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# Accuracy of the Ensemble Kalman Filter in the Near-Linear Setting

E. Calvello<sup>a,1</sup>, P. Monmarché<sup>b,2,3</sup>, A. M. Stuart<sup>c,1</sup>, and U. Vaes<sup>d,4,5</sup>

<sup>a</sup>e.calvello@caltech.edu, <sup>b</sup>pierre.monmarche@sorbonne-universite.fr, <sup>c</sup>astuart@caltech.edu, <sup>d</sup>urbain.vaes@inria.fr

<sup>1</sup>Department of Computing and Mathematical Sciences, Caltech, USA

<sup>2</sup>Laboratoire Jacques-Louis Lions, Sorbonne Université, France

<sup>3</sup>Institut Universitaire de France, France

<sup>4</sup>MATERIALS project-team, Inria Paris, France

<sup>5</sup>CERMICS, École des Ponts, France

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## Abstract

The filtering distribution captures the statistics of the state of a dynamical system from partial and noisy observations. Classical particle filters provably approximate this distribution in quite general settings; however they behave poorly for high dimensional problems, suffering weight collapse. This issue is circumvented by the ensemble Kalman filter which is an equal-weight interacting particle system. However, this finite particle system is only proven to approximate the true filter in the linear Gaussian case. In practice, however, it is applied in much broader settings; as a result, establishing its approximation properties more generally is important. There has been recent progress in the theoretical analysis of the algorithm, establishing stability and error estimates in non-Gaussian settings, but the assumptions on the dynamics and observation models rule out the unbounded vector fields that arise in practice and the analysis applies only to the mean field limit of the ensemble Kalman filter. The present work establishes error bounds between the filtering distribution and the finite particle ensemble Kalman filter when the model exhibits linear growth.

**Keywords.** Ensemble Kalman filter, Stochastic filtering, Weighted total variation metric, Stability estimates, Accuracy estimates, Near-Linear setting.

**AMS subject classification.** 60G35, 62F15, 65C35, 70F45, 93E11.

## 1 Introduction

### 1.1 Literature Review, Contributions and Outline

Algorithms for filtering employ noisy observations arising from a possibly random dynamical system to estimate the distribution of the state of the dynamical system conditional on the observations. The Kalman filter [15] determines the filtering distribution exactly for linear Gaussian dynamics and observations. The extended Kalman filter was developed as an extension of the Kalman filtering technique to nonlinear problems and is based on a linearization approximation; see [14] and [2]. The linearization approximation leads to an inexact distribution for nonlinear problems, and furthermore requires evaluation of covariance matrices, making the methodology impractical for large-scale geophysical applications [13]. Particle filters or sequential Monte Carlo methods [10, 9] offer an alternative methodology for nonlinear filtering problems, allowing recovery of the exact filtering distribution in the large particle limit. However, this class of methods scales poorly with dimension, and in particular suffer weight collapse, making their application to geophysical problems prohibitive; see for example [21, 5, 19, 1].

The ensemble Kalman filter was introduced in the seminal paper [11]. Its success stems from the low-rank approximation of large covariances, cheaply computed using an ensemble of particles and allowing it to be deployed in geophysical applications. However analysis of its accuracy, in relation to the true filtering distribution, remains in its infancy. The papers [17, 18] studied this issue in the linear Gaussian setting where the mean field limit of the Kalman filter is exact; they demonstrate that the ensemble Kalman filter may be viewed as an interacting particle system approximation of the mean field limit, and establish Monte Carlo type error bounds. The recent article [7] overviews the formulation of ensemble Kalman methods using mean field dynamical systems and provides a platform from which the analyses of [17, 18] may be generalized beyond the linear Gaussian setting. In the recent paper [8] the authors establish stability properties of the mean field ensemble Kalman filter and use them to prove accuracy of the filter in a near-Gaussian setting. However the paper does not consider finite particle approximations of the mean field, and the conditions on the dynamics-observation model require boundedness of the vector fields arising. In this paper we address both these issues, establishing error bounds between the finite particle ensemble Kalman filter and the true filtering distribution in settings where the dynamics and observation vector fields grow linearly. We make the following contributions.

1. [Theorem 2.1](#) is a stability result for the mean field ensemble Kalman filter in the setting of dynamical models and Lipschitz observation operators that grow at most linearly at infinity.
2. [Theorem 2.3](#) quantifies the error between the mean field ensemble Kalman filter and the true filter in the setting of dynamical models and Lipschitz observation operators that are near-linear.
3. [Theorem 2.4](#) quantifies the error between the finite particle ensemble Kalman filter (found as an interacting particle system approximation of the mean field) and the true filter in the setting of near-linear, Lipschitz dynamical models and linear observation operators.

In going beyond the work in [8], the current paper simultaneously addresses a more applicable problem class, by allowing linear growth in the dynamics and observations operators, and confronts the substantial technical challenges which arise from doing so. Bounds on moments of the filtering distribution and mean field ensemble Kalman filter must be established; these bounds grow in the number of iterations which is in contrast to [8] where the  $L^\infty$  bounds on model and observation operator ensure a uniform bound on moments. A further challenge is presented by the need to establish stability bounds for the conditioning map, giving rise to the true filter, and the transport map giving rise to the ensemble Kalman filter. These results exhibit dependence on moments in the stability constants and require control of the growth given by the dynamics; this is again in contrast to [8] where the  $L^\infty$  bounds allow for a uniform control.

After discussing notation that will be used throughout the paper in [Subsection 1.2](#), we introduce the filtering problem in [Subsection 1.3](#). In [Section 2](#) we outline the main results of the paper concerning the ensemble Kalman filter. In [Subsection 2.1](#) we formulate the ensemble Kalman filter that we consider in this paper along with the relevant assumptions we will use in the analysis. We state a stability theorem for the mean field formulation of the ensemble Kalman filter in [Subsection 2.2](#), hence [Contribution 1](#). We leverage this result in [Subsection 2.3](#) to derive a theorem quantifying the error between the mean field ensemble Kalman filter and the true filter, [Contribution 2](#). Finally, in [Subsection 2.4](#), we make use of the results of the previous subsections to state a theorem quantifying the error between the (finite particle) ensemble Kalman filter itself and the true filter, yielding [Contribution 3](#). Various technical results, used in the proof of our three main theorems, may be found in the appendices. We conclude with closing remarks in [Section 3](#).

## 1.2 Notation

The Euclidean norm on  $\mathbb{R}^n$  is denoted by  $|\bullet|$ , and the induced operator norm on matrices is denoted by  $\|\bullet\|$ . For a symmetric positive definite matrix  $S \in \mathbb{R}^{n \times n}$ , the notation  $|\bullet|_S$  refers to the weighted norm  $|S^{-1/2}\bullet|$ .

Given an  $L^\infty$  function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  and  $r \geq 0$ , we let  $B_{L^\infty}(f, r)$  denote the  $L^\infty$  ball of radius  $r$  centered at  $f$ . Similarly, for  $(f, g) \in L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^d)$  and  $r \geq 0$ , we denote by  $B_{L^\infty}((f, g), r)$  the  $L^\infty$  ball of radius  $r$  centered at  $(f, g)$ , on the product space, i.e. the set of all  $(\mathbf{f}, \mathbf{g}) \in L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^d)$  satisfying

$$\|\mathbf{f} - f\|_{L^\infty(\mathbb{R}^n)} \leq r, \quad \|\mathbf{g} - g\|_{L^\infty(\mathbb{R}^d)} \leq r.$$

We let  $|\cdot|_{C^{0,1}}$  denote the  $C^{0,1}$  semi-norm, namely the Lipschitz constant.

We use symbol  $\perp$  to denote independence of two random variables. The notation  $\mathcal{N}(m, C)$ , for  $m \in \mathbb{R}^n$  and  $C \in \mathbb{R}^{n \times n}$ , denotes the Gaussian distribution with mean  $m$  and covariance  $C$ . The notation  $\mathcal{P}(\mathbb{R}^n)$  denotes the set of probability measures over  $\mathbb{R}^n$ , while  $\mathcal{G}(\mathbb{R}^n)$  denotes the set of Gaussian probability measures over  $\mathbb{R}^n$ . Throughout this paper all probability measures have a Lebesgue density, because of our assumptions concerning the noise structure in the dynamics model and the data acquisition model. Thus we abuse notation by using the same symbols for probability measures and their densities. For  $\mu \in \mathcal{P}(\mathbb{R}^n)$ , the notation  $\mu(x)$  for  $x \in \mathbb{R}^n$  refers to the Lebesgue density of  $\mu$  evaluated at  $x$ , whereas  $\mu[f]$  for a function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is a short-hand notation for  $\int_{\mathbb{R}^n} f d\mu$ .

For a probability measure  $\mu \in \mathcal{P}(\mathbb{R}^n)$ , the notation  $\mathcal{M}_q(\mu)$  denotes the  $q$ th polynomial moment under the measure  $\mu$ , defined as

$$\mathcal{M}_q(\mu) := \int_{\mathbb{R}^n} |x|^q \mu(dx). \quad (1.1)$$

The notations  $\mathcal{M}(\mu)$  and  $\mathcal{C}(\mu)$  denote respectively the mean and covariance under  $\mu$ :

$$\mathcal{M}(\mu) = \mu[x], \quad \mathcal{C}(\mu) = \mu\left[(x - \mathcal{M}(\mu)) \otimes (x - \mathcal{M}(\mu))\right].$$

The notation  $\mathcal{P}_R(\mathbb{R}^n)$  for  $R \geq 1$  refers to the subset of  $\mathcal{P}(\mathbb{R}^n)$  of probability measures whose mean and covariance satisfy the bound

$$|\mathcal{M}(\mu)| \leq R, \quad \frac{1}{R^2} I_n \preceq \mathcal{C}(\mu) \preceq R^2 I_n. \quad (1.2)$$

Here  $I_n$  denotes the identity matrix in  $\mathbb{R}^{n \times n}$ , and  $\preceq$  is the partial ordering defined by the convex cone of positive semi-definite matrices. Similarly,  $\mathcal{G}_R(\mathbb{R}^n)$  is the subset of  $\mathcal{G}(\mathbb{R}^n)$  of probability measures satisfying (1.2). We also introduce the set  $\mathcal{P}_{\succ 0}^2(\mathbb{R}^{d_u \times d_y})$  of probability measures  $\pi$  with finite second moment satisfying  $\mathcal{C}^{yy}(\pi) \succ 0$ .

For a probability measure  $\pi \in \mathcal{P}(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$  associated with random variable  $(u, y) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_y}$  we use the notation  $\mathcal{M}^u(\pi)$ ,  $\mathcal{M}^y(\pi)$  for the means of the marginal distributions, and the notation  $\mathcal{C}^{uu}(\pi)$ ,  $\mathcal{C}^{yy}(\pi)$  and  $\mathcal{C}^{uy}(\pi)$  for the blocks of the covariance matrix  $\mathcal{C}(\pi)$ . That is to say,

$$\mathcal{M}(\pi) = \begin{pmatrix} \mathcal{M}^u(\pi) \\ \mathcal{M}^y(\pi) \end{pmatrix}, \quad \mathcal{C}(\pi) = \begin{pmatrix} \mathcal{C}^{uu}(\pi) & \mathcal{C}^{uy}(\pi) \\ \mathcal{C}^{uy}(\pi)^\top & \mathcal{C}^{yy}(\pi) \end{pmatrix}. \quad (1.3)$$

For  $h: \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_y}$  we also define  $\mathcal{C}^{hh}(\pi)$  to be the covariance of vector  $h(u)$ , for  $u$  distributed according to the marginal of  $\pi$  on  $u$ , and  $\mathcal{C}^{uh}(\pi)$  to be the covariance between  $u$  and  $h(u)$ .

We also introduce the Gaussian projection operator  $\mathbf{G}: \mathcal{P}(\mathbb{R}^n) \rightarrow \mathcal{G}(\mathbb{R}^n)$  given by  $\mathbf{G}\mu = \mathcal{N}(\mathcal{M}(\mu), \mathcal{C}(\mu))$ . We refer to  $\mathbf{G}$  as a projection because  $\mathbf{G}\mu$  is the Gaussian distribution closest to  $\mu$  with respect to  $\text{KL}(\mu \parallel \bullet)$  [6], where  $\text{KL}(\mu \parallel \nu)$  is the Kullback–Leibler (KL) divergence of  $\mu$  from  $\nu$ . Note that  $\mathbf{G}$  defines a nonlinear mapping. Throughout this work, we employ the following weighted total variation distance, shown to be useful in the analysis of ensemble Kalman methods in [8]:

**Definition 1.1.** Let  $g: \mathbb{R}^n \rightarrow [1, \infty)$  denote  $g(v) := 1 + |v|^2$ . We define the weighted total variation metric  $d_g: \mathcal{P}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \rightarrow \mathbb{R}$

$$d_g(\mu_1, \mu_2) = \sup_{|f| \leq g} |\mu_1[f] - \mu_2[f]|,$$

where the supremum is over all functions  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  which are bounded from above by  $g$  pointwise and in

absolute value.

**Remark 1.2.** *The following remarks concern the distance  $d_g$ :*

- If  $\mu_1, \mu_2$  have Lebesgue densities  $\rho_1, \rho_2$ , then

$$d_g(\mu_1, \mu_2) = \int g(v) |\rho_1(v) - \rho_2(v)| dv.$$

- Unlike the usual total variation distance, the weighted metric in [Definition 1.1](#) enables control of the differences  $|\mathcal{M}(\mu_1) - \mathcal{M}(\mu_2)|$  and  $\|\mathcal{C}(\mu_1) - \mathcal{C}(\mu_2)\|$ . This is the content of [Lemma B.6](#), proved in the appendix, which is used in the proof of a key auxiliary result ([Lemma B.2](#)).

### 1.3 Filtering Distribution

We consider the following stochastic dynamics and data model

$$u_{j+1} = \Psi(u_j) + \xi_j, \quad \xi_j \sim \mathcal{N}(0, \Sigma), \quad (1.4a)$$

$$y_{j+1} = h(u_{j+1}) + \eta_{j+1}, \quad \eta_{j+1} \sim \mathcal{N}(0, \Gamma). \quad (1.4b)$$

Here  $\{u_j\}_{j \in \llbracket 0, J \rrbracket}$  is the unknown state, evolving in  $\mathbb{R}^{d_u}$ , and  $\{y_j\}_{j \in \llbracket 1, J \rrbracket}$  are the observations, evolving in  $\mathbb{R}^{d_y}$ . We assume that the initial state is distributed according to a Gaussian distribution  $u_0 \sim \mathcal{N}(m_0, C_0)$  and that the following independence assumption is satisfied:

$$u_0 \perp \xi_0 \perp \cdots \perp \xi_{J-1} \perp \eta_1 \perp \cdots \perp \eta_J. \quad (1.5)$$

The filtering distribution  $\mu_j$  is the conditional distribution of the state  $u_j$  given a realization  $Y_j^\dagger := \{y_1^\dagger, \dots, y_j^\dagger\}$  of the data process up to step  $j$ . Data  $Y_j^\dagger$  may be thought of as arising from a realization of [\(1.4\)](#); but the case of model misspecification, where  $Y_j^\dagger$  does not necessarily arise from [\(1.4\)](#), is also of interest.

It is well-known [\[16, 20\]](#) that the true filtering distribution evolves according to

$$\mu_{j+1} = \mathsf{L}_j \mathsf{P} \mu_j. \quad (1.6)$$

where  $\mathsf{P}$  and  $\mathsf{L}_j$  are maps on probability measures, referred to respectively as the *prediction* and *analysis* steps in the data assimilation community [\[3\]](#). The prediction operator  $\mathsf{P}$  is linear and defined by the Markov kernel associated with the stochastic dynamics [\(1.4a\)](#). Its action on a probability measure with Lebesgue density  $\mu$  reads

$$\mathsf{P}\mu(u) = \frac{1}{\sqrt{(2\pi)^{d_u} \det \Sigma}} \int_{\mathbb{R}^{d_u}} \exp\left(-\frac{1}{2}|u - \Psi(v)|_\Sigma^2\right) \mu(v) dv. \quad (1.7)$$

The operator  $\mathsf{L}_j$  is a nonlinear map which formalizes the incorporation of the new datum  $y_{j+1}^\dagger$  using Bayes' theorem. Its action reads

$$\mathsf{L}_j \mu(u) = \frac{\exp\left(-\frac{1}{2}|y_{j+1}^\dagger - h(u)|_\Gamma^2\right) \mu(u)}{\int_{\mathbb{R}^{d_u}} \exp\left(-\frac{1}{2}|y_{j+1}^\dagger - h(U)|_\Gamma^2\right) \mu(U) dU} \quad (1.8)$$

The operator  $\mathsf{L}_j$  effects a reweighting of the measure to which it applies, with more weight assigned to the state values that are consistent with the observation. It is convenient in this work to decompose the analysis map  $\mathsf{L}_j$  into the composition  $\mathsf{B}_j \mathsf{Q}$ , where the operators  $\mathsf{Q}: \mathcal{P}(\mathbb{R}^{d_u}) \rightarrow \mathcal{P}(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$  and  $\mathsf{B}_j: \mathcal{P}(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y}) \rightarrow \mathcal{P}(\mathbb{R}^{d_u})$  are defined by

$$\mathsf{Q}\mu(u, y) = \frac{1}{\sqrt{(2\pi)^{d_y} \det \Gamma}} \exp\left(-\frac{1}{2}|y - h(u)|_\Gamma^2\right) \mu(u). \quad (1.9)$$

and

$$\mathsf{B}_j \mu(u) = \frac{\mu(u, y_{j+1}^\dagger)}{\int_{\mathbb{R}^{d_u}} \mu(U, y_{j+1}^\dagger) dU}. \quad (1.10)$$

The operator  $\mathsf{Q}$  maps a probability measure with density  $\mu$  into the density associated with the joint random variable  $(U, h(U) + \eta)$ , where  $U \sim \mu$  is independent of  $\eta \sim \mathcal{N}(0, \Gamma)$ . The operator  $\mathsf{B}_j$  performs conditioning on the data  $y_{j+1}^\dagger$ . Map  $\mathsf{Q}$  is linear whilst  $\mathsf{B}_j$  is nonlinear. We may thus write (1.6) in the form

$$\mu_{j+1} = \mathsf{B}_j \mathsf{Q} \mathsf{P} \mu_j. \quad (1.11)$$

## 2 The Ensemble Kalman Filter

In [Subsection 2.1](#) we define the specific version of the mean field ensemble Kalman filter that we analyze here; other versions may be found in [\[7\]](#) and will be amenable to similar analyses. In [Subsection 2.2](#) we prove our main stability theorem, showing that the error between the true filter and its Gaussian projection, on the joint space of state and observations, may be used to control the error between the true filter and the mean field ensemble Kalman filter. In [Subsection 2.3](#) we prove a corollary to this theorem, establishing that the mean field ensemble Kalman filter accurately approximates the true filter for a specific class of non-Gaussian problems. In [Subsection 2.4](#) we deploy the results of the two preceding subsections to state a theorem quantifying the error between the finite particle ensemble Kalman filter itself and the true filtering distribution. All of the theorems allow for linear growth of the vector fields defining the dynamics and the observation processes.

### 2.1 The Algorithm

The ensemble Kalman filter may be derived as a particle approximation of various mean field dynamics [\[7\]](#). The specific mean field ensemble Kalman filter that we study in this paper reads

$$\hat{u}_{j+1} = \Psi(u_j) + \xi_j, \quad \xi_j \sim \mathcal{N}(0, \Sigma), \quad (2.1a)$$

$$\hat{y}_{j+1} = h(\hat{u}_{j+1}) + \eta_{j+1}, \quad \eta_{j+1} \sim \mathcal{N}(0, \Gamma), \quad (2.1b)$$

$$u_{j+1} = \hat{u}_{j+1} + \mathcal{C}^{uy} (\hat{\pi}_{j+1}^{\text{EK}})^{-1} (y_{j+1}^\dagger - \hat{y}_{j+1}), \quad (2.1c)$$

where  $\hat{\pi}_{j+1}^{\text{EK}} = \text{Law}(\hat{u}_{j+1}, \hat{y}_{j+1})$  and independence of the noise terms (1.5) is still assumed to hold, as in (1.5). See [Subsection 1.2](#) for the definition of the covariance matrices that appear in (2.1c). We denote by  $\mu_j^{\text{EK}}$  the law of  $u_j$ . In order to rewrite the evolution of  $\mu_j^{\text{EK}}$  in terms of maps on probability measures we introduce, for a given  $y_{j+1}^\dagger$ , the map  $\mathsf{T}_j: \mathcal{P}_{>0}^2(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y}) \rightarrow \mathcal{P}(\mathbb{R}^{d_u})$  defined by

$$\mathsf{T}_j \pi = \mathcal{S}(\bullet, \bullet; \pi, y_{j+1}^\dagger)_\# \pi. \quad (2.2)$$

Here the subscript  $\#$  denotes the pushforward and, for given  $\pi \in \mathcal{P}(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$ ,  $z \in \mathbb{R}^{d_y}$ ,

$$\begin{aligned} \mathcal{S}(\bullet, \bullet; \pi, z): \mathbb{R}^{d_u} \times \mathbb{R}^{d_y} &\rightarrow \mathbb{R}^{d_u}; \\ (u, y) &\mapsto u + \mathcal{C}^{uy}(\pi) \mathcal{C}^{yy}(\pi)^{-1} (z - y). \end{aligned} \quad (2.3)$$

Evolution of the probability measure  $\mu_j^{\text{EK}}$  may then be written as [\[7\]](#)

$$\mu_{j+1}^{\text{EK}} = \mathsf{T}_j \mathsf{Q} \mathsf{P} \mu_j^{\text{EK}}. \quad (2.4)$$

We now discuss the preceding map in relation to (1.11). The specific affine map  $\mathcal{T}$  used in (2.1c) is determined by measure  $\widehat{\pi}_{j+1}^{\text{EK}}$  (here equal to  $\text{QP}\mu_j^{\text{EK}}$ ) and data  $y_{j+1}^\dagger$ . The following observations explain why the law of  $u_j$  in (2.1) evolves according to (2.4):

- if  $u_j \sim \mu_j^{\text{EK}}$ , then  $\widehat{u}_{j+1} \sim \text{P}\mu_j^{\text{EK}}$ , by definition of  $\text{P}$ ;
- random vector  $(\widehat{u}_{j+1}, \widehat{y}_{j+1})$  is distributed according to  $\widehat{\pi}_{j+1}^{\text{EK}} = \text{QP}\mu_j^{\text{EK}}$ ;
- equation (2.1c) then implies that  $u_{j+1} \sim \text{T}_j \text{QP}\mu_j^{\text{EK}}$ .

As we show in Lemma B.7, operator  $\text{T}_j$  coincides with conditioning operator  $\text{B}_j$  over the Gaussians  $\mathcal{G}(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y}) \subset \mathcal{P}(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$ . In the particular case where  $\mu_0$  is Gaussian, which is a standing assumption in this paper, and the operators  $\Psi$  and  $h$  are linear, the mean field ensemble Kalman filter (2.4) thus reproduces the exact filtering distribution (1.11), given by the Kalman filter itself. The aim of this paper is to analyze the accuracy of ensemble Kalman methods when  $\Psi$  and  $h$  are not assumed to be linear.

The ensemble Kalman filter as implemented in practice may be derived as a particle approximation of the mean field dynamics as defined by sample paths in (2.1) or as an evolution on measures in (2.4). For any  $\pi$  in the range of  $\text{Q}$ ,  $\mathcal{C}^{yy}(\pi) = \mathcal{C}^{hh}(\pi) + \Gamma$  and  $\mathcal{C}^{uy}(\pi) = \mathcal{C}^{uh}(\pi)$ , using the notation introduced in, and following, (1.3). This is since the noise in the observation component defining  $\pi$  is then independent of the state component defining  $\pi$ . Using this the particle approximation of (2.1) takes the form

$$\widehat{u}_{j+1}^{(i)} = \Psi(u_j^{(i)}) + \xi_j^{(i)}, \quad \xi_j^{(i)} \sim \mathcal{N}(0, \Sigma), \quad (2.5a)$$

$$\widehat{y}_{j+1}^{(i)} = h(\widehat{u}_{j+1}^{(i)}) + \eta_{j+1}^{(i)}, \quad \eta_{j+1}^{(i)} \sim \mathcal{N}(0, \Gamma), \quad (2.5b)$$

$$u_{j+1}^{(i)} = \widehat{u}_{j+1}^{(i)} + \mathcal{C}^{uh} \left( \widehat{\pi}_{j+1}^{\text{EK}, N} \right) \left( \mathcal{C}^{hh} \left( \widehat{\pi}_{j+1}^{\text{EK}, N} \right) + \Gamma \right)^{-1} \left( y_{j+1}^\dagger - \widehat{y}_{j+1}^{(i)} \right). \quad (2.5c)$$

where  $\widehat{\pi}_{j+1}^{\text{EK}, N}$  is the empirical measure

$$\widehat{\pi}_{j+1}^{\text{EK}, N} = \frac{1}{N} \sum_{i=1}^N \delta_{(\widehat{u}_{j+1}^{(i)}, \widehat{y}_{j+1}^{(i)})},$$

and  $\xi_j^{(i)} \sim \mathcal{N}(0, \Sigma)$  i.i.d. in both  $i$  and  $j$  and  $\eta_{j+1}^{(i)} \sim \mathcal{N}(0, \Gamma)$  i.i.d. in both  $i$  and  $j$ ; furthermore, the set of  $\{\xi_j^{(i)}\}$  are independent from the set of  $\{\eta_{j+1}^{(i)}\}$ . Choosing to express the particle approximation of the covariance in observation space through  $\mathcal{C}^{hh}$  and  $\Gamma$  ensures invertibility, provided  $\Gamma$  is invertible. From the particles evolving according to the dynamics in (2.5) we define the empirical measure

$$\mu_{j+1}^{\text{EK}, N} = \frac{1}{N} \sum_{i=1}^N \delta_{u_{j+1}^{(i)}}, \quad (2.6)$$

whose evolution describes the ensemble Kalman filter.

Our theorems in Subsections 2.2 and 2.3 concern relationships between the true filter (1.11) and the mean field ensemble Kalman filter (2.4). We study the setting in which  $\Psi$  and  $h$  are not assumed to be linear, so that the true filter is not Gaussian, in Subsection 2.2; and then we study small perturbations away from the Gaussian setting that arise when the vector fields  $\Psi$  and  $h$  are close to affine in Subsection 2.3. The theorem in Subsection 2.4 concerns the relationship between the true filter (1.11) and the ensemble Kalman filter (2.6). In this subsection, we combine existing analysis on the convergence of the ensemble Kalman filter to the mean field ensemble Kalman filter with the results from the previous subsections to state and prove an error estimate between the ensemble Kalman filter and the true filter in the non-linear setting. Specifically, we study the case of a vector field  $\Psi$  that is a bounded perturbation away from affine and an affine vector field  $h$ . To state our theorems we will use the following set of assumptions:

**Assumption H.** *There exists positive constants  $\kappa_y, \kappa_\Psi, \kappa_h, \ell_h, \sigma$  and  $\gamma$  such that the data  $\{y_j^\dagger\}$ , the vector fields  $(\Psi, h)$  and the covariances  $(\Sigma, \Gamma)$  satisfy:*

(H1) data  $Y^\dagger = \{y_j^\dagger\}_{j=1}^J$  lies in set  $B_y \subset \mathbb{R}^{KJ}$  defined by

$$B_y := \left\{ Y^\dagger \in \mathbb{R}^{KJ} : \max_{j \in \llbracket 1, J \rrbracket} |y_j^\dagger| \leq \kappa_y \right\};$$

(H2) function  $\Psi: \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_u}$  satisfies  $|\Psi(u)| \leq \kappa_\Psi(1 + |u|)$  for all  $u \in \mathbb{R}^{d_u}$ ;

(H3) function  $h: \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_y}$  satisfies  $|h(u)| \leq \kappa_h(1 + |u|)$  for all  $u \in \mathbb{R}^{d_u}$ ;

(H4) function  $h: \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_y}$  satisfies  $|h|_{C^{0,1}} \leq \ell_h < \infty$ ;

(H5) covariance matrices  $\Sigma$  and  $\Gamma$  are positive definite:  $\Sigma \succcurlyeq \sigma I_{d_u}$  and  $\Gamma \succcurlyeq \gamma I_{d_y}$  for positive  $\sigma$  and  $\gamma$ .

**Assumption V.** The vector fields  $(\Psi, h)$  are affine, i.e. they satisfy the following

(V1) The function  $\Psi: \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_u}$  satisfies  $\Psi(u) := Mu + b$ , for some  $M \in \mathbb{R}^{d_u \times d_u}$  and  $b \in \mathbb{R}^{d_u}$ .

(V2) The function  $h: \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_y}$  satisfies  $h(u) := Hu + w$ , for some  $H \in \mathbb{R}^{d_y \times d_u}$  and  $w \in \mathbb{R}^{d_y}$ .

## 2.2 Stability Theorem: Mean Field Ensemble Kalman Filter

Roughly speaking, our stability theorem states that if the true filtering distributions  $(\mu_j)_{j \in \llbracket 0, J-1 \rrbracket}$  are close to Gaussian, after appropriate lifting to the state/data space, then the distribution  $\mu_j^{\text{EK}}$  given by the mean field ensemble Kalman filter (2.4) is close to the true filtering distribution  $\mu_j$  given by (1.11) for all  $j \in \llbracket 0, J \rrbracket$ .

**Theorem 2.1 (Stability: Mean Field Ensemble Kalman Filter).** Assume that the probability measures  $(\mu_j)_{j \in \llbracket 0, J \rrbracket}$  and  $(\mu_j^{\text{EK}})_{j \in \llbracket 0, J \rrbracket}$  are obtained respectively from the dynamical systems (1.11) and (2.4), initialized at the same Gaussian probability measure  $\mu_0 = \mu_0^{\text{EK}} \in \mathcal{G}(\mathbb{R}^{d_u})$ . That is,

$$\mu_{j+1} = \mathbf{B}_j \mathbf{Q} \mathbf{P} \mu_j, \quad \mu_{j+1}^{\text{EK}} = \mathbf{T}_j \mathbf{Q} \mathbf{P} \mu_j^{\text{EK}}.$$

If Assumption H holds, then there exists  $C = C(\mathcal{M}_{\max\{3+d_u, 4+d_y\}}(\mu_0), \kappa_y, \kappa_\Psi, \kappa_h, \ell_h, \Sigma, \Gamma, J)$  such that

$$d_g(\mu_J^{\text{EK}}, \mu_J) \leq C \max_{j \in \llbracket 0, J-1 \rrbracket} d_g(\mathbf{Q} \mathbf{P} \mu_j, \mathbf{G} \mathbf{Q} \mathbf{P} \mu_j).$$

Proof of Theorem 2.1 below relies on the following auxiliary results, all proved in Appendix B.

1. For any probability measure  $\mu$  with finite first and second order polynomial moments  $\mathcal{M}_1(\mu)$  and  $\mathcal{M}_2(\mu)$ , the means of the probability measures  $\mathbf{P}\mu$  and  $\mathbf{Q}\mathbf{P}\mu$  are bounded from above, and their covariances are bounded both from above and from below. The constants in these bounds depend only on the parameters  $\kappa_\Psi, \kappa_h, \Sigma, \Gamma$  and on  $\mathcal{M}_1(\mu)$  and  $\mathcal{M}_2(\mu)$ . See Lemmas B.1 and B.2.
2. Let  $(\mu_j)_{j \in \llbracket 1, J \rrbracket}$  and  $(\mu_j^{\text{EK}})_{j \in \llbracket 1, J \rrbracket}$  denote the probability measures obtained respectively from the dynamical systems (1.11) and (2.4), initialized at the same Gaussian measure  $\mu_0 = \mu_0^{\text{EK}} \in \mathcal{G}(\mathbb{R}^{d_u})$ . Then for any  $q \in \mathbb{N}$ , there exist constants  $M_q, M_q^{\text{EK}} < \infty$  depending on  $\mathcal{M}_q(\mu_0), \kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma, J$  so that

$$\max_{j \in \llbracket 0, J \rrbracket} \mathcal{M}_q(\mu_j) \leq M_q \quad \text{and} \quad \max_{j \in \llbracket 0, J \rrbracket} \mathcal{M}_q(\mu_j^{\text{EK}}) \leq M_q^{\text{EK}}.$$

See Lemmas B.3 and B.5. This will facilitate use of the stability results from items 6 and 7.

3. For any Gaussian measure  $\mu \in \mathcal{G}(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$ , it holds that  $\mathbf{B}_j \mu = \mathbf{T}_j \mu$ . See Lemma B.7 and also [7].
4. The map  $\mathbf{P}$  is Lipschitz on  $\mathcal{P}(\mathbb{R}^{d_u})$  for the metric  $d_g$ , with a Lipschitz constant  $L_{\mathbf{P}}$  depending on the parameters  $\kappa_\Psi$  and  $\Sigma$ . See Lemma B.8.



5. The map  $Q$  is Lipschitz on  $\mathcal{P}(\mathbb{R}^{d_u})$  for the metric  $d_g$ , with a Lipschitz constant  $L_Q$  depending on the parameters  $\kappa_h$  and  $\Gamma$ . See [Lemma B.9](#).
6. The map  $B_j$  satisfies, for any  $\pi \in \{\text{QP}\mu : \mu \in \mathcal{P}(\mathbb{R}^{d_u}) \text{ and } \mathcal{M}_2(\mu) < \infty\} \subset \mathcal{P}(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$ , the bound

$$\forall j \in \llbracket 0, J \rrbracket, \quad d_g(B_j G\pi, B_j \pi) \leq C_B d_g(G\pi, \pi),$$

where  $C_B = C_B(\mathcal{M}_2(\mu), \kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma)$ . This statement concerns the stability of the  $B_j$  operator between a measure and its Gaussian approximation. See [Lemma B.10](#).

7. The map  $T_j$  satisfies the following bound: for all  $R \geq 1$ , it holds for all probability measures  $\pi \in \mathcal{P}_R(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$  and  $p \in \{\text{QP}\mu : \mu \in \mathcal{P}(\mathbb{R}^{d_u}) \text{ and } \mathcal{M}_{\max\{3+d_u, 4+d_y\}}(\mu) < \infty\} \subset \mathcal{P}(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$  that

$$\forall j \in \llbracket 0, J \rrbracket, \quad d_g(T_j \pi, T_j p) \leq L_T d_g(\pi, p),$$

for a constant  $L_T = L_T(R, \mathcal{M}_{\max\{3+d_u, 4+d_y\}}(\mu), \kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma)$ . This statement may be viewed as a local Lipschitz continuity result in the case where the second argument of  $d_g$  is restricted to the range of QP. See [Lemma B.11](#).

*Proof of Theorem 2.1.* In what follows we refer to the preceding itemized list to clarify the proof. For notational simplicity it is helpful to define the following measure of the difference between the true filtering distribution and its Gaussian projection:

$$\varepsilon := \max_{j \in \llbracket 0, J-1 \rrbracket} d_g(\text{QP}\mu_j, \text{GQP}\mu_j). \quad (2.7)$$

Assume throughout the following that  $j \in \llbracket 0, J-1 \rrbracket$ . The proof rests on the following use of the triangle inequality:

$$\begin{aligned} d_g(\mu_{j+1}^{\text{EK}}, \mu_{j+1}) &= d_g(T_j \text{QP}\mu_j^{\text{EK}}, B_j \text{QP}\mu_j) \\ &\leq d_g(T_j \text{QP}\mu_j^{\text{EK}}, T_j \text{QP}\mu_j) + d_g(T_j \text{QP}\mu_j, T_j \text{GQP}\mu_j) + d_g(B_j \text{GQP}\mu_j, B_j \text{QP}\mu_j). \end{aligned} \quad (2.8a)$$

We have used the fact that  $T_j \text{GQP}\mu_j = B_j \text{GQP}\mu_j$  by [Item 3 \(Lemma B.7\)](#). [Item 2 \(Lemmas B.3 and B.5\)](#) shows that, for any  $q \in \mathbb{N}$ , there exist constants  $M_q, M_q^{\text{EK}} < \infty$  depending on  $\mathcal{M}_q(\mu_0), \kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma, J$  so that

$$\max_{j \in \llbracket 0, J \rrbracket} \mathcal{M}_q(\mu_j) \leq M_q \quad \text{and} \quad \max_{j \in \llbracket 0, J \rrbracket} \mathcal{M}_q(\mu_j^{\text{EK}}) \leq M_q^{\text{EK}}.$$

Therefore by [Item 1 \(Lemma B.2\)](#), there is a constant  $R \geq 1$ , depending on the covariance matrices  $\Sigma, \Gamma$ , the bounds  $\kappa_\Psi$  and  $\kappa_h$  from [Assumption H](#), and the moment bounds  $M_2$  and  $M_2^{\text{EK}}$ , such that for any  $j \in \llbracket 0, J-1 \rrbracket$  it holds that  $\text{QP}\mu_j, \text{QP}\mu_j^{\text{EK}} \in \mathcal{P}_R(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$ . By [Items 4, 5 and 7 \(Lemmas B.8, B.9 and B.11\)](#), the first term in [\(2.8a\)](#) satisfies

$$d_g(T_j \text{QP}\mu_j^{\text{EK}}, T_j \text{QP}\mu_j) \leq L_T(R, M_{\max\{3+d_u, 4+d_y\}}) L_Q L_P d_g(\mu_j^{\text{EK}}, \mu_j), \quad (2.9)$$

where, for conciseness, we omitted the dependence of the constants on  $\kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma$ .

Since it holds that  $\text{GQP}\mu_j \in \mathcal{G}_R(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$  by definition of  $R$ , the second term in [\(2.8a\)](#) may be bounded using [Item 7 \(Lemma B.11\)](#) and the definition in [\(2.7\)](#) of  $\varepsilon$ :

$$d_g(T_j \text{QP}\mu_j, T_j \text{GQP}\mu_j) \leq L_T(R, M_{\max\{3+d_u, 4+d_y\}}) d_g(\text{QP}\mu_j, \text{GQP}\mu_j) \leq L_T(R, M_{\max\{3+d_u, 4+d_y\}}) \varepsilon.$$

Finally, the third term in [\(2.8a\)](#) can be bounded using item 6 ([Lemma B.10](#)) and the definition in [\(2.7\)](#) of  $\varepsilon$ :

$$d_g(B_j \text{GQP}\mu_j, B_j \text{QP}\mu_j) \leq C_B d_g(\text{GQP}\mu_j, \text{QP}\mu_j) \leq C_B \varepsilon.$$

Therefore, letting  $\ell = L_{\mathcal{T}}(R, M_{\max\{3+d_u, 4+d_y\}})L_{\mathcal{Q}}L_{\mathcal{P}}$ , we have shown that

$$d_g(\mu_{j+1}^{\text{EK}}, \mu_{j+1}) \leq \ell d_g(\mu_j^{\text{EK}}, \mu_j) + (L_{\mathcal{T}}(R, M_{\max\{3+d_u, 4+d_y\}}) + C_{\mathcal{B}})\varepsilon$$

and the conclusion follows from the discrete Grönwall lemma, using the fact that  $\mu_0^{\text{EK}} = \mu_0$ .  $\square$

### 2.3 Error Estimate: Mean Field Ensemble Kalman Filter

[Theorem 2.1](#) shows that the mean field ensemble Kalman filter error can be made arbitrarily small if the true filter is arbitrarily close to its Gaussian projection, in state-observation space. This “closeness to Gaussian” assumption can be satisfied in our setting of unbounded vector fields by considering small perturbations of affine vector fields, stated and proved in [Proposition 2.2](#). Combining [Proposition 2.2](#) with [Theorem 2.1](#) gives an error estimate for the mean field ensemble Kalman filter, yielding [Theorem 2.3](#).

**Proposition 2.2 (Approximation Result).** *Suppose that the data  $Y^\dagger = \{y_j^\dagger\}_{j=1}^J$  and the matrices  $(\Sigma, \Gamma)$  satisfy [Assumptions \(H1\)](#) and [\(H5\)](#). Fix  $\kappa_\Psi, \kappa_h > 0$  and assume that  $\Psi_0: \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_u}$  and  $h_0: \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_y}$  are affine functions and hence satisfy [Assumptions \(V1\)](#) and [\(V2\)](#), respectively, while also satisfying [Assumptions \(H2\)](#) and [\(H3\)](#) with  $\kappa_\Psi, \kappa_h$ . Let  $(\mu_j)_{j \in \llbracket 0, J \rrbracket}$  denote the true filtering distribution associated with functions  $(\Psi, h)$ , initialized at the Gaussian probability measure  $\mu_0 = \mathcal{N}(m_0, C_0) \in \mathcal{G}(\mathbb{R}^{d_u})$ . Then, for any  $J \in \mathbb{Z}^+$ , there is  $C = C(m_0, C_0, \kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma, J) > 0$  such that for all  $\varepsilon \in [0, 1]$  and all  $(\Psi, h) \in B_{L^\infty}((\Psi_0, h_0), \varepsilon)$ , it holds that*

$$\max_{j \in \llbracket 0, J-1 \rrbracket} d_g(\text{GQP}\mu_j, \text{QP}\mu_j) \leq C\varepsilon. \quad (2.10)$$

*Proof.* In what follows  $(\mu_j^0)_{j \in \llbracket 0, J \rrbracket}$  and  $(\mu_j)_{j \in \llbracket 0, J \rrbracket}$  denote the true filtering distributions associated with functions  $(\Psi_0, h_0)$  and  $(\Psi, h)$ , respectively, initialized at the same Gaussian measure  $\mathcal{N}(m_0, C_0)$ . Furthermore, we let  $\text{P}_0$  and  $\text{Q}_0$  denote the kernel integral operators [\(1.7\)](#) and [\(1.9\)](#) defined by the specific vector fields  $(\Psi_0, h_0)$ . By [Lemma B.3](#), the filtering distributions have bounded second moments. Let

$$\mathcal{R} = \max_{j \in \llbracket 0, J-1 \rrbracket} \left( \left| y_{j+1}^\dagger \right|^2, 1 + \mathcal{M}_2(\mu_j^0), 1 + \mathcal{M}_2(\mu_j) \right).$$

Throughout this proof,  $C$  denotes a constant whose value is irrelevant in the context, depends only on the constants  $m_0, C_0, \kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma, j$  (in particular, it does not depend on  $\varepsilon, \Psi, h$ ), and may change from line to line. Fix  $j \in \llbracket 0, J-1 \rrbracket$ . Note that the filtering distribution defined by  $(\Psi_0, h_0)$  is Gaussian. Then, using the triangle inequality and Gaussianity of  $\text{Q}_0\text{P}_0\mu_j^0$  we obtain

$$\begin{aligned} d_g(\text{GQP}\mu_j, \text{QP}\mu_j) &\leq d_g(\text{GQP}\mu_j, \text{GQ}_0\text{P}_0\mu_j^0) + d_g(\text{GQ}_0\text{P}_0\mu_j^0, \text{QP}\mu_j) \\ &= d_g(\text{GQP}\mu_j, \text{GQ}_0\text{P}_0\mu_j^0) + d_g(\text{Q}_0\text{P}_0\mu_j^0, \text{QP}\mu_j). \end{aligned}$$

We note that since the filters have bounded first and second order polynomial moments, by [Lemmas B.1](#) and [B.2](#) we may deduce that there exists  $R \geq 1$  such that

$$\forall j \in \llbracket 0, J-1 \rrbracket, \quad \text{P}_0\mu_j^0 \in \mathcal{P}_R(\mathbb{R}^{d_u}), \quad \text{QP}\mu_j, \text{Q}_0\text{P}_0\mu_j^0 \in \mathcal{P}_R(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y}). \quad (2.11)$$

The second part of this display allows direct application of [Lemma C.1](#), which concerns the local Lipschitz continuity result for the Gaussian projection operator  $\text{G}$ , to obtain

$$\begin{aligned} d_g(\text{GQP}\mu_j, \text{QP}\mu_j) &\leq C d_g(\text{Q}_0\text{P}_0\mu_j^0, \text{QP}\mu_j) \\ &\leq C d_g(\text{Q}_0\text{P}_0\mu_j^0, \text{QP}_0\mu_j^0) + C d_g(\text{QP}_0\mu_j^0, \text{QP}\mu_j). \end{aligned}$$

Using (C.2) from Lemma C.2, noting that since  $P_0\mu^0 \in \mathcal{P}_R(\mathbb{R}^{d_u})$  by (2.11) it holds that  $\mathcal{M}_2(P_0\mu_j^0) \leq d_u R$ , and using the Lipschitz continuity of Q (Lemma B.9) we deduce that

$$\begin{aligned} d_g(\text{GQP}\mu_j, \text{QP}\mu_j) &\leq C\varepsilon(1 + d_u R) + Cd_g(P_0\mu_j^0, P\mu_j) \\ &\leq C\varepsilon(1 + d_u R) + Cd_g(P_0\mu_j^0, P\mu_j^0) + Cd_g(P\mu_j^0, P\mu_j) \\ &\leq C\varepsilon(1 + d_u R) + C\varepsilon R + Cd_g(\mu_j^0, \mu_j), \end{aligned}$$

where the second inequality follows by the triangle inequality and the third inequality follows from bounding  $d_g(P_0\mu_j^0, P\mu_j)$  using (C.1) from Lemma C.2 and from the Lipschitz continuity of P (Lemma B.9). The statement then follows from Lemma C.3.  $\square$

**Theorem 2.3 (Error Estimate: Mean Field Ensemble Kalman Filter).** *Assume that the probability measures  $(\mu_j)_{j \in [0, J]}$  and  $(\mu_j^{\text{EK}})_{j \in [0, J]}$  are obtained respectively from the dynamical systems (1.11) and (2.4) initialized at the same Gaussian probability measure  $\mu_0 = \mu_0^{\text{EK}} \in \mathcal{G}(\mathbb{R}^{d_u})$ . That is,*

$$\mu_{j+1} = B_j \text{QP}\mu_j, \quad \mu_{j+1}^{\text{EK}} = T_j \text{QP}\mu_j^{\text{EK}}.$$

Suppose that the data  $Y^\dagger = \{y_j^\dagger\}_{j=1}^J$  and the matrices  $(\Sigma, \Gamma)$  satisfy Assumptions (H1) and (H5). Fix  $\kappa_\Psi, \kappa_h, \ell_h > 0$  and let the vector fields  $\Psi_0, h_0$  be affine functions satisfying Assumptions (V1) and (V2), respectively, while also satisfying Assumptions (H2) and (H3) with  $\kappa_\Psi, \kappa_h$ . Then for all  $\varepsilon \in [0, 1]$ , there exists a constant  $C = C(\mathcal{M}_q(\mu_0), \kappa_y, \kappa_\Psi, \kappa_h, \ell_h, \Sigma, \Gamma, J)$ , where  $q := \max\{3 + d_u, 4 + d_y\}$ , such that for any  $(\Psi, h) \in B_{L^\infty}((\Psi_0, h_0), \varepsilon)$  with  $h$  satisfying Assumption (H4), it holds that

$$d_g(\mu_J^{\text{EK}}, \mu_J) \leq C\varepsilon.$$

*Proof.* Since Assumption H is satisfied, it is possible to apply the result of Theorem 2.1 to deduce that there exists  $C = C(\mathcal{M}_{\max\{3+d_u, 4+d_y\}}(\mu_0), J, \kappa_y, \kappa_\Psi, \kappa_h, \ell_h, \Sigma, \Gamma)$  such that

$$d_g(\mu_J^{\text{EK}}, \mu_J) \leq C \max_{j \in [0, J-1]} d_g(\text{QP}\mu_j, \text{GQP}\mu_j).$$

Additionally, since  $(\Psi, h) \in B_{L^\infty}((\Psi_0, h_0), \varepsilon)$  for  $\Psi_0, h_0$  satisfying Assumptions (V1) and (V2), and moreover Assumptions (H2) and (H3), we may apply Proposition 2.2 to deduce the result.  $\square$

## 2.4 Error Estimate: Finite Particle Ensemble Kalman Filter

In this subsection, we combine the results from the work in [17] with stability Theorem 2.1, together with approximation Theorem 2.3, to derive a quantitative error estimate between the finite particle ensemble Kalman filter and the true filter in the non-linear setting. In order to define an appropriate metric we introduce the following class of vector fields.

**Assumption P1.** *The vector field  $\phi : \mathbb{R}^{d_u} \rightarrow \mathbb{R}$  satisfies for any  $u, v \in \mathbb{R}^{d_u}$  the condition*

$$|\phi(u) - \phi(v)| \leq L_\phi |u - v| (1 + |u|^\varsigma + |v|^\varsigma),$$

for some  $\varsigma \geq 0$  and for some  $L_\phi > 0$ . We note that for any such  $\phi$ , there exists  $R_\phi > 0$  which depends on  $L_\phi$  so that  $|\phi(u)| \leq R_\phi(1 + |u|^{\varsigma+1})$  for any  $u \in \mathbb{R}^{d_u}$ .

We will prove various technical lemmas under the Assumption P1; these may be useful beyond the confines of this paper. However for our theorems we use the more specific Assumption P2 which enables the control of first and second moments.

**Assumption P2.** The vector field  $\phi : \mathbb{R}^{d_u} \rightarrow \mathbb{R}$  satisfies for any  $u, v \in \mathbb{R}^{d_u}$  the condition

$$|\phi(u) - \phi(v)| \leq L_\phi |u - v| (1 + |u| + |v|),$$

for some  $L_\phi > 0$ . Note that for any such  $\phi$ , there exists  $R_\phi > 0$  depending on  $L_\phi$  so that  $|\phi(u)| \leq R_\phi (1 + |u|^2)$  for any  $u \in \mathbb{R}^{d_u}$ .

**Theorem 2.4 (Error Estimate: Finite Particle Ensemble Kalman Filter).** Assume that the probability measures  $(\mu_j)_{j \in \llbracket 0, J \rrbracket}$  and  $(\mu_j^{\text{EK}, N})_{j \in \llbracket 0, J \rrbracket}$  are obtained respectively from the dynamical systems (1.11) and (2.6), initialized at the Gaussian probability measure  $\mu_0 \in \mathcal{G}(\mathbb{R}^{d_u})$  and at the empirical measure  $\mu_0^{\text{EK}, N} = \frac{1}{N} \sum_{i=1}^N \delta_{u_0^{(i)}}$  for  $u_0^{(i)} \sim \mu_0$  i.i.d. samples, respectively. That is,

$$\mu_{j+1} = \mathbf{B}_j \mathbf{Q} \mathbf{P} \mu_j, \quad \mu_{j+1}^{\text{EK}, N} = \frac{1}{N} \sum_{i=1}^N \delta_{u_{j+1}^{(i)}},$$

where  $u_{j+1}^{(i)}$  evolve according to the iteration in (2.5). Suppose that the data  $Y^\dagger = \{y_j^\dagger\}_{j=1}^J$  and the matrices  $(\Sigma, \Gamma)$  satisfy Assumptions (H1) and (H5). Assume that the vector field  $h$  is linear, and let  $\kappa_h, \ell_h$  be positive constants such that Assumptions (H3) and (H4) are satisfied. Furthermore, let the vector field  $\Psi_0$  be an affine function satisfying Assumption (V1) as well as Assumption (H2) with  $\kappa_\Psi > 0$ . Additionally, let  $\ell_\Psi > 0$  be a constant. Then, for all  $\varepsilon \in [0, 1]$ , it holds that for any  $\Psi \in B_{L^\infty}(\Psi_0, \varepsilon)$  satisfying  $|\Psi|_{C^{0,1}} \leq \ell_\Psi < \infty$ , there exists a constant  $C = C(\mathcal{M}_q(\mu_0), R_\phi, L_\phi, \kappa_y, \kappa_\Psi, \kappa_h, \ell_h, \ell_\Psi, \Sigma, \Gamma, J)$ , where  $q := \max\{3 + d_u, 4 + d_y, 16 \cdot 3^{J-1}\}$ , such that

$$\left( \mathbb{E} \left| \mu_J^{\text{EK}, N}[\phi] - \mu_J[\phi] \right|^2 \right)^{1/2} \leq C \left( \frac{1}{\sqrt{N}} + \varepsilon \right),$$

for any  $\phi$  satisfying Assumption P2.

The proof presented hereafter relies on the following elements.

1. We apply the triangle inequality in order to employ two distinct results concerning the mean field ensemble Kalman filter. The proof thus involves quantifying the error between finite particle ensemble Kalman filter and the mean field ensemble Kalman filter, as discussed in Item 2, and the error between the mean field ensemble Kalman filter and the true filter, as discussed in Item 3.
2. The work in [17] establishes a Monte Carlo error estimate, with rate of  $1/\sqrt{N}$ , between the empirical measure  $\mu^{\text{EK}, N}$ , representing the particle ensemble Kalman filter, and the measure  $\mu^{\text{EK}}$  describing the evolution of the mean field ensemble Kalman filter. In particular, this holds under Assumptions (H1) and (H5) on the data and covariances of the noise processes, the linearity of the vector field  $h$  and Assumption (H2) on  $\Psi$  with the additional assumption that  $|\Psi|_{C^{0,1}} \leq \ell_\Psi < \infty$ . In Lemma D.1 we mimic the proof of [17, Proposition 4.4] and [17, Theorem 5.2] to gain insight into the dependence of the constant prefactor multiplying  $1/\sqrt{N}$  on the parameters of the Gaussian initial condition and the number of steps  $J$ .
3. We assume that the data  $Y^\dagger = \{y_j^\dagger\}_{j=1}^J$  and the matrices  $(\Sigma, \Gamma)$  satisfy Assumptions (H1) and (H5), and assume that  $h$  satisfies Assumption (V2). Furthermore, we assume  $\Psi$  satisfies Assumption (H2) and  $\Psi \in B_{L^\infty}(\Psi_0, \varepsilon)$  with  $\Psi_0$  satisfying Assumption (V1). These assumptions allow us to apply the result from Theorem 2.3.

*Proof.* Recall that  $\mu_j^{\text{EK}}$  is the mean field ensemble Kalman filter, here initialized at the same Gaussian  $\mu_0$  as the true filter. We fix a function  $\phi$  satisfying Assumption P2 and apply the triangle inequality to deduce that

$$\left( \mathbb{E} \left| \mu_J^{\text{EK}, N}[\phi] - \mu_J[\phi] \right|^2 \right)^{1/2} \leq \left( \mathbb{E} \left| \mu_J^{\text{EK}, N}[\phi] - \mu_J^{\text{EK}}[\phi] \right|^2 \right)^{1/2} + \left( \mathbb{E} \left| \mu_J^{\text{EK}}[\phi] - \mu_J[\phi] \right|^2 \right)^{1/2}. \quad (2.12)$$

Since  $\mu_J^{\text{EK}}$  and  $\mu_J$  are deterministic probability measures, it holds that

$$\left( \mathbb{E} \left| \mu_J^{\text{EK}}[\phi] - \mu_J[\phi] \right|^2 \right)^{1/2} = \left| \mu_J^{\text{EK}}[\phi] - \mu_J[\phi] \right|.$$

Since  $\phi$  satisfies [Assumption P2](#), it follows that

$$\left| \mu_J^{\text{EK}}[\phi] - \mu_J[\phi] \right| \leq \sup_{|\phi| \leq R_\phi(1+|u|^2)} \left| \mu_J^{\text{EK}}[\phi] - \mu_J[\phi] \right| = R_\phi \cdot d_g(\mu_J^{\text{EK}}, \mu_J).$$

We note that [Assumption H](#) holds as  $h$  is assumed to be affine; since we additionally assume that  $\Psi \in B_{L^\infty}(\Psi_0, \varepsilon)$ , where  $\Psi_0$  is an affine function, we may apply the result of [Theorem 2.3](#) to deduce that

$$d_g(\mu_J^{\text{EK}}, \mu_J) \leq C\varepsilon, \tag{2.13}$$

where  $C$  is a constant depending on  $\mathcal{M}_{\max\{3+d_u, 4+d_y\}}(\mu_0)$ ,  $J, \kappa_y, \kappa_\Psi, \kappa_h, \ell_h, \Sigma, \Gamma$ .

The fact that  $h$  is assumed to be linear and the additional assumption that  $|\Psi|_{C^{0,1}} \leq \ell_\Psi < \infty$  allows us to apply the result of [Lemma D.1](#), so that

$$\left( \mathbb{E} \left| \mu_J^{\text{EK},N}[\phi] - \mu_J^{\text{EK}}[\phi] \right|^2 \right)^{1/2} \leq \frac{C}{\sqrt{N}}, \tag{2.14}$$

where  $C$  is a constant depending on  $\mathcal{M}_{16 \cdot 3^{J-1}}(\mu_0)$ ,  $J, R_\phi, L_\phi, \kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma$ . In view of [\(2.12\)](#), combining [\(2.13\)](#) and [\(2.14\)](#) yields the desired result.  $\square$

### 3 Discussion and Future Directions

In this paper we have presented the first analysis, in the setting of a filtering problem defined by nonlinear state space dynamics, quantifying the error between the empirical measure obtained by the ensemble Kalman filter and the true filtering distribution. The analysis for the EnKF outlined is based on the proof methodology developed for particle filters in [\[19\]](#). This work extends the initial results of [\[8\]](#) to a setting of practical interest, for which dynamical models and observation operators are allowed to be unbounded and nonlinear. The work opens a new avenue for analysis in the quantitative understanding of the performance of the ensemble Kalman methodology applied to nonlinear problems. To this end, we identify the following avenues for further work.

1. As surveyed in [\[7\]](#), the ensemble Kalman filtering methodology may be used for solving inverse problems and for sampling; it is of interest to extend this analysis to the ensemble Kalman based inversion algorithms outlined in that paper.
2. There is a substantial body of literature studying the continuous time limits of ensemble Kalman methods; see [\[7\]](#) for a review. Performing analysis analogous to that presented here, but in the continuous time setting, would be of interest.
3. For large scale applications, there has been recent wide interest in replacing the dynamical model  $\Psi$ , representing the solution operator obtained via a high fidelity numerical solver, with a cheap to evaluate surrogate. Multifidelity ensemble methods [\[4\]](#) allow the use of a small number of particles evolved according to the high fidelity solver and a large number evolved according to the surrogate. Extending our error analysis to incorporate the effect of model error would be of interest in this context.

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## A Auxiliary Results

We first state and prove a standard result that will be used throughout the paper.

**Lemma A.1.** *Let  $X$  be a random variable taking values in  $\mathbb{R}^{d_u}$  with finite second moment, and let  $m := \mathbb{E}[X]$ . Then it holds that*

$$\mathbb{E}\left[(X - m)(X - m)^\top\right] = \mathbb{E}[XX^\top] - mm^\top \quad (\text{A.1})$$

and

$$\forall a \in \mathbb{R}^{d_u}, \quad \mathbb{E}\left[(X - a)(X - a)^\top\right] \succcurlyeq \mathbb{E}\left[(X - m)(X - m)^\top\right]. \quad (\text{A.2})$$

*Proof.* It holds that

$$\begin{aligned} \mathbb{E}\left[(X - a)(X - a)^\top\right] &= \mathbb{E}\left[\left((X - m) + (m - a)\right)\left((X - m) + (m - a)\right)^\top\right] \\ &= \mathbb{E}\left[(X - m)(X - m)^\top\right] + (m - a)(m - a)^\top. \end{aligned}$$

Setting  $a = 0$ , we obtain (A.1). The inequality (A.2) follows by noting that the second term in the last expression is positive semidefinite.  $\square$

**Lemma A.2.** *Denote by  $g(\bullet; m, S)$  the Lebesgue density of  $\mathcal{N}(m, S)$  and by  $\mathcal{S}_\alpha^n$  the set of symmetric  $n \times n$  matrices  $M$  satisfying*

$$\frac{1}{\alpha}I_n \preccurlyeq M \preccurlyeq \alpha I_n. \quad (\text{A.3})$$

*Then for all  $n \in \mathbb{N}^+$  and  $\alpha \geq 1$ , there exists  $L_{n,\alpha} > 0$  such that for all parameters  $(c_1, m_1, S_1) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_\alpha^n$  and  $(c_2, m_2, S_2) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_\alpha^n$ ,*

$$\|h\|_\infty \leq L_{n,\alpha} \|h\|_1, \quad h(y) = c_1 g(y; m_1, S_1) - c_2 g(y; m_2, S_2). \quad (\text{A.4})$$

*Proof.* The lemma as stated may be found in [8, Lemma 14], where a complete proof is given.  $\square$

**Lemma A.3.** *Let  $\mathsf{P}$  and  $\mathsf{Q}$  denote the operators on probability measures given respectively in (1.7) and (1.9). Let Assumptions (H2) to (H5) be satisfied and suppose that  $\mu \in \mathcal{P}(\mathbb{R}^{d_u})$  satisfies, for some  $q > 0$ , the moment bound*

$$\mathcal{M}_q(\mu) = \int_{\mathbb{R}^{d_u}} |x|^q \mu(dx) < \infty. \quad (\text{A.5})$$

*Then there is  $L = L(\mathcal{M}_q(\mu), \kappa_\Psi, \kappa_h, \ell_h, \Sigma, \Gamma) > 0$  such that for all  $(u_1, u_2, y) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_y}$ , the probability density of  $p = \mathsf{QP}\mu$  satisfies*

$$|p(u_1, y) - p(u_2, y)| \leq L |u_1 - u_2| \min \left\{ \max \left\{ \frac{1}{1 + |u_1|^q}, \frac{1}{1 + |u_2|^q} \right\}, \frac{1}{1 + |y|^q} \right\}. \quad (\text{A.6})$$

*Proof.* Throughout this proof,  $C$  denotes a constant whose value is irrelevant in the context, depends only on  $\mathcal{M}_q(\mu), \kappa_\Psi, \kappa_h, \ell_h, \Sigma, \Gamma$ , and may change from line to line. Sometimes we write the dependence explicitly to indicate which parameters are involved.

**Step 1. Bounding the density of  $P\mu$ .** We first rewrite

$$(1 + |u|^q)P\mu(u) = C(\Sigma) \int_{\mathbb{R}^{d_u}} \exp\left(-\frac{1}{2}|u - \Psi(v)|_\Sigma^2\right) \frac{1 + |u|^q}{1 + |v|^q} (1 + |v|^q)\mu(dv).$$

We note that

$$S(\Sigma, \kappa_\Psi) := \sup_{(u,v) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_u}} \exp\left(-\frac{1}{2}|u - \Psi(v)|_\Sigma^2\right) \frac{1 + |u|^q}{1 + |v|^q} < \infty;$$

this may be seen observing that

$$\exp\left(-\frac{1}{2}|u - \Psi(v)|_\Sigma^2\right) \frac{1 + |u|^q}{1 + |v|^q} \leq \exp\left(-\frac{1}{2}|u - \Psi(v)|_\Sigma^2\right) \frac{(1 + 2^{q-1}|u - \Psi(v)|^q) + 2^{q-1}|\Psi(v)|^q}{1 + |v|^q}. \quad (\text{A.7})$$

The first term on the righthand side of (A.7) may be bounded by noting that  $1 + 2^{q-1}e^{-x^2}x^q$  is uniformly bounded in  $x \in \mathbb{R}$ ; on the other hand the second term may be bounded by applying [Assumption \(H2\)](#). Now, using (A.5), we obtain that

$$P\mu(u) \leq \frac{C(\Sigma, \kappa_\Psi, \mathcal{M}_q(\mu))}{1 + |u|^q}. \quad (\text{A.8})$$

**Step 2. Establishing Lipschitz continuity of  $u \mapsto P\mu(u)$ .** Since  $g(x) := e^{-x^2}$  has derivative  $2xe^{-x^2}$  and  $|xe^{-x^2}| \leq e^{-\frac{x^2}{2}}$  for all  $x \in \mathbb{R}$ , it holds for some  $\xi$  between  $|a|$  and  $|b|$  that

$$\forall (a, b) \in \mathbb{R} \times \mathbb{R}, \quad \left|e^{-a^2} - e^{-b^2}\right| = |b - a| |g'(\xi)| \leq 2|b - a| \left(e^{-\frac{a^2}{2}} + e^{-\frac{b^2}{2}}\right). \quad (\text{A.9})$$

Using this inequality with  $a^2 = \frac{1}{2}|u_1 - \Psi(v)|_\Sigma^2$  and  $b^2 = \frac{1}{2}|u_2 - \Psi(v)|_\Sigma^2$ , the triangle inequality, and equivalence of norms, we deduce that, for all  $(u_1, u_2, v) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_u}$ ,

$$\begin{aligned} & \left| \exp\left(-\frac{1}{2}|u_1 - \Psi(v)|_\Sigma^2\right) - \exp\left(-\frac{1}{2}|u_2 - \Psi(v)|_\Sigma^2\right) \right| \\ & \leq C|u_2 - u_1| \left( \exp\left(-\frac{1}{4}|u_1 - \Psi(v)|_\Sigma^2\right) + \exp\left(-\frac{1}{4}|u_2 - \Psi(v)|_\Sigma^2\right) \right) \end{aligned}$$

for constant  $C = C(\Sigma)$ . Integrating out the  $v$  variable with respect to  $\mu$  we obtain that

$$\begin{aligned} |P\mu(u_1) - P\mu(u_2)| & \leq C|u_1 - u_2| \int_{\mathbb{R}^{d_u}} \exp\left(-\frac{1}{4}|u_1 - \Psi(v)|_\Sigma^2\right) \mu(dv) \\ & \quad + C|u_1 - u_2| \int_{\mathbb{R}^{d_u}} \exp\left(-\frac{1}{4}|u_2 - \Psi(v)|_\Sigma^2\right) \mu(dv). \end{aligned}$$

The integrals on the right-hand side can be bounded as in the first step, which leads to the inequality

$$\forall (u_1, u_2) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_u}, \quad |P\mu(u_1) - P\mu(u_2)| \leq C|u_1 - u_2| \max\left\{\frac{1}{1 + |u_1|^q}, \frac{1}{1 + |u_2|^q}\right\}. \quad (\text{A.10})$$

**Step 3. Obtaining a coarse estimate.** In view of the elementary inequality (A.9) and the assumed Lipschitz continuity of  $h$ , it holds that, for constant  $C = C(\Gamma, \ell_h)$

$$\begin{aligned} & \left| \mathcal{N}(h(u_1), \Gamma)(y) - \mathcal{N}(h(u_2), \Gamma)(y) \right| \\ & \leq C|u_1 - u_2| \exp\left(-\frac{1}{4}|y - h(u_1)|_\Gamma^2\right) + C|u_1 - u_2| \exp\left(-\frac{1}{4}|y - h(u_2)|_\Gamma^2\right). \quad (\text{A.11}) \end{aligned}$$

Using the decomposition

$$\begin{aligned} p(u_1, y) - p(u_2, y) &= \mathbf{P}\mu(u_1) \mathcal{N}(h(u_1), \Gamma)(y) - \mathbf{P}\mu(u_2) \mathcal{N}(h(u_2), \Gamma)(y) \\ &= (\mathbf{P}\mu(u_1) - \mathbf{P}\mu(u_2)) \mathcal{N}(h(u_1), \Gamma)(y) \\ &\quad + \mathbf{P}\mu(u_2) \left( \mathcal{N}(h(u_1), \Gamma)(y) - \mathcal{N}(h(u_2), \Gamma)(y) \right). \end{aligned}$$

and employing (A.8), (A.10) and (A.11), we deduce that

$$\begin{aligned} |p(u_1, y) - p(u_2, y)| &\leq C |u_1 - u_2| \max \left\{ \frac{1}{1 + |u_1|^q}, \frac{1}{1 + |u_2|^q} \right\} \\ &\quad \times \max \left\{ \exp \left( -\frac{1}{4} |y - h(u_1)|_\Gamma^2 \right), \exp \left( -\frac{1}{4} |y - h(u_2)|_\Gamma^2 \right) \right\}. \end{aligned} \quad (\text{A.12})$$

Note that, for the function  $h(u) = u$ , the quantity multiplying  $|u_1 - u_2|$  on the right-hand does not tend to 0 along the sequence  $(u_1^{(n)}, u_2^{(n)}, y^{(n)}) = (0, n, n)$ . For our purposes in this work, we need the finer estimate (A.6); establishing this bound is the aim of the next two steps.

**Step 4. Bounding the density  $p(u, y)$ .** Recall that  $p(u, y) = \mathbf{P}\mu(u) \mathcal{N}(h(u), \Gamma)(y)$ . We prove in this step the inequality

$$p(u, y) \leq C \min \left\{ \frac{1}{1 + |u|^q}, \frac{1}{1 + |y|^q} \right\}, \quad (\text{A.13})$$

or equivalently

$$\sup_{(u, y) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_y}} p(u, y) \max \{1 + |u|^q, 1 + |y|^q\} < \infty.$$

To this end, note that

$$\begin{aligned} p(u, y) \max \{|u|^q, |y|^q\} &= \mathbf{P}\mu(u) \mathcal{N}(h(u), \Gamma)(y) \max \{|u|^q, |y|^q\} \\ &\leq \mathbf{P}\mu(u) \mathcal{N}(h(u), \Gamma)(y) |u|^q + \mathbf{P}\mu(u) \mathcal{N}(h(u), \Gamma)(y) |y|^q. \end{aligned}$$

The first term is bounded uniformly by **Step 1**, estimate (A.8). For the second term, we use that

$$|y|^q \leq 2^{q-1} |h(u)|^q + 2^{q-1} |y - h(u)|^q, \quad (\text{A.14})$$

to obtain

$$\mathbf{P}\mu(u) \mathcal{N}(h(u), \Gamma)(y) |y|^q \leq C \mathbf{P}\mu(u) \mathcal{N}(h(u), \Gamma)(y) \left( |h(u)|^q + |y - h(u)|^q \right).$$

The first term on the right-hand side is bounded uniformly, again by **Step 1**, estimate (A.8), and using **Assumption (H3)**. The second term is also bounded uniformly because the function  $x \mapsto \mathcal{N}(0, \Gamma)(x) |x|^q$  is uniformly bounded in  $x$ , by a value depending only on  $\Gamma$  and  $q$ .

**Step 5. Obtaining the estimate (A.6).** The claimed inequality is equivalent to

$$\sup_{u_1, u_2, y} \frac{|p(u_1, y) - p(u_2, y)|}{|u_1 - u_2|} \max \left\{ \min \{1 + |u_1|^q, 1 + |u_2|^q\}, 1 + |y|^q \right\} < \infty,$$

where the supremum is over  $\mathbb{R}^{d_u} \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_y}$ . By (A.12), it holds that

$$\sup_{u_1, u_2, y} \frac{|p(u_1, y) - p(u_2, y)|}{|u_1 - u_2|} \min \{1 + |u_1|^q, 1 + |u_2|^q\} < \infty,$$



so it remains to show that

$$\sup_{u_1, u_2, y} \frac{|p(u_1, y) - p(u_2, y)|}{|u_1 - u_2|} |y|^q < \infty.$$

By (A.13), it is clear that the supremum is uniformly bounded if restricted to the set  $|u_1 - u_2| \geq 1$ , so it suffices to show that

$$\sup_{(u, \delta, y) \in \mathbb{R}^{d_u} \times B(0, 1) \times \mathbb{R}^{d_y}} \frac{|p(u, y) - p(u + \delta, y)|}{|\delta|} |y|^q < \infty,$$

where  $B(0, 1)$  is the open ball of radius 1 centered at the origin in  $\mathbb{R}^{d_u}$ . We use again (A.14) in order to bound

$$\begin{aligned} \frac{|p(u, y) - p(u + \delta, y)|}{|\delta|} |y|^q &\leq C \frac{|p(u, y) - p(u + \delta, y)|}{|\delta|} |h(u)|^q \\ &\quad + C \frac{|p(u, y) - p(u + \delta, y)|}{|\delta|} |y - h(u)|^q. \end{aligned}$$

The first term is bounded uniformly by (A.12) and the assumption that  $|h(u)| \leq \kappa_h(1 + |u|)$ . To conclude the proof, it remains to show that

$$\sup_{(u, \delta, y) \in \mathbb{R}^{d_u} \times B(0, 1) \times \mathbb{R}^{d_y}} C \frac{|p(u, y) - p(u + \delta, y)|}{|\delta|} |y - h(u)|^q < \infty. \quad (\text{A.15})$$

By (A.12) it holds that

$$\begin{aligned} &|p(u, y) - p(u + \delta, y)| \\ &\leq \frac{C|\delta|}{1 + |u|^q} \max \left\{ \exp \left( -\frac{1}{4} |y - h(u)|_\Gamma^2 \right), \exp \left( -\frac{1}{4} |y - h(u + \delta)|_\Gamma^2 \right) \right\} \\ &\leq \frac{C|\delta|}{1 + |u|^q} \exp \left( -\frac{1}{8} |y - h(u)|_\Gamma^2 + \frac{1}{4} |h(u) - h(u + \delta)|_\Gamma^2 \right). \end{aligned}$$

In the last line we used the inequality  $|a + b|^2 \geq \frac{1}{2}|a|^2 - |b|^2$ , which follows from Young's inequality. Since the function  $x \mapsto e^{-\frac{x^2}{8} + C} x^q$  is bounded uniformly in  $x \in \mathbb{R}$  and by Assumption (H4), the bound (A.15) easily follows, concluding the proof.  $\square$

## B Technical Results for Theorem 2.1

We show moment bounds in Appendix B.1, we recall that on Gaussian measures the action of the conditioning map and the Kalman transport map are equivalent in Appendix B.2, and we prove stability results in Appendix B.3.

### B.1 Moment Bounds

**Lemma B.1 (Moment Bounds).** *Let  $\mu$  be a probability measure on  $\mathbb{R}^{d_u}$  with bounded first and second order polynomial moments  $\mathcal{M}_1(\mu), \mathcal{M}_2(\mu) < \infty$ . Under Assumption H, it holds that*

$$|\mathcal{M}(\mathbb{P}\mu)| \leq \kappa_\Psi (1 + \mathcal{M}_1(\mu)), \quad \Sigma \preceq \mathcal{C}(\mathbb{P}\mu) \preceq \Sigma + 2\kappa_\Psi^2 (1 + \mathcal{M}_2(\mu)) I_{d_u}. \quad (\text{B.1})$$

*Proof.* Using the definition of  $\mathbf{P}$  in (1.7), it holds that

$$\begin{aligned}\mathcal{M}(\mathbf{P}\mu) &= \int_{\mathbb{R}^{d_u}} u \mathbf{P}\mu(u) \, du = \frac{1}{\sqrt{(2\pi)^{d_u} \det \Sigma}} \int_{\mathbb{R}^{d_u}} \int_{\mathbb{R}^{d_u}} u \exp\left(-\frac{1}{2}|u - \Psi(v)|_{\Sigma}^2\right) \mu(dv) \, du \\ &= \int_{\mathbb{R}^{d_u}} \Psi(v) \mu(dv),\end{aligned}$$

where application of Fubini's theorem yields the last equality. The first inequality in (B.1) then follows from Assumption (H2). For the second inequality in (B.1), we first note that for any  $a, v \in \mathbb{R}^{d_u}$  it holds that

$$\begin{aligned}\int_{\mathbb{R}^{d_u}} (u - a) \otimes (u - a) \exp\left(-\frac{1}{2}|u - \Psi(v)|_{\Sigma}^2\right) \, du \\ \succcurlyeq \int_{\mathbb{R}^{d_u}} (u - \Psi(v)) \otimes (u - \Psi(v)) \exp\left(-\frac{1}{2}|u - \Psi(v)|_{\Sigma}^2\right) \, du;\end{aligned}\tag{B.2}$$

this follows from Lemma A.1 by noting that  $\Psi(v)$  is the mean under the Gaussian probability measure  $\mathcal{N}(\Psi(v), \Sigma)$ . Letting now  $a$  be the mean under the measure  $\mathbf{P}\mu$ , we observe that, by conditioning on  $v$  and using (B.2),

$$\begin{aligned}\mathcal{C}(\mathbf{P}\mu) &= \int_{\mathbb{R}^{d_u}} (u - a) \otimes (u - a) \mathbf{P}\mu(u) \, du \\ &= \frac{1}{\sqrt{(2\pi)^{d_u} \det \Sigma}} \int_{\mathbb{R}^{d_u}} \left( \int_{\mathbb{R}^{d_u}} (u - a) \otimes (u - a) \exp\left(-\frac{1}{2}|u - \Psi(v)|_{\Sigma}^2\right) \, du \right) \mu(dv) \\ &\succcurlyeq \frac{1}{\sqrt{(2\pi)^{d_u} \det \Sigma}} \int_{\mathbb{R}^{d_u}} \left( \int_{\mathbb{R}^{d_u}} (u - \Psi(v)) \otimes (u - \Psi(v)) \exp\left(-\frac{1}{2}|u - \Psi(v)|_{\Sigma}^2\right) \, du \right) \mu(dv) \\ &= \int_{\mathbb{R}^{d_u}} \Sigma \mu(dv) = \Sigma,\end{aligned}$$

hence  $\mathcal{C}(\mathbf{P}\mu) \succcurlyeq \Sigma$ . On the other hand, using Lemma A.1 again and noting that, by the Cauchy-Schwarz inequality,  $ww^{\top} \preccurlyeq (w^{\top}w)I_{d_u}$  for any vector  $w \in \mathbb{R}^{d_u}$ , we deduce that

$$\begin{aligned}\mathcal{C}(\mathbf{P}\mu) &\preccurlyeq \int_{\mathbb{R}^{d_u}} u \otimes u \mathbf{P}\mu(u) \, du \\ &= \frac{1}{\sqrt{(2\pi)^{d_u} \det \Sigma}} \int_{\mathbb{R}^{d_u}} \int_{\mathbb{R}^{d_u}} u \otimes u \exp\left(-\frac{1}{2}|u - \Psi(v)|_{\Sigma}^2\right) \mu(dv) \, du \\ &= \int_{\mathbb{R}^{d_u}} (\Psi(v) \otimes \Psi(v) + \Sigma) \mu(dv) \\ &\preccurlyeq \Sigma + \left( \int_{\mathbb{R}^{d_u}} |\Psi(v)|^2 \mu(dv) \right) I_{d_u} \preccurlyeq \Sigma + 2\kappa_{\Psi}^2 (1 + \mathcal{M}_2(\mu)) I_{d_u},\end{aligned}\tag{B.3}$$

which yields the desired result.  $\square$

**Lemma B.2.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^{d_u}$  with bounded first and second order polynomial moments  $\mathcal{M}_1(\mu), \mathcal{M}_2(\mu) < \infty$ . Under Assumption H, it holds that*

$$|\mathcal{M}^u(\mathbf{QP}\mu)| \leq \kappa_{\Psi} (1 + \mathcal{M}_1(\mu)),\tag{B.4a}$$

$$|\mathcal{M}^y(\mathbf{QP}\mu)| \leq \kappa_h \sqrt{2(1 + \text{tr}(\Sigma) + 2\kappa_{\Psi}^2 (1 + \mathcal{M}_2(\mu)))}.\tag{B.4b}$$

Furthermore, it holds that

$$\mathcal{C}(\text{QP}\mu) \preceq \begin{pmatrix} 4\kappa_\Psi^2(1 + \mathcal{M}_2(\mu))I_{d_u} + 2\Sigma & 0_{d_u \times d_y} \\ 0_{d_y \times d_u} & 4\kappa_h^2(1 + \text{tr}(\Sigma) + 2\kappa_\Psi^2(1 + \mathcal{M}_2(\mu)))I_{d_y} + \Gamma \end{pmatrix}, \quad (\text{B.5a})$$

$$\mathcal{C}(\text{QP}\mu) \succeq \frac{\gamma \cdot \min\left\{2\sigma, \gamma + 4\kappa_h^2(1 + \text{tr}(\Sigma) + 2\kappa_\Psi^2(1 + \mathcal{M}_2(\mu)))\right\}}{2\gamma + 8\kappa_h^2(1 + \text{tr}(\Sigma) + 2\kappa_\Psi^2(1 + \mathcal{M}_2(\mu)))} I_{d_u + d_y}. \quad (\text{B.5b})$$

*Proof.* The inequalities (B.4a) and (B.4b) follow from [Assumption H](#) and the fact that

$$\mathcal{M}(\text{QP}\mu) = \begin{pmatrix} \mathcal{M}(\text{P}\mu) \\ \text{P}\mu[h] \end{pmatrix}. \quad (\text{B.6})$$

Indeed, from [Lemma B.1](#) we know that  $|\mathcal{M}(\text{P}\mu)| \leq \kappa_\Psi(1 + \mathcal{M}_1(\mu))$ , which leads by Jensen's inequality to (B.4a). To deduce (B.4b), we note that

$$\begin{aligned} |\text{P}\mu[h]|^2 &\leq \int_{\mathbb{R}^{d_u}} |h(u)|^2 \text{P}\mu(du) \\ &\leq 2\kappa_h^2 + 2\kappa_h^2 \int_{\mathbb{R}^{d_u}} |u|^2 \text{P}\mu(du) \end{aligned} \quad (\text{B.7a})$$

$$= 2\kappa_h^2 + 2\kappa_h^2 \int_{\mathbb{R}^{d_u}} (|\Psi(v)|^2 + \text{tr}(\Sigma)) \mu(dv), \quad (\text{B.7b})$$

where (B.7a) follows by applying [Assumption \(H3\)](#) and Young's inequality and (B.7b) follows by applying the properties of the Gaussian transition density. The result then follows by applying [Assumption \(H2\)](#).

We now turn our attention to establishing inequality (B.5a). To this end, let  $\phi: \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_u + d_y}$  define the map  $\phi(u) = (u, h(u))$ , and let  $\phi_\#$  denote the associated pushforward map on measures. We note that

$$\mathcal{C}(\text{QP}\mu) = \mathcal{C}(\phi_\# \text{P}\mu) + \begin{pmatrix} 0_{d_u \times d_u} & 0_{d_u \times d_y} \\ 0_{d_y \times d_u} & \Gamma \end{pmatrix} = \begin{pmatrix} \mathcal{C}^{uu}(\phi_\# \text{P}\mu) & \mathcal{C}^{uy}(\phi_\# \text{P}\mu) \\ \mathcal{C}^{yu}(\phi_\# \text{P}\mu) & \mathcal{C}^{yy}(\phi_\# \text{P}\mu) + \Gamma \end{pmatrix}. \quad (\text{B.8})$$

It holds for any  $(a, b) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_y}$  that

$$2 \begin{pmatrix} aa^\top & 0_{d_u \times d_y} \\ 0_{d_y \times d_u} & bb^\top \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix} \otimes \begin{pmatrix} a \\ -b \end{pmatrix} \succeq 0_{(d_u + d_y) \times (d_u + d_y)}.$$

Hence, it follows that

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} a \\ b \end{pmatrix} \preceq 2 \begin{pmatrix} aa^\top & 0_{d_u \times d_y} \\ 0_{d_y \times d_u} & bb^\top \end{pmatrix},$$

which enables to deduce, using [Lemmas A.1](#) and [B.1](#) and [Assumption H](#), that

$$\begin{aligned} \mathcal{C}(\phi_\# \text{P}\mu) &\preceq \int_{\mathbb{R}^{d_u}} \begin{pmatrix} u \\ h(u) \end{pmatrix} \otimes \begin{pmatrix} u \\ h(u) \end{pmatrix} \text{P}\mu(u) du \\ &\preceq 2 \int_{\mathbb{R}^{d_u}} \begin{pmatrix} uu^\top & 0_{d_u \times d_y} \\ 0_{d_y \times d_u} & h(u)h(u)^\top \end{pmatrix} \text{P}\mu(u) du \\ &\preceq 2 \begin{pmatrix} \Sigma + 2\kappa_\Psi^2(1 + \mathcal{M}_2(\mu))I_{d_u} & 0_{d_u \times d_y} \\ 0_{d_y \times d_u} & 2\kappa_h^2(1 + \text{tr}(\Sigma) + 2\kappa_\Psi^2(1 + \mathcal{M}_2(\mu)))I_{d_y} \end{pmatrix}, \end{aligned} \quad (\text{B.9})$$

where we used (B.3) and the calculation resulting in (B.7a) in the last inequality. Noting this inequality in combination with (B.8) yields the upper bound (B.5a). To establish the lower bound (B.5b), we first note

that by the Cauchy–Schwarz inequality, for any  $\pi \in \mathcal{P}(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$  and all  $(a, b) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_y}$  it holds that

$$\begin{aligned} |a^\top \mathcal{C}^{uy}(\pi)b| &= \int_{\mathbb{R}^{d_u} \times \mathbb{R}^{d_y}} \left( a^\top (u - \mathcal{M}^u(\pi)) \right) \left( b^\top (y - \mathcal{M}^y(\pi)) \right) \pi(\mathrm{d}u \mathrm{d}y) \\ &\leq \sqrt{a^\top \mathcal{C}^{uu}(\pi)a} \sqrt{b^\top \mathcal{C}^{yy}(\pi)b}. \end{aligned}$$

Hence, by Young’s inequality, we deduce that for all  $\varepsilon \in (0, 1)$  and for all  $(a, b) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_y}$  it holds that

$$\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix}^\top \mathcal{C}(\phi_\# \mathbf{P}\mu) \begin{pmatrix} a \\ b \end{pmatrix} &\geq (1 - \varepsilon) a^\top \mathcal{C}^{uu}(\phi_\# \mathbf{P}\mu) a - \left( \frac{1}{\varepsilon} - 1 \right) b^\top \mathcal{C}^{yy}(\phi_\# \mathbf{P}\mu) b \\ &\geq (1 - \varepsilon) a^\top \Sigma a - \left( \frac{1}{\varepsilon} - 1 \right) \mathbf{P}\mu[|h|^2], \end{aligned}$$

where we applied (B.1) and the bound  $\mathcal{C}^{yy}(\phi_\# \mathbf{P}\mu) \preceq \mathbf{P}\mu[|h|^2] I_{d_y}$  in the last inequality. We then use (B.8) to deduce that

$$\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix}^\top \mathcal{C}(\mathbf{Q}\mathbf{P}\mu) \begin{pmatrix} a \\ b \end{pmatrix} &\geq (1 - \varepsilon) a^\top \Sigma a - \left( \frac{1}{\varepsilon} - 1 \right) \mathbf{P}\mu[|h|^2] + b^\top \Gamma b \\ &\geq (1 - \varepsilon) \sigma |a|^2 + \left( \gamma - \left( \frac{1}{\varepsilon} - 1 \right) \mathbf{P}\mu[|h|^2] \right) |b|^2. \end{aligned}$$

Taking  $\varepsilon = \frac{2\mathbf{P}\mu[|h|^2]}{\gamma + 2\mathbf{P}\mu[|h|^2]}$ , so that the coefficient of  $|b|^2$  is  $\frac{\gamma}{2}$ , we obtain

$$\begin{pmatrix} a \\ b \end{pmatrix}^\top \mathcal{C}(\mathbf{Q}\mathbf{P}\mu) \begin{pmatrix} a \\ b \end{pmatrix} \geq \frac{\gamma\sigma}{\gamma + 2\mathbf{P}\mu[|h|^2]} |a|^2 + \frac{\gamma}{2} |b|^2 \geq \min \left\{ \frac{\gamma\sigma}{\gamma + 2\mathbf{P}\mu[|h|^2]}, \frac{\gamma}{2} \right\} (|a|^2 + |b|^2). \quad (\text{B.10})$$

Since this is true for any  $(a, b) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_y}$ , it follows that

$$\mathcal{C}(\mathbf{Q}\mathbf{P}\mu) \succeq \gamma \frac{\min \{2\sigma, \gamma + 2\mathbf{P}\mu[|h|^2]\}}{2\gamma + 4\mathbf{P}\mu[|h|^2]} I_{d_u + d_y}.$$

In order to finally obtain (B.5b), we use (B.7b) to deduce that

$$\mathbf{P}\mu[|h|^2] \leq 2\kappa_h^2 \left( 1 + \mathrm{tr}(\Sigma) + 2\kappa_\Psi^2 (1 + \mathcal{M}_2(\mu)) \right),$$

which concludes the proof.  $\square$

**Lemma B.3 (Moment Bound for the True Filtering Distribution).** *Assume that the probability measures  $(\mu_j)_{j \in \llbracket 0, J \rrbracket}$  are obtained from the dynamical system (1.11) initialized at the probability measure  $\mu_0 \in \mathcal{P}(\mathbb{R}^{d_u})$  with bounded  $q$ th order polynomial moment  $\mathcal{M}_q(\mu_0) < \infty$ . If Assumption H holds then there exists  $C = C(\mathcal{M}_q(\mu_0), J, \kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma)$  such that*

$$\max_{j \in \llbracket 0, J \rrbracket} \mathcal{M}_q(\mu_j) \leq C.$$

*Proof.* We have  $\mu_{j+1} = \mathbf{B}_j \mathbf{Q}\mathbf{P}\mu_j$ , for  $j \in \llbracket 0, J - 1 \rrbracket$ . Equivalently, using the notation from (1.6), it holds that  $\mu_{j+1} = \mathbf{L}_j \mathbf{P}\mu_j$  for each  $j \in \llbracket 0, J - 1 \rrbracket$ . Hence, we note that

$$\mu_{j+1} = \frac{\exp\left(-\frac{1}{2}|y_{j+1}^\dagger - h(u)|_\Gamma^2\right) \mathbf{P}\mu_j(u)}{\int_{\mathbb{R}^{d_u}} \exp\left(-\frac{1}{2}|y_{j+1}^\dagger - h(U)|_\Gamma^2\right) \mathbf{P}\mu_j(U) \mathrm{d}U}. \quad (\text{B.11})$$

It is readily observed that

$$\begin{aligned}\mathcal{M}_q(\mu_{j+1}) &= \frac{\int_{\mathbb{R}^{d_u}} |u|^q \exp\left(-\frac{1}{2}|y_{j+1}^\dagger - h(u)|_\Gamma^2\right) \mathbb{P}\mu_j(u) du}{\int_{\mathbb{R}^{d_u}} \exp\left(-\frac{1}{2}|y_{j+1}^\dagger - h(U)|_\Gamma^2\right) \mathbb{P}\mu_j(U) dU} \\ &\leq \frac{\int_{\mathbb{R}^{d_u}} |u|^q \mathbb{P}\mu_j(u) du}{\int_{\mathbb{R}^{d_u}} \exp\left(-\frac{1}{2}|y_{j+1}^\dagger - h(U)|_\Gamma^2\right) \mathbb{P}\mu_j(U) dU}.\end{aligned}\tag{B.12}$$

We first bound from above the numerator of (B.12); indeed, note that

$$\begin{aligned}\int_{\mathbb{R}^{d_u}} |u|^q \mathbb{P}\mu_j(u) du &= \int_{\mathbb{R}^{d_u}} |u|^q \left( \int_{\mathbb{R}^{d_u}} \exp\left(-\frac{1}{2}|u - \Psi(v)|_\Sigma^2\right) \mu_j(dv) \right) du \\ &= \int_{\mathbb{R}^{d_u}} \left( \int_{\mathbb{R}^{d_u}} |u|^q \exp\left(-\frac{1}{2}|u - \Psi(v)|_\Sigma^2\right) du \right) \mu_j(dv)\end{aligned}\tag{B.13a}$$

$$\leq C \int_{\mathbb{R}^{d_u}} (1 + |\Psi(v)|^q) \mu_j(dv)\tag{B.13b}$$

$$\leq C(1 + \mathcal{M}_q(\mu_j)),\tag{B.13c}$$

where (B.13a) follows from Fubini's theorem, the inequality (B.13b) from properties of Gaussians, and (B.13c) from application of Assumption H. We note that in (B.13c) the constant  $C$  depends on  $\kappa_\Psi, \Sigma$ . Now, to obtain a lower bound on the denominator of (B.12), we observe that

$$\begin{aligned}\int_{\mathbb{R}^{d_u}} \exp\left(-\frac{1}{2}|y_{j+1}^\dagger - h(u)|_\Gamma^2\right) \mathbb{P}\mu_j(u) du &\geq \exp\left(-\frac{1}{2} \int_{\mathbb{R}^{d_u}} |y_{j+1}^\dagger - h(u)|_\Gamma^2 \mathbb{P}\mu_j(u)\right) \\ &\geq C \exp\left(-\|\Gamma^{-1}\| \int_{\mathbb{R}^{d_u}} |h(u)|^2 \mathbb{P}\mu_j(u)\right) \\ &\geq C \exp\left(-4\kappa_h^2 \kappa_\Psi^2 \|\Gamma^{-1}\| \mathcal{M}_2(\mu_j)\right),\end{aligned}\tag{B.14}$$

where the first inequality follows by application of Jensen's inequality and (B.14) follows from the calculation leading to (B.7b). We note that the  $C$  in (B.14) is a constant depending on  $\kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma$ . Therefore, by combining (B.14) with (B.13c), it is possible to deduce from (B.12) that

$$\mathcal{M}_q(\mu_{j+1}) \leq C \exp\left(4\kappa_h^2 \kappa_\Psi^2 \|\Gamma^{-1}\| \mathcal{M}_2(\mu_j)\right) (1 + \mathcal{M}_q(\mu_j)),\tag{B.15}$$

where  $C$  is a constant depending on  $\kappa_h, \kappa_\Psi, \kappa_y, \Sigma, \Gamma$ .  $\square$

**Remark B.4.** • *In some situations, the bound (B.15) is overly pessimistic. For example, if  $h$  satisfies Assumption (H3) as well as the inequality  $|h(u) - y_{j+1}^\dagger| \geq c_\ell(|u| - 1)$  for all  $u \in \mathbb{R}^{d_u}$  for some positive  $c_\ell$ , then by [12, Proposition A.3] for all  $q > 0$  there is  $C = C(c_\ell, q)$  such that*

$$\forall \mu \in \mathcal{P}(\mathbb{R}^{d_u}), \quad \mathbb{L}_j \mu \left[ |x|^q \right] \leq C \mu \left[ |x|^q \right].$$

*In this setting, better control of the moments can be achieved than in (B.15).*

- *In obtaining (B.15), we did not use any information on  $\mathbb{P}\mu_j$  other than the moment bound (B.13c). With this approach, the presence of an exponential in the bound (B.15) is to be expected. Indeed, consider the case where  $d_u = 2, d_y = 1, \Gamma = \frac{1}{4}, y_{j+1}^\dagger = 0$ , and*

$$h(u) = u_1, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mu_R = \frac{x_R^2 \delta_{(R,0)} + \delta_{(0,x_R)}}{x_R^2 + 1}, \quad x_R = e^{\frac{R^2}{2}}.$$

Then, as  $R \rightarrow \infty$ ,  $\mu_R [|u|^2] \sim R^2$ , while

$$\mathbb{L}_j \mu_R [|u|^2] = \frac{\int_{\mathbb{R}^2} (u_1^2 + u_2^2) \exp(-2u_1^2) \mu_R(du_1 du_2)}{\int_{\mathbb{R}^2} \exp(-2u_1^2) \mu_R(du_1 du_2)} = \frac{x_R^2 e^{-2R^2} R^2 + x_R^2}{x_R^2 e^{-2R^2} + 1} \sim x_R^2 = e^{R^2}.$$

Thus, for large  $R$ , the second moment of  $\mathbb{L}_j \mu_R$  is approximately equal to the exponential of the second moment of  $\mu_R$ .

**Lemma B.5 (Moment Bound for the Approximate Filtering Distribution).** *Assume that the probability measures  $(\mu_j^{\text{EK}})_{j \in \llbracket 0, J \rrbracket}$  are obtained from the dynamical system (2.4) initialized at the probability measure  $\mu_0^{\text{EK}} \in \mathcal{P}(\mathbb{R}^{d_u})$  with bounded  $q$ th polynomial order moment  $\mathcal{M}_q(\mu_0^{\text{EK}}) < \infty$ . If Assumption H holds then there exists  $C = C(\mathcal{M}_q(\mu_0^{\text{EK}}), J, \kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma)$  such that*

$$\max_{j \in \llbracket 0, J \rrbracket} \mathcal{M}_q(\mu_j^{\text{EK}}) \leq C.$$

*Proof.* We begin by noting that  $\mu_{j+1}^{\text{EK}} = \mathbb{T}_j \mathbb{Q} \mathbb{P} \mu_j^{\text{EK}}$ , for  $j \in \llbracket 0, J-1 \rrbracket$ . Thus

$$\begin{aligned} \mathcal{M}_q(\mu_{j+1}^{\text{EK}}) &= \int_{\mathbb{R}^{d_u}} |u|^q \mathbb{T}_j \mathbb{Q} \mathbb{P} \mu_j^{\text{EK}}(du) \\ &= \int_{\mathbb{R}^{d_u} \times \mathbb{R}^{d_y}} \left| \mathcal{T}(u, y; \mathbb{Q} \mathbb{P} \mu_j^{\text{EK}}, y_{j+1}^\dagger) \right|^q \mathbb{Q} \mathbb{P} \mu_j^{\text{EK}}(u, y) dy du \\ &= \int_{\mathbb{R}^{d_u} \times \mathbb{R}^{d_y}} \left| u + \mathcal{C}^{uy}(\mathbb{Q} \mathbb{P} \mu_j^{\text{EK}}) \mathcal{C}^{yy}(\mathbb{Q} \mathbb{P} \mu_j^{\text{EK}})^{-1} (y_{j+1}^\dagger - y) \right|^q \mathbb{Q} \mathbb{P} \mu_j^{\text{EK}}(u, y) dy du \\ &\leq C \int_{\mathbb{R}^{d_u} \times \mathbb{R}^{d_y}} |u|^q \mathbb{Q} \mathbb{P} \mu_j^{\text{EK}}(u, y) dy du \\ &\quad + C \left( 1 + \mathcal{M}_2(\mu_j^{\text{EK}}) \right)^{2q} \cdot \left( 1 + \int_{\mathbb{R}^{d_u} \times \mathbb{R}^{d_y}} |y|^q \mathbb{Q} \mathbb{P} \mu_j^{\text{EK}}(u, y) dy du \right), \end{aligned} \quad (\text{B.16})$$

where in (B.16) the constant  $C$  depends on  $\kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma$  and where the dependence on the second moment of  $\mu_j^{\text{EK}}$  in the second term is derived from (B.5a) and (B.5b). By using the definitions of  $\mathbb{Q}$  and  $\mathbb{P}$  and by applying reasoning analogous to (B.13c), we deduce that

$$\begin{aligned} \mathcal{M}_q(\mu_{j+1}^{\text{EK}}) &\leq C \left( 1 + \mathcal{M}_2(\mu_j^{\text{EK}}) \right)^{2q} \cdot \left( 1 + \mathcal{M}_q(\mu_j^{\text{EK}}) \right), \\ &\leq C \left( 1 + \mathcal{M}_q(\mu_j^{\text{EK}}) \right)^{2q+1} \end{aligned}$$

where  $C$  is a constant depending on  $\kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma$ . Iteration gives the desired result.  $\square$

**Lemma B.6.** *For  $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^n)$  with finite second moments, it holds that*

$$\begin{aligned} |\mathcal{M}(\mu_1) - \mathcal{M}(\mu_2)| &\leq \frac{1}{2} d_g(\mu_1, \mu_2), \\ \|\mathcal{C}(\mu_1) - \mathcal{C}(\mu_2)\| &\leq \left( 1 + \frac{1}{2} |\mathcal{M}(\mu_1) + \mathcal{M}(\mu_2)| \right) d_g(\mu_1, \mu_2). \end{aligned}$$

*Proof.* The lemma as stated may be found in [8, Lemma 20], where a complete proof is given.  $\square$

## B.2 Action of $\mathbb{T}_j$ on Gaussians

**Lemma B.7** ( $\mathbb{B}_j \mathbb{G} = \mathbb{T}_j \mathbb{G}$ ). *Fix  $y_{j+1}^\dagger \in \mathbb{R}^{d_y}$  and let  $\pi$  be a Gaussian measure over  $\mathbb{R}^{d_u} \times \mathbb{R}^{d_y}$ . Then the probability measure  $\mathbb{B}_j \pi$ , with  $\mathbb{B}_j$  defined in (1.10), coincides with the probability measure  $\mathbb{T}_j \pi = \mathcal{T}(\bullet, \bullet; \pi, y_{j+1}^\dagger) \sharp \pi$ , where  $\mathcal{T}$  is defined in (2.3).*

*Proof.* The lemma as stated may be found in [8, Lemma 21], where a complete proof is given.  $\square$

### B.3 Stability Results

**Lemma B.8 (Map P is Lischitz).** *Suppose that  $\Psi$  and  $\Sigma$  satisfy [Assumptions \(H2\)](#) and [\(H5\)](#), respectively. Then it holds that*

$$\forall (\mu, \nu) \in \mathcal{P}(\mathbb{R}^{d_u}) \times \mathcal{P}(\mathbb{R}^{d_u}), \quad d_g(\mathbb{P}\mu, \mathbb{P}\nu) \leq \left(1 + 2\kappa_\Psi^2 + \text{tr}(\Sigma)\right) d_g(\mu, \nu).$$

*Proof.* By definition of  $\mathbb{P}$ , it holds that

$$\mathbb{P}\mu(\text{d}u) = \int_{\mathbb{R}^{d_u}} p(v, \text{d}u) \mu(\text{d}v), \quad \text{where } p(v, \text{d}u) := \frac{\exp\left(-\frac{1}{2}|u - \Psi(v)|_\Sigma^2\right)}{\sqrt{(2\pi)^{d_u} \det \Sigma}} \text{d}u.$$

Let  $g(u) = 1 + |u|^2$ , as in [Definition 1.1](#), and take any function  $f: \mathbb{R}^{d_u} \rightarrow \mathbb{R}$  such that  $|f| \leq g$ . [Assumption \(H2\)](#) implies that

$$\begin{aligned} \left| \int_{\mathbb{R}^{d_u}} f(u) p(v, \text{d}u) \right| &\leq \int_{\mathbb{R}^{d_u}} g(u) p(v, \text{d}u) = 1 + |\Psi(v)|^2 + \text{tr}(\Sigma) \\ &\leq 1 + 2\kappa_\Psi^2 + 2\kappa_\Psi^2 |v|^2 + \text{tr}(\Sigma) \leq \left(1 + 2\kappa_\Psi^2 + \text{tr}(\Sigma)\right) g(v). \end{aligned}$$

By Fubini's theorem, it follows that

$$\begin{aligned} \left| \mathbb{P}\mu[f] - \mathbb{P}\nu[f] \right| &= \left| \int_{\mathbb{R}^{d_u}} \left( \int_{\mathbb{R}^{d_u}} f(u) p(v, \text{d}u) \right) (\mu(\text{d}v) - \nu(\text{d}v)) \right| \\ &= \left(1 + 2\kappa_\Psi^2 + \text{tr}(\Sigma)\right) \left| \mu[g] - \nu[g] \right| \leq \left(1 + 2\kappa_\Psi^2 + \text{tr}(\Sigma)\right) d_g(\mu, \nu), \end{aligned}$$

which concludes the proof.  $\square$

**Lemma B.9 (Map Q is Lipschitz).** *Suppose that  $h$  and  $\Gamma$  satisfy [Assumptions \(H3\)](#) and [\(H5\)](#), respectively. Then it holds that*

$$\forall (\mu, \nu) \in \mathcal{P}(\mathbb{R}^{d_u}) \times \mathcal{P}(\mathbb{R}^{d_u}), \quad d_g(\mathbb{Q}\mu, \mathbb{Q}\nu) \leq \left(1 + 2\kappa_h^2 + \text{tr}(\Gamma)\right) d_g(\mu, \nu). \quad (\text{B.17})$$

*Proof.* Take  $f: \mathbb{R}^{d_u} \times \mathbb{R}^{d_y} \rightarrow \mathbb{R}$  satisfying  $|f| \leq g$ , where  $g(u, v) = 1 + |u|^2 + |y|^2$ . By Fubini's theorem, it holds that

$$\mathbb{Q}\mu[f] - \mathbb{Q}\nu[f] = \int_{\mathbb{R}^{d_u}} \Pi f(u) (\mu(\text{d}u) - \nu(\text{d}u)), \quad \text{where } \Pi f(u) := \int_{\mathbb{R}^{d_y}} f(u, y) \mathcal{N}(h(u), \Gamma)(\text{d}y).$$

The function  $\Pi f: \mathbb{R}^{d_u} \rightarrow \mathbb{R}$  satisfies

$$\begin{aligned} \forall u \in \mathbb{R}^{d_u}, \quad |\Pi f(u)| &\leq \int_{\mathbb{R}^{d_y}} |f(u, y)| \mathcal{N}(h(u), \Gamma)(\text{d}y) \\ &\leq \int_{\mathbb{R}^{d_y}} (1 + |u|^2 + |y|^2) \mathcal{N}(h(u), \Gamma)(\text{d}y) \\ &= 1 + |u|^2 + |h(u)|^2 + \text{tr}(\Gamma) \leq \left(1 + 2\kappa_h^2 + \text{tr}(\Gamma)\right) (1 + |u|^2). \end{aligned}$$

Therefore, we deduce that

$$\left| \mathbb{Q}\mu[f] - \mathbb{Q}\nu[f] \right| \leq \left(1 + 2\kappa_h^2 + \text{tr}(\Gamma)\right) d_g(\mu, \nu),$$

which concludes the proof.  $\square$

**Lemma B.10 (Stability of Map B<sub>j</sub>).** *Under [Assumption H](#), for any probability measure  $\mu \in \mathcal{P}(\mathbb{R}^{d_u})$  with*

$\mathcal{M}_2(\mu) < \infty$  there exists a constant  $C_B = C_B(\mathcal{M}_2(\mu), \kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma) > 0$  such that

$$\forall j \in \llbracket 0, J \rrbracket, \quad d_g(\mathbf{B}_j \text{GQP}\mu, \mathbf{B}_j \text{QP}\mu) \leq C_B d_g(\text{GQP}\mu, \text{QP}\mu).$$

*Proof.* For ease of exposition we write  $y^\dagger = y_{j+1}^\dagger$ . We define the  $y$ -marginal densities

$$\alpha_\mu(y) := \int_{\mathbb{R}^{d_u}} \text{GQP}\mu(u, y) du, \quad \beta_\mu(y) := \int_{\mathbb{R}^{d_u}} \text{QP}\mu(u, y) du.$$

It holds that

$$\begin{aligned} d_g(\mathbf{B}_j \text{GQP}\mu, \mathbf{B}_j \text{QP}\mu) &= \int_{\mathbb{R}^{d_u}} (1 + |u|^2) \left| \frac{\text{GQP}\mu(u, y^\dagger)}{\alpha_\mu(y^\dagger)} - \frac{\text{QP}\mu(u, y^\dagger)}{\beta_\mu(y^\dagger)} \right| du \\ &\leq \frac{1}{\alpha_\mu(y^\dagger)} \int_{\mathbb{R}^{d_u}} (1 + |u|^2) |\text{GQP}\mu(u, y^\dagger) - \text{QP}\mu(u, y^\dagger)| du \\ &\quad + \left| \frac{\alpha_\mu(y^\dagger) - \beta_\mu(y^\dagger)}{\alpha_\mu(y^\dagger)\beta_\mu(y^\dagger)} \right| \int_{\mathbb{R}^{d_u}} (1 + |u|^2) \text{QP}\mu(u, y^\dagger) du. \end{aligned} \quad (\text{B.18})$$

**Step 1: bounding  $\alpha_\mu(y^\dagger)$  and  $\beta_\mu(y^\dagger)$  from below.** The distribution  $\alpha_\mu(\bullet)$  is Gaussian with mean  $\mathcal{M}^y(\text{QP}\mu)$  and covariance

$$\mathcal{C}^{yy}(\text{QP}\mu) = \Gamma + \text{P}\mu[h(\bullet) \otimes h(\bullet)] - \text{P}\mu[h(\bullet)] \otimes \text{P}\mu[h(\bullet)], \quad (\text{B.19})$$

which by [Assumption H](#) is bounded from below by  $\Gamma$  and from above by  $\Gamma + 2\kappa_h^2(1 + \text{tr}(\Sigma) + 2\kappa_\Psi^2(1 + \mathcal{M}_2(\mu)))I_{d_y}$ . To deduce this we have again made use of the fact that  $aa^\top \preceq (a^\top a)I_d$  for any vector  $a \in \mathbb{R}^{d_u}$ , by the Cauchy-Schwarz inequality. Therefore, noting that [\(B.4b\)](#) implies that  $|\mathcal{M}^y(\text{QP}\mu)| \leq C$  for a constant  $C$  depending only on the parameters  $\mathcal{M}_2(\mu), \kappa_h, \kappa_\Psi, \Sigma$ , we obtain

$$\begin{aligned} \alpha_\mu(y) &= \frac{1}{\sqrt{(2\pi)^{d_u} \det(\mathcal{C}^{yy}(\text{QP}\mu))}} \exp\left(-\frac{1}{2}(y - \mathcal{M}^y(\text{QP}\mu))^\top \mathcal{C}^{yy}(\text{QP}\mu)^{-1}(y - \mathcal{M}^y(\text{QP}\mu))\right) \\ &\geq \frac{\exp\left(-\frac{1}{2}(|y| + C)^2 \|\Gamma^{-1}\|\right)}{\sqrt{(2\pi)^{d_y} \det(\Gamma + 2\kappa_h^2(1 + \text{tr}(\Sigma) + 2\kappa_\Psi^2(1 + \mathcal{M}_2(\mu)))I_{d_y})}}. \end{aligned} \quad (\text{B.20})$$

The function  $\beta_\mu$  can be bounded from below using similar reasoning. Indeed, by [Assumption H](#) we have that for all  $y \in \mathbb{R}^{d_y}$ ,

$$\begin{aligned} \beta_\mu(y) &= \int_{\mathbb{R}^{d_u}} \text{QP}\mu(u, y) du = \int_{\mathbb{R}^{d_u}} \frac{\exp\left(-\frac{1}{2}(y - h(u))^\top \Gamma^{-1}(y - h(u))\right)}{\sqrt{(2\pi)^{d_y} \det(\Gamma)}} \text{P}\mu(u) du \\ &\geq \frac{C \exp\left(-\|\Gamma^{-1}\| |y|^2 - \|\Gamma^{-1}\| \kappa_h^2 \int_{\mathbb{R}^{d_u}} |u|^2 \text{P}\mu du\right)}{\sqrt{(2\pi)^{d_y} \det(\Gamma)}} \end{aligned} \quad (\text{B.21a})$$

$$\geq C \exp\left(-\|\Gamma^{-1}\| |y|^2\right), \quad (\text{B.21b})$$

where we applied Jensen's inequality in [\(B.21a\)](#) and the constant in [\(B.21b\)](#) depends on  $\mathcal{M}_2(\mu), \kappa_h, \kappa_\Psi, \Sigma, \Gamma$ .

**Step 2: bounding the first term in [\(B.18\)](#).** We begin by noting that for fixed  $u \in \mathbb{R}^{d_u}$ , the functions  $y \mapsto \text{QP}\mu(u, y)$  and  $y \mapsto \text{GQP}\mu(u, y)$  are Gaussians up to constant factors. The covariance matrix of the former is  $\Gamma$ ; it is possible to deduce that the covariance of the latter is given by

$$\mathcal{C}^{yy}(\text{QP}\mu) - \mathcal{C}^{yu}(\text{QP}\mu)\mathcal{C}^{uu}(\text{QP}\mu)^{-1}\mathcal{C}^{uy}(\text{QP}\mu), \quad (\text{B.22})$$



where we have used the formula for the covariance of the conditional distribution of a Gaussian. We note that  $\mathcal{C}^{yu}(\mathbf{QP}\mu)\mathcal{C}^{uu}(\mathbf{QP}\mu)^{-1}\mathcal{C}^{uy}(\mathbf{QP}\mu)$  is positive semi-definite, hence by (B.19) and its upper bound it follows that the matrix (B.22) is bounded from above by  $2\kappa_h^2(1 + \text{tr}(\Sigma) + 2\kappa_\Psi^2(1 + \mathcal{M}_2(\mu)))I_{d_y} + \Gamma$ . With the same notation as in (B.8), we find that

$$\mathcal{C}^{yy}(\mathbf{QP}\mu) - \mathcal{C}^{yu}(\mathbf{QP}\mu)\mathcal{C}^{uu}(\mathbf{QP}\mu)^{-1}\mathcal{C}^{uy}(\mathbf{QP}\mu) \quad (\text{B.23})$$

$$= \Gamma + \left( \mathcal{C}^{yy}(\phi_{\sharp}\mathbf{P}\mu) - \mathcal{C}^{yu}(\phi_{\sharp}\mathbf{P}\mu)\mathcal{C}^{uu}(\phi_{\sharp}\mathbf{P}\mu)^{-1}\mathcal{C}^{uy}(\phi_{\sharp}\mathbf{P}\mu) \right) \succcurlyeq \Gamma; \quad (\text{B.24})$$

here, to deduce that the term in brackets is positive semi-definite, we observe that the term is the Schur complement of the block  $\mathcal{C}^{uu}(\phi_{\sharp}\mathbf{P}\mu)$  of the matrix  $\mathcal{C}(\phi_{\sharp}\mathbf{P}\mu)$ . Hence, (B.22) is bounded from below by  $\Gamma$ . Therefore the integral in the first term of (B.18) may be bounded from above by applying the result of Lemma A.2 in Appendix A with parameter  $\alpha = \alpha(\mathcal{M}_2(\mu), \kappa_h, \kappa_\Psi, \Sigma, \Gamma)$ , yielding

$$\begin{aligned} & \int_{\mathbb{R}^{d_u}} (1 + |u|^2) |\text{GQP}\mu(u, y) - \text{QP}\mu(u, y)| \, du \\ & \leq C \int_{\mathbb{R}^{d_y}} \int_{\mathbb{R}^{d_u}} (1 + |u|^2) |\text{GQP}\mu(u, y) - \text{QP}\mu(u, y)| \, du \, dy \\ & \leq C \int_{\mathbb{R}^{d_y}} \int_{\mathbb{R}^{d_u}} (1 + |u|^2 + |y|^2) |\text{GQP}\mu(u, y) - \text{QP}\mu(u, y)| \, du \, dy \\ & = Cd_g(\text{GQP}\mu, \text{QP}\mu), \end{aligned} \quad (\text{B.25})$$

where  $C$  depends on  $\mathcal{M}_2(\mu), \kappa_h, \kappa_\Psi, \Sigma, \Gamma$ .

**Step 3: bounding the second term in (B.18).** By (B.25), it holds that

$$\begin{aligned} |\alpha_\mu(y) - \beta_\mu(y)| &= \left| \int_{\mathbb{R}^{d_u}} \text{GQP}\mu(u, y) - \text{QP}\mu(u, y) \, du \right| \\ &\leq \int_{\mathbb{R}^{d_u}} (1 + |u|^2) |\text{GQP}\mu(u, y) - \text{QP}\mu(u, y)| \, du \leq Cd_g(\text{GQP}\mu, \text{QP}\mu). \end{aligned}$$

Now, since  $y \mapsto \text{QP}\mu(u, y)/\text{P}\mu(u)$  is a Gaussian density with covariance  $\Gamma$ , which is bounded uniformly from above by  $((2\pi)^{d_y} \det(\Gamma))^{-1/2}$ , we also have that

$$\begin{aligned} \int_{\mathbb{R}^{d_u}} (1 + |u|^2) \text{QP}\mu(u, y) \, du &= \beta_\mu(y) + \int_{\mathbb{R}^{d_u}} |u|^2 \text{QP}\mu(u, y) \, du \\ &\leq \beta_\mu(y) + \int_{\mathbb{R}^{d_u}} \frac{|u|^2 \text{P}\mu(u)}{\sqrt{(2\pi)^{d_y} \det(\Gamma)}} \, du \\ &= \beta_\mu(y) + \frac{\text{tr}(\mathcal{C}(\text{P}\mu)) + |\mathcal{M}(\text{P}\mu)|^2}{\sqrt{(2\pi)^{d_y} \det(\Gamma)}}. \end{aligned}$$

We proceed by noting the result of Lemma B.1, which gives that the first and second moments of  $\text{P}\mu$  are bounded from above by a constant which depends on  $\mathcal{M}_2(\mu), \kappa_\Psi, \Sigma$ .

**Step 4: concluding the proof.** Putting together the above bounds, we conclude that

$$d_g(\mathbb{B}_j \text{GQP}\mu, \mathbb{B}_j \text{QP}\mu) \leq \frac{C(\mathcal{M}_2(\mu), \kappa_\Psi, \kappa_h, \Sigma, \Gamma)}{\alpha_\mu(y_{j+1}^\dagger)} \left( 1 + \frac{1}{\beta_\mu(y_{j+1}^\dagger)} \right) d_g(\text{GQP}\mu, \text{QP}\mu).$$

Applying the inequalities (B.20) and (B.21b) yields the desired result.  $\square$

**Lemma B.11 (Stability of Map  $\Upsilon_j$ ).** *Suppose that Assumption H is satisfied. Then, for all  $R \geq 1$ , it holds that for any  $\pi \in \mathcal{P}_R(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$  and  $p \in \{\text{QP}\mu : \mu \in \mathcal{P}(\mathbb{R}^{d_u}) \text{ and } \mathcal{M}_{\max\{3+d_u, 4+d_y\}}(\mu) < \infty\} \subset \mathcal{P}(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$ ,*

there is  $L_\Gamma := L_\Gamma(R, \mathcal{M}_{\max\{3+d_u, 4+d_y\}}(\mu), \kappa_y, \kappa_\Psi, \kappa_h, \ell_h, \Sigma, \Gamma)$ , such that

$$\forall j \in \llbracket 1, J \rrbracket, \quad d_g(\mathbb{T}_j \pi, \mathbb{T}_j p) \leq L_\Gamma d_g(\pi, p).$$

*Proof.* By the results of [Lemma B.2](#), it holds that the probability measure  $\mathbb{Q}\mathbb{P}\mu$  belongs to  $\mathcal{P}_{\tilde{R}}(\mathbb{R}^{d_u} \times \mathbb{R}^{d_y})$  for some  $\tilde{R} \geq 1$ . Define

$$r = \max\{R, \tilde{R}, \kappa_y\}.$$

We denote by  $\mathcal{T}^\pi$  and  $\mathcal{T}^p$  the affine maps corresponding to evaluation of covariance information at the probability measures  $\pi$  and  $p = \mathbb{Q}\mathbb{P}\mu$ , respectively. Namely, letting  $K = \mathcal{C}(\pi)$ ,  $S = \mathcal{C}(p)$  and  $y^\dagger = y_{j+1}^\dagger$ , we write

$$\begin{aligned} \mathcal{T}^\pi(u, y) &= u + A_\pi(y^\dagger - y), & A_\pi &:= K_{uy} K_{yy}^{-1}, \\ \mathcal{T}^p(u, y) &= u + A_p(y^\dagger - y), & A_p &:= S_{uy} S_{yy}^{-1}. \end{aligned}$$

By a straightforward application of the triangle inequality, it holds that

$$d_g(\mathbb{T}_j \pi, \mathbb{T}_j p) \leq d_g(\mathcal{T}_\#^\pi \pi, \mathcal{T}_\#^\pi p) + d_g(\mathcal{T}_\#^\pi p, \mathcal{T}_\#^p p). \quad (\text{B.26})$$

Before bounding each term on the right-hand side of [\(B.26\)](#), we derive some inequalities that we will use in the proof. It holds that

$$\|A_\pi\| \leq \|K_{uy}\| \|K_{yy}^{-1}\| \leq \|K\| \|K^{-1}\| \leq r^4, \quad (\text{B.27})$$

and similarly for  $A_p$ . This can easily be seen by noting that the matrix norm (induced by the Euclidean vector norm) of any submatrix is bounded from above by the norm of the full matrix. Using this bound again and assuming without loss of generality that  $r > 1$ , we deduce that

$$\begin{aligned} \|A_\pi - A_p\| &= \|K_{uy} K_{yy}^{-1} - S_{uy} S_{yy}^{-1}\| \leq \|(K_{uy} - S_{uy}) K_{yy}^{-1}\| + \|S_{uy} (K_{yy}^{-1} - S_{yy}^{-1})\| \\ &\leq r^2 \left( \|K_{uy} - S_{uy}\| + \|K_{yy}^{-1} - S_{yy}^{-1}\| \right) \\ &= r^2 \left( \|K_{uy} - S_{uy}\| + \|K_{yy}^{-1} (S_{yy} - K_{yy}) S_{yy}^{-1}\| \right) \\ &\leq r^2 \left( \|K_{uy} - S_{uy}\| + \|K_{yy}^{-1}\| \|S_{yy} - K_{yy}\| \|S_{yy}^{-1}\| \right) \\ &\leq 2r^6 \|K - S\| \leq 2r^6 (1 + 2r) d_g(\pi, p), \end{aligned} \quad (\text{B.28})$$

where the last line follows from the result in [Lemma B.6](#).

**Bounding the first term in [\(B.26\)](#).** Let  $f: \mathbb{R}^{d_u} \rightarrow \mathbb{R}$  be a function such that  $|f| \leq g$  with  $g(u) = 1 + |u|^2$ . It follows from the definition of the pushforward measure that

$$|\mathcal{T}_\#^\pi \pi[f] - \mathcal{T}_\#^p p[f]| = |\pi[f \circ \mathcal{T}^\pi] - p[f \circ \mathcal{T}^p]|.$$

For any  $(u, y) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_y}$ , we have that

$$\begin{aligned} |f \circ \mathcal{T}^\pi(u, y)| &= \left| f\left(u + A_\pi(y^\dagger - y)\right) \right| \leq g\left(u + A_\pi(y^\dagger - y)\right) \\ &= 1 + \left|u + A_\pi(y^\dagger - y)\right|^2 \leq 1 + 3|u|^2 + 3|A_\pi y^\dagger|^2 + 3|A_\pi y|^2 \\ &\leq 3(1 + |A_\pi y^\dagger|^2) \max\{1, \|A_\pi\|^2\} g(u, y). \end{aligned}$$

Hence, by [\(B.27\)](#) we conclude that

$$d_g(\mathcal{T}_\#^\pi \pi, \mathcal{T}_\#^p p) \leq 3(1 + r^{10}) r^8 d_g(\pi, p). \quad (\text{B.29})$$

**Bounding the second term in (B.26).** Let  $f: \mathbb{R}^{d_u} \rightarrow \mathbb{R}$  again be a function such that the inequality  $|f| \leq g$ , with  $g(u) = 1 + |u|^2$ , is satisfied. We note that

$$\left| \mathcal{T}_\#^\pi p[f] - \mathcal{T}_\#^p p[f] \right| = \left| p[f \circ \mathcal{T}^\pi] - p[f \circ \mathcal{T}^p] \right| = \left| p[f \circ \mathcal{T}^\pi - f \circ \mathcal{T}^p] \right|.$$

The right-hand side may be rewritten as

$$\left| p[f \circ \mathcal{T}^\pi - f \circ \mathcal{T}^p] \right| = \left| \int_{\mathbb{R}^{d_y}} \int_{\mathbb{R}^{d_u}} f(u + A_\pi(y^\dagger - y)) - f(u + A_p(y^\dagger - y)) p(u, y) du dy \right|.$$

A change of variable yields

$$\begin{aligned} & \int_{\mathbb{R}^{d_y}} \int_{\mathbb{R}^{d_u}} (f(u + A_\pi z) - f(u + A_p z)) p(u, y^\dagger - z) du dz \\ &= \int_{\mathbb{R}^{d_y}} \int_{\mathbb{R}^{d_u}} f(v) \left( p(v - A_\pi z, y^\dagger - z) - p(v - A_p z, y^\dagger - z) \right) dv dz. \end{aligned} \quad (\text{B.30})$$

Since  $\mathcal{M}_{\max\{3+d_u, 4+d_y\}}(\mu) < \infty$  by assumption, from [Lemma A.3](#) it follows that

$$\begin{aligned} & \left| p(v - A_\pi z, y^\dagger - z) - p(v - A_p z, y^\dagger - z) \right| \\ & \leq C |A_\pi z - A_p z| \cdot \max \left\{ \frac{1}{1 + |v - A_\pi z|^{\max\{3+d_u, 4+d_y\}}}, \frac{1}{1 + |v - A_p z|^{\max\{3+d_u, 4+d_y\}}} \right\} \\ & \quad \cdot \frac{1}{1 + |y^\dagger - z|^{\max\{3+d_u, 4+d_y\}}}. \end{aligned}$$

We apply this inequality to bound for fixed  $z \in \mathbb{R}^{d_y}$  the inner integral in (B.30). Keeping only the terms that depend on  $v$ , we obtain that

$$\begin{aligned} & \int_{\mathbb{R}^{d_u}} |f(v)| \max \left\{ \frac{1}{1 + |v - A_\pi z|^{\max\{3+d_u, 4+d_y\}}}, \frac{1}{1 + |v - A_p z|^{\max\{3+d_u, 4+d_y\}}} \right\} dv \\ & \leq \int_{\mathbb{R}^{d_u}} \frac{|f(v)|}{1 + |v - A_\pi z|^{\max\{3+d_u, 4+d_y\}}} dv + \int_{\mathbb{R}^{d_u}} \frac{|f(v)|}{1 + |v - A_p z|^{\max\{3+d_u, 4+d_y\}}} dv. \end{aligned}$$

Since  $|f(v)| \leq 1 + |v|^2$ , proceeding by substitution gives that this expression is bounded from above by

$$C \left( 1 + |A_\pi z|^2 + |A_p z|^2 \right) \leq Cr^8 (1 + |z|^2),$$

where the last inequality follows from (B.27). The remaining integral in the  $z$  variable can be bounded in a similar way:

$$\begin{aligned} d_g(\mathcal{T}_\#^\pi p, \mathcal{T}_\#^p p) & \leq Cr^8 \int_{\mathbb{R}^{d_y}} \frac{1 + |z|^2}{1 + |y^\dagger - z|^{\max\{3+d_u, 4+d_y\}}} |A_\pi z - A_p z| dz \\ & \leq Cr^{11} \|A_\pi - A_p\| \leq Cr^{18} d_g(\pi, p), \end{aligned} \quad (\text{B.31})$$

where the last inequality follows from (B.28). Combining (B.29) and (B.31) yields the desired result.  $\square$

## C Technical Results for Approximation Result in [Proposition 2.2](#)

In [Lemma C.1](#) we recall the local Lipschitz continuity result for the operator  $G$  established in [8]. [Lemma C.3](#) establishes that the filtering distribution is a locally Lipschitz function of  $(\Psi, h)$ , viewed as a mapping from Banach space equipped with the  $\|\cdot\|_\infty$  norm into the space of probability measures metrized using the  $d_g$  distance. This is preceded by [Lemma C.2](#) which establishes bounds used to prove this Lipschitz property. The two lemmas do not require  $\Psi_0$  and  $h_0$  to be affine, but simply require that they both satisfy [Assumptions \(H2\)](#)

and (H3). This is in contrast with the more specific setting of Proposition 2.2, which imposes an affine assumption on  $(\Psi_0, h_0)$ .

**Lemma C.1.** *For all  $R \geq 1$ , there exists  $L_G = L_G(R, n)$  so that for any  $\mu_1, \mu_2 \in \mathcal{P}_R(\mathbb{R}^n)$  it holds that*

$$d_g(\mathbf{G}\mu_1, \mathbf{G}\mu_2) \leq L_G(R, n) \cdot d_g(\mu_1, \mu_2).$$

*Proof.* The lemma as stated may be found in [8, Lemma 28], where a complete proof is given.  $\square$

**Lemma C.2.** *Suppose that the matrices  $(\Sigma, \Gamma)$  satisfy Assumption (H5). Fix  $\kappa_\Psi, \kappa_h > 0$  and assume that  $\Psi_0: \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_u}$  and  $h_0: \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_y}$  are functions satisfying Assumptions (H2) and (H3). Then the following statements hold:*

- There is a constant  $C_p = C_p(\kappa_\Psi, \Sigma)$  such that for all  $\varepsilon \in [0, 1]$  and all  $(\Psi, h) \in B_{L^\infty}((\Psi_0, h_0), \varepsilon)$ ,

$$\forall \mu \in \mathcal{P}(\mathbb{R}^{d_u}), \quad d_g(\mathbf{P}_0\mu, \mathbf{P}\mu) \leq C_p \varepsilon \cdot (1 + \mathcal{M}_2(\mu)) \quad (\text{C.1})$$

- There is a constant  $C_q = C_q(\kappa_h, \Gamma)$  such that for all  $\varepsilon \in [0, 1]$  and all  $(\Psi, h) \in B_{L^\infty}((\Psi_0, h_0), \varepsilon)$ ,

$$\forall \mu \in \mathcal{P}(\mathbb{R}^{d_u}), \quad d_g(\mathbf{Q}_0\mu, \mathbf{Q}\mu) \leq C_q \varepsilon \cdot (1 + \mathcal{M}_2(\mu)) \quad (\text{C.2})$$

- There is  $C_b = C_b(\kappa_h, \Gamma)$  such that for all  $y^\dagger \in \mathbb{R}^{d_y}$ , all  $\varepsilon \in [0, 1]$ , all  $(\Psi, h) \in B_{L^\infty}((\Psi_0, h_0), \varepsilon)$ , and all probability measures  $(\mu_0, \mu) \in \mathcal{P}(\mathbb{R}^{d_u}) \times \mathcal{P}(\mathbb{R}^{d_u})$ ,

$$d_g\left(\mathbf{B}(\mathbf{Q}_0\mu_0; y^\dagger), \mathbf{B}(\mathbf{Q}\mu, y^\dagger)\right) \leq \exp(C_b \mathcal{R}) \left(\varepsilon + d_g(\mu_0, \mu)\right), \quad (\text{C.3})$$

where  $\mathcal{R} \in [1, \infty]$  is given by

$$\mathcal{R} = \max \left\{ |y^\dagger|^2, 1 + \mathcal{M}_2(\mu_0), 1 + \mathcal{M}_2(\mu) \right\}.$$

Here  $\mathbf{P}_0$  and  $\mathbf{Q}_0$  denote the maps associated to  $(\Psi_0, h_0)$ , and  $\mathbf{P}$  and  $\mathbf{Q}$  are the maps associated to  $(\Psi, h)$ .

*Proof.* Throughout this proof,  $C_p$  denotes a constant depending only on  $(\kappa_\Psi, \Sigma)$ , and  $C_q$  is a constant that depends only on  $(\kappa_h, \Gamma)$ . Both may change from line to line.

**Proof of (C.1).** It holds that

$$\mathbf{P}_0\mu(u) - \mathbf{P}\mu(u) = C_p \int_{\mathbb{R}^{d_u}} \exp\left(-\frac{1}{2}|u - \Psi_0(v)|_\Sigma^2\right) - \exp\left(-\frac{1}{2}|u - \Psi(v)|_\Sigma^2\right) \mu(dv).$$

By the elementary inequality (A.9), the integrand on the right-hand side is bounded in absolute value by

$$2|\Psi_0(v) - \Psi(v)| \left( \exp\left(-\frac{1}{4}|u - \Psi_0(v)|_\Sigma^2\right) + \exp\left(-\frac{1}{4}|u - \Psi(v)|_\Sigma^2\right) \right).$$

By Young's inequality, it holds that  $|a + b|^2 \geq \frac{1}{2}|a|^2 - |b|^2$ , and so this is bounded by

$$4|\Psi_0(v) - \Psi(v)| \left( \exp\left(-\frac{1}{8}|u - \Psi_0(v)|_\Sigma^2 + \frac{1}{4}|\Psi_0(v) - \Psi(v)|_\Sigma^2\right) \right) \leq C_p \varepsilon \exp\left(-\frac{1}{8}|u - \Psi_0(v)|_\Sigma^2\right).$$

It follows, by Fubini's theorem, that

$$\begin{aligned} d_g(\mathbf{P}_0\mu, \mathbf{P}\mu) &\leq C_p\varepsilon \int_{\mathbb{R}^{d_u}} \int_{\mathbb{R}^{d_u}} (1 + |u|^2) \exp\left(-\frac{1}{8}|u - \Psi_0(v)|_\Sigma^2\right) du \mu(dv) \\ &\leq C_p\varepsilon \int_{\mathbb{R}^{d_u}} (1 + |\Psi_0(v)|^2) \mu(dv) \leq C_p\varepsilon (1 + \kappa_\Psi^2) \int_{\mathbb{R}^{d_u}} (1 + |v|^2) \mu(dv). \end{aligned}$$

This concludes the proof of (C.1).

**Proof of (C.2).** Recall that

$$d_g(\mathbf{Q}_0\mu, \mathbf{Q}\mu) = C_q \int_{\mathbb{R}^{d_u}} \int_{\mathbb{R}^{d_y}} g(u, y) \left( \exp\left(-\frac{1}{2}|y - h_0(u)|_\Gamma^2\right) - \exp\left(-\frac{1}{2}|y - h(u)|_\Gamma^2\right) \right) dy \mu(du),$$

where  $g(u, y) = 1 + |u|^2 + |y|^2$ . Using the same reasoning as above we obtain that

$$\left| \exp\left(-\frac{1}{2}|y - h_0(u)|_\Gamma^2\right) - \exp\left(-\frac{1}{2}|y - h(u)|_\Gamma^2\right) \right| \leq C_q\varepsilon \exp\left(-\frac{1}{8}|y - h_0(u)|_\Gamma^2\right). \quad (\text{C.4})$$

Therefore, we deduce that

$$d_g(\mathbf{Q}_0\mu, \mathbf{Q}\mu) \leq C_q\varepsilon \int_{\mathbb{R}^{d_u}} (1 + |h_0(u)|^2) \mu(du) \leq C_q\varepsilon(1 + \kappa_h^2) \int_{\mathbb{R}^{d_u}} (1 + |u|^2) \mu(du),$$

which proves (C.2).

**Proof of (C.3).** We assume for simplicity that  $\mu_0$  and  $\mu$  have densities, but note that this is not required. Let  $\nu_0 = \mathbf{B}(\mathbf{Q}_0\mu_0; y^\dagger)$  and  $\nu = \mathbf{B}(\mathbf{Q}\mu; y^\dagger)$  and recall that

$$\nu_0(u) = \frac{\exp\left(-\frac{1}{2}|y^\dagger - h_0(u)|_\Gamma^2\right) \mu_0(u)}{\int_{\mathbb{R}^{d_u}} \exp\left(-\frac{1}{2}|y^\dagger - h_0(U)|_\Gamma^2\right) \mu_0(U) dU} =: \frac{f_0(u)}{\int_{\mathbb{R}^{d_u}} f_0(U) dU} =: \frac{f_0(u)}{Z_0}$$

and similarly

$$\nu(u) = \frac{\exp\left(-\frac{1}{2}|y^\dagger - h(u)|_\Gamma^2\right) \mu(u)}{\int_{\mathbb{R}^{d_u}} \exp\left(-\frac{1}{2}|y^\dagger - h(U)|_\Gamma^2\right) \mu(U) dU} =: \frac{f(u)}{\int_{\mathbb{R}^{d_u}} f(U) dU} =: \frac{f(u)}{Z}.$$

Note that

$$\begin{aligned} d_g(\nu_0, \nu) &= \int_{\mathbb{R}^{d_u}} (1 + |u|^2) \left| \frac{f_0(u)}{Z_0} - \frac{f(u)}{Z} \right| du \\ &= \frac{1}{Z} \int_{\mathbb{R}^{d_u}} (1 + |u|^2) |f_0(u) - f(u)| du + \left| \frac{1}{Z_0} - \frac{1}{Z} \right| \int_{\mathbb{R}^{d_u}} (1 + |u|^2) f_0(u) du. \end{aligned}$$

In order to bound the first term, we write

$$\begin{aligned} f_0(u) - f(u) &= \left( \exp\left(-\frac{1}{2}|y^\dagger - h_0(u)|_\Gamma^2\right) - \exp\left(-\frac{1}{2}|y^\dagger - h(u)|_\Gamma^2\right) \right) \mu_0(u) \\ &\quad + \exp\left(-\frac{1}{2}|y^\dagger - h(u)|_\Gamma^2\right) (\mu_0(u) - \mu(u)). \end{aligned} \quad (\text{C.5})$$

Using (C.4), we obtain that

$$|f_0(u) - f(u)| \leq C_q\varepsilon \exp\left(-\frac{1}{8}|y^\dagger - h_0(u)|_\Gamma^2\right) \mu_0(u) + \exp\left(-\frac{1}{2}|y^\dagger - h(u)|_\Gamma^2\right) |\mu_0(u) - \mu(u)|, \quad (\text{C.6})$$

and so

$$\int_{\mathbb{R}^{d_u}} (1 + |u|^2) |f_0(u) - f(u)| \, du \leq C_q \varepsilon \mathcal{R} + d_g(\mu_0, \mu).$$

Therefore it holds that

$$d_g(\nu_0, \nu) \leq \mathcal{R} \left( \frac{C_q \varepsilon}{Z} + \frac{|Z_0 - Z|}{Z_0 Z} \right) + \frac{1}{Z} d_g(\mu_0, \mu).$$

By (C.6) it holds that

$$|Z_0 - Z| \leq \int_{\mathbb{R}^{d_u}} |f_0(U) - f(U)| \, dU \leq C_q \varepsilon + d_g(\mu_0, \mu),$$

and so we obtain finally

$$d_g(\nu_0, \nu) \leq \mathcal{R} \left( \frac{1}{Z} + \frac{1}{Z_0 Z} \right) (C_q \varepsilon + d_g(\mu_0, \mu)).$$

Furthermore  $Z_0$  is bounded from below because, by Jensen's inequality,

$$\begin{aligned} Z_0 &= \int_{\mathbb{R}^{d_u}} \exp \left( -\frac{1}{2} |y^\dagger - h_0(U)|_\Gamma^2 \right) \mu_0(U) \, dU \geq \exp \left( -\frac{1}{2} \int_{\mathbb{R}^{d_u}} |y^\dagger - h_0(U)|_\Gamma^2 \mu_0(U) \, dU \right) \\ &\geq \exp \left( -|y^\dagger|_\Gamma^2 - C_q \kappa_h^2 \int_{\mathbb{R}^{d_u}} (1 + |u|^2) \mu_0(U) \, dU \right) = \exp(-C_q \mathcal{R}). \end{aligned}$$

The same bound holds for  $Z$ , and so we obtain the result.  $\square$

**Lemma C.3.** *Suppose that the data  $Y^\dagger = \{y_j^\dagger\}_{j=1}^J$  and the matrices  $(\Sigma, \Gamma)$  satisfy [Assumptions \(H1\)](#) and [\(H5\)](#). Fix  $\kappa_\Psi, \kappa_h > 0$  and assume that  $\Psi_0: \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_u}$  and  $h_0: \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_y}$  are functions satisfying [Assumptions \(H2\)](#) and [\(H3\)](#), respectively. Let  $(\mu_j^0)_{j \in [1, J]}$  and  $(\mu_j)_{j \in [1, J]}$  denote the true filtering distributions associated with functions  $(\Psi_0, h_0)$  and  $(\Psi, h)$ , respectively, initialized at the same Gaussian probability measure  $\mu_0 = \mathcal{N}(m_0, C_0) \in \mathcal{G}(\mathbb{R}^{d_u})$ . For all  $J \in \mathbb{Z}^+$  there is  $C = C(m_0, C_0, \kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma, J) > 0$  such that for all  $\varepsilon \in [0, 1]$  and all  $(\Psi, h) \in B_{L^\infty}((\Psi_0, h_0), \varepsilon)$ , it holds that*

$$\max_{j \in [0, J]} d_g(\mu_j^0, \mu_j) \leq C \varepsilon. \quad (\text{C.7})$$

*Proof.* By [Lemma B.3](#), the filtering distributions have bounded second moments. Let

$$\mathcal{R} = \max_{j \in [0, J-1]} \left( \left| y_{j+1}^\dagger \right|^2, 1 + \mathcal{M}_2(\mu_j^0), 1 + \mathcal{M}_2(\mu_j) \right).$$

Throughout this proof,  $C$  denotes a constant whose value is irrelevant in the context, depends only on the constants  $m_0, C_0, \kappa_y, \kappa_\Psi, \kappa_h, \Sigma, \Gamma, k$  (but neither on  $\varepsilon$ , nor on  $\Psi$  and  $h$ ) and may change from line to line.

The statement is obviously true for  $J = 0$ . Reasoning by induction, we assume that the statement is true up to  $J = k$  and show that there is  $C > 0$  such that

$$\forall \varepsilon \in [0, 1], \quad \forall (\Psi, h) \in B_{L^\infty}((\Psi_0, h_0), \varepsilon), \quad d_g(\mu_{k+1}^0, \mu_{k+1}) \leq C \varepsilon.$$

To this end, let  $P_0$  and  $Q_0$  denote the maps associated to  $(\Psi_0, h_0)$ , fix  $\varepsilon \in [0, 1]$ , and fix  $(\Psi, h) \in B_{L^\infty}((\Psi_0, h_0), \varepsilon)$ . Using [\(C.3\)](#), then the triangle inequality, and finally [\(C.1\)](#) and [Lemma B.8](#), we have that

$$\begin{aligned} d_g(\mu_{k+1}^0, \mu_{k+1}) &= d_g(\mathbf{B}_k \mathbf{Q}_0 P_0 \mu_k^0, \mathbf{B}_k \mathbf{Q} P \mu_k) \\ &\leq e^{C\mathcal{R}} \left( \varepsilon + d_g(P_0 \mu_k^0, P \mu_k) \right) \\ &\leq e^{C\mathcal{R}} \left( \varepsilon + d_g(P_0 \mu_k^0, P \mu_k^0) + d_g(P \mu_k^0, P \mu_k) \right) \\ &\leq e^{C\mathcal{R}} \left( \varepsilon + C \varepsilon \mathcal{R} + C d_g(\mu_k^0, \mu_k) \right). \end{aligned}$$

Using the induction hypothesis gives us the desired bound.  $\square$

## D Technical Result for Theorem 2.4

In [17] machinery is established to prove Monte Carlo error estimates between the finite particle ensemble Kalman filter and its mean field limit. We use such results as a component in proving Theorem 2.4 and, in so doing, explicit dependence on moments must be tracked. In this section we detail proofs of the result in [17, Proposition 4.4] and [17, Theorem 5.2], following the analysis closely and, in addition, tracking dependence on moments. This leads to the following error estimate stating the desired Monte Carlo error estimate.

**Lemma D.1.** *Assume that the probability measures  $(\mu_j^{\text{EK}})_{j \in [0, J]}$  and  $(\mu_j^{\text{EK}, N})_{j \in [0, J]}$  are obtained, respectively, from the dynamical systems (2.4) and (2.6), initialized at the Gaussian probability measure  $\mu_0^{\text{EK}} \in \mathcal{G}(\mathbb{R}^{d_u})$  and at the empirical measure  $\mu_0^{\text{EK}, N} = \frac{1}{N} \sum_{i=1}^N \delta_{u_0^{(i)}}$  for  $u_0^{(i)} \sim \mu_0^{\text{EK}}$  i.i.d. samples. That is,*

$$\mu_{j+1}^{\text{EK}} = \mathsf{T}_j \mathsf{QP} \mu_j^{\text{EK}}, \quad \mu_{j+1}^{\text{EK}, N} = \frac{1}{N} \sum_{i=1}^N \delta_{u_{j+1}^{(i)}},$$

where  $u_{j+1}^{(i)}$  evolve according to the iteration in (2.5). Suppose that the data  $Y^\dagger = \{y_j^\dagger\}_{j=1}^J$  and the matrices  $(\Sigma, \Gamma)$  satisfy Assumptions (H1) and (H5). We assume that the vector field  $h$  is a linear transformation, i.e. that Assumption (V2) is satisfied. Furthermore, if the vector field  $\Psi$  additionally satisfies  $|\Psi|_{C^{0,1}} \leq \ell_\Psi < \infty$ , then for all  $\phi$  satisfying Assumption P1, there exists a constant  $C = C(\mathcal{M}_q(\mu_0^{\text{EK}}), J, R_\phi, L_\phi, \kappa_y, \kappa_\Psi, \kappa_h, \ell_\Psi, \Sigma, \Gamma)$  where  $q := \max\{16 \cdot 3^{J-1}, 8(\varsigma + 1)3^{J-1}\}$  such that

$$\left( \mathbb{E} \left| \mu_j^{\text{EK}, N}[\phi] - \mu_j^{\text{EK}}[\phi] \right|^2 \right)^{1/2} \leq \frac{C}{\sqrt{N}}.$$

*Proof.* To prove the proposition, we apply the coupling argument used in [17]. Using similar notation to the one in [17], to the interacting  $N$ -particle system  $\{u_j^{(i)}\}_{n=1}^N$  evolving according to the ensemble Kalman dynamics (2.5), we couple  $N$  copies of the mean field dynamics  $\{\bar{u}_j^{(j)}\}_{n=1}^N$  evolving according to the mean field ensemble Kalman dynamics (2.1). The mean field replicas are synchronously coupled to the interacting particle system, in the sense that they are initialized at the same initial condition and driven by the same noises; namely, the two particle systems are initialized at i.i.d. samples  $u_0^{(i)} = \bar{u}_0^{(i)} \sim \mu_0^{\text{EK}}$  for  $i = 1, \dots, N$ . For simplicity of notation, the forecast particles are denoted by the letter  $v$ , and we drop the hat notation from the forecast and simulated observations. Furthermore, we add a bar  $\bar{\cdot}$  to all the variables related to the synchronously coupled mean field particles, including the probability measures  $\bar{\pi}_j^{\text{EK}}$ . We define for  $\pi \in \mathcal{P}(\mathbb{R}^{d_u \times d_y})$  the Kalman gain

$$\mathcal{K}(\pi) := \mathcal{C}^{uh}(\pi) \left( \mathcal{C}^{hh}(\pi) + \Gamma \right)^{-1}.$$

With this notation, the interacting particle system, and synchronously coupled system read as follows:

*Interacting particle system*

$$\begin{aligned} \textbf{Initialization: } u_0^{(i)} &= \bar{u}_0^{(i)} \\ v_{j+1}^{(i)} &= \Psi(u_j^{(i)}) + \xi_j^{(i)}, \\ y_{j+1}^{(i)} &= h(v_{j+1}^{(i)}) + \eta_{j+1}^{(i)}, \\ u_{j+1}^{(i)} &= v_{j+1}^{(i)} + \mathcal{K}(\pi_{j+1}^{\text{EK}, N}) \left( y_{j+1}^\dagger - y_{j+1}^{(i)} \right). \end{aligned}$$

*Synchronous coupling*

$$\begin{aligned} \textbf{Initialization: } \bar{u}_0^{(i)} &= u_0^{(i)} \\ \bar{v}_{j+1}^{(i)} &= \Psi(\bar{u}_j^{(i)}) + \xi_j^{(i)}, \\ \bar{y}_{j+1}^{(i)} &= h(\bar{v}_{j+1}^{(i)}) + \eta_{j+1}^{(i)}, \\ \bar{u}_{j+1}^{(i)} &= \bar{v}_{j+1}^{(i)} + \mathcal{K}(\bar{\pi}_{j+1}^{\text{EK}}) \left( y_{j+1}^\dagger - \bar{y}_{j+1}^{(i)} \right). \end{aligned}$$

Note that the synchronously coupled particles are independent and identically distributed. With this

set-up we proceed with the proof. We first define

$$D_j^{N,p} := \left( \mathbb{E} \left| u_j^{(i)} - \bar{u}_j^{(i)} \right|^p \right)^{1/p}.$$

We now let  $\phi$  satisfy [Assumption P1](#). By applying the triangle inequality, we deduce that

$$\left( \mathbb{E} \left| \mu_J^{\text{EK},N}[\phi] - \mu_J^{\text{EK}}[\phi] \right|^2 \right)^{1/2} \leq \left( \mathbb{E} \left| \mu_J^{\text{EK},N}[\phi] - \frac{1}{N} \sum_{i=1}^N \phi(\bar{u}_J^{(i)}) \right|^2 \right)^{1/2} + \left( \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \phi(\bar{u}_J^{(i)}) - \mu_J^{\text{EK}}[\phi] \right|^2 \right)^{1/2}. \quad (\text{D.3})$$

The proof of [[17](#), Theorem 5.2] uses the law of large numbers in  $L^2$  to establish the bound for the second term of the right hand side of [\(D.3\)](#)

$$\left( \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N \phi(\bar{u}_J^{(i)}) - \mu_J^{\text{EK}}[\phi] \right|^2 \right)^{1/2} \leq \frac{C \left( \mathcal{M}_{2(\varsigma+1)}(\mu_J^{\text{EK}}), R_\phi \right)}{\sqrt{N}}; \quad (\text{D.4})$$

furthermore, in the proof of [[17](#), Theorem 5.2], the following bound on the first term of the right hand side of [\(D.3\)](#) may be deduced:

$$\left( \mathbb{E} \left| \mu_J^{\text{EK},N}[\phi] - \frac{1}{N} \sum_{i=1}^N \phi(\bar{u}_J^{(i)}) \right|^2 \right)^{1/2} \leq L_\phi D_J^{N,4} \cdot \left( 1 + |\mathcal{M}_{4\varsigma}^{1/4}(\mu_J^{\text{EK}})| \right) + L_\phi |D_J^{N,2(\varsigma+1)}|^{\varsigma+1}. \quad (\text{D.5})$$

Under the assumptions of the proposition, the result of [[17](#), Proposition 4.4] shows by induction that for any  $j \in \mathbb{N}$ , it holds that <sup>1</sup>

$$\begin{aligned} D_j^{N,p} &\leq C D_{j-1}^{N,p} + C D_{j-1}^{N,2\nu(pr)} \left( D_{j-1}^{N,2\nu(pr')} + (\mathcal{M}(\mathbb{P}\mu_{j-1}^{\text{EK}}))^{2\nu(pr')} \right) \cdot \left( D_{j-1}^{N,2\nu(pr'')} + \bar{S}_j^{pr''} \right) \\ &\quad + C \left( \mathbb{E} |Z_j^N|^{pq} \right)^{1/pq} \cdot \left( D_{j-1}^{N,pq'} + \bar{S}_j^{pq'} \right), \end{aligned} \quad (\text{D.6})$$

for any  $j \in \mathbb{N}$ , where  $C$  is a constant depending on  $\Sigma, \Gamma, \ell_\Psi, \kappa_\Psi, \kappa_h, \mathcal{M}_2(\mu_{j-1}^{\text{EK}})$ . We note that the specific form of  $C$  may be deduced using the proof of [[17](#), Lemma 4.1]. Furthermore, we note that

$$\frac{1}{q} + \frac{1}{q'} = 1, \quad \frac{1}{r} + \frac{1}{r'} + \frac{1}{r''} = 1,$$

and that  $\bar{S}_j^t$  and  $Z_j^N$  are defined as

$$\begin{aligned} \bar{S}_j^t &:= \left( \mathbb{E} |y_j^\dagger - h(\bar{v}_j^{(i)}) - \eta_j^{(i)}|^t \right)^{1/t}, \\ Z_j^N &:= \left\| \mathcal{C} \left( \frac{1}{N} \sum_{i=1}^N \delta_{\bar{v}_j^{(i)}} \right) - \mathcal{C}(\mathbb{P}\mu_{j-1}^{\text{EK}}) \right\|. \end{aligned}$$

By applying the triangle inequality it is straightforward to deduce that

$$\bar{S}_j^t \leq \left( \int_{\mathbb{R}^{d_u}} |y_j^\dagger - h(u)|^t \mathbb{P}\mu_{j-1}^{\text{EK}}(du) \right)^{1/t} + (\mathbb{E} |\eta_j|^t)^{1/t} \leq C \left( \mathcal{M}_t(\mu_{j-1}^{\text{EK}}), \kappa_\Psi, \kappa_h, \kappa_y, \Sigma, \Gamma \right), \quad (\text{D.7})$$

<sup>1</sup>The result of [[17](#), Proposition 4.4] holds more generally for functions  $\Psi$  that are locally Lipschitz and that grow at most polynomially at infinity [[17](#), Assumption B]. However, to obtain [\(D.6\)](#) from the result of [[17](#), Proposition 4.4] we use that  $\Psi$  is assumed to be globally Lipschitz.



for any  $t \in \mathbb{N}$ . On the other hand, [17, Lemma 3.3] shows that

$$\left(\mathbb{E}|Z_j^N|^t\right)^{1/t} \leq \frac{C\left(\mathcal{M}_{2t}(\mu_{j-1}^{\text{EK}}), \kappa_\Psi, \Sigma\right)}{\sqrt{N}}, \quad (\text{D.8})$$

for any  $N \in \mathbb{N}$  and any  $t \in \mathbb{N}$ . Combining the results from (D.7) and (D.8) with the one in (D.6), and choosing for convenience  $q = q' = 2$  and  $r = r' = r'' = 3$  for any  $j \in \mathbb{N}$  we obtain

$$D_j^{N,p} \leq CD_{j-1}^{N,p} + CD_{j-1}^{N,3p} \left(D_{j-1}^{N,3p} + 1\right) \cdot \left(D_{j-1}^{N,3p} + 1\right) + \frac{C}{\sqrt{N}} \cdot \left(D_{j-1}^{N,2p} + 1\right), \quad (\text{D.9})$$

for a constant  $C$  depending on  $\mathcal{M}_{4p}(\mu_{j-1}^{\text{EK}}), \kappa_\Psi, \kappa_h, \kappa_y, \ell_\Psi, \Sigma, \Gamma$ . Since  $D_0^{N,p} = 0$  for any  $p \in \mathbb{N}$ , by iterating in  $j$  the result in (D.9), we find that

$$D_J^{N,4} \leq \frac{C}{\sqrt{N}},$$

where  $C$  is a constant depending on  $\mathcal{M}_{16,3^{J-1}}(\mu_0^{\text{EK}}), J, \kappa_\Psi, \kappa_h, \kappa_y, \ell_\Psi, \Sigma, \Gamma$ . Hence, by combining the results of (D.5) and (D.4) with the decomposition (D.3), we deduce that

$$\left(\mathbb{E} \left| \mu_J^{\text{EK},N}[\phi] - \mu_J^{\text{EK}}[\phi] \right|^2\right)^{1/2} \leq \frac{C}{\sqrt{N}},$$

where  $C$  is a constant depending on  $\mathcal{M}_q(\mu_0^{\text{EK}}), J, R_\phi, L_\phi, \varsigma, \kappa_\Psi, \kappa_h, \kappa_y, \ell_\Psi, \Sigma, \Gamma$  for a constant  $q$  defined as  $q := \max\{16 \cdot 3^{J-1}, 8(\varsigma + 1)3^{J-1}\}$ .  $\square$

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