = \rho^{k-1}\}$$

for a crossing from the inner to the outer and similarly for a crossing in the opposite direction.

Claim 30. *For all $n \geq 1$, $\{n \geq N_2\} \cap \text{Bad}_n \subset \bigcup_{k \geq n+1} \text{SmallBad}_k$.*

Proof of Claim 30. Let $n \geq 1$. Assume that Bad_n occurs. Let $x \in \xi$ be as in its definition. Assume also $n \geq N_1$ and $n \geq N_2$. The idea is that, wherever x is:

- $G^{\sharp\text{-fat}}(x)$ must cross three annuli toward 0 or toward ∞ .
- $g_x([0, 1])$ cross no annuli (because $n \geq N_2$).
- The fattening $+B(3)$ is responsible for no annuli crossing.

As a consequence $g_x([1, \tau^\sharp(x)])$ must cross at least one annuli and there can be no obstacle on the crossing.

Let us detail for example the case $x \in B(\rho^{n-1})$. As $n \geq N_2$, $g_x([0, 1])$ does not cross A_n and therefore remains inside $B(\rho^n)$. As 3 is smaller than the width of all annuli (the minimal width is $\rho^{1-1}(\rho - 1) = 99$) and as $G^\sharp(x) + B(3)$ touches A_{n+3} , $g_x([0, \tau^\sharp(x)])$ touches A_{n+2} . In particular, g_x crosses A_{n+1} during the time interval $[1, \tau^\sharp(x)]$. As a consequence (by definition of $\tau^\sharp(x)$) there is no point of \mathcal{O} [ref] on $\text{Crossing}(g_x, A_{n+1})$. Therefore there is no obstacle on any point of $\text{Crossing}(g_x, A_{n+1})$ (otherwise this would contradict the previous statement by the separating property of a loop and by the fact that a loop is contained in a ball of radius much smaller than the width of the annuli). We thus get that SmallBad_{n+1} holds.

The cases $x \in A_n$, $x \in A_{n+1}$, $x \in A_{n+2}$ are proven similarly. We get (according to the case considered) that SmallBad_{n+2} , SmallBad_{n+3} or SmallBad_{n+4} hold. The cases $x \in A_{n+k}$ for some $k \geq 3$ are proven in the same way except that we look for a crossing from the outer to the inner. We then get that SmallBad_{n+k-2} holds. \square

Claim 31. *There exists $\alpha > 0$ such that, for all $\ell \geq 2$, $\mathbb{P}[\{\ell \geq N_1\} \cap \text{SmallBad}_\ell] \leq \pi \rho^{2\ell+4} \exp(-\alpha(4/3)^\ell)$.*

Proof of Claim 31. Write $D = B(\rho^{\ell-2}) \cup A_{\ell+2}$ to shorten notations. Then,

$$\begin{aligned} & \mathbb{P}[\{\ell \geq N_1\} \cap \text{SmallBad}_\ell] \\ & \leq \mathbb{E} \left[\sum_{x \in \xi \cap D} \mathbb{1}_{\{g_x \text{ crosses } A_\ell \text{ and there is no obstacle at any point of } \text{Crossing}(g_x, A_\ell)\}} \mathbb{1}_{\ell \geq N_1} \right] \\ & \leq \mathbb{E} \left[\sum_{x \in \xi \cap D} \mathbb{1}_{\{g_x \text{ crosses } A_\ell \text{ and there is no local obstacle at any point of } W(\text{Crossing}(g_x, A_\ell))\}} \mathbb{1}_{\ell \geq N_1} \right] \end{aligned}$$

where $W(\text{Crossing}(g_x, A_\ell))$ is a maximiser of $S(\text{Crossing}(g_x, A_\ell))$. The previous inequality is a consequence of $W(\text{Crossing}(g_x, A_\ell)) \subset \text{Crossing}(g_x, A_\ell)$, $W(\text{Crossing}(g_x, A_\ell)) \subset \Sigma^c$ and (20).

By Slivnyak-Mecke formula, we then get

$$\begin{aligned} & \mathbb{P}[\{\ell \geq N_1\} \cap \text{SmallBad}_\ell] \\ & \leq \int_D \mathbb{E} [\mathbb{1}_{\{g_x \text{ crosses } A_\ell \text{ and there is no local obstacle at any point of } W(\text{Crossing}(g_x, A_\ell))\}} \mathbb{1}_{\ell \geq N_1}] dx \end{aligned}$$

where the underlying point process is $\{(x, g_x)\} \cup \bar{\xi}$, where $g_x = x + h$ and where h is independent of $\bar{\xi}$ and distributed according to μ . Write $\bar{\xi}$ as the disjoint union of $\bar{\xi}^{\geq V_0}$ (the high-speed grains) and $\bar{\xi}^{< V_0}$ (the low-speed grains) which can be assumed from each other:

$$\bar{\xi}^{\geq V_0} = \{(y, g_y) \in \bar{\xi} : V(y) \geq V_0\} \text{ and } \bar{\xi}^{< V_0} = \{(y, g_y) \in \bar{\xi} : V(y) < V_0\}.$$

Note that $W(\text{Crossing}(g_x, A_\ell))$, the family $(T_n)_n$ and N_1 are measurable with respect to $\mathcal{F} := \sigma(g_x, \bar{\xi}^{\geq V_0})$. Moreover, on $\{\ell \geq N_1\}$, we have $T_\ell \geq (4/3)^\ell$ and thus $W(\text{Crossing}(g_x, A_\ell)) \geq (4/3)^\ell$. For $x \in D$, we have

$$\begin{aligned} & \mathbb{E} [\mathbb{1}_{\{g_x \text{ crosses } A_\ell \text{ and there is no local obstacle at any point of } W(\text{Crossing}(g_x, A_\ell))\}} \mathbb{1}_{\ell \geq N_1}] \\ & = \mathbb{E} [\mathbb{E} [\mathbb{1}_{\{g_x \text{ crosses } A_\ell \text{ and there is no local obstacle at any point of } W(\text{Crossing}(g_x, A_\ell))\}} \mathbb{1}_{\ell \geq N_1} \mid \mathcal{F}]] \\ & \leq \mathbb{E} [\mathbb{E} [\mathbb{1}_{\{g_x \text{ crosses } A_\ell \text{ and there is no local low-speed obstacle at any point of } W(\text{Crossing}(g_x, A_\ell))\}} \mathbb{1}_{\ell \geq N_1} \mid \mathcal{F}]] \\ & \leq \exp(-\alpha(4/3)^\ell) \end{aligned}$$

where $\exp(-\alpha)$ is the probability of having no local low-speed obstacle at 0 which is smaller than 1 by (21). The claim follows. \square

From Claim 31 and Borel-Cantelli Lemma we get that

$$N_3 = \inf\{n_0 \geq 1 : \forall \ell \geq n_0, \{\ell \geq N_1\} \cap \text{SmallBad}_\ell \text{ does not occur}\}$$

is finite with probability 1. Let $n \geq \max(N_1, N_2, N_3)$. Let $\ell \geq n$. As $\ell \geq n \geq \max(N_1, N_3)$, SmallBad_ℓ does not occur. By Claim 30 we deduce that $\{n \geq N_2\} \cap \text{Bad}_n$ does not occur. But $n \geq N_2$. Therefore Bad_n does not occur. As a consequence, $N \leq \max(N_1, N_2, N_3)$ which is finite with probability 1. \square

4.5 Proof of Proposition 18

In this section ρ becomes a free variable again (we do not fix $\rho = 100$ anymore). By Lemma 29, N is finite with probability 1. But for $n \geq N$, there exists no $x \in \xi$ such that $G^{\#-\text{fat}}(x)$ crosses the big annulus

$$\bigcup_{i=0}^5 A_{n+i}(100) = B(100^{n+5}) \setminus \dot{B}(100^{n-1}).$$

Therefore for any $\rho \geq 100^{13}$, with probability 1, there exists n_0 such that for all $n \geq n_0$ and all $x \in \xi$, $G^{\#-\text{fat}}(x)$ does not cross $A_n(\rho)$. The first item of Proposition 18 holds for any choice of ρ_0 larger than 100^{13} .

Let $\rho \geq 100^{13}$. Let

$$N' = \inf\{n_0 \geq 1 : \forall n \geq n_0, G^{\sharp-\text{fat}}(x) \text{ does not cross } A_n(\rho)\}$$

and

$$N'_2 = \inf\{n_0 \geq 1 : \forall n \geq n_0, \text{ there exists no } x \in \xi \text{ such that } B(x, V^+(x)) \text{ crosses } A_n(\rho)\}.$$

We know by the first item that N' is almost surely finite. By the same argument as for N_2 [ref] (we only change the value of ρ) we know that N'_2 is also almost surely finite. Let $n \geq \max(N', N'_2)$ and $x \in \xi \cap B(\rho^n)$. Let us check, for such n and x ,

$$G^{\sharp}(x; \bar{\xi}) = G^{\sharp}(x; \bar{\xi}|_{B(\rho^{n+3})}). \quad (30)$$

This will establish the second item – and thus Proposition 18 – with $\rho_0 = (100^{13})^3$.

As $n \geq N'$ we have $G^{\sharp}(x) \subset B(\rho^{n+1})$. Therefore, in order to prove (30), we only have to check that the obstacles touching $B(\rho^{n+1})$ are the same in the configuration $\bar{\xi}$ and in the configuration $\bar{\xi}|_{B(\rho^{n+3})}$. If $c \in \mathbb{R}^2 \setminus B(\rho^{n+1} + 1)$ then, even if there is an obstacle at c , the obstacle does not touch $B(\rho^{n+1})$. Therefore it is sufficient to consider $c \in B(\rho^{n+1} + 1)$. Let us inspect to two conditions defining "there is an obstacle at c " (we write it for $\bar{\xi}$) :

1. $\bar{\xi} - c \in \mathcal{L}(1)$: this only depends on $\bar{\xi} \cap B(c, 1)$ and $B(c, 1) \subset B(\rho^{n+3})$.
2. For all $x \in \xi \setminus B(c, 1)$, $G_1(x) \cap B(c, 1) = \emptyset$: this only depends on $x \in \xi$ such that $B(x, V^+(x))$ touches $B(c, 1)$ and (as $n \geq N'_2$) all such x belongs to $B(\rho^{n+3})$.

This proves (30) and ends the proof. \square

5 Proof of Proposition 19: study of \sharp -good points

The blurred approach with scenarios. We want to witness α -goodness of x_0 by looking only in a limited neighborhood of the graph associated with the \sharp -model in Section 3.5.2 (for short we will talk of \sharp -neighborhood). With so few information, we have no access to what really happens. For example we do not know $s(x_0)$ or $\tau(x_0)$. We therefore have to accept to lose some information. We develop what we call a blurred approach with scenarios in which we agree to lose information of two kinds:

- We accept to lose the exact time at which some grain stops and, instead, keep only the knowledge of the interval of time in which the grain stops. We refer to this as blurring.
- We accept to only know that what happens is described (up to the above blurring) by one of a finite number of scenarios. Basically, a scenario consists of some meta-information giving the list of particles stopping in each interval of time of the above time discretization.

We emphasize the fact that some of these scenarios may have no sense (for example making a particle stop at some empty place). The point is that one of these scenarios corresponds to the true situation. Moreover, this blurred approach with scenarios is local and contains nevertheless enough information to witness α -goodness. The basic idea is that x_0 is declared (α, \sharp) -good if it is good in each of the scenarios. This is developed in Section 5.1.

The dominant time. The number of scenarios is finite partly because we only consider a finite number of particles (the ones in a relevant \sharp -neighborhood of x_0) and partly because we only consider what happens up to some finite time which we call the dominant time. We therefore need to define this dominant time in a local way. The first idea would be to define it as a (local) upper bound on all possible hitting times of the particles which may stop x_0 or be stopped by x_0 . One could think that knowing the local configuration (up to blurring and scenarios) up to that time would be sufficient to place a loop stopping x_0 without interfering with its backward component. This is true up to the following subtlety: there may exist non stopped grains $G^{\sharp}(x) = G(x)$ with $\tau^{\sharp}(x) = \tau(x) = +\infty$ which are not compact and could be tangent to $G^{\sharp}(x_0)$. The presence of such pathological grains could make impossible the existence of a suitable small ball $B(v, 2\alpha)$ avoiding all the grains other than x_0 in the final configuration. Overcome this second obstacle requires considering potential hitting times in a larger \sharp -neighborhood to make sure that, even if some grains other than x_0 are modified, this will not reduce the backward of x_0 . See Section 5.2.

Conclusion. Finally, Section 5.3 contains the definition of \sharp -good points and the proof of Proposition 19, i.e. any point is (α, \sharp) -good for α small enough, the notion of \sharp -goodness is local and any (α, \sharp) -good point stopped in a regular way is an α -good point.

5.1 A blurred approach

In this section, we develop our blurred approach, which generally implies that the stopping character of some particles during an interval is indeterminate. We retain only the information that the particle is stopped at some point within this interval. Some information is lost, forcing us to work with a finite sequence of scenarios, but this enables us to achieve locality.

Setting for the whole Section 5.1. Recall that $x \sim^\sharp y$ means that $G^\sharp(x) + B(3)$ and $G^\sharp(y) + B(3)$ overlap. We denote by $B^\sharp(x, 1)$ the ball with center x and radius 1 w.r.t. the graph distance associated to \sim^\sharp . Proposition 18 asserts that any ball $B^\sharp(x, 1)$ admits finitely many elements with probability 1. In Section 5, the word *locality* has to be understood in terms of \sharp -neighborhood.

For the whole Section 5.1, we consider an element $x_0 \in \xi$ and we denote by x_0, x_1, \dots, x_K the elements of $B^\sharp(x_0, 1)$. For the moment, no hypothesis is required about the grain x_0 (such as stopped in a regular way or with a finite lifetime $\tau(x_0)$).

Scenarios and modified dynamical algorithm. Let $T_0 > 0$ be a real number. Let us select the integer $N > 0$ as the smallest integer such that

$$\forall 0 \leq i \leq K \text{ and } \forall t, t' \in [0, T_0] \text{ with } |t - t'| \leq \frac{T_0}{N}, \text{ we have } |g_{x_i}(t) - g_{x_i}(t')| \leq \frac{1}{K+1}. \quad (31)$$

Such an integer exists by uniform continuity of the g_{x_i} 's on the compact time interval $[0, T_0]$. Remark that N only depends on T_0 and the vertices x_0, x_1, \dots, x_K .

Let us set for any $k \in \{1, \dots, N\}$,

$$I_k(T_0, N) = I_k := \left[(k-1) \frac{T_0}{N}, k \frac{T_0}{N} \right).$$

Respectively to x_0, x_1, \dots, x_K , T_0 and N , a **scenario** \mathbf{s} is a couple $(J, (I(j) : j \in J))$ where $J \subset \{0, 1, \dots, K\}$ and for any $j \in J$, $I(j)$ is taken among the intervals I_1, \dots, I_N . According to the scenario \mathbf{s} , the grain x_j is declared **blurred** if and only if $j \in J$. In this case, $g_{x_j}(I(j))$ represents its blurred part. Note that there are finitely many scenarios.

Given a scenario $\mathbf{s} = (J, (I(j) : j \in J))$, we define the **modified dynamical algorithm** for the growth of the grains x_0, x_1, \dots, x_K . The dynamics is the same as for the dynamical algorithm defined in Section 2.1.3 except for the following points: for any $0 \leq i \leq K$,

- if $i \in J$ and if x_i is still alive at time $t(i) := \inf I(i)$, then it is (artificially) stopped at time $t(i)$;
- if $i \notin J$ and if x_i is still alive at time T_0 , then it is (artificially) stopped at time T_0 ;

In particular, any blurred part $g_{x_j}(I(j))$ is invisible to the other grains x_i , $i \neq j$: it cannot stop them. This modified dynamics could be formalized as we did in Section 2.1.3 by working successively on time intervals of length T_0/N and adding the above extra rules. Note that, in the modified dynamics, a grain x_i with $i \in J$ may be stopped before $t(i) = \inf I(i)$, in this case it does not reach its blurred part. However it can never be stopped after $t(i)$.

This modified dynamics, restricted to the grains x_0, x_1, \dots, x_K and to the time interval $[0, T_0]$, is realized according to the scenario \mathbf{s} . It allows to associate to the scenario \mathbf{s} a **blurred configuration**, i.e. the union of the grains x_0, \dots, x_K produced by the modified dynamics with their blurred parts. We denote this union of blurred grains by $C(\mathbf{s})$. Note also that the modified dynamics involves only the elements of $B^\sharp(x_0, 1)$. A blurred configuration is thus locally determined.

Locality through admissible scenarios. Respectively to x_0, x_1, \dots, x_K and T_0 , a scenario $\mathbf{s} = (J, \{I(i) : i \in J\})$ (or the corresponding blurred configuration) is said **admissible** when its set of blurred parts $\cup_{i \in J} g_{x_i}(I(i))$ does not overlap the set

$$\text{Heart} := g_{x_0}([0, \tau^\sharp(x_0)]) + B(1) .$$

As claimed previously, the true configuration of grains around x_0 (and till time T_0) is known up to a finite number of admissible scenarios.

Lemma 32. *Using the previous notations, the following holds:*

$$\text{Heart} \cap \left(\bigcup_{x \in \xi} g_x([0, \tau(x) \wedge T_0]) \right) \in \{ \text{Heart} \cap C(\mathbf{s}) : \mathbf{s} \text{ is admissible} \} \quad (32)$$

The left hand side of (32) is the true configuration of grains in Heart and till time T_0 : this is a non-local information. However, the right hand side of (32) is a finite collection of blurred configurations which are all locally determined. Roughly speaking, to get locality we have weakened our knowledge of what really happens inside Heart and until time T_0 .

The rest of this section is devoted to the proof of Lemma 32.

Proof. The proof is splitted into three steps.

Step 1 : Blurring algorithm. Our first goal is to deteriorate the lifetime function τ (non-local information) into a **blurred lifetime function** τ^b (local information), leading to a scenario \mathbf{s}_{τ^b} .

Let $x_0, x_1, \dots, x_K, T_0, N$ as before. Given also a configuration $\bar{S} \in \bar{\mathcal{S}}$, a blurred lifetime function $\tilde{\tau}$ is defined as a function

$$\tilde{\tau} : S \rightarrow [0, T_0] \cup \{I_1, \dots, I_N\} \cup \{\emptyset\} .$$

Each of the possible values of $\tilde{\tau}(x)$, with $x \in S$, will be interpreted as follows:

- $\tilde{\tau}(x) \in [0, T_0]$: the grain x stops growing at time $\tilde{\tau}(x)$.
- $\tilde{\tau}(x) = I_k$ for some k : the grain x stops growing at some *unknown* time in the interval I_k . It is blurred.
- $\tilde{\tau}(x) = \emptyset$: the grain x is discarded.

Let $\bar{S} \in \bar{\mathcal{S}}$ be a tempered configuration and let τ be its lifetime function. **The Blurring algorithm** allows to blur the lifetime function τ around the grain x_0 and until time T_0 , leading to a blurred lifetime function denoted by τ^b .

Start by setting $\tau^b = \tau$. We will then change the values of some of the $\tau^b(x)$ by performing the following steps. We first need some notations. If $t \in [0, T_0]$, $\text{blur}(t)$ is defined as the unique interval I_k containing t . We define the blurred region by

$$\text{blurred-region}(\bar{S}, \tau^b, x_0, T_0, N) := (g_{x_0}([0, \tau^\sharp(x_0)]) + B(6))^c \cup \left(\bigcup_{x \in S: \tau^b(x) \in \{I_1, \dots, I_N\}} g_x(\tau^b(x)) \right) .$$

Now we perform the algorithm to deteriorate τ^b . The first two steps below reduces the algorithm to grains of $B^\sharp(x_0, 1)$ and to the time interval $[0, T_0]$. The third steps is repeated until the algorithm stop.

1. For all $x \notin B^\sharp(x_0, 1)$, set $\tau^b(x) = \emptyset$.
2. For all $x \in B^\sharp(x_0, 1)$ with $\tau(x) \geq T_0$, set $\tau^b(x) = T_0$.
3. Consider the set of all $x \in B^\sharp(x_0, 1)$ such that $\tau^b(x)$ is a real number in $[0, T_0)$ and $g_x(\tau^b(x)) \in \text{blurred-region}(\bar{S}, \tau^b, x_0, T_0, N)$.
 - If this set is not empty:
 - For each x in this set, blur $\tau^b(x)$ by setting $\tau^b(x) = \text{blur}(\tau^b(x))$.

– Go back to Step 3.

- If this set is empty, the algorithm stops.

When Step 3 is realized for the first time, the blurred region is equal to $(g_{x_0}([0, \tau^\sharp(x_0)]) + B(6))^c$ and at each new Step 3, it increases by adding blurred parts of grains, overlapping the current blurred region. Hence, when the blurred algorithm stops, any connected component of the set

$$\bigcup_{i \leq K: \tau^b(x_i) \in \{I_1, \dots, I_N\}} g_{x_i}(\tau^b(x_i))$$

overlaps $(g_{x_0}([0, \tau^\sharp(x_0)]) + B(6))^c$. See Fig. 6.

Besides, the blurred lifetime function τ^b provides a unique scenario \mathbf{s}_{τ^b} to x_0, x_1, \dots, x_K and T_0 , indicating which are the blurred grains (among the x_i 's) and what is their blurred part. More formally, $\mathbf{s}_{\tau^b} = (J, (I(j) : j \in J))$ with

$$J = \{j \in \{0, \dots, K\} : \tau^b(x_j) \text{ is a time interval}\} \text{ and for all } j \in J, I(j) = \tau^b(x_j) .$$

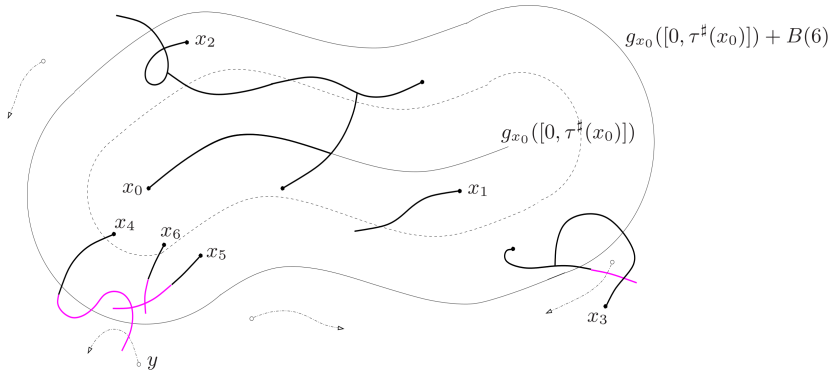


Figure 6: Here is the blurred configuration associated to the scenario \mathbf{s}_{τ^b} , itself provided by the blurred lifetime function τ^b . The dotted trajectories (as y) are some discarded grains, i.e. with $\tau^b(y) = \emptyset$, by Step 1 of the blurred algorithm. The pink pieces are the blurred parts of grains. Some grains, as x_1 and x_2 , are stopped at time T_0 by Step 2. In the whole configuration $\bar{\xi}$, the grain x_4 is stopped before T_0 and outside the set $(g_{x_0}([0, \tau^\sharp(x_0)]) + B(6))$ by y . Because y is discarded, this information is lost. The role of the blurred part of the grain x_4 is to encode this lack of information. Since, we do not know when x_4 is stopped (before or after crossing x_5 ?), this uncertainty is extended to x_5 , then x_6 during Step 3.

Step 2. Let us prove that the modified dynamics generates from the scenario \mathbf{s}_{τ^b} , a blurred configuration which is consistent with the blurred lifetime function τ^b . In other words, the blurred lifetime function implicitly given by the modified dynamics with scenario \mathbf{s}_{τ^b} is τ^b .

Since the list of discarded grains is the same for the scenario \mathbf{s}_{τ^b} and the blurred lifetime function τ^b , we only need to focus on the grains x_0, x_1, \dots, x_K . Let us consider the blurred configuration provided by the modified dynamics from the scenario \mathbf{s}_{τ^b} and the grains (x_i, g_{x_i}) . Let us call the cleaned configuration the previous blurred configuration without its blurred parts. The key remark is that this cleaned configuration is also provided by the Dynamical algorithm (defined in Section 2.1.3) from the modified grains (x_i, f_{x_i}) where the trajectories f_{x_i} are defined as follows. For any index i such that $\tau^b(x_i) = T_0$, we set $t(i) := T_0$. For any i such that $\tau^b(x_i) \in \{I_1, \dots, I_N\}$, we set $t(i) := \inf \tau^b(x_i)$. In both cases, we define

$$f_{x_i}(t) := g_{x_i}(t) \mathbf{1}_{t \leq t(i)} + g_{x_i}(t(i)) \mathbf{1}_{t > t(i)} ,$$

Otherwise, $f_{x_i} := g_{x_i}$. Now, the Reconciliation lemma (Lemma 3) asserts that the Dynamical algorithm provides a unique grain configuration– which is the cleaned configuration –with a unique lifetime function, say τ_0 . Hence, the lifetime function τ_0 satisfies $\tau_0(x_i) = t(i)$ whenever $\tau^b(x_i)$ equals to T_0 or one of the intervals I_1, \dots, I_N , and $\tau_0(x_i) = \tau^b(x_i)$ otherwise. Finally, adding the blurred parts to that unique

grain configuration and the values I_1, \dots, I_N and \emptyset to the lifetime function τ_0 , we just have proved that to the blurred configuration (associated to \mathbf{s}_{τ^b}), it corresponds a unique blurred lifetime function which is τ^b .

Step 3: Let us prove that the scenario \mathbf{s}_{τ^b} is admissible. This means that the set of blurred parts

$$\bigcup_{x \in B^\sharp(x_0, 1) : \tau^b(x) \in \{I_1, \dots, I_N\}} g_x(\tau^b(x)) \quad (33)$$

does not overlap $\text{Heart} = g_{x_0}([0, \tau^\sharp(x_0)]) + B(1)$. Recall that the scenario \mathbf{s}_{τ^b} is associated to the blurred lifetime function τ^b .

Thanks to (31), each blurred part $g_{x_i}(\tau^b(x_i))$ is included in a ball with diameter $\frac{1}{K+1}$. Moreover each connected component D of the set given in (33) is made up with at most $K+1$ such blurred parts (since each grain is blurred at most one time). So D is included in a ball with diameter 1. Combining with the fact that D overlaps $(g_{x_0}([0, \tau^\sharp(x_0)]) + B(6))^c$ by construction of the Blurring algorithm, we get the result.

Let us now conclude. Recall that $C(\mathbf{s}_{\tau^b})$ denotes the union of (possibly blurred) grains of the blurred configuration provided by the modified dynamics from the scenario \mathbf{s}_{τ^b} . Let us write:

$$\text{Heart} \cap \left(\bigcup_{x \in \xi} g_x([0, \tau(x) \wedge T_0]) \right) = \text{Heart} \cap \left(\bigcup_{x \in B^\sharp(x_0, 1)} g_x([0, \tau(x) \wedge T_0]) \right) = \text{Heart} \cap C(\mathbf{s}_{\tau^b}). \quad (34)$$

The second equality of (34) is due to Steps 2 and 3. The true configuration of grains around x_0 and described by the lifetime function τ is corrupted into a blurred configuration obtained from τ^b and corresponding to the scenario \mathbf{s}_{τ^b} thanks to Step 2. Hence, the true configuration and the blurred one, i.e. $C(\mathbf{s}_{\tau^b})$, differ only on the set of blurred parts given in (33). By Step 3, the scenario \mathbf{s}_{τ^b} is admissible which means that these blurred parts are outside Heart . As a consequence, the true configuration and the blurred one coincide on Heart . This achieves the proof of Lemma 32. ■

5.2 The dominant time $\mathbb{T}(x_0)$

Up to a finite number of admissible scenarios, we know what happens around the grain x_0 until time T_0 . The goal of this section is to ensure that it is possible to choose the time T_0 in a local way and large enough so that, up to time T_0 , we have enough information to place a loop stopping x_0 without reducing its backward. We call this time the **dominant time** $\mathbb{T}(x_0)$. The basic idea is that $\mathbb{T}(x_0)$ depends on a suitable \sharp -neighbourhood of x_0 and provides an upper bound on all possible impact between grains in a relevant \sharp -neighbourhood of x_0 .

Remark that for the whole Section 5.2, no hypothesis is required about the grain x_0 (such as stopped in a regular way or with a finite lifetime $\tau(x_0)$).

Proposition 33. *Let $x_0 \in \xi$. Then there a.s. exists $\mathbb{T}(x_0) < \infty$, called the dominant time for x_0 , such that:*

(i) *Domination: For any $y \in B^\sharp(x_0, 2)$ such that $\tau(y) < \infty$ then $\tau(y) \leq \mathbb{T}(x_0)$.*

(ii) *Locality: $\mathbb{T}(x_0)$ only depends on grains (z, g_z) with $z \in B^\sharp(x_0, 4)$.*

Note that Item (i) above concerns not only the elements of $B^\sharp(x_0, 1)$, i.e. the x 's such that $x \sim^\sharp x_0$, but all those in $B^\sharp(x_0, 2)$. We will see why in Section 5.3.3. Determining the dominant time $\mathbb{T}(x_0)$ in a local way (Item (ii)) will be needed so that the \sharp -goodness is a local notion.

Proposition 33 is an immediate consequence of the next result.

Lemma 34. *Let $y \in \xi$. Then there a.s. exists $T(y) \in [0, +\infty]$ only depending on grains (z, g_z) with $z \in B^\sharp(y, 2)$ and such that:*

$$\tau(y) < +\infty \Rightarrow \tau(y) \leq T(y) < +\infty.$$

Proof of Proposition 33. Let $x_0 \in \xi$. The set $\{y \in \xi : y \in B^\sharp(x_0, 2) \text{ and } T(y) < +\infty\}$ is a.s. finite by Proposition 18. If it is empty then we set $\mathbb{T}(x_0) := 0$. Otherwise

$$\mathbb{T}(x_0) := \max\{T(y) : y \in B^\sharp(x_0, 2) \text{ and } T(y) < +\infty\}$$

is well defined and finite. By Lemma 34, it clearly depends only on grains (z, g_z) with $z \in B^\sharp(x_0, 4)$. Moreover, for any $y \in B^\sharp(x_0, 2)$ with $\tau(y) < \infty$, Lemma 34 asserts that $\tau(y) \leq T(y)$ which is itself smaller than $\mathbb{T}(x_0)$ by definition. ■

Proof of Lemma 34. Let $y \in \xi$ such that $\tau(y) < +\infty$. By the Hitting property, there exists $z \in \xi$, $z \neq y$, such that y hits z (at time $\tau(y)$). But knowing precisely who is the stopping grain z of y is a non-local information. We only know that it belongs to $B^\sharp(y, 1)$. We are going to prove:

Claim: For all $z \in B^\sharp(y, 1)$, there exists $T(y, z) \in [0, +\infty]$ depending only on grains of $B^\sharp(y, 2)$ such that “ y hits z ” implies $\tau(y) \leq T(y, z) < +\infty$.

We conclude easily from the Claim with setting

$$T(y) := \sup \{T(y, z) : z \in B^\sharp(y, 1) \text{ and } T(y, z) < +\infty\} .$$

First, the variable $T(y)$ only depends on grains of $B^\sharp(y, 2)$. Moreover, if $\tau(y) < +\infty$ then y hits some z in $B^\sharp(y, 1)$ and, by the Claim, the variable $T(y, z)$ is finite and bounds $\tau(y)$. The set over which the supremum $T(y)$ is taken is then non-empty. It is also finite by Proposition 18. So $\tau(y) \leq T(y, z) \leq T(y) < +\infty$.

Hence our goal is to prove the Claim. Let $z \in B^\sharp(y, 1)$ and assume that y hits z . Let us consider

$$t_{z,y} := \inf\{t \geq 0 : g_z(t) \in g_y(\mathbb{R}_+)\}$$

which is finite since y hits z . First remark that $t_{z,y} \leq \tau(z)$: the grain z is still alive at time $t_{z,y}$. Otherwise $\tau(z) < t_{z,y}$ and $g_z([0, \tau(z)])$ would not overlap $g_y(\mathbb{R}_+)$ which would contradict that y hits z . Two cases must be distinguished depending on whether the infimum $t_{z,y}$ is reached or not. At a first stage, the reader may only focus on Case 1. The second case requires more work.

Case 1: The infimum $t_{z,y}$ is reached, i.e. $g_z(t_{z,y}) \in g_y(\mathbb{R}_+)$. Let us introduce the first passage time of the trajectory g_y at $g_z(t_{z,y})$:

$$T_{z,y}^{(1)} := \inf\{t \geq 0 : g_y(t) = g_z(t_{z,y})\} .$$

By construction, $T_{z,y}^{(1)}$ is finite and reached. If $T_{z,y}^{(1)} > t_{z,y}$, i.e. the trajectory g_z visits $g_z(t_{z,y})$ before g_y , then $\tau(y) \leq T_{z,y}^{(1)}$ by the Stopping property. On the contrary, if $T_{z,y}^{(1)} \leq t_{z,y}$, the trajectory g_y visits $g_z(t_{z,y})$ before g_z . We then have $T_{z,y}^{(1)} \leq \tau(y)$. Otherwise $\tau(y) < T_{z,y}^{(1)} \leq t_{z,y} \leq \tau(z)$ would mean that the grain y is stopped before arriving at $g_z(t_{z,y})$: $g_y([0, \tau(y)])$ and $g_z([0, \tau(y)])$ would be disjoint that would contradict the hypothesis “ y hits z ”. From $T_{z,y}^{(1)} \leq \tau(y)$ combined with $g_z(t_{z,y}) = g_y(T_{z,y}^{(1)})$, $t_{z,y} \leq \tau(z)$ and $T_{z,y}^{(1)} \leq t_{z,y}$, the Stopping property asserts that $t_{z,y} = \tau(z)$, i.e. the grain z is stopped by y at time $t_{z,y}$. The equality $\tau(z) = t_{z,y}$ yields, by definition of $t_{z,y}$, that if y hits z it happens at the point $g_z(t_{z,y})$. Therefore, “ y hits z ” forces the grain y to visit $g_z(t_{z,y})$ at some time $t \geq t_{z,y}$ to be stopped. As a consequence, in this case,

$$\tau(y) = T_{z,y}^{(2)} := \inf\{t \geq t_{z,y} : g_y(t) = g_z(t_{z,y})\} < +\infty .$$

To sum up, in Case 1, the lifetime $\tau(y)$ is a.s. bounded by

$$T(y, z) := T_{z,y}^{(1)} \mathbf{1}_{T_{z,y}^{(1)} > t_{z,y}} + T_{z,y}^{(2)} \mathbf{1}_{T_{z,y}^{(1)} \leq t_{z,y}}$$

which is finite and only depends on grains (z, g_z) and (y, g_y) .

Case 2: The infimum $t_{z,y}$ is not reached. In this case, we have $\tau(z) > t_{z,y}$. Otherwise $\tau(z) \leq t_{z,y}$ would mean that $g_z([0, \tau(z)])$ and $g_y(\mathbb{R}_+)$ are disjoint since $t_{z,y}$ is not reached, and this would contradict the fact that y hits z . For the same reason, we know that $g_z([t_{z,y}, \tau(z)])$ is not reduced to the singleton $\{g_z(t_{z,y})\}$, i.e. the grain z continues to grow after time $t_{z,y}$. Actually, we need a local version of what precedes:

$$\begin{aligned} \text{A.s. } \exists \delta > 0 \text{ only depending on grains } (z', g_{z'}) \text{ with } z' \in B^\sharp(z, 1) \\ \text{such that } \tau(z) \geq t_{z,y} + \delta \text{ and } g_z(t_{z,y} + \delta) \neq g_z(t_{z,y}). \end{aligned} \tag{35}$$

In what follows, we first prove Statement (35) by a blurring approach and thus how it allows to conclude.

It could be possible that the grain z remains stuck at $g_z(t_{z,y})$ before restarting. To take into account this possibility in our proof, we have to introduce the time $t'_{z,y}$ at which the trajectory g_z leaves the point $g_z(t_{z,y})$: $t'_{z,y} := \sup\{t' \geq t_{z,y} : g_z([t_{z,y}, t']) = \{g_z(t_{z,y})\}\}$. As $t_{z,y}$, we know that $t'_{z,y}$ is finite and strictly smaller than $\tau(z)$. At first reading, the reader may assume that $t'_{z,y} = t_{z,y}$.

Let us use a blurring approach around the grain z until time $t'_{z,y}$ (instead of x_0 and T_0 w.r.t. Section 5.1). By Proposition 18, the set $B^\sharp(z, 1)$ is a.s. finite, say $z'_0 = z, z'_1, \dots, z'_K$ be its elements, with K random and finite. Thus, by uniform continuity of the $g_{z'_i}$'s on $[0, t'_{z,y}]$, we choose an integer $N' > 0$ such that, for any $0 \leq i \leq K$ and for any $t, t' \in [0, t'_{z,y}]$ with $|t - t'| \leq t'_{z,y}/N'$, one has $|g_{z'_i}(t) - g_{z'_i}(t')| \leq (K + 1)^{-1}$. The time interval $[0, t'_{z,y}]$ is then splitted into the disjoint intervals $I_1, \dots, I_{N'}$. So, any scenario $\mathbf{s} = (J, (I(j) : j \in J))$, where $J \subset \{0, 1, \dots, K\}$ and for any $j \in J$, $I(j)$ is taken among the I_k 's, generates a blurred configuration of grains z'_0, z'_1, \dots, z'_K using the modified dynamics. As before, the number of admissible scenarios is a.s. finite and, among them, there is the one corresponding to the blurred lifetime function τ^b (thanks to the choice of N') and then to the true grain configuration inside $g_z([0, \tau^\sharp(z)]) + B(1)$.

Let us consider an admissible scenario \mathbf{s} according to which the grain z has been stopped at time $t'_{z,y}$ (artificially by the modified dynamics) and, in the corresponding blurred configuration, any other grain of $B^\sharp(z, 1)$ is at positive distance from $g_z(t'_{z,y})$. Let \mathbf{S} be this set of scenarios. Recall that the true scenario \mathbf{s}_{τ^b} belongs to \mathbf{S} . Now, let us restart the growth of grains of $B^\sharp(z, 1)$ which have been stopped at time $t'_{z,y}$ according to \mathbf{s} . Then, by continuity of trajectories, there exists $\delta(\mathbf{s}) > 0$ small enough such that $g_z([0, t'_{z,y} + \delta(\mathbf{s})])$ does not intersect any of the sets $g_{z'}([0, t'_{z,y} + \delta(\mathbf{s})])$ for z' stopped at time $t'_{z,y}$. The parameter $\delta(\mathbf{s}) > 0$ can be chosen small enough so that $g_z([0, t'_{z,y} + \delta(\mathbf{s})])$ also does not intersect the grains stopped before $t'_{z,y}$ or blurred according to the scenario \mathbf{s} (since \mathbf{s} is admissible). Roughly speaking, according to the scenario \mathbf{s} , the grain z is still alive at time $t'_{z,y} + \delta(\mathbf{s})$. By definition of $t'_{z,y}$, decreasing $\delta(\mathbf{s}) > 0$ if it is needed, one can also assume that $g_z(t'_{z,y} + \delta(\mathbf{s})) \neq g_z(t'_{z,y})$.

Let $\bar{\delta} := \min \delta(\mathbf{s}) > 0$ where the minimum is taken over the finite set \mathbf{S} . Then,

$$\tau(z) \geq t'_{z,y} + \delta(\mathbf{s}_{\tau^b}) \geq t'_{z,y} + \bar{\delta} = t_{z,y} + \delta$$

where $\delta := \bar{\delta} + t'_{z,y} - t_{z,y} > 0$ and

$$g_z(t_{z,y} + \delta) = g_z(t'_{z,y} + \bar{\delta}) \neq g_z(t'_{z,y}) = g_z(t_{z,y}).$$

Moreover the construction of δ (and $\bar{\delta}$) only depends on time $t_{z,y}, t'_{z,y}$ and on grains $(z', g_{z'})$ with $z' \in B^\sharp(z, 1)$.

It remains to explain how Statement (35) implies the result. Because the infimum $t_{z,y}$ is not reached and by definition of δ , there exist s_0, s_1 such that $t_{z,y} < s_0 < t_{z,y} + \delta$ and $g_z(s_0) = g_y(s_1)$. By continuity of trajectories g_y, g_z , we can also require that $s_1 > t_{z,y} + \delta$. Henceforth

$$T_{z,y}^{(3)} := \inf\{t \geq t_{z,y} + \delta : g_y(t) \in g_z([t_{z,y}, t_{z,y} + \delta])\}$$

is well defined and finite (it is also reached). Since the grain z is still alive at time $t_{z,y} + \delta$, we necessarily have $\tau(y) \leq T_{z,y}^{(3)}$ by the Stopping property. Let us add that $T_{z,y}^{(3)}$ only depends on grains (y, g_y) and (z, g_z) , and on the parameter δ , i.e. only on grains $z' \in B^\sharp(y, 2)$. In conclusion, $T(y, z) := T_{z,y}^{(3)}$ satisfies the Claim. ■

5.3 Proof of Proposition 19

5.3.1 All grains are \sharp -good points

Let $x_0 \in \xi$. Since we only want to use local information, we have to forget the true configuration around x_0 and work with admissible scenarios and blurred configurations. So, let us use a blurred approach around the grain x_0 and until the dominant time $\mathbb{T}(x_0)$, involving grains of $B^\sharp(x_0, 1)$: as before they

are denoted by x_0, x_1, \dots, x_K . We then pick the integer N as in (31) but this time w.r.t. the vertices x_0, x_1, \dots, x_K and the dominant time $\mathbb{T}(x_0)$: N is the smallest integer such that

$$\forall 0 \leq i \leq K \text{ and } \forall t, t' \in [0, \mathbb{T}(x_0)] \text{ with } |t - t'| \leq \frac{\mathbb{T}(x_0)}{N}, \text{ we have } |g_{x_i}(t) - g_{x_i}(t')| \leq \frac{1}{K+1}. \quad (36)$$

In the sequel, we focus our attention on the (finite) set \mathbf{S} of admissible scenarios (w.r.t. x_0, x_1, \dots, x_K and $\mathbb{T}(x_0)$) in which x_0 is stopped in a regular way at time $0 < t(\mathbf{s}, x_0) \leq \tau^\sharp(x_0)$. Let us point out here that we do not assume that x_0 is stopped in a regular way in the true grain configuration or that x_0 has a finite lifetime ($\tau(x_0) = \tau^\sharp(x_0) = +\infty$ is allowed). The set \mathbf{S} could be empty. Besides, when $\mathbf{S} \neq \emptyset$, either x_0 is not stopped in a regular way in the true grain configuration and then none of the scenarios in \mathbf{S} correspond to reality. Or x_0 is stopped in a regular way and then \mathbf{S} contains the scenario \mathbf{s}_{τ^b} provided by the blurred lifetime function τ^b . However, with only local informations, we cannot distinguish these two alternatives, nor the the scenario \mathbf{s}_{τ^b} in \mathbf{S} if there is. This is why we focus our efforts on all the scenarios of the set \mathbf{S} .

Let $\mathbf{s} \in \mathbf{S}$ such a scenario. Our strategy consists in determining a small ball $B(v, 2\alpha)$ located *just before* the impact of x_0 on its stopping grain and overlapping no other grains (for the blurred configuration corresponding to \mathbf{s}). To do it, we need extra notations. Let $i(\mathbf{s})$ be the impact point of x_0 on its stopping grain. Since x_0 is stopped in a regular way (for the scenario \mathbf{s}), it passes only one time through $i(\mathbf{s})$. Hence,

$$t(\mathbf{s}, x_0) = \inf\{t : g_{x_0}(t) = i(\mathbf{s})\} < +\infty.$$

Since the scenario \mathbf{s} is admissible, there is no blurred parts inside $g_{x_0}([0, t(\mathbf{s}, x_0)]) + B(1)$ included in Heart since $t(\mathbf{s}, x_0) \leq \tau^\sharp(x_0)$. We can then read on the blurred configuration the impact points on $g_{x_0}([0, t(\mathbf{s}, x_0)])$ created by some of the x_i 's. Each impact point on $g_{x_0}([0, t(\mathbf{s}, x_0)]) = g_{x_0}([0, t(\mathbf{s}, x_0)]) \setminus \{i(\mathbf{s})\}$ is created by a x_i which is stopped in a regular way by x_0 and then belongs to the Backward set of x_0 . Nothing prevents the occurrence of an impact point exactly at $i(\mathbf{s})$. But the corresponding stop is not regular by definition and the grain creating this impact point is not in the Backward set of x_0 . See Fig. 7. Let us denote by $i^-(\mathbf{s})$ the last impact point— when the grain x_0 is browsed from time 0 to $t(\mathbf{s}, x_0)$ — on $g_{x_0}([0, t(\mathbf{s}, x_0)])$. We set $i^-(\mathbf{s}) := x_0$ if there is no such impact points. Notice that $i^-(\mathbf{s})$ is necessarily different from $i(\mathbf{s})$, otherwise the grain creating this last impact point would not be stopped in a regular way by x_0 (or because $t(\mathbf{s}, x_0) > 0$ in the case where $i^-(\mathbf{s}) = x_0$). Let us finally define $t^-(\mathbf{s}, x_0)$ as the last passage time before $t(\mathbf{s}, x_0)$ of the grain x_0 at $i^-(\mathbf{s})$:

$$t^-(\mathbf{s}, x_0) := \sup\{t < t(\mathbf{s}, x_0) : g_{x_0}(t) = i^-(\mathbf{s})\}.$$

By construction, $i(\mathbf{s})$ does not belong to the compact set $g_{x_0}([0, t^-(\mathbf{s}, x_0)])$: $i(\mathbf{s})$ is at positive distance to $g_{x_0}([0, t^-(\mathbf{s}, x_0)])$. We can then choose

$$v(\mathbf{s}) \in g_{x_0}([0, t(\mathbf{s}, x_0)]) \setminus g_{x_0}([0, t^-(\mathbf{s}, x_0)]) \subset G^\sharp(x_0)$$

(since $t(\mathbf{s}, x_0) \leq \tau^\sharp(x_0)$) and $0 < \alpha < 1/2$ small enough such that $B(v(\mathbf{s}), 2\alpha)$ avoids $g_{x_0}([0, t^-(\mathbf{s}, x_0)])$ and $i(\mathbf{s})$ (Items (a) and (b) of Definition 35 below).

For $1 \leq i \leq K$, let $G(\mathbf{s}, x_i)$ be the (possibly partially blurred) grain of x_i in the blurred configuration corresponding to the scenario \mathbf{s} . By construction, $v(\mathbf{s})$ does not belong to the compact set $\cup_{i \leq K} G(\mathbf{s}, x_i)$ (all these grains are compact sets since they are at the latest stopped at time $\mathbb{T}(x_0)$). Hence, $\alpha > 0$ can be chosen small enough so that $B(v(\mathbf{s}), 2\alpha)$ avoids $\cup_{i \leq K} G(\mathbf{s}, x_i)$. This is Item (c) of Definition 35.

Since x_0 does not belong to $B(v(\mathbf{s}), 2\alpha)$, the parameter $\alpha > 0$ can be further reduced so that $\inf\{t : g_{x_0}(t) \in B(v(\mathbf{s}), 2\alpha)\}$ is larger than α (Item (d) of Definition 35).

Clearly, the value $\alpha = \alpha(\mathbf{s}) > 0$ previously obtained depends on the scenario \mathbf{s} . Let us minimize $\alpha(\mathbf{s})$ over the (finite) set of scenarios in \mathbf{S} :

$$\alpha^* := \min_{\mathbf{s} \in \mathbf{S}} \alpha(\mathbf{s}) > 0.$$

In a last step, we know by Proposition 18 that a.s. any \sharp -grain is bounded (but not necessarily stopped). Henceforth, the radius

$$R_{x_0}^\sharp := \inf\{R > 0 : G_{x_0}^\sharp \subset B(x_0, R)\}$$

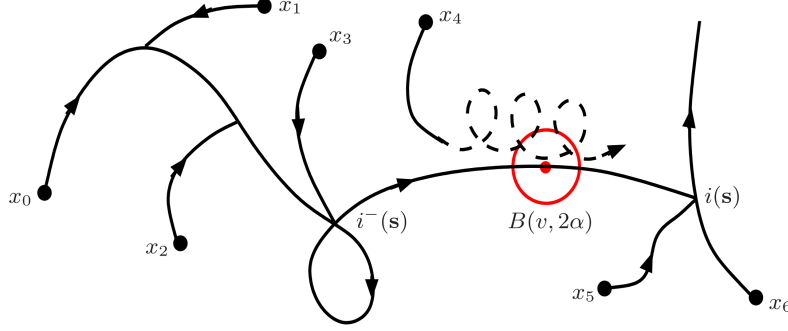


Figure 7: On this picture x_0 is stopped in a regular way by x_6 at $i(\mathbf{s})$. x_1 , x_2 and x_3 are all stopped by x_0 in a regular way, but not x_5 (actually x_5 is stopped in a regular way by x_6 and is in its Backward set). The small ball $B(v, 2\alpha)$ (in red) has to be located between $i^-(\mathbf{s})$ and $i(\mathbf{s})$. Let us point out the presence of the grain x_4 , with $\tau(x_4) = \infty$, which is non-compact and tangent to the grain x_0 . By construction of the dominant time $\mathbb{T}(x_0)$, the piece of trajectory $g_{x_4}((\mathbb{T}(x_0), +\infty))$ (the dotted curve) is irrelevant and can then be ignored without reducing the backward set of x_0 . This is why $B(v, 2\alpha)$ may overlap $g_{x_4}((\mathbb{T}(x_0), +\infty))$.

of the \sharp -grain $G_{x_0}^\sharp$ is a.s. finite. We can then choose $\alpha^* > 0$ sufficiently small such that $R_{x_0}^\sharp \leq 1/\alpha^*$.

We are now ready to introduce the notion of \sharp -good point.

Definition 35. Let $0 < \alpha < 1/2$ and $x_0 \in \xi$. Let us denote by x_0, x_1, \dots, x_K the elements of the (almost surely finite) set $B^\sharp(x_0, 1)$. The point x_0 is said (α, \sharp) -good if $R_{x_0}^\sharp \leq \alpha^{-1}$ and, for any scenario \mathbf{s} (resp. to x_0, x_1, \dots, x_K and the dominant time $\mathbb{T}(x_0)$) in \mathbf{S} (previously defined in this section), there exists a ball $B(v(\mathbf{s}), 2\alpha)$ satisfying the following properties:

- (a) $v(\mathbf{s}) \in g_{x_0}([0, t(\mathbf{s}, x_0))) \setminus g_{x_0}([0, t^-(\mathbf{s}, x_0)]) \subset G^\sharp(x_0)$.
- (b) $B(v(\mathbf{s}), 2\alpha)$ avoids the set $g_{x_0}([0, t^-(\mathbf{s}, x_0)]) \cup \{i(\mathbf{s})\}$.
- (c) $B(v(\mathbf{s}), 2\alpha)$ avoids all the (possibly blurred) grains $G(\mathbf{s}, x_i)$ of the blurred configuration corresponding to the scenario \mathbf{s} .
- (d) $\inf\{t : g_{x_0}(t) \in B(v(\mathbf{s}), 2\alpha)\} \geq \alpha$.

Finally, we denote by $\text{Good}^\sharp(\alpha) \subset \{\bar{S} \in \bar{\mathcal{S}} : 0 \in S\}$ the following event: $\bar{\xi} - x_0$ belongs to $\text{Good}^\sharp(\alpha)$ iff x_0 is an (α, \sharp) -good point.

In the case where the set \mathbf{S} is empty, it becomes easy to be an (α, \sharp) -good point since it suffices to check if $R_{x_0}^\sharp \leq \alpha^{-1}$. However, the arguments developed before Definition 35 prove that any $x_0 \in \xi$ is (α^*, \sharp) -good for the (random) value $\alpha^* = \alpha^*(x_0)$ previously obtained. This is the fourth item of Proposition 19:

Proposition 36. A.s. any $x_0 \in \xi$ is a (α, \sharp) -good point for α small enough.

Whereas the radius of the ball $B(v(\mathbf{s}), 2\alpha)$ is the same for any scenario considered in Definition 35 and is locally determined, it is not the case for its center $v(\mathbf{s})$, which may then depend on the scenario \mathbf{s} . This means that our strategy does not allow us to specify locally where the suitable ball $B(v, 2\alpha)$ is located.

5.3.2 The \sharp -goodness is a local notion

Unlike the notion of α -good point, the one of (α, \sharp) -good point is local in the following sense. This is the third item of Proposition 19.

Proposition 37. *Let $0 < \alpha < 1/2$ and $x_0 \in \xi$. The property “ x_0 is an (α, \sharp) -good point” only depends on grains $(z, g_z) \in \bar{\xi}$ such that $z \in B^\sharp(x_0, 4)$.*

Proof. Let $x_0 \in \xi$. We develop a blurred approach respectively to the elements of $B^\sharp(x_0, 1)$ and the dominant time $\mathbb{T}(x_0)$. By Proposition 33, $\mathbb{T}(x_0)$ only depends on grains of $B^\sharp(x_0, 4)$. The same holds for the choice of N in (36) depending on g_{x_i} , with $x_i \in B^\sharp(x_0, 1)$, and on $\mathbb{T}(x_0)$. This leads to a finite number of scenarios. For each scenario \mathbf{s} in the set \mathbf{S} (defined in the previous section), the locations $i(\mathbf{s}), i^-(\mathbf{s})$, the times $t(\mathbf{s}, x_0), t^-(\mathbf{s}, x_0)$ and the grains $G(\mathbf{s}, x_i)$ (also defined in the previous section) only depend on grains of $B^\sharp(x_0, 1)$ and $\mathbb{T}(x_0)$. These quantities allow to check if it is possible or not to find a ball $B(v(\mathbf{s}), 2\alpha)$ satisfying Items (a) – (d) of Definition 35. Finally, let us add that the radius $R_{x_0}^\sharp$ also depends on $B^\sharp(x_0, 1)$. ■

5.3.3 \sharp -good implies good

Let us prove the second item of Proposition 19.

Proposition 38. *Let $0 < \alpha < 1/2$. A.s. any (α, \sharp) -good point stopped in a regular way is also an α -good point.*

Proof. Let $x_0 \in \xi$ which is **stopped in a regular way**. Respectively to the elements x_0, x_1, \dots, x_K of $B^\sharp(x_0, 1)$ and the dominant time $\mathbb{T}(x_0)$, the Blurring algorithm deteriorates the lifetime function τ into the blurring lifetime function τ^b (see Section 5.1). Associated to τ^b , we consider the scenario \mathbf{s}_{τ^b} which is admissible thanks to the choice of N in (36). Since the corresponding blurred configuration coincides with the true configuration inside $\text{Heart} = g_{x_0}([0, \tau^\sharp(x_0)]) + B(1)$ thanks to Lemma 32, then x_0 is still stopped in a regular way in the blurred configuration (w.r.t. \mathbf{s}_{τ^b}) and $t(\mathbf{s}_{\tau^b}, x_0) = \tau(x_0) \leq \tau^\sharp(x_0) \wedge \mathbb{T}(x_0)$ (recall that $\tau(x_0) < \infty$ implies $\tau(x_0) \leq \mathbb{T}(x_0)$ by Proposition 33). Hence, the scenario \mathbf{s}_{τ^b} belongs to the set \mathbf{S} defined in Section 5.3.1. By assumption, x_0 is an (α, \sharp) -good point, i.e. $R_x^\sharp \leq \alpha^{-1}$ and there exists a ball $B(v(\mathbf{s}_{\tau^b}), 2\alpha)$ satisfying Items (a) – (d) of Definition 35. We can first deduce the existence of a deterministic $u \in U(\alpha)$ such that

$$v(\mathbf{s}_{\tau^b}) \in B(x_0 + u, r(\alpha)) \subset B(x_0 + u, \alpha) \subset B(v(\mathbf{s}_{\tau^b}), 2\alpha) .$$

We are going to prove that x_0 is an (α, u) -good point (in $\bar{\xi}$) which means that x_0 is an α -good point, then proving Proposition 38.

Let us resample the PPP $\bar{\xi}$ in $B(x_0 + u, \alpha)$. Let $\bar{\xi}'$ be an independent copy of $\bar{\xi}$ and set

$$r_{x_0}(\bar{\xi}, \bar{\xi}') := (\bar{\xi} \cap B(x_0 + u, \alpha)^c) \cup (\bar{\xi}' \cap B(x_0 + u, \alpha)) .$$

Moreover let us assume that $\bar{\xi}' \in x_0 + u + \mathcal{L}(\alpha)$: for the configuration $\bar{\xi}' \cap B(x_0 + u, \alpha)$, a small loop is created inside $B(x_0 + u, \alpha)$ and surrounding $B(x_0 + u, r(\alpha))$. We have to investigate which grains are modified by this resampling; we inventory which grains of $\bar{\xi}$ may visit the ball $B(v(\mathbf{s}_{\tau^b}), 2\alpha)$ and at what time such visits occur.

First remark that, by Item (b) – (c), the ball $B(v(\mathbf{s}_{\tau^b}), 2\alpha)$ avoids the point process ξ : no grains of $\bar{\xi}$ are deleted by the resampling. Hence, replacing $\bar{\xi}$ with $r_{x_0}(\bar{\xi}, \bar{\xi}')$ adds only some points inside $B(x_0 + u, \alpha)$ (i.e. those of $\bar{\xi}' \cap B(x_0 + u, \alpha)$). Let us gather various facts:

- Since $\bar{\xi}' \in x_0 + u + \mathcal{L}(\alpha)$, the new grains provided by $\bar{\xi}' \cap B(x_0 + u, \alpha)$ are stopped before time α and remains completely included in $B(x_0 + u, \alpha) \subset B(v(\mathbf{s}_{\tau^b}), 2\alpha)$.
- By Item (d), the trajectory g_{x_0} visits for the first time $B(v(\mathbf{s}_{\tau^b}), 2\alpha)$ at time

$$t_0 := \inf\{t : g_{x_0}(t) \in B(v(\mathbf{s}_{\tau^b}), 2\alpha)\} \geq \alpha .$$

- By Item (c), no grains of $B^\sharp(x_0, 1) \setminus \{x_0\}$ may visit $B(v(\mathbf{s}_{\tau^b}), 2\alpha)$ before the dominant time $\mathbb{T}(x_0) \geq \tau(x_0) > t_0$.
- Since $\alpha < 1/2$, no grains (or \sharp -grains) of $B^\sharp(x_0, 1)^c$ may visit $B(v(\mathbf{s}_{\tau^b}), 2\alpha)$.

Hence all the grains of $\xi \setminus \{x_0\}$ evolve in the same way until time $\mathbb{T}(x_0)$ for both configurations $\bar{\xi}$ and $r_{x_0}(\bar{\xi}, \bar{\xi}')$. This has two consequences. First, for $r_{x_0}(\bar{\xi}, \bar{\xi}')$, nothing prevents grains of $\bar{\xi}' \cap B(x_0 + u, \alpha)$ to form a loop (before time α) separating $v(\mathbf{s}_{\tau^b})$ from $B(v(\mathbf{s}_{\tau^b}), 2\alpha)^c$. Thus, nothing prevents the grain x_0 to grow until t_0 , time at which it comes into the ball $B(v(\mathbf{s}_{\tau^b}), 2\alpha)$, and to be stopped before $t_1 := \inf\{t : g_{x_0}(t) = v(\mathbf{s}_{\tau^b})\} \leq \mathbb{T}(x_0)$ by the loop created by the grains of $\bar{\xi}' \cap B(x_0 + u, \alpha)$. In other words, for the configuration $r_{x_0}(\bar{\xi}, \bar{\xi}')$, x_0 is followed by a loop.

Let us now check that $\#\text{Back}(x_0, \bar{\xi}) \leq \#\text{Back}(x_0, r_{x_0}(\bar{\xi}, \bar{\xi}'))$. Let us first recall that the piece of trajectory $g_{x_0}([0, t^-(\mathbf{s}_{\tau^b}, x_0)])$ which gathers all the impacts of the grains of $\bar{\xi}$ whose x_0 is the successor, remains unchanged when $\bar{\xi}$ is replaced with $r_{x_0}(\bar{\xi}, \bar{\xi}')$ ($t^-(\mathbf{s}_{\tau^b}, x_0)$ is defined in Section 5.3.1). Moreover, by Item (b) – (c) and the construction of the dominant time, only the pieces of grains $g_{x_i}((\mathbb{T}(x_0), \infty))$, with $1 \leq i \leq K$ and $\tau(x_i) = +\infty$ on the one hand and $g_{x_0}((t^-(\mathbf{s}_{\tau^b}, x_0), \tau(x_0)))$ on the other hand, may overlap $B(v(\mathbf{s}_{\tau^b}), 2\alpha)$ and could be possibly removed by the resampling. Let us set W the union of these pieces of grains.

We claim that no grains in $\bar{\xi} \setminus \{x_0\}$ are stopped by the set W . Indeed, by absurd, assume that such a grain, say z , exists. First, no grains are stopped by $g_{x_0}((t^-(\mathbf{s}_{\tau^b}, x_0), \tau(x_0)))$ by construction. So the grain z is stopped by some $g_{x_i}((\mathbb{T}(x_0), \infty))$ which means $\tau(z) > \mathbb{T}(x_0)$. Besides, z being stopped by one of the x_i 's, it necessarily belongs to $B^\sharp(x_0, 2)$. Hence, $\tau(z) < +\infty$ implies $\tau(z) \leq \mathbb{T}(x_0)$ by Proposition 33. This is our contradiction. Actually, this is the only place where we use that the dominant time $\mathbb{T}(x_0)$ involves all the grains of $B^\sharp(x_0, 2)$, and not only those of $B^\sharp(x_0, 1)$.

Hence, the (possible) deletion of any element in W has no effect on the grains of $z \in \bar{\xi} \setminus \{x_0\}$ with $\tau(z)(\bar{\xi}) < +\infty$, which includes in particular the elements of $\text{Back}(x_0, \bar{\xi})$. In other words the backward set of x_0 may only increase when the configuration $\bar{\xi}$ is replaced with $r_{x_0}(\bar{\xi}, \bar{\xi}')$. Let us specify that the only grains which could belong to $\text{Back}(x_0, r_{x_0}(\bar{\xi}, \bar{\xi}')) \setminus \text{Back}(x_0, \bar{\xi})$ are the x_i 's with $i \leq K$ and $\tau(x_i) = +\infty$.

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6 Appendix

6.1 Proof of Proposition 4

Let us consider a probability measure μ on $(\mathcal{C}, \mathcal{F}^{\mathcal{C}})$ satisfying the moment condition (3): for all $t \geq 0$,

$$\mathbb{E} \left[\sup_{s \in [0, t]} \|h(s)\|^2 \right] < \infty$$

where h is a random variable with distribution μ . Our goal is to define, for a.e. realization of $\bar{\xi}$, a function $\tau_{\text{dyn}}^\infty : \xi \rightarrow (0, +\infty]$ specifying the lifetimes of all the grains of ξ . In a second time we will prove that the constructed function τ_{dyn}^∞ is a.s. a $\bar{\xi}$ -lifetime function and this is the only one, proving that μ is tempered.

The first step is the following technical result that we will prove later.

Lemma 39. *Under the moment condition (3), for any $\alpha > 0$, there exists an increasing sequence $(t_n)_{n \geq 0}$ in \mathbb{R}_+ (depending on α) starting at $t_0 = 0$ and tending to ∞ such that, for all n ,*

$$\mathbb{E} \left[\sup_{s \in [t_n, t_{n+1}]} \|h(s) - h(t_n)\|^2 \right] \leq \alpha .$$

The key idea of the proof of Theorem 3.1 of [4] is to subdivide \mathbb{R}_+ into small time intervals $[t_n, t_{n+1})$ during which the growth of grains can be compared to a subcritical Boolean model. Reducing the model

with infinitely many grains to (infinitely many) finite clusters, i.e. each of them involving only a finite number of grains, will allow us to define step by step a lifetime function τ_{dyn}^∞ (see further).

Let us first emphasize the comparison with subcritical Boolean models. Let us consider a Boolean model in \mathbb{R}^2 whose centers are given by an homogeneous Poisson point process ζ with intensity 1 and i.i.d. radii $\{R_x : x \in \zeta\}$:

$$\Sigma := \bigcup_{x \in \zeta} B(x, R_x) .$$

The Boolean model Σ is known to be subcritical whenever $\mathbb{E}R^2$ is small enough (where R denotes the common distribution of the R_x 's). Let $(t_n)_{n \geq 0}$ be the sequence given by Lemma 39 and let us set

$$R_{x,n} := \sup_{s \in [t_n, t_{n+1}]} \|g_x(s) - g_x(t_n)\| = \sup_{s \in [t_n, t_{n+1}]} \|h_x(s) - h_x(t_n)\| .$$

Then, for any n , the random set

$$\Sigma_n := \bigcup_{x \in \xi} B(g_x(t_n), R_{x,n})$$

is a Boolean model—notice that $\{g_x(t_n) : x \in \xi\}$ is still a Poisson point process with intensity 1—which can be made subcritical by Lemma 39 if α is chosen small enough.

Let us now build step by step a function $\tau_{\text{dyn}}^\infty : \xi \rightarrow (0, +\infty]$. We proceed by induction with assuming that τ_{dyn}^∞ has been already built until time t_n . I.e. for any grain $x \in \xi$, $\tau_{\text{dyn}}^\infty(x) < t_n$ means that the grain x has been stopped before time t_n while $\tau_{\text{dyn}}^\infty(x) = t_n$ indicates that x is still alive at time t_n . Let us set $A_n := \{x \in \xi : \tau_{\text{dyn}}^\infty(x) = t_n\}$. The random set

$$\Sigma'_n := \bigcup_{x \in A_n} B(g_x(t_n), R_{x,n})$$

is included in the Boolean model Σ_n and then is subcritical; all its clusters are a.s. finite. Let \mathcal{C} be one of them. So we can apply the dynamical algorithm introduced in Section 2.1.3 to the finite set of (alive) grains involved in \mathcal{C} and during the time interval $[t_n, t_{n+1}]$. Let's be careful since for this step we also have to take into account the set of grains already realized till time t_n and possibly important for the cluster \mathcal{C} , i.e. the set

$$\{x \in \xi : g_x([0, \tau_{\text{dyn}}^\infty(x)]) \cap \mathcal{C} \neq \emptyset\}$$

which could include grains already stopped at time t_n or still alive but not in \mathcal{C} . Thanks to Lemma 4.2 of [4] which only uses hypothesis (3), this set is a.s. finite and then the dynamical algorithm applies without extra difficulties. Henceforth, we treat independently each (finite) cluster of Σ'_n which allows us to extend the construction of the function τ_{dyn}^∞ until time t_{n+1} .

The last step consists in proving that the function τ_{dyn}^∞ previously built is a.s. a $\bar{\xi}$ -lifetime function and this is the only one. First consider $x \neq y \in \xi$ and $t_x \geq 0$ such that x τ_{dyn}^∞ -hits y at time t_x . Let t_n be such that $t_x \in [t_n, t_{n+1})$ and consider the cluster \mathcal{C} of the random set Σ'_n (defined above) containing x —with a slight abuse of notations, we use the same symbol \mathcal{C} for the subset of Σ'_n and the set of grains involved in \mathcal{C}). Remark that y may not belong to \mathcal{C} but necessarily $g_y([0, t_n \wedge \tau_{\text{dyn}}^\infty(y)])$ overlaps \mathcal{C} . The dynamical algorithm applies to the (alive) grains of \mathcal{C} . A very similar proof to that of the Reconciliation Lemma implies that $t_x = \tau_{\text{dyn}}^\infty(x)$, i.e. the Stopping property holds. The same arguments work for the Hitting property.

In order to prove that there is uniqueness of the $\bar{\xi}$ -lifetime function, we proceed as in the proof of the Reconciliation Lemma by absurd. Assume that there exist two $\bar{\xi}$ -lifetime function, say τ_1 and τ_2 . Let $[t_n, t_{n+1})$ be the first interval on which a discrepancy between τ_1 and τ_2 occurs (the growths of all the grains until t_n coincide for τ_1 and τ_2). Hence, consider a cluster \mathcal{C} of Σ'_n containing such discrepancy. Since \mathcal{C} is a.s. finite, we can select a grain x minimizing $\tau_1(x) \wedge \tau_2(x)$ among the grains of

$$\{x' \in \mathcal{C} : \tau_1(x') \neq \tau_2(x') \text{ and } t_n \leq \tau_1(x') \wedge \tau_2(x') < t_{n+1}\} .$$

By symmetry, let us assume that $\tau_1(x) < \tau_2(x)$ which implies that $\tau_1(x) < \infty$. By the Hitting property (for τ_1), there exists a grain $y \neq x$ such that x τ_1 -hits y at time $\tau_1(x)$. As in the proof of the Reconciliation

Lemma, we can prove that $\tau_2(y) < t_y \leq \tau_1(y)$ and then deduce that y satisfies $y \in \mathcal{C}$, $\tau_1(y) \neq \tau_2(y)$ and

$$t_n \leq \tau_1(y) \wedge \tau_2(y) < \tau_1(x) \wedge \tau_2(x) < t_{n+1} .$$

contradicting the minimality of x . This achieves the proof of Proposition 4.

Proof of Lemma 39. Let $t, t' \geq 0$. Let us first prove that

$$\lim_{t' \rightarrow t} \mathbb{E} \left[\sup_{s \in [t^-, t^+]} \|h(t) - h(s)\|^2 \right] = 0 \quad (37)$$

where $t^- := t \wedge t'$ and $t^+ := t \vee t'$. By continuity of the random variable h , $\sup_{s \in [t^-, t^+]} \|h(t) - h(s)\|^2$ a.s. tends to 0 as $t' \rightarrow t$. It then remains to check the domination hypothesis to apply the Lebesgue's dominated convergence theorem and conclude that (37) holds. Restricting our attention to $|t' - t| \leq 1$,

$$f(t) := \sup_{s \in [t-1, t+1]} \|h(t) - h(s)\|^2$$

will play the role of the dominating function. It is not difficult to see that f is integrable writing

$$f(t) \leq \|h(t)\|^2 + \sup_{s \in [t-1, t+1]} \|h(s)\|^2 + 2\|h(t)\| \sup_{s \in [t-1, t+1]} \|h(s)\| ,$$

and using the Cauchy–Schwarz inequality and the moment condition (3).

Now let us define by induction a non-decreasing sequence $(t_n)_{n \geq 0}$ as follows: $t_0 = 0$ and for any integer n , given t_n , we set

$$t_{n+1} := \sup \left\{ t \geq t_n : \mathbb{E} \left[\sup_{s \in [t_n, t]} \|h(s) - h(t_n)\|^2 \right] \leq \alpha \right\} \in [t_n, \infty]$$

with convention that $t_n = \infty$ implies $t_{n+1} = \infty$. Let us assume for the moment that

$$\sup_{n \geq 0} t_n = \infty . \quad (38)$$

Either all the t_n 's are finite and by the monotone convergence theorem, we have

$$\mathbb{E} \left[\sup_{s \in [t_n, t_{n+1}]} \|h(s) - h(t_n)\|^2 \right] \leq \alpha$$

leading to Lemma 39. Or the sequence $(t_n)_n$ is infinite from some index $k \geq 1$ and, in this case, the sequence $(s_n)_{n \geq 0}$ defined by $s_n = t_n$ for any $n \leq k-1$ and $s_n = s_{n-1} + 1$ for any $n \geq k$, will work.

It then remains to prove (38). Let us proceed by absurd with assuming that $t^* := \sup_n t_n$ is finite. Each time t_n is then finite and smaller than t^* . Thus (37) gives the existence of $\varepsilon > 0$ such that

$$\mathbb{E} \left[\sup_{s \in [t^* - \varepsilon, t^* + \varepsilon]} \|h(s) - h(t^*)\|^2 \right] \leq \frac{\alpha}{4} . \quad (39)$$

Let us pick $t_n \in [t^* - \varepsilon, t^*]$ and set $t := t^* + \varepsilon$. Proceeding as before with the Cauchy–Schwarz inequality and (39), we get

$$\mathbb{E} \left[\sup_{s \in [t_n, t]} \|h(s) - h(t_n)\|^2 \right] \leq \alpha$$

proving that, by definition of t_{n+1} , $t^* \geq t_{n+1} \geq t = t^* + \varepsilon$. This is absurd and proves (38). \square

Let us point out that w.r.t. Theorem 3.1 of [4], two hypotheses have been deleted : 1. the Lebesgue measure of $(h - h')(\mathbb{R}_+)$ equals 0 a.s. and 2. the Lebesgue measure of $h(\mathbb{R}_+)$ equals 0 a.s., where h, h' are two independent r.v.'s with distribution μ . The first one was required to prevent collisions of grains at the same time and at the same location. Actually such collisions are allowed by the dynamical algorithm and thence they are absolutely not an obstacle to the existence of the model. The second hypothesis concerned the multi-branching case (i.e. various grain emanating from the same germ) which is not considered here.

6.2 Stability of tempered configurations by resampling

In this section, we provide a technical justification of an item in the definition of "good point" given in Section 3.4.1. Precisely we have the following lemma which claims that after resampling the point process in a ball then almost surely the configuration remains tempered.

Lemma 40. 1. With probability one, $\bar{\xi}^0 \in \bar{\mathcal{A}}$.

2. With probability one, $\bar{\xi}^0$ belongs to $\{\bar{S} \in \bar{\mathcal{S}}^0 : \mathbb{P}[\text{resample}(\bar{S}, \bar{\xi}') \in \bar{\mathcal{A}}] = 1\}$.

3. With probability one, for all $x \in \xi$, $\bar{\xi} - x$ belongs to the set $\{\bar{S} \in \bar{\mathcal{S}}^0 : \mathbb{P}[\text{resample}(\bar{S}, \bar{\xi}') \in \bar{\mathcal{A}}] = 1\}$.

Proof. Let us prove the first item. Fix $C = [0, 1]^d$. Then

$$\mathbb{P}[\bar{\xi}^0 \in \bar{\mathcal{A}}] = \mathbb{E} \left[\sum_{x \in \xi \cap C} \mathbf{1}_{\bar{\xi} - x \in \bar{\mathcal{A}}} \right] = \mathbb{E} \left[\sum_{x \in \xi \cap C} \mathbf{1}_{\bar{\xi} \in \bar{\mathcal{A}}} \right] = \mathbb{E} \left[\sum_{x \in \xi \cap C} \mathbf{1} \right] = 1.$$

The second equality is due to the invariance of $\bar{\mathcal{A}}$ under translations. The third equality holds because $\bar{\xi}$ belongs almost surely to $\bar{\mathcal{A}}$.

Let us consider the second item. The point processes $\bar{\xi}^0$ and $\text{resample}(\bar{\xi}^0, \bar{\xi}')$ have the same distribution. By the first item, we then know that $\text{resample}(\bar{\xi}^0, \bar{\xi}')$ belongs almost surely to $\bar{\mathcal{A}}$. The second item follows by Fubini theorem.

Set $\bar{\mathcal{R}}^0 = \{\bar{S} \in \bar{\mathcal{S}}^0 : \mathbb{P}[\text{resample}(\bar{S}, \bar{\xi}') \in \bar{\mathcal{A}}] = 1\}$. We have just shown $\mathbb{P}[\bar{\xi}^0 \in \bar{\mathcal{R}}^0] = 1$. As a consequence,

$$\mathbb{E} \left[\sum_{x \in \xi \cap C} \mathbf{1}_{\bar{\xi} - x \in \bar{\mathcal{R}}^0} \right] = \mathbb{P}[\bar{\xi}^0 \in \bar{\mathcal{R}}^0] = 1.$$

Therefore with probability one, for any $x \in \xi \cap C$, we have $\bar{\xi} - x \in \bar{\mathcal{R}}^0$. The third item follows by stationarity. \square

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