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Modeling the operational capability of a system in relation to a defined mission [★]

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Abstract: This paper suggests a model for evaluating the operational capability of a system to accomplish a mission. This model involves a dynamic equation defined on the space of multi-graphs, associated with a weak discrete metric. It could be analysed further using the framework of control theory. This modeling is based on a definition of the operational capability that includes three keywords: *system*, *mission* and *ability*. These keywords are defined and represented by appropriate mathematical objects. The paper focuses on justifying the use of the proposed mathematical objects to represent these keywords.

Keywords: Operational capability, Dynamical system, Mission, Ability.

1. INTRODUCTION

What all fields of science have in common is the construction of models. Models are used to describe a phenomenon or an object. In particular, a model can be used to assess whether an object is capable of successfully accomplishing a mission. In the literature, this refers to operational capability (Najgebauer et al. (2015)). Knowledge of the operational capability of a system is useful for several achievements. For example, it can be used to authorize and plan a mission to be carried out by the system. It is an important piece of information for decision-making in both the military and industrial sectors. The aim of this paper is to propose a generic model for assessing the operational capability of a system.

In capability-based planning (CBP), a framework in which the concept of capability is often referred to, the capability of a system is directly linked to a mission (Davis et al. (2002)). The latter defines the Required Operational Capability (ROC), which is a transcription of a mission scenario into requirements. The system, by satisfying these requirements, maximizes the mission success rate (Cho et al. (2022)). The ROC can be established by experts, by a combination of expert opinion and analytical methods (Hristov et al. (2010)), and by a combined optimization and simulation approach (Cho et al. (2022)). It can be in quantitative or descriptive form. The quantitative form summarizes mission-related observations into a score. This score is often determined by expert analysis, which is subject to interpretation. While, the descriptive form indicates the state in which the system, or a part of it, should be.

This is directly linked to the technical characteristics of the system.

At any given time, the state of a system defines its exact topological structure, *i.e.*, the organisation of its components, including their number and the various links between them. Additionally, it defines the values taken by its attribute, which is a vector of quantitative, qualitative, and boolean variables of the system. Its state allows it to produce an effect which can be synonymous with the success or failure of a mission. It evolves over time, *i.e.*, its topological structure and its attribute. Its evolution describes a trajectory in a given state space. In this paper, the assessment of operational capability involves verifying if the states indicated by the system trajectory are appropriate for the mission.

To assess the operational capability of an armed force, Najgebauer et al. (2015) treat a capabilities allocation problem in which they determine the ROC in quantitative form, and then assess operational capability as a measure of satisfaction, which is the ratio between the ROC and the current capability of the armed force. In contrast to Najgebauer et al. (2015), in this paper we deal with a generic problem where the ROC is defined in a descriptive form, and to the best of our knowledge, there is no mathematical model for evaluating operational capability with a descriptive form of the ROC.

In other contexts, capability refers to performance (Cho et al. (2022)), effectiveness (Lee and Lee (2014)) and robustness (Li et al. (2018)). The degradation of a system due to use and the operating environment (Peysson et al. (2008a)) is also an operational capability. These qualifiers, to which capability refers, are for the most part evaluated using indicators that are a function of either the system attribute or the elements of the topological structure

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of the system (components and links). However, these indicators alone are insufficient to determine if the system can successfully carry out a mission, as they are not evaluated against a success criterion such as the ROC. For example, degradation (Peysson et al. (2008a)) is not sufficient in the sense that the absence of degradation in a system does not imply that it is capable of carrying out a mission. Indeed, this one assumes that the system is in the required state to accomplish its mission and focuses on its degradation. While, in this paper, there is already an effort to determine if the system is in an appropriate state and if it will maintain it throughout the completion of the mission.

Operational capability should not be confused with reliability, or even availability (Birolini (2007)). Despite having similar intentions, they remain distinct. Reliability serves as a statistical indicator tied to a group of identical systems operating under the same conditions. It precisely gauges the percentage of systems within the group that have completed the mission up to a given time t . On the contrary, operational capability evaluates if the state of a single system is suitable for the mission at each instance.

The purpose of this paper is to propose a mathematical model for evaluating the operational capability of a system in relation to the ROC defined in descriptive form. The ROC defines a specific region within a state space, referred to as the mission target. If the system state is within the target, it is deemed capable; otherwise, it is not regarded as such. To the question of whether a system is capable of carrying out a mission, two complementary answers can be given. The first would be to simply say yes or no, and the second would be to indicate how much capable it is. This paper proposes a method to address the first question at a given time t during the mission. This naturally poses the problem of operational capability as a decision problem, which is a mathematical question whose answer is yes or no for a given instance (Kozen (2007)).

In the following, the model is presented in Section 2. This section starts with the definition of operational capability, which comprises three key words that form the basis of the model: *System*, *Mission* and *Ability*. These three keywords are defined and associated with suitable mathematical objects to represent them respectively in sub-sections 2.1, 2.2 and 2.3. Finally, the conclusion and perspectives are given in Section 3.

2. OPERATIONAL CAPABILITY MODELING

There are several definitions of capability in the literature. Whatever the field, Martin et al. (2022) point out that each definition includes either all or some of the following types of keywords: *i)* ability, capacity, power; *ii)* effects; *iii)* standards, conditions; and *iv)* tasks, missions, function and action. The definition adopted here is as follows:

Definition 2.1. Operational capability is the ability of a system to carry out a mission.

This definition implicitly includes all the types of keywords mentioned above. It is assumed that the keywords *effects* and *conditions* are the prerogatives of the mission. They appear explicitly in the modeling of the mission (*cf.*

subsection 2.2). The three keywords *System*, *Mission* and *Ability* in this definition form the basis of the model:

$$\langle \mathcal{S}, \mathcal{M}, \rho \rangle, \quad (1)$$

where \mathcal{S} , \mathcal{M} and ρ represent the system, mission and ability respectively. In the following, these keywords are defined in unambiguous terms, and their mathematical representation is proposed.

2.1 System modeling

The mathematical definition of the system has evolved over time. Mesarovic (1964), considered as one of the first to have established the mathematical foundations of general systems theory, conceived of a system as a single link connecting several objects. This conception is generalized by Lin (1987), who believes that Mesarovic's conception has limitations insofar as it is reduced to a single relationship between objects. So, he proposes a design that consists of having a set of object on the one hand, and a set of links on the other. This conception is mentioned in the book of Klir (1991) as that of common sense. However, in his paper entitled "*The definition of system*", Backlund (2000) underlines the weakness of this conception and demonstrates that it can lead to undesirable consequences. He thus proposes the following definition of the system:

Definition 2.2. A system \mathcal{S} consists of a set \mathcal{V} , and a non-empty set of relations on \mathcal{V} , \mathcal{E} , satisfying the following conditions:

- (1) $|\mathcal{V}| \geq 2$;
- (2) from every member of \mathcal{V} there is a path to every other member of \mathcal{V} .

To instantiate this definition, Torres et al. (2021) highlight three mathematical frameworks commonly used by researchers to model systems, namely: graph theory, simplicial complexes, and hypergraphs. Graph is probably the most commonly used representation for modeling system. The relationships shared by the elements of graph are dyadic, *i.e.*, shared by two elements, whereas simplicial complexes and hypergraphs allow to model systems whose relationships between elements are polyadic, *i.e.* shared by several elements, for example, a biochemical network (Klamt and Gilles (2004)). In this paper, the relationships shared between elements are considered to be dyadic. A system is therefore represented here by a graph.

It should be noted that the notion of a system is widely used in the theory of control systems, but its structure is not always emphasised. There are, however, works that focus on the theory of structural systems, using graphs to represent dynamic systems (Pequito et al. (2015), Ramos et al. (2020)).

Modeling a system using a graph. Graph theory is a powerful tool for structuring data with their correlations and representing interactions between system components. A system can be represented by a simple graph or by a multi-graph. The latter allows for taking into account the different relationships existing between two elements of the system, which corresponds fairly well to real-world observations. These relationships in an engineering system can be, for example, mechanical, electrical or simply

hierarchical. To distinguish these different relationships, a labeled graph or multi-graph is considered.

Definition 2.3. A labeled multi-graph \mathcal{S} is defined as 4-tuple $\mathcal{S} = (\mathcal{V}, \mathcal{E}, \mathcal{L}_v, \mathcal{L}_e)$, where \mathcal{V} is the set of vertex, \mathcal{E} , the set of dyadic edges between vertex of \mathcal{V} , \mathcal{L}_v is the set of vertex labels and \mathcal{L}_e , the set of edge labels. To \mathcal{S} , the following functions are associated:

- $p : \mathcal{E} \mapsto \mathcal{V}^2$, which associates with each edge, the vertices at the ends of the edge considered;
- $l_e : \mathcal{E} \mapsto \mathcal{L}_e$, which associates with each edge its label, *i.e.*, the specification of the link or interaction between the vertices of the edge in question;
- $l_v : \mathcal{V} \mapsto \mathcal{L}_v$, which associates each vertex with its label;
- $\lambda : \mathcal{V} \mapsto \mathbb{R}^{n_i}$, which associates an attribute with each vertex, where i and $n_i \in \mathbb{N}$ are respectively the index of the vertex and the dimension of the attribute.

By extension, $\lambda(\mathcal{S}) \in \mathbb{R}^n$ refers to the system attribute, which is a concatenation of the vertex attributes, with $n = \sum_{i=1}^N n_i$, where N is the number of vertices. A graph \mathcal{S} is simple if and only if the incidence mapping p is an injective function; otherwise \mathcal{S} is a multi-graph. More simply, \mathcal{S} is a multi-graph if there are vertices that share more than one edge, as shown in Fig. 1. In the following, only multi-graphs are discussed, with clarification provided if a simple graph is intended. Edges can be both directed and undirected. Each edge has a label determined by the function l_e , which is not necessarily unique to it. Furthermore, each edge can have a weight representing the interaction between the involved vertices.

Each vertex of the multi-graph has a unique label and an attribute vector determined respectively by the functions l_v and λ . In reality, the attribute vector of a vertex can be made up of quantitative, qualitative and boolean variables. Given that the sets to which the last two types of variables belong can be put into bijection with a subset of \mathbb{R} , it is therefore considered here that λ directly returns the state vector of a vertex in \mathbb{R}^{n_i} .

For a graph to claim to represent a system, it must have at least two vertices and be connected. These two criteria satisfy the requirements of Definition 2.2.

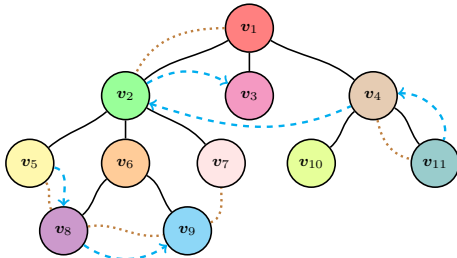


Fig. 1. Example of system modeling using a connected multi-graph, where vertex labels are depicted using colors, while edge labels are distinguished by lines of various shapes.

2.2 Mission modeling

Silva et al. (2014) recommends looking at the mission as a goal, a functionality, a set of tasks to be executed.

For example, to make a prognosis on the degradation of system resources, Peysson et al. (2008b) consider a mission as a succession of tasks, each associated with a given environment. Intending to assess the success of a mission concerning a defined outcome, the mission is formally defined as follows:

Definition 2.4. A given mission \mathcal{M} , is a target \mathcal{T}_{ar} to be achieved in a given context \mathcal{C}_{xt} .

Mission target. In the military domain, an objective, synonymous here with target, is defined in terms of the desired effect (Biltgen (2007)). This effect is mirrored by the ROC, specifying the states for the system or a part of it, to produce the effect. The target is defined as follows:

Definition 2.5. The target \mathcal{T}_{ar} denotes the state in terms of both topology and attribute values, which a system \mathcal{S} , or a part of it, should possess or converge towards to achieve an effect consistent with mission success. It is characterized by the pair:

$$\mathcal{T}_{ar} = (\mathbf{s}^*, T^*), \quad (2)$$

where \mathbf{s}^* is a graph whose attribute is $\lambda(\mathbf{s}^*)$ of dimension $n^* \in \mathbb{N}$, and T^* , the region in which the attribute $\lambda(\mathbf{s}^*)$ resides, *i.e.*, $\lambda(\mathbf{s}^*) \in T^* \subseteq \mathbb{R}^{n^*}$.

The target \mathcal{T}_{ar} provides on the one hand, through the graph \mathbf{s}^* , the suitable topology that one or a part of a system \mathcal{S} must have, and on the other hand, through $\lambda(\mathbf{s}^*)$, the decisive variables of the mission, and by T^* , the region in which the attribute $\lambda(\mathbf{s}^*)$ must reside. Thus, any graph \mathbf{s} which satisfy:

- $\mathbf{s} \cong \mathbf{s}^*$, *i.e.*, \mathbf{s} and \mathbf{s}^* are isomorphic graphs (*cf.* similarity of topologies in sub-section 2.3),
- $\lambda(\mathbf{s}) \sim \lambda(\mathbf{s}^*) \iff \lambda(\mathbf{s})$ and $\lambda(\mathbf{s}^*) \in T^*$, where \sim is an equivalence relation,

is considered as one of the states of the target \mathcal{T}_{ar} .

Mission context. The context defines the conditions in which the mission is carried out. It constitutes a set of relevant elements \mathcal{C}_{xt} that interact with the system. These elements are not part of the system, but can change the state of the system, that is to say, the topological structure of the system and its attribute. Amongst these elements, examples include the environment and the operating procedure (Peysson et al. (2008b)). Under the action of the elements of the context, the evolution of a system \mathcal{S} can be described by a time series

$$\begin{cases} \mathcal{S}_t = f(\mathcal{S}_{t-1}, \mathcal{C}_{t-1}^1, \mathcal{C}_{t-1}^2, \dots), \\ \mathbf{s}_t = g(\mathcal{S}_t), \end{cases} \quad (3)$$

where $(\mathcal{C}_{t-1}^k)_{k=1:K} \in \mathcal{C}_{xt}^K$ with $K \in \mathbb{N}$, are input and \mathbf{s}_t is the output. The function f describes the combined action of the elements of the context $(\mathcal{C}_{t-1}^k)_{k=1:K}$ on the system, causing it to evolve from the initial condition \mathcal{S}_0 to the state \mathcal{S}_t . The inputs $(\mathcal{C}_{t-1}^k)_{k=1:K}$ and the outputs \mathbf{s}_t can be represented as graphs. There has been work on dynamic graphs (Šiljak (2008)) and from a control perspective (Zecevic and Šiljak (2010)). However, these studies assume that the number of vertices is fixed. There is a need to extend this work to scenarios where the structure can evolve over time (Alippi and Zambon (2023)). In our specific case, the evolution of our system falls within the domain of hybrid dynamics, involving continuous flows

for attribute variables and discrete jumps for topology changes (Goebel et al. (2012)).

2.3 Ability modeling

Ability is the information that indicates the capability of a system to perform a mission. This information is acquired by comparing the state of system \mathcal{S} with that indicated by the target \mathcal{T}_{ar} . This comparison takes place at two levels: topology structure, for multi-graph similarity between \mathcal{S} and \mathbf{s}^* , and attribute, for attribute equivalence between $\lambda(\mathcal{S})$ and $\lambda(\mathbf{s}^*)$. Therefore, in this context, ability is associated with the concept of state proximity.

Definition 2.6. Let \mathbb{S} be the set of systems described by multi-graphs, ability is the distance $\rho : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{R}^+$ between the state of system $\mathcal{S} \in \mathbb{S}$ with the state of target \mathcal{T}_{ar} , denoted by $\mathbf{s}^* \in \mathbb{S}$ and $\lambda(\mathbf{s}^*) \in \mathbb{R}^{n^*}$.

For the function ρ to be considered as a distance, it must satisfy the axioms of a metric (Arkhangel'skii and Fedorchuk (2012)). Under these conditions, the pair (\mathbb{S}, ρ) forms a metric space. In the following, we show that ρ verifies all the axioms except those of separation and symmetry, which makes ρ a weak distance and (\mathbb{S}, ρ) a weak metric space (Papadopoulos and Troyanov (2006)). In this paper, operational capability is approached as a decision problem (Kozen (2007)). Hence, the distance of interest is discrete, often called trivial. Once the topological similarity and attribute equivalence are defined, they will be linked to distances, and the satisfied axioms will be determined.

Similarity of topologies. The topology similarity problem boils down to checking whether the system $\mathcal{S} = (\mathcal{V}, \mathcal{E}, \mathcal{L}, \mathcal{C})$ has the topology required to carry out the mission, *i.e.*, that of $\mathbf{s}^* = (\mathcal{V}^*, \mathcal{E}^*, \mathcal{L}^*, \mathcal{C}^*)$. The target topology \mathbf{s}^* will always be smaller than the system \mathcal{S} , in the sense that $|\mathcal{V}^*| \leq |\mathcal{V}|$ and $|\mathcal{E}^*| \leq |\mathcal{E}|$. This difference in size is due to the fact that \mathbf{s}^* describes the minimal topology required for the mission, while \mathcal{S} describes the topology of entire system. The search for a required topology \mathbf{s}^* in the system \mathcal{S} is very similar to the well-known sub-graph isomorphism problem. This is an algorithmic problem for which the complexity is NP-complete (Garey and Johnson (1979)).

The sub-graph isomorphism problem, illustrated in Fig. 2, consists in finding a sub-graph \mathbf{s} of \mathcal{S} , note $\mathbf{s} \preceq \mathcal{S}$, such that \mathbf{s} is isomorphic to \mathbf{s}^* , denoted $\mathbf{s} \cong \mathbf{s}^*$. The symbol \preceq designates an order relation that is reflexive, anti-symmetrical and transitive. The sub-graph \mathbf{s} is called a *signal*. For \mathbf{s}^* and \mathbf{s} to be isomorphic, there must exist a sub-graph isomorphism between \mathbf{s}^* and \mathcal{S} . The sub-graph isomorphism is defined in a slightly modified form of the definition of Moorman et al. (2021) as follows:

Definition 2.7. An injective function ψ is a sub-graph isomorphism from a target $\mathbf{s}^* = (\mathcal{V}^*, \mathcal{E}^*, \mathcal{L}^*, \mathcal{C}^*)$ to a system $\mathcal{S} = (\mathcal{V}, \mathcal{E}, \mathcal{L}, \mathcal{C})$ if

- $l_v(\mathbf{v}) = l_v(\psi(\mathbf{v}))$, $\forall \mathbf{v} \in \mathcal{V}^*$;
- $|p^{-1}(\mathbf{u}, \mathbf{v})| \leq |p^{-1}(\psi(\mathbf{u}), \psi(\mathbf{v}))|$, $\forall (\mathbf{u}, \mathbf{v}) \in \mathcal{V}^* \times \mathcal{V}^*$;
- For all $\epsilon \in \mathcal{E}^*$ and $(\mathbf{u}, \mathbf{v}) \in \mathcal{V}^* \times \mathcal{V}^*$, such that $p(\epsilon) = (\mathbf{u}, \mathbf{v})$, there exists $\varepsilon \in \mathcal{E}$ verifying $p(\varepsilon) = (\psi(\mathbf{u}), \psi(\mathbf{v}))$, such that $l_\varepsilon(\epsilon) = l_\varepsilon(\varepsilon)$,

where p^{-1} is the inverse of the function p and returns the set of edges shared between any two vertices $u, v \in \mathcal{V}^*$ or their image $\psi(u), \psi(v) \in \mathcal{V}$. Thus, $|p^{-1}(\dots)|$ returns the cardinal of the antecedents of the argument of p^{-1} .

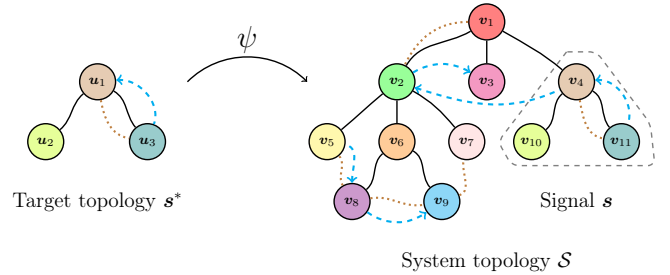


Fig. 2. Illustration of a sub-graph isomorphism ψ between the topology \mathbf{s}^* of target \mathcal{T}_{ar} and that of system \mathcal{S} to determine signal \mathbf{s} , such that, $\mathbf{s} \cong \mathbf{s}^*$.

Applying a brute force algorithm to find a sub-graph isomorphism between \mathbf{s}^* and \mathcal{S} can be relatively practical for small graphs. As larger systems are considered, time increases exponentially with the size of the graphs. In this case, more elaborate algorithms can be of great help. The paper of Foggia et al. (2001) compares the performance of five different algorithms for simple graphs. The works of Ingalalli et al. (2016), Micale et al. (2020) and Moorman et al. (2021) concern to the development of efficient algorithms for multi-graphs.

Discrete distance of graph similarity. The existence of a sub-graph isomorphism between the target \mathbf{s}^* and the system \mathcal{S} is an essential element in defining the discrete distance ρ_s of similarity of topological structures. Thus, for all couple $(\mathbf{s}_i, \mathbf{s}_j) \in \mathbb{S}^2$, the following function introduced:

$$\rho_s(\mathbf{s}_i, \mathbf{s}_j) = \begin{cases} 0, & \text{if } \exists \mathbf{s} \preceq \mathbf{s}_i, \text{ such as } \mathbf{s} \cong \mathbf{s}_j; \\ 1, & \text{otherwise.} \end{cases} \quad (4)$$

Note that $\rho_s(\mathbf{s}_i, \mathbf{s}_j) = 0$ is structurally or topologically equivalent to $\mathbf{s}_j \preceq \mathbf{s}_i$. Thus, $|\mathcal{V}_j| \leq |\mathcal{V}_i|$ and $|\mathcal{E}_j| \leq |\mathcal{E}_i|$.

Proposition 2.1. ρ_s is a weak discrete distance.

Proof. For all $\mathbf{s}_i, \mathbf{s}_j$ and $\mathbf{s}_k \in \mathbb{S}$ such as $|\mathcal{V}_j| < |\mathcal{V}_i|$, we check that

- (1) ρ_s is defined: $\rho_s(\mathbf{s}_i, \mathbf{s}_i) = 0$, because $\mathbf{s}_i \cong \mathbf{s}_i$;
- (2) ρ_s does not satisfy the separation: $\rho_s(\mathbf{s}_i, \mathbf{s}_j) = 0$ does not necessarily imply $\mathbf{s}_i = \mathbf{s}_j$. The verification is direct through the definition of ρ_s ;
- (3) ρ_s is non-symmetrical: $\rho_s(\mathbf{s}_i, \mathbf{s}_j) \neq \rho_s(\mathbf{s}_j, \mathbf{s}_i)$. This assertion can be verified by reasoning by absurdity. Assume that $\rho_s(\mathbf{s}_i, \mathbf{s}_j) = \rho_s(\mathbf{s}_j, \mathbf{s}_i)$. Then if $\rho_s(\mathbf{s}_i, \mathbf{s}_j) = 0$, this implies that, on the one hand, $\mathbf{s}_j \preceq \mathbf{s}_i$, on the other hand, $\rho_s(\mathbf{s}_j, \mathbf{s}_i) = 0$. The latter implies that $\mathbf{s}_i \preceq \mathbf{s}_j$. As a result, $\mathbf{s}_j = \mathbf{s}_i$. However, this is impossible because $|\mathcal{V}_j| < |\mathcal{V}_i|$.

So, it holds that $\rho_s(\mathbf{s}_i, \mathbf{s}_j) \neq \rho_s(\mathbf{s}_j, \mathbf{s}_i)$;

- (4) ρ_s satisfies the triangular inequality: $\rho_s(\mathbf{s}_i, \mathbf{s}_j) \leq \rho_s(\mathbf{s}_i, \mathbf{s}_k) + \rho_s(\mathbf{s}_k, \mathbf{s}_j)$. Given that ρ_s is within the countable set $\{0, 1\}$, verification involves simply determining if the paradoxical case $1 \leq 0$ is ruled out. Suppose that $\rho_s(\mathbf{s}_i, \mathbf{s}_k) = 0 \iff \mathbf{s}_k \preceq \mathbf{s}_i$, and $\rho_s(\mathbf{s}_k, \mathbf{s}_j) = 0 \iff \mathbf{s}_j \preceq \mathbf{s}_k$. So, this leads to the fact that $\mathbf{s}_j \preceq \mathbf{s}_k \preceq \mathbf{s}_i \implies \rho_s(\mathbf{s}_i, \mathbf{s}_j) = 0$.

Subsequently, it is impossible to have $1 \leq 0$. Hence, the triangular inequality holds true.

For all these reasons, ρ_s is a weak discrete distance (Papadopoulos and Troyanov (2006)). Thus, (\mathbb{S}, ρ_s) form a weak space metric.

Equivalence of attributes. The attribute equivalence problem here consists in determining whether the values taken by the system attribute $\lambda(\mathcal{S}) \in \mathbb{R}^n$ are those required by the mission. The attribute vector $\lambda(\mathbf{s}^*)$ indicates the decisive variables involved in mission execution, whose values are contained in $T^* \subseteq \mathbb{R}^{n^*}$. It is carried by a canonical basis $(\mathbf{e}_i^*)_{i=1:n^*}$, such that $\lambda(\mathbf{s}^*) = \sum_{i=1}^{n^*} \lambda_i^* \mathbf{e}_i^*$, with $\lambda_i^* \in \mathbb{R}$. The belonging of $\lambda(\mathbf{s}^*)$ to T^* comes from the fact it satisfies a certain number of constraints. For example, the k -th constraint is defined as

$$\phi_k(\mathbf{x}) \in I_k \subseteq \mathbb{R}, \text{ with } \mathbf{x} \in \mathbb{R}^{n^*}, \quad (5)$$

where ϕ_k , is a function which gives a combination of the coordinates of any attribute $\mathbf{x} \in \mathbb{R}^{n^*}$. According to set theory, constraint (5) can be written using an indicator function so that,

$$\phi_k(\mathbf{x}) \in I_k \iff \mathbb{1}_{I_k}(\phi_k(\mathbf{x})) = 1, \quad (6)$$

where $\mathbb{1}_{I_k}(\cdot)$ represents the indicator function and is 1 if $\phi_k(\mathbf{x}) \in I_k$, otherwise it is 0. Thus, the set of mission constraints on an attribute $\mathbf{x} \in \mathbb{R}^{n^*}$ is written as,

$$\mathcal{C}_{\text{st}} = \{(\phi_k, I_k) : \text{s.t. } \mathbb{1}_{I_k}(\phi_k(\mathbf{x})) = 1, \forall k = 1 : n_{\text{st}}\}, \quad (7)$$

where n_{st} is the number of constraints.

To determine the number of constraints satisfied by the attributes $\mathbf{x} \in \mathbb{R}^{n^*}$, the following function is introduced,

$$h: \mathbb{R}^{n^*} \longrightarrow \mathbb{N}$$

$$\mathbf{x} \longmapsto \sum_{k=1}^{n_{\text{st}}} (1 - \mathbb{1}_{I_k}(\phi_k(\mathbf{x}))). \quad (8)$$

This function is a *complete invariant* (Fortnow and Grochow (2011)) defined using the set of constraints \mathcal{C}_{st} . When $\mathbf{x} \in \mathbb{R}^{n^*}$ satisfies all constraints \mathcal{C}_{st} , $h(\mathbf{x}) = 0$. The attributes $\mathbf{x}, \tilde{\mathbf{x}} \in \mathbb{R}^{n^*}$ are equivalent through h , denoted $\mathbf{x} \sim_h \tilde{\mathbf{x}}$, if and only if $h(\mathbf{x}) = h(\tilde{\mathbf{x}})$. Then the region T^* of eq. (2) is nothing other than $\ker(h)$, defined as,

$$\ker(h) = \{\forall \mathbf{z} \in \mathbb{R}^{n^*} : h(\mathbf{z}) = 0\}, \quad (9)$$

hence, $\mathcal{T}_{\text{ar}} = (\mathbf{s}^*, \ker(h))$.

The aim here is to verify if $\lambda(\mathcal{S}) \in \mathbb{R}^n$ satisfies the set of constraints \mathcal{C}_{st} , or in other words, is equivalent to the $\lambda(\mathbf{s}^*) \in \mathbb{R}^{n^*}$. Note that $\lambda(\mathcal{S})$ is carried by a canonical basis $(\mathbf{e}_i)_{i=1:n}$, such that $\lambda(\mathcal{S}) = \sum_{i=1}^n \lambda_i \mathbf{e}_i$, with $\lambda_i \in \mathbb{R}$ and n , potentially greater than n^* . Therefore, it is precisely a matter of verifying if the projection of $\lambda(\mathcal{S})$ onto \mathbb{R}^{n^*} , denoted as $\lambda(\mathcal{S})|_{\mathbf{e}^*}$, is equivalent to $\lambda(\mathbf{s}^*)$. This projection is precisely determined as follows,

$$\lambda(\mathcal{S})|_{\mathbf{e}^*} = \sum_{j=1}^{n^*} \langle \lambda(\mathcal{S}), \mathbf{e}_j^* \rangle \mathbf{e}_j^*, \quad (10)$$

where $\langle \cdot, \cdot \rangle$ is a scalar product.

The equivalence relation is an algorithmic problem belonging to complexity class P (Fortnow and Grochow (2011)). A relatively simple algorithm can therefore be applied.

Note that if there exists $\mathbf{s} \preceq \mathcal{S}$ such as $\mathbf{s} \cong \mathbf{s}^*$, then $\lambda(\mathbf{s})|_{\mathbf{e}^*} = \lambda(\mathcal{S})|_{\mathbf{e}^*}$.

Discrete distance of attribute equivalence. The attribute equivalence problem between system \mathcal{S} and target \mathcal{T}_{ar} can be transformed as a distance search problem between the attribute of the system $\lambda(\mathcal{S})$ and that of the target $\lambda(\mathbf{s}^*)$, or even, $\ker(h)$. For all $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^{n^*}$ the following function is defined:

$$\rho_e^h(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \text{if } \mathbf{x}|_{\mathbf{e}^*} \sim_h \mathbf{y}|_{\mathbf{e}^*}; \\ 1, & \text{otherwise,} \end{cases} \quad (11)$$

where $\mathbf{x}|_{\mathbf{e}^*}$ and $\mathbf{y}|_{\mathbf{e}^*}$ are respectively the projections \mathbf{x} and \mathbf{y} on \mathbb{R}^{n^*} .

Proposition 2.2. ρ_e^h is a discrete pseudo-distance.

Proof. For $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x}', \mathbf{z} \in \mathbb{R}^{n^*}$, we check that

- (1) ρ_e^h is defined: $\rho_e^h(\mathbf{x}, \mathbf{x}) = 0$, since $\mathbf{x}|_{\mathbf{e}^*} \sim_h \mathbf{x}|_{\mathbf{e}^*}$;
- (2) ρ_e^h does not satisfy the separation: $\rho_e^h(\mathbf{x}, \mathbf{x}') = 0$ does not imply $\mathbf{x} = \mathbf{x}'$. Verification is straightforward, especially when $n \neq n^*$;
- (3) ρ_e^h is symmetrical: $\rho_e^h(\mathbf{x}, \mathbf{x}') = \rho_e^h(\mathbf{x}', \mathbf{x})$. The verification is direct through the definition of ρ_e^h ;
- (4) ρ_e^h satisfies the triangular inequality: $\rho_e^h(\mathbf{x}, \mathbf{x}') \leq \rho_e^h(\mathbf{x}, \mathbf{z}) + \rho_e^h(\mathbf{z}, \mathbf{x}')$. This axiom is verified simply by showing that the paradoxical case $1 \leq 0$ cannot occur. Suppose that if $\rho_e^h(\mathbf{x}, \mathbf{z}) = 0 \iff \mathbf{x}|_{\mathbf{e}^*} \sim_h \mathbf{z}$, and if $\rho_e^h(\mathbf{z}, \mathbf{x}') = 0 \iff \mathbf{z} \sim_h \mathbf{x}'$. Since, \sim_h being a transitive relation, then $\mathbf{x}|_{\mathbf{e}^*} \sim_h \mathbf{x}' \implies \rho_e^h(\mathbf{x}, \mathbf{x}') = 0$. So, it is impossible to have $1 \leq 0$. Then the triangle inequality holds.

So ρ_e^h is indeed a discrete pseudo-distance (Howes (2012)).

Discrete distance associated with ability. This distance pertains to the state of both the system and the target, encompassing their respective topological structures and attributes. It takes the following form:

$$\rho: \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{N}$$

$$(\mathbf{s}_i, \mathbf{s}_j) \longmapsto \rho_s(\mathbf{s}_i, \mathbf{s}_j) + \rho_e^h(\lambda(\mathbf{s}_i), \lambda(\mathbf{s}_j)) \quad (12)$$

Proposition 2.3. ρ is a weak discrete distance.

Proof. The proof is simple. Indeed, ρ inherits properties common to ρ_s and ρ_e^h , which are respectively a weak discrete distance and a discrete pseudo-distance.

The pair (\mathbb{S}, ρ) constitutes a weak metric space, providing a framework enabling the utilization of mathematical tools associated with weak metric spaces to address the notion of proximity. Assuming that the mission target \mathcal{T}_{ar} is constant, the decision problem therefore boils down to knowing at a given time t during the mission whether, $\rho(\mathcal{S}_t, \mathbf{s}^*) = 0$, the system is capable of carrying out mission \mathcal{M} , and if $\rho(\mathcal{S}_t, \mathbf{s}^*) \neq 0$, the system is not capable.

3. CONCLUSION

The operational capability model proposed in this paper is based on three keywords: *System*, *Mission* and *Ability*. The mission consists of a *Target* and a *Context*. All these keywords are explained and modeled using appropriate mathematical objects. The system and the target of mission are states, each modeled by a multi-graph. The

context consists of elements that exert an influence on the system and are incorporated into the evolution equation of the system. And, the ability is seen as the distance between the system and the target. The distance proposed in this paper is discrete, and we show that it satisfies the axioms of a weak distance. When the distance is equal to zero, the system is considered capable of performing the mission at the instant t when the distance is evaluated. Future work will involve extending this distance to a continuous distance and applying the model to real-life scenarios. In this paper, we justify that the evolution of the system is described by a dynamic equation in the space of multi-graphs. This motivates the need to develop control theory for the evolution equation on the multi-graph space.

REFERENCES

- Alippi, C. and Zambon, D. (2023). Graph Kalman Filters. doi:10.48550/arXiv.2303.12021.
- Arkhangel'Skii, A. and Fedorchuk, V. (2012). *General topology I: basic concepts and constructions dimension theory*, volume 17. Springer Science & Business Media.
- Backlund, A. (2000). The definition of system. *Kybernetes*, 29(4), 444–451.
- Biltgen, P.T. (2007). *A methodology for capability-based technology evaluation for systems-of-systems*, volume 68. Citeseer.
- Birolini, A. (2007). *Reliability engineering*, volume 5. Springer.
- Cho, N., Moon, H., Cho, J., Han, S., and Pyun, J. (2022). A framework for determining required operational capabilities: A combined optimization and simulation approach. *J Defense Manag*, 12, 234.
- Davis, P.K. et al. (2002). *Analytic architecture for capabilities-based planning, mission-system analysis, and transformation*. Rand.
- Foggia, P., Sansone, C., Vento, M., et al. (2001). A performance comparison of five algorithms for graph isomorphism. In *Proceedings of the 3rd IAPR TC-15 Workshop on Graph-based Representations in Pattern Recognition*, 188–199. Citeseer.
- Fortnow, L. and Grochow, J.A. (2011). Complexity classes of equivalence problems revisited. *Information and Computation*, 209(4), 748–763.
- Garey, M.R. and Johnson, D.S. (1979). Computers and intractability. *A Guide to the*.
- Goebel, R., Sanfelice, R.G., and Teel, A.R. (2012). *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press.
- Howes, N.R. (2012). *Modern analysis and topology*. Springer Science & Business Media.
- Hristov, N., Radulov, I., Iliev, P., and Andreeva, P. (2010). Prioritization methodology for development of required operational capabilities. Available at SSRN 3135696.
- Ingalalli, V., Ienco, D., and Poncelet, P. (2016). Sumgra: Querying multigraphs via efficient indexing. In *International Conference on Database and Expert Systems Applications*, 387–401. Springer.
- Klamt, S. and Gilles, E.D. (2004). Minimal cut sets in biochemical reaction networks. *Bioinformatics*, 20(2), 226–234.
- Klir, G.J. (1991). Facets of systems science. *Springer Book Archive-Mathematics*.
- Kozen, D.C. (2007). *Automata and computability*. Springer Science & Business Media.
- Lee, Y. and Lee, T. (2014). Network-based metric for measuring combat effectiveness. *Defence Science Journal*, 64(2), 115–122.
- Li, J., Jiang, J., Yang, K., and Chen, Y. (2018). Research on functional robustness of heterogeneous combat networks. *IEEE Systems Journal*, 13(2), 1487–1495.
- Lin, Y. (1987). A model of general systems. *Mathematical Modelling*, 9(2), 95–104.
- Martin, J., Axelsson, J., Carlson, J., and Suryavedara, J. (2022). The capability concept in the context of systems of systems: A systematic literature review. In *2022 IEEE International Symposium on Systems Engineering (ISSE)*, 1–8. IEEE.
- Mesarovic, M.D. (1964). Foundations for a general systems theory. *Views on general systems theory*, 1–24.
- Micale, G., Bonnici, V., Ferro, A., Shasha, D., Giugno, R., and Pulvirenti, A. (2020). Multiri: Fast subgraph matching in labeled multigraphs. *arXiv preprint arXiv:2003.11546*.
- Moorman, J.D., Tu, T.K., Chen, Q., He, X., and Bertozzi, A.L. (2021). Subgraph matching on multiplex networks. *IEEE Transactions on Network Science and Engineering*, 8(2), 1367–1384.
- Najgebauer, A., Antkiewicz, R., Chmielewski, M., Dyk, M., Kasprzyk, R., Pierzchała, D., Rulka, J., and Tarapata, Z. (2015). The qualitative and quantitative support method for capability based planning of armed forces development. In *Intelligent Information and Database Systems: 7th Asian Conference, ACIIDS 2015, Bali, Indonesia, March 23-25, 2015, Proceedings, Part II 7*, 212–223. Springer.
- Papadopoulos, A. and Troyanov, M. (2006). Weak metrics on euclidean domains. *arXiv preprint math/0609236*.
- Pequito, S., Kar, S., and Aguiar, A.P. (2015). A framework for structural input/output and control configuration selection in large-scale systems. *IEEE Transactions on Automatic Control*, 61(2), 303–318.
- Peysson, F., Ouladsine, M., Noura, H., Leger, J.B., and Allemand, C. (2008a). New approach to prognostic system failures. *IFAC Proceedings Volumes*, 41(2), 12861–12866.
- Peysson, F., Ouladsine, M., Outbib, R., Leger, J.B., Myx, O., and Allemand, C. (2008b). Damage trajectory analysis based prognostic. In *2008 International Conference on Prognostics and Health Management*, 1–8. IEEE.
- Ramos, G., Aguiar, A.P., and Pequito, S. (2020). Structural systems theory: an overview of the last 15 years. *arXiv preprint arXiv:2008.11223*.
- Šiljak, D. (2008). Dynamic graphs. *Nonlinear Analysis: Hybrid Systems*, 2(2), 544–567.
- Silva, E., Cavalcante, E., Batista, T., Oquendo, F., Delicato, F.C., and Pires, P.F. (2014). On the characterization of missions of systems-of-systems. In *Proceedings of the 2014 European Conference on Software Architecture Workshops*, 1–8.
- Torres, L., Blevins, A.S., Bassett, D., and Eliassi-Rad, T. (2021). The why, how, and when of representations for complex systems. *SIAM Review*, 63(3), 435–485.
- Zecevic, A.I. and Šiljak, D.D. (2010). Control of dynamic graphs. *SICE Journal of Control, Measurement, and System Integration*, 3(1), 1–9.