Fast inference for stationary time series

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Abstract: This paper considers the statistical inference for stationary time series under weak assumptions. Firstly, a frequency domain approach is proposed for fast estimation based on a one step procedure. This method correct an initial Whittle guess estimator on a subsample by a single Fisher scoring step. The resulting estimator shares the same asymptotic properties of the Whittle estimator on the whole sample and reduce drastically the computation time. Secondly, the asymptotic covariance matrix of the Whittle estimator is estimated for full inference solving an open question raised by [40].

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1. Introduction

The Whittle log-likelihood is an approximation of Gaussian log-likelihood used for parametric estimation. The Whittle estimator (WE) is obtained by optimizing the Whittle log-likelihood. Consistency and asymptotic normality of this estimator were obtained in [21, 18, 19] under some regularity assumptions on the spectral density. In the Gaussian setting, this estimator is asymptotically efficient and shares the same asymptotic properties as the maximum likelihood estimator (MLE) (see [35] for more details on the MLE). Quite recently, the Whittle approach was adapted for non-Gaussian time series by [40]. He showed under some assumptions on the spectral density and the fourth order cumulant spectra of the dependent errors that the WE is asymptotically Gaussian with classical $\sqrt{n}$ rate. Unfortunately, the WE and the MLE are not in closed form and numerical optimization is necessary to obtain a feasible estimation. For these estimators, the numerical implementation is time consuming and could be unstable for the MLE. It is therefore interesting to construct an estimator which shares the same asymptotic properties as the WE and reduces the computation time.

To achieve this goal, we will adapt the one-step procedure presented in [34] for independent and identically distributed random variables. In this method, a unique Fisher scoring step on the log-likelihood is performed, starting from an initial $\sqrt{n}$-consistent estimator. The obtained estimator, called one-step, is quickly computed while retaining the same limit properties of the WE. This method has been successfully extended for diffusion processes [31, 25], ergodic Markov chains [33], high frequency fractional Gaussian noise [15], stable noise [14] and more recently for temporal domain weak FARIMA models [4]. Roughly speaking, the asymptotic properties of the one-step estimator are obtained under the Sweeting conditions [44].

In our approach, the WE on a subsample of size $\lfloor n^{\delta} \rfloor$ is considered as an initial guess estimator. Since the initial guess estimator is only $n^{\delta/2}$-consistent, a stochastic Lipschitz condition of the Hessian matrix is needed to obtain the asymptotic properties of the one-step estimator. The limit properties will be proved under the same standard conditions on the spectral density (as in [18, 35]) and under some assumptions on the integrability of the moment and the summability of the joint cumulant of the noise process.

In order to establish full inference tools, we also consider the estimation of the covariance matrix which remained an open problem raised by [40]. To this end, we adapt the estimators of [45] and [32], initially developed for short-memory time series data, to include long-memory models.

The paper is structured as follow. Section 2 is devoted to the presentation of the main results on the estimation procedure. In this section, we also propose a consistent estimator of the asymptotic covariance matrix of the Whittle estimator. A Monte-Carlo study is conducted in Section 3 in order to illustrate the performances of the one-step Whittle estimator via numerous examples. Conclusion and perspectives are given in Section 4. The technical proofs and auxiliary results are collected in Section 5.

2. Main results

2.1. One-step estimation

Let $(X_t)_{t\in\mathbb{Z}}$ be a second order stationary process. By the Wold representation, the process can be written as

$$X_t = \mu + \sum_{j=0}^{\infty} b_{j,\theta}\varepsilon_{t-j}, \tag{2.1}$$

where $b_{j,\theta} \in \ell^2(\mathbb{N})$, $\theta \in \Theta \subset \mathbb{R}^p$ is an unknown parameter to be estimated and $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a weak white noise of variance σ^2, admitting the following representation:

$$\varepsilon_t = G(\ldots, w_{t-1}, w_t),$$

where $(w_t)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. random variables, and G is a measurable function for which ε_t is well-defined. The parameter space Θ is assumed to be compact in $\mathbb{R}^p$. It is worth mentioning that the noise $(\varepsilon_t)_{t\in\mathbb{Z}}$ is assumed to be only uncorrelated, not necessarily independent nor a martingale difference. The parameter μ is considered a nuisance parameter which is not restrictive here and can be easily estimated.

The spectral density of $(X_t)_{t\in\mathbb{Z}}$ is therefore given for any $\lambda \in [-\pi, \pi]$ by

$$f_\vartheta(\lambda) = \frac{\sigma^2}{2\pi} g_\theta(\lambda), \tag{2.2}$$

where $\vartheta = \left(\theta, \sigma^2\right)$ and

$$g_\theta(\lambda) = \left| \sum_{j=0}^{\infty} \exp(ij\lambda)\, b_{j,\theta} \right|^2. \tag{2.3}$$

Given a realization $(X_1, X_2, \ldots, X_n)$ of length n, we consider the following Whittle log-likelihood contrast function

$$\mathcal{L}_n^W(\vartheta) = \frac{1}{2n} \sum_{j=1}^{n-1} \log f_\vartheta\left(\lambda_j\right) + \frac{I_n\left(\lambda_j\right)}{f_\vartheta\left(\lambda_j\right)}, \tag{2.4}$$

with $\lambda_j = 2\pi j/n$, $j \in \{1, \ldots, n-1\}$ and $I_n(\cdot)$ is the periodogram defined by

$$I_n(\lambda) = \frac{1}{2\pi n} \left| \sum_{k=1}^{n} \exp(ik\lambda)\, X_k \right|^2. \tag{2.5}$$

We make the following standard assumptions on the spectral density of the process and the joint cumulants of the noise:

(A1) We assume that:

- for any $\vartheta \in \Theta$, $f_\vartheta(\lambda)$ is three-times continuously differentiable on Θ. In addition, for any $0 \leqslant \ell \leqslant 3$, the partial derivative

$$\frac{\partial^\ell}{\partial\vartheta_{j_1} \ldots \partial\vartheta_{j_\ell}} f_\vartheta(\lambda),$$

is continuous on $\Theta \times [-\pi, \pi] \backslash \{0\}$, continuously differentiable with respect to λ and its partial derivative

$$\frac{\partial^{\ell+1}}{\partial\lambda\partial\vartheta_{j_1} \ldots \partial\vartheta_{j_\ell}} f_\vartheta(\lambda),$$

is continuous on $\Theta \times [-\pi, \pi] \backslash \{0\}$.

- There exists a continuous function $\alpha : \Theta \longrightarrow]-1,1[$, such that for any compact set $\Theta^* \subset \Theta$ and $\delta > 0$, the following conditions hold for all $(\vartheta, \lambda) \in \Theta^* \times [-\pi, \pi] \backslash \{0\}$:

$$c_{1,\delta,\Theta^*} |\lambda|^{-\alpha(\vartheta)+\delta} \leqslant f_\vartheta(\lambda) \leqslant c_{2,\delta,\Theta^*} |\lambda|^{-\alpha(\vartheta)-\delta}.$$

Moreover, for any $\ell \in \{0,1,2,3\}$ and any $j \in \{1,\dots,p+1\}^\ell$,

$$\left| \frac{\partial^{\ell+1}}{\partial\lambda\partial\vartheta_{j_1}\dots\partial_{j_\ell}} f_\vartheta(\lambda) \right| \leqslant c_{3,\delta,\Theta^*} |\lambda|^{-\alpha(\vartheta)-1-\delta}$$

and

$$\left| \frac{\partial^{\ell}}{\partial\vartheta_{j_1}\dots\partial_{j_\ell}} f_\vartheta(\lambda) \right| \leqslant c_{4,\delta,\Theta^*} |\lambda|^{-\alpha(\vartheta)-\delta},$$

where c_{i,δ,Θ^*} are some positive finite constants that depend only on δ and Θ^*.

(A1') The spectral density f_ϑ verifies the same assumptions as in **(A1)** with $\ell = 4$.

(A2) $\vartheta_1 \neq \vartheta_2$ implies that the set $\left\{\lambda \,\middle|\, f_{\vartheta_1}(\lambda) \neq f_{\vartheta_2}(\lambda)\right\}$ has positive Lebesgue measure.

(A3) Suppose that $\mathbb{E}[\varepsilon_t^4] < \infty$ and that $\sum_{t\geq 0}(\mathbb{E}[\left|\varepsilon_t - \varepsilon_t'\right|^4])^{1/4} < \infty$, where $\varepsilon_t' = G(\dots, w_{-1}, w_0', w_1, \dots, w_t)$ with $(w_t')_{t\in\mathbb{Z}}$ being an i.i.d. copy of $(w_t)_{t\in\mathbb{Z}}$.

(A4) $\sum_{(j,k,\ell)\in\mathbb{Z}^3} \left|\mathrm{cum}\left(\varepsilon_0, \varepsilon_j, \varepsilon_k, \varepsilon_\ell\right)\right| < \infty$.

(A5) For any $1 \leqslant \ell \leqslant 7$ and $1 \leqslant k \leqslant \ell$, $\sum_{(j_1,\dots,j_\ell)\in\mathbb{Z}^\ell} \left|j_k \mathrm{cum}\left(\varepsilon_0, \varepsilon_{j_1}, \dots, \varepsilon_{j_\ell}\right)\right| < \infty$.

***Remark* 2.1.** *Obviously, **(A5)** implies **(A4)**. Assumption **(A5)** is technical and only needed for proving the consistency of the estimator of the limiting covariance matrix.*

***Remark* 2.2.** *Under **(A4)**, the bispectrum of the process $(\varepsilon_t)_{t\in\mathbb{Z}}$ is given by*

$$f_4\left(\underline{\lambda}\right) = \frac{1}{(2\pi)^3} \sum_{\underline{k}=(k_1,k_3,k_3)\in\mathbb{Z}^3} cum\left(\varepsilon_0, \varepsilon_{k_1}, \varepsilon_{k_2}, \varepsilon_{k_3}\right) \exp\left(-i\left\langle\underline{\lambda}, \underline{k}\right\rangle\right), \tag{2.6}$$

*where $\underline{\lambda} \in [-\pi,\pi]^3$. Note that under **(A4)**, f_4 is continuous and bounded.*

***Remark* 2.3.** *Assumption **(A1)** is satisfied for a large class of long memory processes including the well known FARIMA models (see for example [18]).*

The following proposition is a slight modification of [40, Theorem 2.1] and its proof is postponed to Subsection 5.2.

Proposition 2.1. *Consider*

$$\widehat{\vartheta}_n = \arg\min_{\vartheta\in\Theta} \mathcal{L}_n^W(\vartheta). \tag{2.7}$$

*Under **(A1)-(A4)**, we have*

$$\sqrt{n}\left(\widehat{\vartheta}_n - \vartheta\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \mathcal{I}(\vartheta)^{-1} \mathcal{W}(\vartheta) \mathcal{I}(\vartheta)^{-1}\right), \tag{2.8}$$

where

$$\mathcal{I}(\vartheta) = \frac{1}{4\pi}\left(\int_{-\pi}^{\pi} \frac{\partial \log f_\vartheta(\lambda)}{\partial \vartheta_k} \frac{\partial \log f_\vartheta(\lambda)}{\partial \vartheta_j} d\lambda\right)_{1\leqslant k,j\leqslant d} \tag{2.9}$$

and

$$\mathcal{W}(\vartheta) = \frac{\pi}{2\sigma_2^2}\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_4(\lambda_1, -\lambda_2, \lambda_2) \frac{\partial \log f_\vartheta(\lambda_1)}{\partial \vartheta_k} \frac{\partial \log f_\vartheta(\lambda_2)}{\partial \vartheta_j} d\lambda_1 d\lambda_2\right)_{1\leqslant k,j\leqslant d} + \mathcal{I}(\vartheta). \tag{2.10}$$

***Remark* 2.4.** *When the process is Gaussian, $f_4 \equiv 0$ and the limiting covariance matrix is reduced to $\{\mathcal{I}(\vartheta)\}^{-1}$. This is a well-known result presented in [21, 18].*

In practice, the computation of the Whittle estimator $\widehat{\vartheta}_n$ requires a numerical optimization of the objective function (2.4) which can be time consuming for large samples. To improve significantly the computation costs, we introduce a one-step estimator.

Starting from an initial guess estimator $\widetilde{\vartheta}_n$ of ϑ, the one-step corrected estimator is defined by

$$\overline{\vartheta}_n = \widetilde{\vartheta}_n - \mathcal{I}\left(\widetilde{\vartheta}_n\right)^{-1} \nabla \mathcal{L}_n^W\left(\widetilde{\vartheta}_n\right). \tag{2.11}$$

Theorem 2.1. *Let $\widetilde{\vartheta}_n$ such that*

$$n^{\frac{\delta}{2}}\left(\widetilde{\vartheta}_n - \vartheta\right) = O_{\mathbb{P}}(1) \text{ for some } \frac{1}{2} < \delta \leqslant 1.$$

*Under Assumptions **(A1)-(A4)**, we have*

$$\sqrt{n}\left(\overline{\vartheta}_n - \vartheta\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \mathcal{I}(\vartheta)^{-1} \mathcal{W}(\vartheta) \mathcal{I}(\vartheta)^{-1}\right). \tag{2.12}$$

The proof of this theorem is given in Subsections 5.3.

***Remark* 2.5.** *One possible initial estimator which satisfies our assumptions is the Whittle estimator evaluated on a subsample of size $\lfloor n^\delta \rfloor$, $1/2 < \delta \leq 1$. If $\delta \leq 1/2$, the one-step estimator remains consistent but needs multiple Fisher scoring steps to reach similar asymptotic normality as the one given in Theorem 2.1. Furthermore, the limiting variance matrix $\mathcal{I}(\vartheta)$ can be replaced by the second derivative of the Whittle log-likelihood function.*

2.2. *Estimation of the asymptotic covariance matrix*

The limit covariance matrix of the one-step estimator $\overline{\vartheta}_n$ involves the bispectral density in an integral form, namely

$$T = \left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_4(\lambda_1, -\lambda_2, \lambda_2) \frac{\partial \log f_\vartheta(\lambda_1)}{\partial \vartheta_k} \frac{\partial \log f_\vartheta(\lambda_2)}{\partial \vartheta_j} d\lambda_1 d\lambda_2\right)_{1\leqslant k,j\leqslant d}. \tag{2.13}$$

This quantity depends on the unobservable noise and needs to be estimated from the observations. This is necessary to achieve full inference analysis. When the process is directly observable, [45, 32] proposed an estimation of this integral term. We extend there results to the unobservable noise in order to estimate the matrix in (2.13). Here we consider that the process satisfies

$$(1-L)^{d_0} X_t = u_t, \tag{2.14}$$

where

$$u_t = A_{\theta_0}(L)\,\varepsilon_t = \sum_{j\geqslant 0} a_j(\theta_0)\,\varepsilon_{t-j}. \tag{2.15}$$

The real number d_0 is the long memory parameter, L stands for back-shift operator and $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a weak white noise. The fractional difference operator $(1-L)^{d_0}$ admits the following expansion:

$$(1-L)^{d_0} = \sum_{j\geqslant 0} \alpha_j(d_0)\,L^j,$$

where

$$\alpha_j(d_0) = \frac{\Gamma(j-d_0)}{\Gamma(j+1)\,\Gamma(-d_0)}$$

and $\Gamma(\cdot)$ is the Gamma function. The filter $A_{\theta_0}(L)$ describes the short-memory part of the process $(X_t)_{t\in\mathbb{Z}}$ while the filter $(1-L)^{d_0}$ is for, eventually, the long-range dependence. The parameter $\vartheta = (d,\theta)$ is assumed to belong to $\Theta = [d_1,d_2]\times\Xi$ where $[d_1,d_2]$, Ξ are compact subsets of $(-1/2,1/2)$ and $\mathbb{R}^{p-1}$ respectively. Ξ is a compact subspace of $\mathbb{R}^{p-1}$ such that the filter $A_\theta(L)$ is invertible for any $\theta\in\Xi$. The inverse of $A_\theta(L)$ is denoted by $\mathbf{A}_\theta(\mathbf{L})$ (generic conditions to ensure the inversibility of the filter $A_\theta(L)$ can be found in [26]). Since we only observe $X_1,\ldots,X_n$, we define, for any $\vartheta\in\Theta$ and $1\leqslant t\leqslant n$, the process

$$\widetilde{\varepsilon}_t(\vartheta) = \mathbf{A}_\theta(\mathbf{L})\,(1-L)^{d_0}\{X_t\mathbb{1}_{t\geqslant 1}\}. \tag{2.16}$$

Finally, let H be a continuous function with support in $[-\pi,\pi]$ such that

$$\int_{-\pi}^{\pi} H(\lambda)\,d\lambda = 1$$

and consider

$$w_n(\lambda) = \sum_{j=1}^{n} \exp(-ij\lambda)\,\widetilde{\varepsilon}_j\left(\overline{\vartheta}_n\right).$$

Let $I = \left\{-\left[\frac{n}{2}\right],\left[\frac{n}{2}\right]\right\}\setminus\{0\}$, we denote

$$A_{1,n}^{(k,\ell)} = \left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3\in I^3} H_n\left(\frac{2\pi(j_2+j_3)}{n}\right)\phi_{\overline{\vartheta}_n}^{(k)}\left(\frac{2\pi j_1}{n}\right)\phi_{\overline{\vartheta}_n}^{(\ell)}\left(\frac{2\pi j_2}{n}\right)\times\frac{1}{8\pi^3 n}w_n\left(\frac{2\pi j_1}{n}\right)w_n\left(\frac{2\pi j_2}{n}\right)$$

$$\times w_n\left(\frac{2\pi j_3}{n}\right) w_n\left(-\frac{2\pi\left(j_1+j_2+j_3\right)}{n}\right),$$

$$A_{2,n}^{(k,\ell)} = \frac{\pi}{n}\sum_{j\in I}\left(\phi_{\bar{\vartheta}_n}^{(k)}\left(\frac{2\pi j}{n}\right)\phi_{\bar{\vartheta}_n}^{(\ell)}\left(\frac{2\pi j}{n}\right) + \phi_{\bar{\vartheta}_n}^{(k)}\left(\frac{2\pi j}{n}\right)\phi_{\bar{\vartheta}_n}^{(\ell)}\left(\frac{-2\pi j}{n}\right)\right)\frac{1}{4\pi^2 n^2}\left|w_n\left(\frac{2\pi j}{n}\right)\right|^4$$

and

$$A_{3,n}^{(k,\ell)} = \frac{1}{k_n}\frac{4\pi^2}{n^2}\sum_{j_1,j_2\in I^2} H(0)\,\phi_{\bar{\vartheta}_n}^{(k)}\left(\frac{2\pi j_1}{n}\right)\phi_{\bar{\vartheta}_n}^{(\ell)}\left(\frac{2\pi j_2}{n}\right)\frac{1}{4\pi^2 n^2}\left|w_n\left(\frac{2\pi j_1}{n}\right)w_n\left(\frac{2\pi j_2}{n}\right)\right|^2.$$

In this expression,

$$\phi_{\bar{\vartheta}_n}^{(k)}(\lambda) = \frac{\partial \log f_{\bar{\vartheta}_n}(\lambda)}{\partial \vartheta_k}, \tag{2.17}$$

k_n is a bandwidth and $H_n(\lambda) = k_n^{-1}H\left(k_n^{-1}\lambda\right)$. An estimator of the term (k,ℓ) in (2.13) is given by

$$V_n^{(k,\ell)} = A_{1,n}^{(k,\ell)} - A_{2,n}^{(k,\ell)} - A_{3,n}^{(k,\ell)}. \tag{2.18}$$

Roughly speaking, the last two terms in (2.18) cancel out the contribution of f_2 in the first term, allowing us to recover (2.13).

Theorem 2.2. *Assume that the bandwidth satisfies $k_n \longrightarrow 0$, $nk_n^2 \longrightarrow \infty$ and $d_1 - d_0 > -\frac{1}{2}$. Then, under Assumptions **(A1')-(A5)**, we have*

$$V_n^{(k,\ell)} - T^{(k,\ell)} \xrightarrow[n\to\infty]{\mathbb{P}} 0,$$

for any $1 \leqslant k, \ell \leqslant p$.

The proof is postponed to Subsection 5.4.

***Remark* 2.6.** *This last result allows to obtain confidence intervals for the parameters of Model (2.14)-(2.15). The previous two subsections therefore provide a complete fast Whittle-based inference for nonlinear long-memory processes.*

The estimator proposed by [45] includes the sum over all Fourier frequencies, including those that satisfy the relation $\sum_{j=p}^{s}\lambda_j = 0\,[2\pi]$. To improve the estimator, [32] proposes a variant where the sum over these indices is excluded. We prove the convergence of the Keenan estimator for our model in the following theorem.

Theorem 2.3. *For fixed $(j_1, j_2, j_3) \in I^3$, denote by $\mathcal{P}(j_1, j_2, j_3)$ the set of all subsets of $\{j_1, j_2, j_3\}$. Consider the following subset of I^3:*

$$\Lambda^3 = \left\{(j_1, j_2, j_3) \in I^3 \;\middle|\; \sum_{j_k\in E}\frac{2\pi j_k}{n} \neq 0\,[2\pi],\ \forall\, E \in \mathcal{P}(j_1, j_2, j_3)\right\}. \tag{2.19}$$

Let

$$B_n^{(k,\ell)} = \left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3 \in \Lambda^3} H_n\left(\frac{2\pi\,(j_2+j_3)}{n}\right) \phi_{\vartheta_n}^{(k)}\left(\frac{2\pi j_1}{n}\right) \phi_{\vartheta_n}^{(\ell)}\left(\frac{2\pi j_2}{n}\right) \times \frac{1}{8\pi^3 n} w_n\left(\frac{2\pi j_1}{n}\right) w_n\left(\frac{2\pi j_2}{n}\right) \times w_n\left(\frac{2\pi j_3}{n}\right) w_n\left(-\frac{2\pi\,(j_1+j_2+j_3)}{n}\right).$$

Under the assumptions of Theorem 2.2, we have

$$B_n^{(k,\ell)} - T^{(k,\ell)} \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

The proof is given in Subsection 5.5.

3. Simulations

In this section, we perform simulations for FARIMA models of the form:

$$a_{\theta_0}(L)\,(1-L)^{d_0}\,X_t = b_{\theta_0}(L)\,\varepsilon_t, \tag{3.1}$$

where $a_{\theta_0}(L)$ and $b_{\theta_0}(L)$ are respectively the autoregressive and the moving-average operators and represent the short memory part of the model. Note that these models verify the standard assumptions of the parameter estimation part and are particular case of the model adopted in the covariance estimation subsection. The operators $a_{\theta_0}(L)$ and $b_{\theta_0}(L)$ are defined by

$$a_{\theta_0}(L) = 1 - \sum_{i=1}^{p} a_i L^i \text{ and } b_{\theta_0}(L) = 1 - \sum_{j=1}^{q} b_j L^j.$$

Let Θ^* be the parameter space

$$\Theta^* = \Big\{\theta = (\theta_1, \theta_2, \ldots, \theta_{p+q}) \in \mathbb{R}^{p+q}; a_\theta(z) = 1 - \sum_{i=1}^{p} \theta_i z^i \text{ and } b_\theta(z) = 1 - \sum_{j=1}^{q} \theta_{p+j} z^j \text{ have all their zeros outside the unit disk}\Big\}.$$

We consider therefore the following parameter space $\Theta = \Theta^* \times]-\frac{1}{2}, \frac{1}{2}[$ and we assume that the unknown parameter $\vartheta_0 = \left(a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q, d_0\right)^\top$ belongs to Θ. Under these conditions, the process $(X_t)_{t\in\mathbb{Z}}$ is causal and invertible. For all $\vartheta \in \Theta$ we define the stationary process $\varepsilon_t(\vartheta)$ as solution of

$$\varepsilon_t(\vartheta) = \sum_{j\geqslant 0} \alpha_j(d)\, X_{t-j} - \sum_{i=1}^{p} \theta_i \sum_{j\geqslant 0} \alpha_j(d)\, X_{t-i-j} + \sum_{j=1}^{q} \theta_{p+j} \varepsilon_{t-j}(\vartheta).$$

Since we only have n observations $X_1, \ldots, X_n$ of the process $(X_t)_{t\in\mathbb{Z}}$, we approximate $\varepsilon_t(\vartheta)$, for $1 \le t \le n$, by $\widetilde{\varepsilon}_t(\vartheta)$ defined recursively by

$$\widetilde{\varepsilon}_t(\vartheta) = \sum_{j=0}^{t-1} \alpha_j(d) X_{t-j} - \sum_{i=1}^{p} \theta_i \sum_{j=0}^{t-i-1} \alpha_j(d) X_{t-i-j} + \sum_{j=1}^{q} \theta_{p+j} \widetilde{\varepsilon}_{t-j}(\vartheta)$$

with $\widetilde{\varepsilon}_t(\vartheta) = X_t = 0$ if $t \leqslant 0$. These initial values are asymptotically negligible, in particular $\varepsilon_t(\vartheta) - \widetilde{\varepsilon}_t(\vartheta) \longrightarrow 0$ in $\mathbb{L}^2$ as $t \longrightarrow \infty$. For the Monte-Carlo simulations, we consider FARIMA$(1, d, 1)$ model of the form

$$(1-L)^d (X_t - aX_{t-1}) = \varepsilon_t - b\varepsilon_{t-1} \tag{3.2}$$

for different type of noise $(\varepsilon_t)_{t\in\mathbb{Z}}$. We denote by $\vartheta = (a, b, d)$.

3.1. *Numerical illustrations for the one-step estimator*

For different values of the parameter $\vartheta = (a, b, d)$, we illustrate the empirical distribution of the OS estimator of the memory parameter d. We perform $3,000$ Monte-Carlo simulations on a sample of size $n = 2^{12}$. The initial estimator is the Whittle one carried out on a subsample of size $n^{5/6}$. In this subsection, we consider the noise process $\varepsilon_t = \eta_t\eta_{t-1}$ where $(\eta_t)_{t\in\mathbb{Z}}$ is i.i.d. sequence of standard Gaussian distribution. Figure 1 illustrates the concordance between the empirical and the theoretical distributions derived in Theorem 2.1, even when we approach the boundary of the parameter space.

***Remark* 3.1.** *To reduce the computational complexity of the Discrete Fourier Transform, we consider sample sizes that are powers of 2 (see for example [17]).*

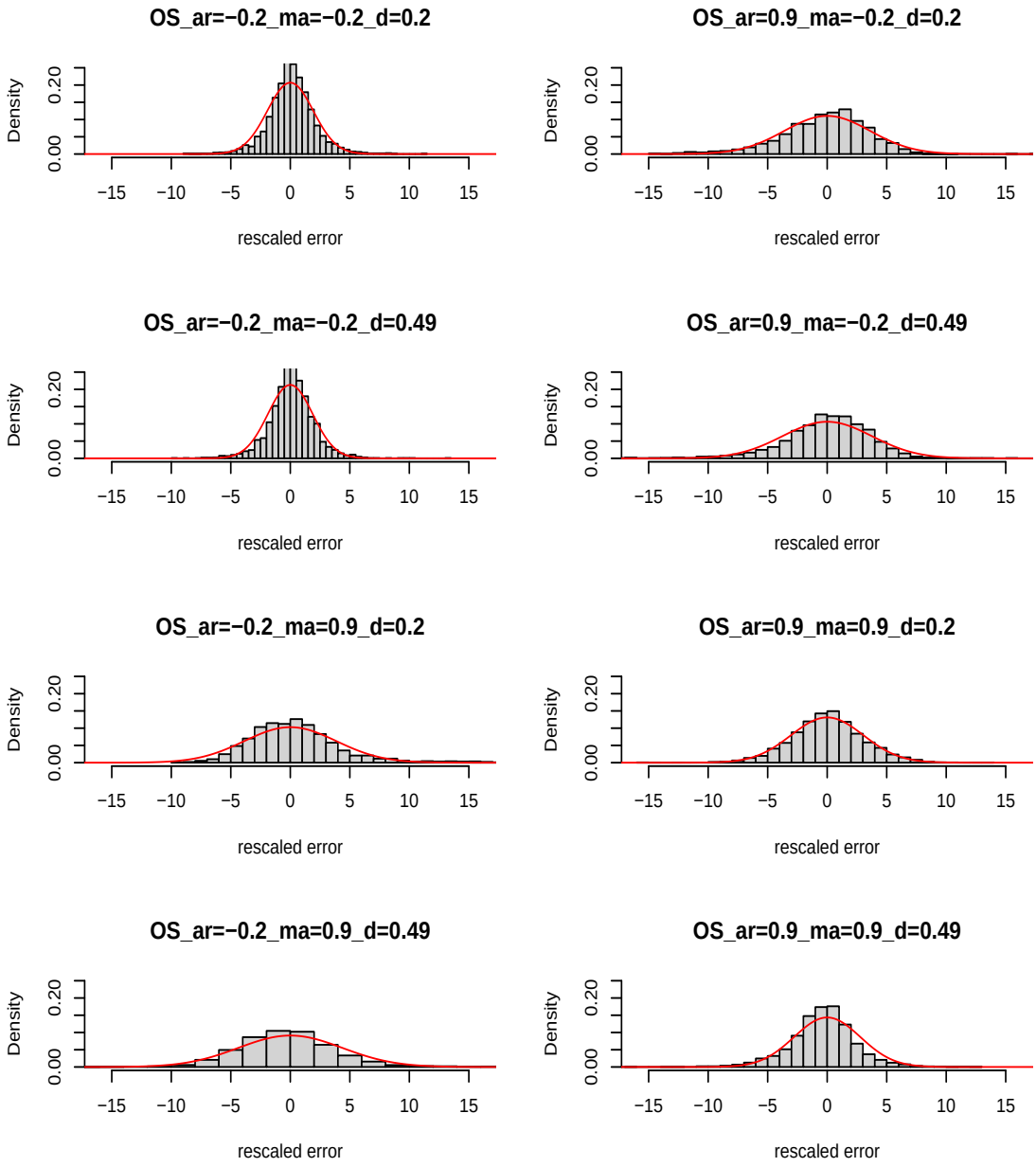

Fig 1: Empirical distribution of the rescaled statistical error of the one-step estimator of the memory parameter for FARIMA$(1, d, 1)$ model.

We now compare the evolution of the computation time between the Whittle

and the one-step estimators for different sample sizes. We perform 1,000 Monte-Carlo simulations for $n = 2^j$ where $j \in \{11, 12, 13\}$. The sub-Whittle estimator is carried out on a $2^{[j^\delta]}$ subsample where $\delta \in \{0.7, 0.9\}$.

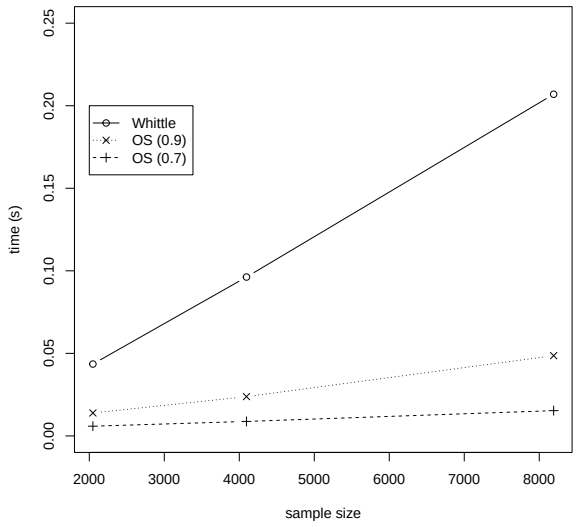

Fig 2: Comparison of the computation times (in seconds) with respect to the sample size of the Whittle and the one-step estimators of the parameters of Model (3.2) with $(a, b, d)^\top = (0.2, 0.5, 0.3)^\top$. For each size n, 1,000 replications are generated.

To finish this subsection, we give the evolution of the root mean square error (RMSE) for different estimators (OS and Whittle) as a function of sample size.

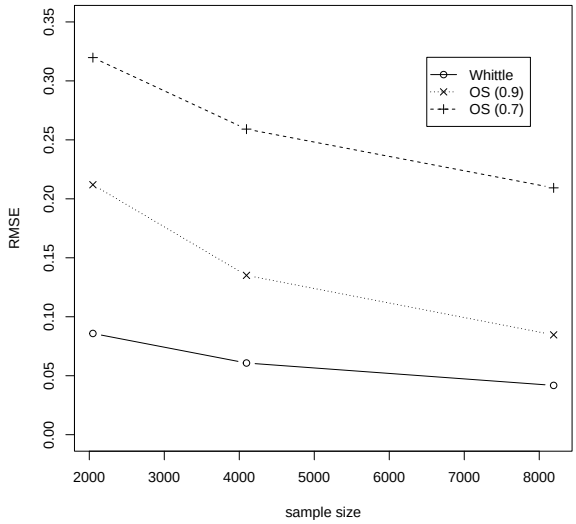

Fig 3: Comparison of the RMSE with respect to the sample size of the Whittle and the one-step estimators of the parameters of Model (3.2) with $(a, b, d)^\top = (0.2, 0.5, 0.3)^\top$. For each size n, 1,000 replications are generated.

We observe that the one-step estimator is more efficient than the LSE in terms of computation time. Additionally, using a small fraction δ further decreases the calculation time, though it slightly increases the corresponding RMSE.

3.2. *Numerical experiments for the estimation of the limit covariance matrix*

In this subsection, we numerically illustrate the convergence of our estimators given in Theorems 2.2 and 2.3 to the matrix T defined in (2.13). We consider two examples of noise: the first is Gaussian, while the second is more complex and results in a non-constant bispectrum. It should be noted that the performance of the estimator, particularly its variance, depends on the nonlinear dependence of the noise. For each set of parameters, we perform $1,000$ Monte Carlo simulations. The sample size n and the bandwidth k_n are specified in the tables. We choose the rectangular kernel $H(\lambda) = \frac{1}{2\pi}\mathbb{1}_{[-\pi,\pi]}(\lambda)$ to compute the estimators. For simplicity of reading, we adopt the following notations hereafter:

- B: the bias;
- SD: the standard deviation;
- RSME: the root mean square error;
- AR: the autoregressive parameter (*i.e.* a);
- MA: the moving average parameter (*i.e.* b).

Gaussian noise. We begin with a straightforward case where the FARIMA model (3.2) is driven by Gaussian noise, leading to (2.13) being the zero matrix.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	0.1804	0.2329	0.3332	0.0314	0.0341	0.0804	0.1831	0.2354	0.3427
$n^{-0.15}$	0.1048	0.1596	0.2290	0.0360	0.0370	0.0871	0.1108	0.1639	0.2450
$n^{-0.2}$	0.0604	0.1053	0.1532	0.0420	0.0412	0.0975	0.0736	0.1131	0.1816

TABLE 1

Bias, SD and RMSE when $\vartheta = (0.2, -0.2, 0.2)^\top$ and $n = 2^{12}$ for Taniguchi's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	0.1659	0.2204	0.3233	0.0238	0.0254	0.0644	0.1676	0.2218	0.3297
$n^{-0.15}$	0.0919	0.1449	0.2156	0.0275	0.0273	0.0693	0.0959	0.1475	0.2265
$n^{-0.2}$	0.0512	0.0919	0.1410	0.0312	0.0309	0.0769	0.0603	0.0969	0.1606

TABLE 2

Bias, SD and RMSE when $\vartheta = (0.2, -0.2, 0.2)^\top$ and $n = 2^{13}$ for Taniguchi's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	-0.0009	-0.0011	-0.0009	0.0228	0.0199	0.0336	0.0228	0.0199	0.0336
$n^{-0.15}$	-0.0015	-0.0016	-0.0016	0.0299	0.0264	0.0469	0.0299	0.0265	0.0470
$n^{-0.2}$	-0.0013	-0.0012	-0.0021	0.0371	0.0338	0.0614	0.0371	0.0338	0.0614

TABLE 3

Bias, SD and RMSE when $\vartheta = (0.2, -0.2, 0.2)^\top$ and $n = 2^{12}$ for Keenan's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	-0.0005	-0.0005	-0.0008	0.0173	0.0149	0.0260	0.0173	0.0149	0.0261
$n^{-0.15}$	-0.0008	-0.0008	0.0019	0.0230	0.0201	0.0372	0.0230	0.0291	0.0372
$n^{-0.2}$	-0.0008	-0.0007	-0.0019	0.0284	0.0258	0.0491	0.0284	0.0259	0.0492

TABLE 4

Bias, SD and RMSE when $\vartheta_0 = (0.2, -0.2, 0.2)^\top$ and $n = 2^{13}$ for Keenan's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	0.4323	0.2325	0.3242	0.1242	0.0334	0.0793	0.4498	0.2349	0.3388
$n^{-0.15}$	0.2877	0.1591	0.2194	0.1254	0.0358	0.0854	0.3139	0.1631	0.2355
$n^{-0.2}$	0.1692	0.1048	0.1432	0.1278	0.0406	0.0936	0.2120	0.1123	0.1710

TABLE 5

Bias, SD and RMSE when $\vartheta = (0.7, -0.2, -0.3)^\top$ and $n = 2^{12}$ for Taniguchi's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	0.4059	0.2199	0.3159	0.0848	0.0242	0.0618	0.4147	0.2212	0.3219
$n^{-0.15}$	0.2525	0.1445	0.2078	0.0848	0.0264	0.0659	0.2664	0.1469	0.2179
$n^{-0.2}$	0.1365	0.0914	0.1329	0.0862	0.0303	0.0717	0.1614	0.0963	0.1510

TABLE 6

Bias, SD and RMSE when $\vartheta = (0.7, -0.2, -0.3)^\top$ and $n = 2^{13}$ for Taniguchi's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	-0.0007	-0.0011	-0.0011	0.0415	0.0200	0.0338	0.0415	0.0201	0.0338
$n^{-0.15}$	-0.0014	-0.0016	-0.0018	0.0600	0.0265	0.0472	0.0600	0.0265	0.0472
$n^{-0.2}$	-0.0019	-0.0013	-0.0025	0.0817	0.0339	0.0618	0.0817	0.0339	0.0618

TABLE 7

Bias, SD and RMSE when $\vartheta = (0.7, -0.2, 0.3)^\top$ and $n = 2^{12}$ for Keenan's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	-0.0011	-0.0005	-0.0010	0.0313	0.0149	0.0259	0.0313	0.0149	0.0260
$n^{-0.15}$	-0.0020	-0.0008	-0.0020	0.0463	0.0201	0.0371	0.463	0.0201	0.0372
$n^{-0.2}$	-0.0026	-0.0008	-0.0023	0.0628	0.0258	0.0491	0.0628	0.0258	0.0492

TABLE 8

Bias, SD and RMSE when $\vartheta = (0.7, -0.2, 0.3)^\top$ and $n = 2^{13}$ for Keenan's estimator.

Nonlinear ARMA type noise. We consider an example of weak white noise with non constant bispectrum. For any $t \in \mathbb{Z}$, the processes is defined by

$$z_t = \alpha z_{t-1} + w_t - \frac{1}{\alpha} w_{t-1}, \tag{3.3}$$

where $|\alpha| < 1, \alpha \neq 0$ and $(w_t)_{t\in\mathbb{Z}}$ is a strong white noise. The process $(z_t)_{t\in\mathbb{Z}}$ is an ARMA(1, 1) process with constant spectral density

$$f_2(\lambda) = \frac{\mathbb{V}ar(w_0)}{2\pi} \frac{\left|1 - \frac{1}{\alpha}\exp(-i\lambda)\right|^2}{\left|1 - \alpha\exp(-i\lambda)\right|^2} = \frac{\mathbb{V}ar(w_0)}{2\pi\alpha^2},$$

so $(z_t)_{t\in\mathbb{Z}}$ is a weak white noise. Let, for any $t \in \mathbb{Z}$,

$$\varepsilon_t = z_{2t}. \tag{3.4}$$

Straightforward computation gives the following lemma:

Lemma 3.1. *We suppose that $\mathbb{E}\left(w_0^4\right) < \infty$. If $(w_t)_{t\in\mathbb{Z}}$ is not Gaussian then $(\varepsilon_t)_{t\in\mathbb{Z}}$ is not a martingale difference. Moreover, for any $(\lambda_1, \lambda_2) \in [-\pi, \pi]^2$,*

$$f_4(\lambda_1, -\lambda_2, \lambda_2) = \frac{1}{8\pi^3}\left(K_1(\lambda_1, \lambda_2) + K_2(\lambda_1, \lambda_2)\right)\left(\mathbb{E}\left(w_0^4\right) - 3\mathbb{E}\left(w_0^2\right)^2\right)$$

where f_4 is the fourth-order cumulant spectral density of $(\varepsilon_t)_{t\in\mathbb{Z}}$. In the last definition,

$$K_1(\lambda_1, \lambda_2) = \frac{2 - 2\cos(\lambda_1)}{1 - 2\alpha^2\cos(\lambda_1) + \alpha^4} \times \frac{2 - 2\cos(\lambda_2)}{1 - 2\alpha^2\cos(\lambda_2) + \alpha^4}$$

and

$$K_2(\lambda_1, \lambda_2) = \left(\frac{\alpha^2 - 1}{\alpha}\right)^4 \frac{1}{\left(1 - 2\alpha^2\cos(\lambda_1) + \alpha^4\right)\left(1 - 2\alpha^2\cos(\lambda_2) + \alpha^4\right)}.$$

The proof of this lemma is postponed to Subsection 5.1.

***Remark* 3.2.** *The same computations as those presented in the proof of Lemma 3.1 show that the fourth-order cumulant spectral density of the process $(z_t)_{t\in\mathbb{Z}}$ is constant.*

In the tables below, we consider the FARIMA model (3.2) generated by the noise (3.4), where w_t follows a Student distribution with 10 degrees of Freedom. The parameter α is fixed at $\alpha = 0.7$. According to Lemma 3.1, the integrand in (2.13) is in closed form. We compare the estimation to the numerical integration of the integrand in (2.13) in order to evaluate numerically the bias.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	1.0516	1.3900	2.1167	0.1281	0.1280	0.2887	1.0594	1.3959	2.1362
$n^{-0.15}$	0.5596	0.8787	1.3551	0.1559	0.1459	0.3242	0.5809	0.8907	1.3934
$n^{-0.2}$	0.3025	0.5339	0.8447	0.1875	0.1747	0.3814	0.3559	0.5618	0.9268

TABLE 9

Bias, SD and RMSE when $\vartheta = (0.2, -0.2, 0.2)^\top$ and $n = 2^{14}$ for Taniguchi's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	0.9570	1.3069	2.0052	0.0961	0.0982	0.2318	0.9619	1.3106	2.0185
$n^{-0.15}$	0.4832	0.7897	1.2379	0.1163	0.1104	0.2603	0.4969	0.7974	1.2694
$n^{-0.2}$	0.2512	0.4609	0.7496	0.1376	0.1288	0.3004	0.2864	0.4785	0.8078

TABLE 10

Bias, SD and RMSE when $\vartheta = (0.2, -0.2, 0.2)^\top$ and $n = 2^{15}$ for Taniguchi's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	0.0585	0.0451	0.1165	0.1095	0.0932	0.1613	0.1242	0.1036	0.1990
$n^{-0.15}$	0.0312	0.0279	0.0645	0.1436	0.1257	0.2251	0.1469	0.1287	0.2342
$n^{-0.2}$	0.0189	0.0172	0.0369	0.1771	0.1613	0.2941	0.1782	0.1622	0.2964

TABLE 11

Bias, SD and RMSE when $\vartheta = (0.2, -0.2, 0.2)^\top$ and $n = 2^{14}$ for Keenan's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	0.0506	0.0413	0.1028	0.0812	0.0701	0.1190	0.0957	0.0814	0.1573
$n^{-0.15}$	0.0240	0.0239	0.0512	0.1058	0.0933	0.1672	0.1085	0.0963	0.1749
$n^{-0.2}$	0.0129	0.0137	0.0269	0.1300	0.1185	0.2219	0.1307	0.1193	0.2235

TABLE 12

Bias, SD and RMSE when $\vartheta = (0.2, -0.2, 0.2)^\top$ and $n = 2^{15}$ for Keenan's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	2.6886	1.3951	2.1040	0.4124	0.1249	0.2954	2.7200	1.4007	2.1246
$n^{-0.15}$	1.5760	0.8827	1.3437	0.4304	0.1446	0.3254	1.6338	0.8944	1.3825
$n^{-0.2}$	0.7928	0.5366	0.8368	0.4713	0.1744	0.3728	0.9223	0.5643	0.9161

TABLE 13

Bias, SD and RMSE when $\vartheta = (0.7, -0.2, -0.3)^\top$ and $n = 2^{14}$ for Taniguchi's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	2.4965	1.3098	1.9966	0.2811	0.0971	0.2257	2.0093	1.3134	2.0091
$n^{-0.15}$	1.3569	0.7917	1.2301	0.2937	0.1093	0.2493	1.3883	0.7999	1.2551
$n^{-0.2}$	0.6278	0.4621	0.7441	0.3302	0.1283	0.2834	0.7094	0.4796	0.7962

TABLE 14
Bias, SD and RMSE when $\vartheta = (0.7, -0.2, -0.3)^\top$ and $n = 2^{15}$ for Taniguchi's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	0.1465	0.0453	0.1162	0.1974	0.0935	0.1615	0.2458	0.1039	0.1989
$n^{-0.15}$	0.0787	0.0282	0.0643	0.2872	0.1260	0.2255	0.2978	0.1291	0.2345
$n^{-0.2}$	0.0412	0.0175	0.0368	0.3831	0.1619	0.2948	0.3853	0.1628	0.2971

TABLE 15
Bias, SD and RMSE when $\vartheta = (0.7, -0.2, -0.3)^\top$ and $n = 2^{14}$ for Keenan's estimator.

k_n	B AR	B MA	B d_0	SD AR	SD MA	SD d_0	RMSE AR	RMSE MA	RMSE d_0
$n^{-0.1}$	0.1297	0.0413	0.1024	0.1430	0.0702	0.1190	0.1931	0.0814	0.1570
$n^{-0.15}$	0.0618	0.0237	0.0506	0.2096	0.0934	0.1672	0.2186	0.0964	0.1747
$n^{-0.2}$	0.0289	0.0137	0.0263	0.2832	0.1188	0.2217	0.2848	0.1196	0.2232

TABLE 16
Bias, SD and RMSE when $\vartheta = (0.7, -0.2, -0.3)^\top$ and $n = 2^{15}$ for Keenan's estimator.

We observe that the estimator presented in Theorem 2.2 present a strong bias which leads to very poor estimates. The second estimator presented in Theorem 2.3 is more suitable but nevertheless requires a calibration of k_n.

4. Conclusion and perspectives

In this paper, we propose a one-step estimator with asymptotic properties similar to the Whittle estimator, but computed more quickly, making it particularly relevant for large sample sizes.

Additionally, we have constructed a consistent covariance estimator based on an unobservable process in the Wold representation. It would be interesting to study the influence of the kernel and the window (k_n) on this estimator. However, this topic is beyond the scope of the current work and will be investigated in future studies.

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5. Technical results

Lemma 5.1. *Under Assumptions **A1**-**A4**, we have*

$$\sqrt{n}\nabla\mathcal{L}_n^W(\vartheta) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,\mathcal{W}(\vartheta)),$$

where $\mathcal{W}(\vartheta)$ is defined in (2.10).

Proof. For $k \in \{1,\dots,p\}$, the k-th component of $\sqrt{n}\nabla\mathcal{L}_n^W(\vartheta)$ is given by

$$\sqrt{n}\frac{\partial\mathcal{L}_n^W(\vartheta)}{\partial\vartheta_k} = \frac{1}{2\sqrt{n}}\sum_{j=1}^{n-1}\frac{\partial f_\vartheta\left(\lambda_j\right)}{\partial\vartheta_k}f_\vartheta\left(\lambda_j\right)^{-1} - \frac{\partial f_\vartheta\left(\lambda_j\right)}{\partial\vartheta_k}\frac{I_n\left(\lambda_j\right)}{f_\vartheta\left(\lambda_j\right)^2}.$$

Denote by

$$\widetilde{I}_n(\lambda) = \frac{1}{2\pi n}\left|\sum_{k=1}^{n}\exp(ik\lambda)\,\varepsilon_k\right|^2.$$

One can use similar arguments as those in the proof of Theorem 2.1 in [40] to obtain

$$\frac{1}{2\sqrt{n}}\sum_{j=1}^{n-1}\frac{\partial f_\vartheta\left(\lambda_j\right)}{\partial\vartheta_k}\frac{I_n\left(\lambda_j\right)}{f_\vartheta\left(\lambda_j\right)^2} - 2\pi\frac{\partial f_\vartheta\left(\lambda_j\right)}{\partial\vartheta_k}\frac{\widetilde{I}_n\left(\lambda_j\right)}{\sigma_2 f_\vartheta\left(\lambda_j\right)} = o_{\mathbb{P}}(1). \tag{5.1}$$

For simplicity, let

$$V_n(\vartheta) = \frac{1}{2\sqrt{n}}\sum_{j=1}^{n-1}\frac{\partial f_\vartheta\left(\lambda_j\right)}{\partial\vartheta}f_\vartheta\left(\lambda_j\right)^{-1} - 2\pi\frac{\partial f_\vartheta\left(\lambda_j\right)}{\partial\vartheta}\frac{\widetilde{I}_n\left(\lambda_j\right)}{\sigma_2 f_\vartheta\left(\lambda_j\right)}.$$

In view of (5.1), the proof of the lemma will be completed once we show that

$$V_n(\vartheta) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,\mathcal{W}(\vartheta)). \tag{5.2}$$

Consider

$$\phi_\vartheta(\lambda) = \frac{\partial f_\vartheta(\lambda)}{\partial\vartheta}f_\vartheta(\lambda)^{-1},$$

$$\phi_{1,\vartheta,c}(\lambda) = \phi_\vartheta(\lambda)\,\mathbb{1}_{\{|\lambda|\geqslant c\}} + \phi_\vartheta(c)\,\mathbb{1}_{\{|\lambda|<c\}}$$

and

$$\phi_{2,\vartheta,c}(\lambda) = \phi_\vartheta(\lambda) - \phi_{1,\vartheta,c}(\lambda)$$

for some $c > 0$. We define

$$V_{n,1}(\vartheta) = \frac{1}{2\sqrt{n}}\sum_{j=1}^{n-1}\phi_{1,\vartheta,c}\left(\lambda_j\right) - 2\pi\phi_{1,\vartheta,c}\left(\lambda_j\right)\frac{\widetilde{I}_n\left(\lambda_j\right)}{\sigma_2}$$

and

$$V_{n,2}(\vartheta) = \frac{1}{2\sqrt{n}} \sum_{j=1}^{n-1} \phi_{2,\vartheta,c}\left(\lambda_j\right) - 2\pi\phi_{2,\vartheta,c}\left(\lambda_j\right) \frac{\widetilde{I}_n\left(\lambda_j\right)}{\sigma_2}.$$

It is easy to see that $\mathbb{E}\left(V_{n,1}(\vartheta)\right) = \mathbb{E}\left(V_{n,2}(\vartheta)\right) = 0$ and Equation (A.11) in [40] gives

$$\limsup_{c\to 0} \limsup_{n\to\infty} \mathbb{V}ar\left(V_{n,2}(\vartheta)\right) = 0.$$

Let $a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_{1,\vartheta,c}(\lambda)\exp(ik\lambda)\, d\lambda$, for a fixed $h \in \mathbb{N}^*$, let

$$\psi_h(\lambda) = \sum_{|k|<h} \left(1 - \frac{|k|}{h}\right) a_k \exp(-ik\lambda)$$

and $\overline{\psi}_h(\lambda) = \phi_{1,\vartheta,c}(\lambda) - \psi_h(\lambda)$ be the remainder. We get from Lemma A.8 of [40] that

$$\mathbb{V}ar\left(\frac{2\pi}{\sigma_2\sqrt{n}} \sum_{j=1}^{n-1} \overline{\psi}_h\left(\lambda_j\right)\widetilde{I}_n\left(\lambda_j\right)\right) \leqslant \frac{K}{n}\left(\sum_{j=1}^{\left[\frac{n}{2}\right]} \overline{\psi}_h\left(\lambda_j\right)^2 + \frac{1}{n}\sum_{j\neq k=1}^{\left[\frac{n}{2}\right]} f_4\left(\lambda_j, -\lambda_k, \lambda_j\right)\overline{\psi}_h\left(\lambda_j\right)\overline{\psi}_h\left(\lambda_j\right)\overline{\psi}_h\left(\lambda_k\right)\right)$$

for some constant $K > 0$. It follows then that

$$\limsup_{n\to\infty} \mathbb{V}ar\left(\frac{2\pi}{\sigma_2\sqrt{n}} \sum_{j=1}^{n-1} \overline{\psi}_h\left(\lambda_j\right)\widetilde{I}_n\left(\lambda_j\right)\right) \leqslant \sup_{\lambda\in[0;2\pi]} \left|\overline{\psi}_h(\lambda)\right|^2$$

and by Fejér's Theorem,

$$\limsup_{h\to\infty} \sup_{\lambda\in[0;2\pi]} \left|\overline{\psi}_h(\lambda)\right|^2 = 0.$$

Let $A_h = (2\sigma_2)^{-1}\left(a_0, 2a_1\left(1-\frac{1}{h}\right), \ldots, \frac{2a_{h-1}}{h}\right)^*$, $\widehat{\gamma}_\varepsilon(k) = \frac{1}{n}\sum_{j=1}^{n-|k|} \varepsilon_j\varepsilon_{j+k}$, and $\widehat{\boldsymbol{\gamma}}_{\boldsymbol{\varepsilon}}(\boldsymbol{h}) = \left(\widehat{\gamma}_\varepsilon(0), \widehat{\gamma}_\varepsilon(1), \ldots, \widehat{\gamma}_\varepsilon(h-1)\right)^*$. We have

$$\begin{aligned}\frac{1}{2\sqrt{n}} \sum_{j=1}^{n-1} \psi_h\left(\lambda_j\right) - 2\pi\psi_h\left(\lambda_j\right)\frac{\widetilde{I}_n\left(\lambda_j\right)}{\sigma_2} &= -\frac{\sqrt{n}}{2\sigma_2}\sum_{|k|<h} a_k\left(1-\frac{|k|}{h}\right)\left(\widehat{\gamma}_\varepsilon(k) - \mathbb{E}\left(\widehat{\gamma}_\varepsilon(k)\right)\right) + o_{\mathbb{P}}(1) \\ &= -\sqrt{n}A_h^*\left[\widehat{\boldsymbol{\gamma}}_{\boldsymbol{\varepsilon}}(\boldsymbol{h}) - \mathbb{E}\left(\widehat{\boldsymbol{\gamma}}_{\boldsymbol{\varepsilon}}(\boldsymbol{h})\right)\right] + o_{\mathbb{P}}(1).\end{aligned}$$

We obtain from Lemma 1 in [47] that for fixed h,

$$\sqrt{n}\left[\widehat{\boldsymbol{\gamma}}_{\boldsymbol{\varepsilon}}(\boldsymbol{h}) - \mathbb{E}\left(\widehat{\boldsymbol{\gamma}}_{\boldsymbol{\varepsilon}}(\boldsymbol{h})\right)\right] \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, \boldsymbol{\Gamma}_{\boldsymbol{h}}),$$

where

$$\boldsymbol{\Gamma}_{\boldsymbol{h}} = \left(\sum_{j\in\mathbb{Z}} \mathbb{E}\left(\varepsilon_j\varepsilon_0\right)\mathbb{E}\left(\varepsilon_{j+u-v}\varepsilon_0\right) + \mathbb{E}\left(\varepsilon_{j+v}\varepsilon_0\right)\mathbb{E}\left(\varepsilon_0\varepsilon_{j-u}\right) + \mathrm{cum}\left(\varepsilon_0, \varepsilon_u, \varepsilon_j, \varepsilon_{j+v}\right)\right)_{0\leqslant u,v\leqslant h-1}.$$

Finally, we use similar calculations as those used in the proof of Corollary 2 on page 61 of [39] to show that

$$V_{n,1}(\vartheta) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, \mathcal{W}_c(\vartheta))$$

where

$$\mathcal{W}_c(\vartheta) = \frac{\pi}{2\sigma_2^2}\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_4(\lambda_1, -\lambda_2, \lambda_2)\,\phi_{1,\vartheta_k,c}(\lambda)\,\phi_{1,\vartheta_j,c}(\lambda)\,d\lambda_1\,d\lambda_2\right)_{1\leqslant k,j\leqslant d} + \frac{1}{4\pi}\left(\int_{-\pi}^{\pi}\phi_{1,\vartheta_k,c}(\lambda)\,\phi_{1,\vartheta_j,c}(\lambda)\,d\lambda\right)_{1\leqslant k,j\leqslant d}.$$

The dominated convergence theorem when $c \to 0$ gives Equality (5.2) and the proof is completed. □

Lemma 5.2. *Let ϕ_ϑ be a 2π-periodic and even function with respect to λ, satisfying Assumptions **A1-A4**. Consider*

$$J_n(\vartheta) = \frac{1}{2n}\sum_{j=1}^{n-1}\log\phi_\vartheta\left(\lambda_j\right) + \frac{I_n\left(\lambda_j\right)}{\phi_\vartheta\left(\lambda_j\right)}$$

and

$$J(\vartheta) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\log\phi_\vartheta(\lambda) + \frac{f_\vartheta(\lambda)}{\phi_\vartheta(\lambda)}\,d\lambda\,.$$

Then, one has

$$\sup_{\vartheta\in\Theta}|J_n(\vartheta) - J(\vartheta)| = o_{\mathbb{P}}(1)\,. \tag{5.3}$$

Proof. The mean value theorem gives

$$\begin{aligned} I(\vartheta) &= \frac{1}{4\pi}\int_0^{2\pi}\log\phi_\vartheta(\lambda)\,d\lambda \\ &= \frac{1}{4\pi}\int_0^{\frac{2\pi}{n}}\log\phi_\vartheta(\lambda)\,d\lambda + \frac{2}{n}\sum_{j=1}^{n-2}\log\phi_\vartheta\left(\widetilde{\lambda}_j\right) + \frac{1}{4\pi}\int_{\frac{2\pi(n-1)}{n}}^{2\pi}\log\phi_\vartheta(\lambda)\,, \end{aligned}$$

where $\widetilde{\lambda}_j \in \left[\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}\right]$ for any $j \in \{1, n-2\}$. Under Assumption **A1** and using the mean value theorem, we have

$$\left|\frac{1}{2n}\sum_{j=1}^{n-2}\log\phi_\vartheta\left(\widetilde{\lambda}_j\right) - \frac{1}{2n}\sum_{j=1}^{n-1}\log\phi_\vartheta\left(\lambda_j\right)\right| \leqslant \frac{K}{n}\left(\sum_{j=1}^{n-2}\frac{2\pi}{n}\left|\overline{\lambda}_j\right|^{-1-\delta} + \left|\overline{\lambda}_{n-1}\right|^{\delta}\right)$$

for some constant K that depends only on Θ and $\delta > 0$. In the last inequality, the frequencies $\overline{\lambda}_j$ and $\overline{\lambda}_n$ are taken in $[-\pi, \pi]$, which is possible since the function is 2π-periodic. Hence, for some $C > 0$,

$$\left|\frac{1}{2n}\sum_{j=1}^{n-2}\log\phi_\vartheta\left(\widetilde{\lambda}_j\right) - \frac{1}{2n}\sum_{j=1}^{n-1}\log\phi_\vartheta\left(\lambda_j\right)\right| \leqslant \frac{C}{n^{-1+\delta}} + \frac{C}{n}\int_{\frac{2\pi}{n}}^{\pi}\lambda^{-1-\delta}\,\mathrm{d}\lambda$$

$$= O\left(n^{-1+\delta}\right) \text{ uniformly in } \theta \in \Theta.$$

Finally, under Assumption **A1**, we obtain uniformly in $\vartheta \in \Theta$ that

$$\frac{1}{4\pi}\int_0^{\frac{2\pi}{n}} \log \phi_\vartheta(\lambda) = O\left(n^{-1+\delta}\right), \ \frac{1}{4\pi}\int_{\frac{2\pi(n-1)}{n}}^{2\pi} \log \phi_\vartheta(\lambda) = O\left(n^{-1+\delta}\right)$$

and

$$\left| I(\vartheta) - \frac{1}{2n}\sum_{j=1}^{n-1} \log \phi_\vartheta\left(\lambda_j\right)\right| = O\left(n^{-1+\delta}\right).$$

The other terms are handled similarly to Lemma A1 in [40], which completes the proof of this lemma. □

Lemma 5.3. *Under Assumptions **A1**-**A4**, we have*

$$\Delta\mathcal{L}_n^W(\vartheta) - \mathcal{I}(\vartheta) = O_{\mathbb{P}}\left(n^{-\frac{1}{2}}\right). \tag{5.4}$$

Proof. A similar reasoning to that developed in the proof of Lemma 5.1 (replacing $\phi_\vartheta(\lambda)$ by $\frac{\partial^2}{\partial\vartheta_j\partial\vartheta_k}\log f_\vartheta(\lambda)$, which satisfy the same estimation by Assumption **A1**) leads us to

$$\sqrt{n}\left(\Delta\mathcal{L}_n^W(\vartheta) - T_n(\vartheta)\right) = o_{\mathbb{P}}(1),$$

where

$$T_n(\vartheta) = \frac{1}{2n}\sum_{j=1}^{n-1}\frac{\partial^2}{\partial\vartheta\partial\vartheta^\top}\log f_\vartheta\left(\lambda_j\right) - 2\pi\widetilde{\phi}_\vartheta\left(\lambda_j\right)\frac{\widetilde{I_n}\left(\lambda_j\right)}{\sigma_2}$$

and

$$\widetilde{\phi}_\vartheta(\lambda) = \frac{\partial^2}{\partial\vartheta\partial\vartheta^\top}f_\vartheta(\lambda)\times f_\vartheta(\lambda)^{-1} - 2\frac{\partial\log f_\vartheta(\lambda)}{\partial\vartheta}\times\frac{\partial\log f_\vartheta(\lambda)}{\partial\vartheta^\top}.$$

We use similar arguments as those in the proof of Lemma 5.1 to deduce

$$\sqrt{n}\left(\mathbb{E}\left(T_n(\vartheta)\right) - \mathcal{I}(\vartheta)\right) \xrightarrow[n\to\infty]{} 0 \text{ and } \mathbb{V}ar\left(\sqrt{n}T_n(\vartheta)\right) = O(1)$$

which allows us to conclude the Lemma. It is also possible to use this reasoning to obtain the asymptotic normality of $\sqrt{n}\left(\Delta\mathcal{L}_n^W(\vartheta) - \mathcal{I}(\vartheta)\right)$ as given in Lemma 5.1. However, this conclusion is unnecessary to establish this lemma and is stronger than the tighness condition. □

Lemma 5.4. *Under Assumptions **A1**-**A4** and for any $\vartheta_1 \in \Theta$, there exists a constant K_{ϑ_1} and an open ball B_{ϑ_1} containing ϑ_1 such that for any $\vartheta_2 \in B_{\vartheta_1}$, we have*

$$\|\mathcal{I}(\vartheta_1) - \mathcal{I}(\vartheta_2)\| \leqslant K_{\vartheta_1}\|\vartheta_1 - \vartheta_2\|. \tag{5.5}$$

Proof. The proof is similar to that of Lemma A.1 in [3], so it is omitted. □

Lemma 5.5. *Let $(\widehat{\vartheta}_n^{(1)})_n$ be a random sequence such that $\widehat{\vartheta}_n^{(1)} - \vartheta = o_{\mathbb{P}}(1)$. Under Assumptions **A1**-**A4**, we have*

$$\Delta\mathcal{L}_n^W\left(\widehat{\vartheta}_n^{(1)}\right) - \Delta\mathcal{L}_n^W(\vartheta) = O_{\mathbb{P}}\left(\widehat{\vartheta}_n^{(1)} - \vartheta\right). \tag{5.6}$$

Proof. We fix $(k, j) \in \{1, 2, \ldots, p\}^2$. Conditionally to $\widehat{\vartheta}_n^{(1)} \in B(\vartheta, \delta)$, the mean value theorem implies

$$\left|\frac{\partial^2}{\partial\vartheta_k\partial\vartheta_j}\mathcal{L}_n^W\left(\widehat{\vartheta}_n^{(1)}\right) - \frac{\partial^2}{\partial\vartheta_k\partial\vartheta_j}\mathcal{L}_n^W(\vartheta)\right| \leqslant \sup_{\vartheta^*\in B(\vartheta,\delta)}\left\|\frac{\partial^3}{\partial\vartheta_\ell\partial\vartheta_k\partial\vartheta_j}\mathcal{L}_n^W(\vartheta^*)\right\|\left\|\widehat{\vartheta}_n^{(1)} - \vartheta\right\|.$$

Lemma 5.2 ensures that

$$\sup_{\vartheta^*\in B(\vartheta,\delta)}\left\|\frac{\partial^3}{\partial\vartheta_\ell\partial\vartheta_k\partial\vartheta_j}\mathcal{L}_n^W(\vartheta^*)\right\|_{1\leqslant\ell\leqslant d} = O_{\mathbb{P}}(1)$$

and the condition $\widehat{\vartheta}_n^{(1)} - \vartheta = o_{\mathbb{P}}(1)$ gives the expected result. □

Lemma 5.6. *Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ be a strictly stationary process (not necessarily a white noise) and denote by*

$$w_n^0(\lambda) = \sum_{j=1}^n \exp(-ij\lambda)\,\varepsilon_j \text{ and } D_n(\lambda) = \sum_{j=1}^n \exp(-i\lambda j).$$

*Consider $(j_1, \ldots, j_k)$ as k non-negative integers and $(\lambda_1, \ldots, \lambda_k) \in [-\pi, \pi]^k$. Then, if $(\varepsilon_t)_{t\in\mathbb{Z}}$ satisfies **(A5)**, one has*

$$cum\left(w_{j_1}^0(\lambda_1)\ldots w_{j_k}^0(\lambda_k)\right) = (2\pi)^{k-1} f_k\left(\lambda_1, \ldots, \check{\lambda}_{[j_{(k)}]}, \ldots, \lambda_k\right) D_{j_{(k)}}\left(\sum_{\ell=1}^k \lambda_\ell\right) + O(1)$$

where the error term $O(1)$ is uniform in $(\lambda_1, \ldots, \lambda_k)$ and $(j_1, \ldots, j_k)$. In this lemma, we use the notations $j_{(k)} = \min(j_1, \ldots, j_k)$ and f_k the k-th order spectrum of $(\varepsilon_t)_{t\in\mathbb{Z}}$. The frequencies $\lambda_{[j_{(k)}]}$ associated to the term $w_{[j_{(k)}]}^0(\cdot)$ is omitted in the computation of f_k.

Proof. The proof is similar to the one of Lemma 1 in [8]. We have

$$\operatorname{cum}\left(w_{j_1}^0(\lambda_1)\ldots w_{j_k}^0(\lambda_k)\right) = \sum_{t_1=1}^{j_1}\cdots\sum_{t_k=1}^{j_k}\exp\left(-i\left(\lambda_1 t_1 + \cdots + \lambda_k t_k\right)\right)\operatorname{cum}\left(\varepsilon_{t_1}\ldots\varepsilon_{t_k}\right).$$

In order to lighten the notations, we suppose (without loss of generality) that $j_k = \min(j_1, \ldots, j_k)$. By the stationarity of $(\varepsilon_t)_{t\in\mathbb{Z}}$, we have

$$\operatorname{cum}\left(w_{j_1}^0(\lambda_1)\ldots w_{j_k}^0(\lambda_k)\right) = \sum_{t_1=1-j_k}^{j_1-1}\cdots\sum_{t_{k-1}=1-j_k}^{j_{k-1}-1}\exp\left(-i\left(\lambda_1 t_1 + \cdots + \lambda_{k-1}t_{k-1}\right)\right)$$

$$\times \operatorname{cum}\left(\varepsilon_{t_1} \ldots \varepsilon_{t_{k-1}} \varepsilon_0\right) \sum_{t_k=t_{\min}}^{t_{\max}} \exp\left(-i\overline{\lambda}_k t_k\right),$$

where $t_{\min} = \max\left(1, 1-t_1, \ldots, 1-t_{k-1}\right)$, $t_{\max} = \min\left(j_1 - t_1, \ldots, j_{k-1} - t_{k-1}, j_k\right)$ and $\overline{\lambda}_k = \lambda_1 + \cdots + \lambda_k$. We observe that $1 \leqslant t_{\min} \leqslant t_{\max} \leqslant j_k$. Now as in Equation (4.10) of [8] and thanks to **(A4)**, we have

$$\left| f_k\left(\lambda_1, \ldots, \lambda_{k-1}\right) - (2\pi)^{-k+1} \sum_{t_1=1-j_k}^{j_1-1} \cdots \sum_{t_{k-1}=1-j_k}^{j_{k-1}-1} \exp\left(-i\left(\lambda_1 t_1 + \cdots + \lambda_{k-1} t_{k-1}\right)\right) \operatorname{cum}\left(\varepsilon_0 \varepsilon_{t_1} \ldots \varepsilon_{t_{k-1}}\right) \right| = O\left(\frac{1}{j_k}\right)$$

uniformly in $\left(\lambda_1, \ldots, \lambda_{k-1}\right) \in [-\pi, \pi]^{k-1}$. Let

$$B^{(1)}_{(j_1,\ldots,j_k)}\left(\lambda_1, \ldots, \lambda_k\right) = \sum_{t_1=1-j_k}^{j_1-1} \cdots \sum_{t_{k-1}=1-j_k}^{j_{k-1}-1} \exp\left(-i\left(\lambda_1 t_1 + \cdots + \lambda_{k-1} t_{k-1}\right)\right) \operatorname{cum}\left(\varepsilon_0 \varepsilon_{t_1} \ldots \varepsilon_{t_{k-1}}\right) D_{j_k}\left(\overline{\lambda_k}\right)$$

and

$$B^{(2)}_{(j_1,\ldots,j_k)}\left(\lambda_1, \ldots, \lambda_k\right) = \sum_{t_1=1-j_k}^{j_1-1} \cdots \sum_{t_{k-1}=1-j_k}^{j_{k-1}-1} \exp\left(-i\left(\lambda_1 t_1 + \cdots + \lambda_{k-1} t_{k-1}\right)\right) \operatorname{cum}\left(\varepsilon_0 \varepsilon_{t_1} \ldots \varepsilon_{t_{k-1}}\right) \sum_{t_k=t_{\min}}^{t_{\max}} \exp\left(-i\overline{\lambda}_k t_k\right).$$

Observe that $t_{\min} \leqslant |t_1| + \cdots + |t_{k-1}|$ and $j_k - t_{\max} \leqslant |t_1| + \cdots + |t_{k-1}|$. Hence, it can be shown that

$$\left| B^{(1)}_{(j_1,\ldots,j_k)}\left(\lambda_1, \ldots, \lambda_k\right) - B^{(2)}_{(j_1,\ldots,j_k)}\left(\lambda_1, \ldots, \lambda_k\right) \right| \leqslant 2 \sum_{t_1=1-j_k}^{j_1-1} \cdots \sum_{t_{k-1}=1-j_k}^{j_{k-1}-1} \left(|t_1| + \cdots + |t_{k-1}|\right) \left| \operatorname{cum}\left(\varepsilon_0 \varepsilon_{t_1} \ldots \varepsilon_{t_{k-1}}\right) \right|.$$

Thanks to **(A4)**, the right-hand side of the previous inequality is bounded. The proof is completed. □

5.1. *Proof of Lemma 3.1*

One can see from the recursive equation (3.3) that, for any $t \in \mathbb{Z}$,

$$z_t = \sum_{j \in \mathbb{Z}} h(j)\, w_{t-j}, \tag{5.7}$$

where

$$h(j) = \begin{cases} 0 \text{ if } j < 0 \\ 1 \text{ if } j = 0 \\ \frac{\alpha^2-1}{\alpha} \alpha^{j-1} \text{ if } j > 0. \end{cases} \tag{5.8}$$

Since $(w_t)_{t \in \mathbb{Z}}$ is a strong white noise with fourth-order moment and the sequence $h(\cdot)$ decays at geometric rate, we obtain via the Brillinger-Rosenblatt formula that for any $(t_1, t_2, t_3) \in \mathbb{Z}^3$,

$$cum\left(\varepsilon_0, \varepsilon_{t_1}, \varepsilon_{t_2}, \varepsilon_{t_3}\right) = cum\left(z_0, z_{2t_1}, z_{2t_2}, z_{2t_3}\right)$$

$$= \sum_{j\in\mathbb{Z}} h(j)\,h(j+2t_1)\,h(j+2t_2)\,h(j+2t_3)\left(\mathbb{E}\left(w_0^4\right) - 3\mathbb{E}\left(w_0^2\right)^2\right)$$

and hence

$$f_4(\lambda_1,\lambda_2,\lambda_3) = \frac{1}{8\pi^3}\sum_{(t_1,t_2,t_3)\in\mathbb{Z}^3}\exp\left(-i\left(\lambda_1 t_1+\lambda_2 t_2+\lambda_3 t_3\right)\right)$$
$$\times\sum_{j\in\mathbb{Z}} h(j)\,h(j+2t_1)\,h(j+2t_2)\,h(j+2t_3)\left(\mathbb{E}\left(w_0^4\right) - 3\mathbb{E}\left(w_0^2\right)^2\right).$$

Assume that $j = 2\ell_1$ for some $\ell_1 \in \mathbb{N}$. By changing the order of summation, one can see that

$$\begin{aligned}\sum_{t_1\in\mathbb{Z}}\exp\left(-i\lambda_1 t_1\right)h(j+2t_1) &= \sum_{t_1\geqslant -\ell_1}\exp\left(-i\lambda_1 t_1\right)h(2\ell_1+2t_1)\\ &= \exp\left(i\lambda_1\ell_1\right) + \sum_{t_1\geqslant -\ell_1+1}\exp\left(-i\lambda_1 t_1\right)h(2\ell_1+2t_1)\\ &= \exp\left(i\lambda_1\ell_1\right)\left(1+\exp\left(-i\lambda_1\right)\frac{\alpha^2-1}{1-\alpha^2\exp\left(-i\lambda_1\right)}\right).\end{aligned}$$

Similarly, we have

$$\sum_{t_2\in\mathbb{Z}}\exp\left(-i\lambda_1 t_2\right)h(2\ell_1+2t_2) = \exp\left(i\lambda_2\ell_1\right)\left(1+\exp\left(-i\lambda_2\right)\frac{\alpha^2-1}{1-\alpha^2\exp\left(-i\lambda_2\right)}\right)$$

and

$$\sum_{t_3\in\mathbb{Z}}\exp\left(-i\lambda_1 t_3\right)h(2\ell_1+2t_3) = \exp\left(i\lambda_3\ell_1\right)\left(1+\exp\left(-i\lambda_3\right)\frac{\alpha^2-1}{1-\alpha^2\exp\left(-i\lambda_3\right)}\right).$$

We also obtain

$$\sum_{\ell_1\in\mathbb{N}}\exp\left(i\ell_1\left(\lambda_1+\lambda_2+\lambda_3\right)\right)h(2\ell_1) = 1+\exp\left(i\left(\lambda_1+\lambda_2+\lambda_3\right)\right)\frac{\alpha^2-1}{1-\alpha^2\exp\left(i\left(\lambda_1+\lambda_2+\lambda_3\right)\right)}.$$

If $j = 2\ell_2+1$ for some $\ell_2 \in \mathbb{N}$, a similar reasoning as above leads us to

$$\begin{aligned}\sum_{t_1\in\mathbb{Z}}\exp\left(-i\lambda_1 t_1\right)h(j+2t_1) &= \sum_{t_1\geqslant -\ell_2}\exp\left(-i\lambda_1 t_1\right)h(2\ell_2+1+2t_1)\\ &= \frac{\alpha^2-1}{\alpha}\times\exp\left(i\lambda_1\ell_2\right)\times\frac{1}{1-\alpha^2\exp\left(-i\lambda_1\right)},\end{aligned}$$

$$\sum_{t_2\in\mathbb{Z}}\exp\left(-i\lambda_2 t_2\right)h(j+2t_2) = \frac{\alpha^2-1}{\alpha}\times\exp\left(i\lambda_2\ell_2\right)\times\frac{1}{1-\alpha^2\exp\left(-i\lambda_2\right)},$$

$$\sum_{t_3\in\mathbb{Z}} \exp\left(-i\lambda_3 t_3\right) h\left(j+2t_3\right) = \frac{\alpha^2-1}{\alpha} \times \exp\left(i\lambda_3\ell_2\right) \times \frac{1}{1-\alpha^2\exp\left(-i\lambda_3\right)},$$

and

$$\sum_{\ell_2\in\mathbb{N}} \exp\left(i\ell_2\left(\lambda_1+\lambda_2+\lambda_3\right)\right) h\left(2\ell_2+1\right) = \frac{\alpha^2-1}{\alpha} \times \frac{1}{1-\alpha^2\exp\left(i\left(\lambda_1+\lambda_2+\lambda_3\right)\right)}.$$

By combining the previous equations, we get

$$f_4\left(\lambda_1,\lambda_2,\lambda_3\right) = \frac{1}{8\pi^3}\left(K_1\left(\lambda_1,\lambda_2,\lambda_3\right)+K_2\left(\lambda_1,\lambda_2,\lambda_3\right)\right)\left(\mathbb{E}\left(w_0^4\right)-3\mathbb{E}\left(w_0^2\right)^2\right),$$

where

$$\begin{aligned} K_1\left(\lambda_1,\lambda_2,\lambda_3\right) =& \left(1+\exp\left(i\left(\lambda_1+\lambda_2+\lambda_3\right)\right)\frac{\alpha^2-1}{1-\alpha^2\exp\left(i\left(\lambda_1+\lambda_2+\lambda_3\right)\right)}\right) \\ &\times\left(1+\exp\left(-i\lambda_1\right)\frac{\alpha^2-1}{1-\alpha^2\exp\left(-i\lambda_1\right)}\right)\left(1+\exp\left(-i\lambda_2\right)\frac{\alpha^2-1}{1-\alpha^2\exp\left(-i\lambda_2\right)}\right) \\ &\times\left(1+\exp\left(-i\lambda_3\right)\frac{\alpha^2-1}{1-\alpha^2\exp\left(-i\lambda_3\right)}\right) \end{aligned}$$

and

$$\begin{aligned} K_2\left(\lambda_1,\lambda_2,\lambda_3\right) =& \left(\frac{\alpha^2-1}{\alpha}\right)^4 \frac{1}{1-\alpha^2\exp\left(i\left(\lambda_1+\lambda_2+\lambda_3\right)\right)}\frac{1}{1-\alpha^2\exp\left(-i\lambda_1\right)} \\ &\times\frac{1}{1-\alpha^2\exp\left(-i\lambda_2\right)}\frac{1}{1-\alpha^2\exp\left(-i\lambda_3\right)}. \end{aligned}$$

It follows that

$$K_1\left(\lambda_1,-\lambda_2,\lambda_2\right) = \frac{2-2\cos\left(\lambda_1\right)}{1-2\alpha^2\cos\left(\lambda_1\right)+\alpha^4} \times \frac{2-2\cos\left(\lambda_2\right)}{1-2\alpha^2\cos\left(\lambda_2\right)+\alpha^4}$$

and

$$K_2\left(\lambda_1,-\lambda_2,\lambda_2\right) = \left(\frac{\alpha^2-1}{\alpha}\right)^4 \frac{1}{\left(1-2\alpha^2\cos\left(\lambda_1\right)+\alpha^4\right)\left(1-2\alpha^2\cos\left(\lambda_2\right)+\alpha^4\right)}.$$

The proof is then completed.

5.2. *Proof of Proposition 2.1*

The consistency follows from Lemma 5.2 and the identifiability condition, with a proof similar to that of Theorem 1 in [46]. A Taylor expansion of $\nabla\mathcal{L}_n^W\left(\widehat{\vartheta}_n\right)$ (which is equal to 0) around ϑ gives

$$\nabla \mathcal{L}_n^W\left(\widehat{\vartheta}_n\right) = \nabla \mathcal{L}_n^W(\vartheta) + \Delta \mathcal{L}_n^W\left(\widehat{\vartheta}_n^{(1)}\right)\left(\widehat{\vartheta}_n - \vartheta\right),$$

where $\widehat{\vartheta}_n^{(1)} = v\vartheta + (1-v)\,\widehat{\vartheta}_n$ for some $0 < v < 1$.

In view of Lemma 5.1, the sequence $\Delta \mathcal{L}_n^W\left(\widehat{\vartheta}_n^{(1)}\right)\sqrt{n}\left(\widehat{\vartheta}_n - \vartheta\right)$ converges in distribution to $\mathcal{N}(0, \mathcal{W}(\vartheta))$. We can now use Lemma 5.2 and Slutsky's theorem to show that

$$\left(\Delta \mathcal{L}_n^W\left(\widehat{\vartheta}_n^{(1)}\right), \Delta \mathcal{L}_n^W\left(\widehat{\vartheta}_n^{(1)}\right)\sqrt{n}\left(\widehat{\vartheta}_n - \vartheta\right)\right)$$

converges in distribution to $(\mathcal{I}(\vartheta), Z)$ with $\mathbb{P}_Z = \mathcal{N}(0, \mathcal{W}(\vartheta))$. The continuous mapping theorem completes the proof of the proposition.

5.3. *Proof of Theorem 2.1*

By the mean value theorem, observe that

$$\overline{\vartheta}_n - \vartheta = \widetilde{\vartheta}_n - \vartheta - \mathcal{I}(\widetilde{\vartheta}_n)^{-1}\left(\nabla \mathcal{L}_n^W(\vartheta) + \Delta \mathcal{L}_n^W(\widetilde{\vartheta}_n^{(1)})\left(\widetilde{\vartheta}_n - \vartheta\right)\right),$$

where $\widetilde{\vartheta}_n^{(1)} = v\vartheta + (1-v)\,\widetilde{\vartheta}_n$ for some $0 < v < 1$. Furthermore, it can easily be shown that

$$\begin{aligned}\sqrt{n}\left(\overline{\vartheta}_n - \vartheta\right) &= -\mathcal{I}(\widetilde{\vartheta}_n)^{-1}\sqrt{n}\nabla \mathcal{L}_n^W(\vartheta) - \mathcal{I}(\widetilde{\vartheta}_n)^{-1}\left(-\mathcal{I}(\widetilde{\vartheta}_n) + \Delta \mathcal{L}_n^W(\widetilde{\vartheta}_n^{(1)})\right)\sqrt{n}\left(\widetilde{\vartheta}_n - \vartheta\right)\\ &= A_n - B_n,\end{aligned}$$

where

$$A_n = -\mathcal{I}\left(\widetilde{\vartheta}_n\right)^{-1}\sqrt{n}\nabla \mathcal{L}_n^W(\vartheta)$$

and

$$B_n = \mathcal{I}\left(\widetilde{\vartheta}_n\right)^{-1}\left(-\mathcal{I}\left(\widetilde{\vartheta}_n\right) + \Delta \mathcal{L}_n^W\left(\widetilde{\vartheta}_n^{(1)}\right)\right)\sqrt{n}\left(\widetilde{\vartheta}_n - \vartheta\right).$$

Let us first focus on the second term B_n. We can write

$$B_n = \mathcal{I}\left(\widetilde{\vartheta}_n\right)^{-1}\left(-\mathcal{I}\left(\widetilde{\vartheta}_n\right) + \Delta \mathcal{L}_n^W\left(\widetilde{\vartheta}_n^{(1)}\right)\right)n^{\frac{1}{2}-\delta}n^{\delta}\left(\widetilde{\vartheta}_n - \vartheta\right)$$

and

$$-\mathcal{I}\left(\widetilde{\vartheta}_n\right) + \Delta \mathcal{L}_n^W\left(\widetilde{\vartheta}_n^{(1)}\right) = \mathcal{I}(\vartheta) - \mathcal{I}\left(\widetilde{\vartheta}_n\right) + \Delta \mathcal{L}_n^W(\vartheta) - \mathcal{I}(\vartheta) + \Delta \mathcal{L}_n^W\left(\widetilde{\vartheta}_n^{(1)}\right) - \Delta \mathcal{L}_n^W(\vartheta).$$

Lemma 5.4 ensures that

$$\mathcal{I}(\vartheta) - \mathcal{I}\left(\widetilde{\vartheta}_n\right) = O_{\mathbb{P}}\left(n^{-\delta/2}\right)$$

and Lemma 5.3 gives

$$\Delta \mathcal{L}_n^W(\vartheta) - \mathcal{I}(\vartheta) = O_{\mathbb{P}}\left(n^{-\frac{1}{2}}\right).$$

Moreover, we can use Lemma 5.5 to get

$$\Delta\mathcal{L}_n^W\left(\widetilde{\vartheta}_n^{(1)}\right) - \Delta\mathcal{L}_n^W(\vartheta) = O_{\mathbb{P}}\left(n^{-\delta/2}\right).$$

Consequently, the condition

$$n^{\delta/2}\left(\widetilde{\vartheta}_n - \vartheta\right) = O_{\mathbb{P}}(1) \text{ for some } \frac{1}{2} < \delta \leqslant 1$$

implies that

$$B_n = O_{\mathbb{P}}\left(n^{\frac{1}{2}-\delta}\right) + O_{\mathbb{P}}\left(n^{-\frac{1}{2}}\right)$$

and hence

$$B_n \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

The first term can be rewritten in form

$$\begin{aligned} A_n &= -\mathcal{I}\left(\widetilde{\vartheta}_n\right)^{-1} \sqrt{n}\nabla\mathcal{L}_n^W(\vartheta) \\ &= -\left(\mathcal{I}\left(\widetilde{\vartheta}_n\right)^{-1} - \mathcal{I}(\vartheta)^{-1}\right)\sqrt{n}\nabla\mathcal{L}_n^W(\vartheta) - \mathcal{I}(\vartheta)^{-1}\sqrt{n}\nabla\mathcal{L}_n^W(\vartheta). \end{aligned}$$

We have from Lemma 5.1 and the continuity of $\mathcal{I}(\cdot)$ that

$$A_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \mathcal{I}(\vartheta)^{-1}\mathcal{W}(\vartheta)\mathcal{I}(\vartheta)^{-1}\right)$$

and the theorem is proved.

5.4. *Proof of Theorem 2.2*

The key element in the proof is to show that the estimator with the residual process (2.16) is asymptotically equivalent to the one directly applied to the noise in the decomposition (2.14)-(2.15). We split the proof into the several steps for clarity.

Step 1: Taylor expansion of the generic terms $\phi_\theta^{(\cdot)}(\lambda)$

For $\left(\alpha_1, \ldots, \alpha_p\right) \in \mathbb{N}^p$ and $x = \left(x_1, \ldots, x_p\right) \in \mathbb{R}^p$, we denote in all the proof by n_1 a multi-index such that

$$|n_1| = \sum_{j=1}^p \alpha_j,\ n_1! = \prod_{j=1}^p \alpha_j! \text{ and } x^{n_1} = \prod_{j=1}^p x_j^{\alpha_j}.$$

Consider the first term $A_{1,n}^{(k,\ell)}$. A Taylor expansion around ϑ_0 gives

$$A_{1,n}^{(k,\ell)} = \sum_{|n_1|=0}^{2} \frac{1}{n_1!}\left(\overline{\vartheta}_n - \vartheta_0\right)^{n_1} A_{1,n}^{(k,\ell),n_1}(\vartheta_0) + \sum_{|n_1|=3} \frac{1}{n_1!}\left(\overline{\vartheta}_n - \vartheta_0\right)^{n_1} A_{1,n}^{(k,\ell),n_1}\left(\overline{\vartheta}_n^{n_1}\right), \quad (5.9)$$

where

$$A_{1,n}^{(k,\ell),n_1}(\vartheta_0) = \left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3 \in I^3} H_n\left(\frac{2\pi(j_2+j_3)}{n}\right) \frac{\partial^{|n_1|}}{\partial \vartheta^{n_1}} \left(\phi_{\vartheta_0}^{(k)}\left(\frac{2\pi j_1}{n}\right)\phi_{\vartheta_0}^{(\ell)}\left(\frac{2\pi j_2}{n}\right)\right) \times \frac{1}{8\pi^3 n} w_n\left(\frac{2\pi j_1}{n}\right) \times w_n\left(\frac{2\pi j_2}{n}\right) w_n\left(\frac{2\pi j_3}{n}\right) w_n\left(-\frac{2\pi(j_1+j_2+j_3)}{n}\right)$$

and $\overline{\vartheta}_n^{n_1} = \nu\vartheta_0 + (1-\nu)\overline{\vartheta}_n$ for some $0 < \nu < 1$.
Similarly, we have

$$A_{2,n}^{(k,\ell)} = \sum_{|n_1|=0}^{2} \frac{1}{n_1!}\left(\overline{\vartheta}_n - \vartheta_0\right)^{n_1} A_{2,n}^{(k,\ell),n_1}(\vartheta_0) + \sum_{|n_1|=3} \frac{1}{n_1!}\left(\overline{\vartheta}_n - \vartheta_0\right)^{n_1} A_{2,n}^{(k,\ell),n_1}\left(\overline{\vartheta}_n^{n_1}\right)$$

and

$$A_{3,n}^{(k,\ell)} = \sum_{|n_1|=0}^{2} \frac{1}{n_1!}\left(\overline{\vartheta}_n - \vartheta_0\right)^{n_1} A_{3,n}^{(k,\ell),n_1}(\vartheta_0) + \sum_{|n_1|=3} \frac{1}{n_1!}\left(\overline{\vartheta}_n - \vartheta_0\right)^{n_1} A_{3,n}^{(k,\ell),n_1}\left(\overline{\vartheta}_n^{n_1}\right).$$

Step 2: Expansion of the discrete Fourier transform of the residual process

Let n_2 be an another multi-index. As in [26, 6], we can write

$$\widetilde{\varepsilon}_t\left(\overline{\vartheta}_n\right) = \sum_{j\geqslant 0} \xi_j\left(\overline{\vartheta}_n\right)\varepsilon_{t-j} = \varepsilon_t + \sum_{|n_2|=2} \left(\overline{\vartheta}_n - \vartheta_0\right)^{n_2} \left(\sum_{j=0}^{t-1} \xi_j^{n_2}\left(\overline{\vartheta}_n^{n_2}\right)\varepsilon_{t-j} + \sum_{j\geqslant t} \xi_j^{n_2}\left(\overline{\vartheta}_n^{n_2}\right)\varepsilon_{t-j}\right) \tag{5.10}$$

where

$$\xi_j^{n_2}(\vartheta_0) = \frac{\partial^{n_2}}{\partial\theta^{n_2}}\xi_j(\vartheta_0)$$

and $\overline{\vartheta}_n^{n_2} = \nu\vartheta_0 + (1-\nu)\overline{\vartheta}_n$ for some $0 < \nu < 1$. It is shown in [6] that

$$\mathbb{E}\left(\left(\sum_{j\geqslant t} \xi_j^{n_2}\left(\overline{\vartheta}_n^{n_2}\right)\varepsilon_{t-j}\right)^2\right) = O\left(t^{-1-2(d_1-d_0)}\right). \tag{5.11}$$

We recall that $w_n^0(\cdot)$ is the discrete Fourier transform with respect to the true noise defined in (2.15). We obtain from (5.10) and (5.11) that

$$w_n(\lambda) = w_n^0(\lambda) + \sum_{|n_2|=2} \left(\overline{\vartheta}_n - \vartheta_0\right)^{n_2} \sum_{j=0}^{n-1} \exp(-ij\lambda)\,\xi_j^{n_2}\left(\overline{\vartheta}_n^{n_2}\right) w_{n-j}^0(\lambda) + o_{\mathbb{P}}(1) \tag{5.12}$$

where $o_{\mathbb{P}}(1)$ is uniform in λ and n_2. Let for any $(i_1,i_2,i_3,i_4) \in \{0,1,\ldots,n-1\}^4$, $\vartheta \in \Theta_\delta$ and any multi-index $|n_1|$,

$$A_{1,n,(i_1,i_2,i_3,i_4)}^{(k,\ell),n_1}(\vartheta) = \left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3 \in I^3} \exp\left(-i\left(\frac{2\pi j_1}{n} i_1 + \frac{2\pi j_2}{n} i_2 + \frac{2\pi j_3}{n} i_3 - \frac{2\pi (j_1+j_2+j_3)}{n} i_4\right)\right)$$
$$\times H_n\left(\frac{2\pi (j_2+j_3)}{n}\right) \frac{\partial^{n_1}}{\partial \vartheta^{n_1}} \left(\phi_\vartheta^{(k)}\left(\frac{2\pi j_1}{n}\right) \phi_\vartheta^{(\ell)}\left(\frac{2\pi j_2}{n}\right)\right) \frac{1}{8\pi^3 n} w_{n-i_1}^0\left(\frac{2\pi j_1}{n}\right)$$
$$\times w_{n-i_2}^0\left(\frac{2\pi j_2}{n}\right) w_{n-i_3}^0\left(\frac{2\pi j_3}{n}\right) w_{n-i_4}^0\left(-\frac{2\pi (j_1+j_2+j_3)}{n}\right).$$

We use the same construction for the other terms $A_{2,n,(i_1,i_2,i_3,i_4)}^{(k,\ell)}(\vartheta)$ and $A_{3,n,(i_1,i_2,i_3,i_4)}^{(k,\ell)}(\vartheta)$. Then, by plugging (5.12) into $A_{1,n}^{k,\ell}$, we get

$$A_{1,n}^{(k,\ell),n_1}(\vartheta_0) = A_{1,n,(0,0,0,0)}^{(k,\ell),n_1}(\vartheta_0) + \sum_{|n_2|=2} \left(\overline{\vartheta}_n - \vartheta_0\right)^{n_1} \sum_{i_1=0}^{n-1} \xi_{i_1}^{n_2}\left(\overline{\vartheta}_n^{n_2}\right) A_{1,n,(i_1,0,0,0)}^{(k,\ell),n_1}(\vartheta_0)$$
$$+ \sum_{|n_2|,|n_3|=2} \left(\overline{\vartheta}_n - \vartheta_0\right)^{n_2+n_3} \sum_{i_1,i_2=0}^{n-1} \xi_{i_1}^{n_2}\left(\overline{\vartheta}_n^{n_2}\right) \xi_{i_2}^{n_3}\left(\overline{\vartheta}_n^{n_3}\right) A_{1,n,(i_1,i_2,0,0)}^{(k,\ell),n_1}(\vartheta_0)$$
$$+ \sum_{|n_2|,|n_3|,|n_4|=2} \left(\overline{\vartheta}_n - \vartheta_0\right)^{n_2+n_3+n_4} \sum_{i_1,i_2,i_3=0}^{n-1} \xi_{i_1}^{n_2}\left(\overline{\vartheta}_n^{n_2}\right) \xi_{i_2}^{n_3}\left(\overline{\vartheta}_n^{(n2)}\right) \xi_{i_3}^{n_4}\left(\overline{\vartheta}_n^{n_4}\right) A_{1,n,(i_1,i_2,i_3,0)}^{(k,\ell),n_1}(\vartheta_0)$$
$$+ \sum_{|n_2|,|n_3|,|n_4|,|n_5|=2} \left(\overline{\vartheta}_n - \vartheta_0\right)^{n_2+n_3+n_4+n_5} \sum_{i_1,i_2,i_3,i_4=0}^{n-1} \xi_{i_1}^{n_2}\left(\overline{\vartheta}_n^{(n_2)}\right) \xi_{i_2}^{n_3}\left(\overline{\vartheta}_n^{(n_3)}\right) \xi_{i_3}^{n_4}\left(\overline{\vartheta}_n^{(n_4)}\right)$$
$$\times \xi_{i_4}^{n_5}\left(\overline{\vartheta}_n^{(n_5)}\right) A_{1,n,(i_1,i_2,i_3,i_4)}^{(k,\ell),n_1}(\vartheta_0)$$
$$+ \text{ eleven terms of the same nature}$$
$$+ R_n^{(k,\ell),n_1}(\vartheta_0).$$

The remainder term $R_n^{(k,\ell),n_1}(\vartheta_0)$ is a combination of 65 terms, each containing at least one $o_{\mathbb{P}}(1)$ as a factor induced by the decomposition (5.12).

Step 3: Treatment of a generic term of the form $A_{1,n,(i_1,i_2,i_3,i_4)}^{(k,\ell),n_1}(\vartheta_0)$, $A_{2,n,(i_1,i_2,i_3,i_4)}^{(k,\ell),n_1}(\vartheta_0)$ and $A_{3,n,(i_1,i_2,i_3,i_4)}^{(k,\ell),n_1}(\vartheta_0)$.

Case 1: $i_1 = i_2 = i_3 = i_4 = 0$.

The same proof as [45] combined with his Remark 2 and Lemma 5.6 gives for any multi-index

$$\mathbb{E}\left(A_{1,n,(0,0,0,0)}^{(k,\ell),n_1}(\vartheta_0) - A_{2,n,(0,0,0,0)}^{(k,\ell),n_1}(\vartheta_0) - A_{3,n,(0,0,0,0)}^{(k,\ell),n_1}(\vartheta_0)\right) = O(1) \tag{5.13}$$

and (in the worst case)

$$\mathbb{V}ar\left(A_{m,n,(0,0,0,0)}^{(k,\ell),n_1}(\vartheta_0)\right) = O\left(\frac{1}{nk_n^2}\right) \tag{5.14}$$

for any $m \in \{1,2,3\}$. To be more precise, let us consider the term $A^{(k,\ell)}_{1,n,(0,0,0,0)}(\vartheta_0)$. We obtain using direct computations that

$$\begin{aligned}
\mathbb{E}\left(A^{(k,\ell),n_1}_{1,n,(0,0,0,0)}(\vartheta_0)\right) = &\left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3 \in I^3} H_n\left(\frac{2\pi(j_2+j_3)}{n}\right) \frac{\partial^{n_1}}{\partial \vartheta^{n_1}} \left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right)\phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right) \frac{1}{8\pi^3 n} \\
&\times \left(cum\left(w^0_n\left(\frac{2\pi j_1}{n}\right) w^0_n\left(\frac{2\pi j_2}{n}\right) w^0_n\left(\frac{2\pi j_3}{n}\right) w^0_n\left(-\frac{2\pi(j_1+j_2+j_3)}{n}\right)\right)\right. \\
&+ cum\left(w^0_n\left(\frac{2\pi j_1}{n}\right) w^0_n\left(\frac{2\pi j_2}{n}\right)\right) cum\left(w^0_n\left(\frac{2\pi j_3}{n}\right) w^0_n\left(-\frac{2\pi(j_1+j_2+j_3)}{n}\right)\right) \\
&+ cum\left(w^0_n\left(\frac{2\pi j_1}{n}\right) w^0_n\left(\frac{2\pi j_3}{n}\right)\right) cum\left(w^0_n\left(\frac{2\pi j_2}{n}\right) w^0_n\left(-\frac{2\pi(j_1+j_2+j_3)}{n}\right)\right) \\
&\left. + cum\left(w^0_n\left(\frac{2\pi j_1}{n}\right) w^0_n\left(-\frac{2\pi(j_1+j_2+j_3)}{n}\right)\right) cum\left(w^0_n\left(\frac{2\pi j_2}{n}\right) w^0_n\left(\frac{2\pi j_3}{n}\right)\right)\right).
\end{aligned}$$

Hence, one can use Lemma 5.6 to get

$$\begin{aligned}
\mathbb{E}\left(A^{(k,\ell),n_1}_{1,n,(0,0,0,0)}(\vartheta_0)\right) = &\left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3 \in I^3} H_n\left(\frac{2\pi(j_2+j_3)}{n}\right) \frac{\partial^{n_1}}{\partial \vartheta^{n_1}} \left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right)\phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right) \frac{1}{8\pi^3 n} \\
&\times \left((2\pi)^3 f_4\left(\frac{2\pi j_1}{n}, \frac{2\pi j_2}{n}, \frac{2\pi j_3}{n}\right) n + O(1)\right. \\
&+ (2\pi)^2 f_2\left(\frac{2\pi j_1}{n}\right) D_n\left(\frac{2\pi(j_1+j_2)}{n}\right) f_2\left(\frac{2\pi j_3}{n}\right) D_n\left(\frac{-2\pi(j_1+j_2)}{n}\right) \\
&+ (2\pi)^2 f_2\left(\frac{2\pi j_1}{n}\right) D_n\left(\frac{2\pi(j_1+j_3)}{n}\right) f_2\left(\frac{2\pi j_2}{n}\right) D_n\left(\frac{-2\pi(j_1+j_3)}{n}\right) \\
&+ (2\pi)^2 f_2\left(\frac{2\pi j_1}{n}\right) D_n\left(\frac{-2\pi(j_2+j_3)}{n}\right) f_2\left(\frac{2\pi j_2}{n}\right) D_n\left(\frac{2\pi(j_2+j_3)}{n}\right) \\
&+ O\left(D_n\left(\frac{2\pi(j_1+j_2)}{n}\right)\right) + O\left(D_n\left(\frac{2\pi(j_1+j_3)}{n}\right)\right) \\
&\left. + O\left(D_n\left(\frac{2\pi(j_2+j_3)}{n}\right)\right)\right).
\end{aligned}$$

The first term of the last equality is

$$\begin{aligned}
A^{(k,\ell),n_1}_{1,1,n,(0,0,0,0)}(\vartheta_0) = &\left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3 \in I^3} H_n\left(\frac{2\pi(j_2+j_3)}{n}\right) \frac{\partial^{n_1}}{\partial \vartheta^{n_1}} \left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right)\phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right) \frac{1}{8\pi^3 n} \\
&\times \left(f_4\left(\frac{2\pi j_1}{n}, \frac{2\pi j_2}{n}, \frac{2\pi j_3}{n}\right) n + O(1)\right).
\end{aligned}$$

We recall that the support of H is $[-\pi,\pi]$. Then it is necessary that $\frac{2\pi(j_2+j_3)}{n} \in$

$[-k_n\pi, k_n\pi]$ to ensure that $H_n\left(\frac{2\pi(j_2+j_3)}{n}\right) \neq 0$. Therefore, we have $\frac{2\pi j_3}{n} = -\frac{2\pi j_2}{n} + \frac{2\pi j_4}{n}$ where $j_4 \in \widetilde{I} = [-[k_n n], [k_n n]]$. Note that $\frac{2\pi j_4}{n} \longrightarrow 0$ uniformly in $j_4 \in \widetilde{I}$ so, by **(A4)**, $f_4\left(\frac{2\pi j_1}{n}, \frac{2\pi j_2}{n}, \frac{2\pi j_3}{n}\right) = f_4\left(\frac{2\pi j_1}{n}, \frac{2\pi j_2}{n}, -\frac{2\pi j_2}{n}\right) + O(k_n)$, where the error term $O(k_n)$ is uniform in $(\lambda_1, \lambda_2, \lambda_3)$. Thus, one has

$$\begin{aligned}
\left|A^{(k,\ell),n_1}_{1,1,n,(0,0,0,0)}(\vartheta_0)\right| &\leqslant \left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3 \in I^3} \left|H_n\left(\frac{2\pi(j_2+j_3)}{n}\right) \frac{\partial^{n_1}}{\partial \vartheta^{n_1}} \left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right) \phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right)\right| \frac{1}{8\pi^3 n} \\
&\quad \times \left|\left(f_4\left(\frac{2\pi j_1}{n}, \frac{2\pi j_2}{n}, \frac{2\pi j_3}{n}\right) n + O(1)\right)\right| \\
&\leqslant \left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_4 \in I^2 \times \widetilde{I}} \left|H_n\left(\frac{2\pi j_4}{n}\right) \phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right) \phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right| \frac{1}{8\pi^3 n} \\
&\quad \times \left(\left|f_4\left(\frac{2\pi j_1}{n}, \frac{2\pi j_2}{n}, \frac{-2\pi j_2}{n}\right)\right| n + O(k_n n) + O(1)\right)
\end{aligned}$$

and so $A^{(k,\ell),n_1}_{1,1,n,(0,0,0,0)}(\vartheta_0) = O(1)$. The same approach can be used to show that the term $A^{(k,\ell),n_1}_{1,2,n,(0,0,0,0)}(\vartheta_0)$ (respectively $A^{(k,\ell),n_1}_{1,3,n,(0,0,0,0)}(\vartheta_0)$) which contributes when $j_1 = -j_2$ (respectively $j_1 = -j_3$) is $O(1)$. The most delicate term is the last one, namely

$$\begin{aligned}
A^{(k,\ell),n_1}_{1,4,n,(0,0,0,0)}(\vartheta_0) &= \left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3 \in I^3} H_n\left(\frac{2\pi(j_2+j_3)}{n}\right) \frac{\partial^{n_1}}{\partial \vartheta^{n_1}} \left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right) \phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right) \frac{1}{2\pi n} \\
&\quad \times \left(f_2\left(\frac{2\pi j_1}{n}\right) D_n\left(\frac{-2\pi(j_2+j_3)}{n}\right) f_2\left(\frac{2\pi j_2}{n}\right) D_n\left(\frac{2\pi(j_2+j_3)}{n}\right)\right. \\
&\quad \left. + O\left(D_n\left(\frac{2\pi(j_2+j_3)}{n}\right)\right)\right)
\end{aligned}$$

which contributes when $j_3 = -j_2$. This term is unbounded but is compensated by the expectation of the term $A^{(k,\ell),n_1}_{3,n,(0,0,0,0)}$. This can be shown using the same reasoning as above, and we obtain (5.13). Now we have

$$\mathbb{V}ar\left(A^{(k,\ell),n_1}_{m,n,(0,0,0,0)}(\vartheta_0)\right) = cum\left(A^{(k,\ell),n_1}_{m,n,(0,0,0,0)}(\vartheta_0), \overline{A^{(k,\ell),n_1}_{m,n,(0,0,0,0)}(\vartheta_0)}\right)$$

and by Remark 2 of [45] (when $m = 1$),

$$\begin{aligned}
\mathbb{V}ar\left(A^{(k,\ell),n_1}_{1,n,(0,0,0,0)}(\vartheta_0)\right) &= \left(\frac{2\pi}{n}\right)^6 \sum_{j_1,j_2,j_3,j_4,j_5,j_6 \in I^6} H_n\left(\frac{2\pi(j_2+j_3)}{n}\right) \frac{\partial^{n_1}}{\partial \vartheta^{n_1}} \left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right) \phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right) \\
&\quad \times H_n\left(\frac{2\pi(j_5+j_6)}{n}\right) \frac{\partial^{n_1}}{\partial \vartheta^{n_1}} \left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_4}{n}\right) \phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_5}{n}\right)\right) \frac{1}{64\pi^6 n^2}
\end{aligned}$$

$$\times \sum_{V} cum\left(w_n^0\left(\frac{2\pi j_{r_1}}{n}\right); r_1 \in V_1\right) \ldots cum\left(w_n^0\left(\frac{2\pi j_{r_p}}{n}\right); r_p \in V_p\right)$$

where the summation is over all irreductible partitions $V = V_1 \cup \cdots \cup V_p$ of the table

$$\begin{pmatrix} j_1 & j_2 & j_3 & -(j_1+j_2+j_3) \\ -j_4 & -j_5 & -j_6 & (j_4+j_5+j_6) \end{pmatrix}.$$

It remains to apply Lemma 5.6 in order to get (5.14).

Case 2: at least one index is non-negative

Again the same proof as [45] combined with his Remark 2 and Lemma 5.6 gives, for any (i_1, i_2, i_3, i_4) and $m \in \{1, 2, 3\}$ (in the worst case),

$$\mathbb{E}\left(A^{(k,\ell),n_1}_{m,n,(i_1,i_2,i_3,i_4)}(\vartheta_0)\right) = O\left(\frac{1}{k_n}\right) \tag{5.15}$$

uniformly in $(i_1, i_2, i_3, i_4) \in \{0, 1, \ldots, n-1\}^4$ and

$$\mathbb{V}ar\left(A^{(k,\ell),n_1}_{m,n,(i_1,i_2,i_3,i_4)}(\vartheta_0)\right) = O\left(\frac{1}{nk_n^2}\right) \tag{5.16}$$

uniformly in $(i_1, i_2, i_3, i_4) \in \{0, 1, \ldots, n-1\}^4$ for any $m \in \{1, 2, 3\}$. To be more precise, let us consider the term $A^{(k,\ell)}_{1,n,(i_1,i_2,i_3,i_4)}(\vartheta_0)$ and assume (without loss of generality) that $i_4 = \max(i_1, i_2, i_3, i_4)$. Then, we obtain using direct computations that

$$\begin{aligned}
\mathbb{E}\left(A^{(k,\ell),n_1}_{1,n,(i_1,i_2,i_3,i_4)}(\vartheta_0)\right) = &\left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3 \in I^3} \exp\left(-i\left(\frac{2\pi j_1}{n} i_1 + \frac{2\pi j_2}{n} i_2 + \frac{2\pi j_3}{n} i_3 - \frac{2\pi(j_1+j_2+j_3)}{n} i_4\right)\right) \\
&\times H_n\left(\frac{2\pi(j_2+j_3)}{n}\right) \frac{\partial^{n_1}}{\partial \vartheta^{n_1}}\left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right)\phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right)\frac{1}{8\pi^3 n} \\
&\times \left(cum\left(w^0_{n-i_1}\left(\frac{2\pi j_1}{n}\right) w^0_{n-i_2}\left(\frac{2\pi j_2}{n}\right) w^0_{n-i_3}\left(\frac{2\pi j_3}{n}\right) w^0_{n-i_4}\left(-\frac{2\pi(j_1+j_2+j_3)}{n}\right)\right)\right. \\
&+ cum\left(w^0_{n-i_1}\left(\frac{2\pi j_1}{n}\right) w^0_{n-i_2}\left(\frac{2\pi j_2}{n}\right)\right) cum\left(w^0_{n-i_3}\left(\frac{2\pi j_3}{n}\right) w^0_{n-i_4}\left(-\frac{2\pi(j_1+j_2+j_3)}{n}\right)\right) \\
&+ cum\left(w^0_{n-i_1}\left(\frac{2\pi j_1}{n}\right) w^0_{n-i_3}\left(\frac{2\pi j_3}{n}\right)\right) cum\left(w^0_{n-i_2}\left(\frac{2\pi j_2}{n}\right) w^0_{n-i_4}\left(-\frac{2\pi(j_1+j_2+j_3)}{n}\right)\right) \\
&\left.+ cum\left(w^0_{n-i_1}\left(\frac{2\pi j_1}{n}\right) w^0_{n-i_4}\left(-\frac{2\pi(j_1+j_2+j_3)}{n}\right)\right) cum\left(w^0_{n-i_2}\left(\frac{2\pi j_2}{n}\right) w^0_{n-i_3}\left(\frac{2\pi j_3}{n}\right)\right)\right).
\end{aligned}$$

Hence, one can use Lemma 5.6 to get

$$\mathbb{E}\left(A^{(k,\ell),n_1}_{1,n,(i_1,i_2,i_3,i_4)}(\vartheta_0)\right) = \left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3 \in I^3} \exp\left(-i\left(\frac{2\pi j_1}{n} i_1 + \frac{2\pi j_2}{n} i_2 + \frac{2\pi j_3}{n} i_3 - \frac{2\pi(j_1+j_2+j_3)}{n} i_4\right)\right)$$

$$\times H_n\left(\frac{2\pi\left(j_2+j_3\right)}{n}\right)\frac{\partial^{n_1}}{\partial\vartheta^{n_1}}\left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right)\phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right)\frac{1}{8\pi^3 n}$$
$$\times\left((2\pi)^3 f_4\left(\frac{2\pi j_1}{n},\frac{2\pi j_2}{n},\frac{2\pi j_3}{n}\right)(n-i_4)+O(1)\right.$$
$$+(2\pi)^2 f_2\left(\frac{2\pi j_1}{n}\right)D_{\min(n-i_1,n-i_2)}\left(\frac{2\pi(j_1+j_2)}{n}\right)f_2\left(\frac{2\pi j_3}{n}\right)D_{n-i_4}\left(\frac{-2\pi(j_1+j_2)}{n}\right)$$
$$+(2\pi)^2 f_2\left(\frac{2\pi j_1}{n}\right)D_{\min(n-i_1,n-i_3)}\left(\frac{2\pi(j_1+j_3)}{n}\right)f_2\left(\frac{2\pi j_2}{n}\right)D_{n-i_4}\left(\frac{-2\pi(j_1+j_3)}{n}\right)$$
$$+(2\pi)^2 f_2\left(\frac{2\pi j_1}{n}\right)D_{n-i_4}\left(\frac{-2\pi(j_2+j_3)}{n}\right)f_2\left(\frac{2\pi j_2}{n}\right)D_{\min(n-i_2,n-i_3)}\left(\frac{2\pi(j_2+j_3)}{n}\right)$$
$$+O\left(D_{\min(n-i_1,n-i_2)}\left(\frac{2\pi(j_1+j_2)}{n}\right)\right)+O\left(D_{\min(n-i_1,n-i_3)}\left(\frac{2\pi(j_1+j_3)}{n}\right)\right)$$
$$\left.+O\left(D_{\min(n-i_2,n-i_3)}\left(\frac{2\pi(j_2+j_3)}{n}\right)\right)\right).$$

The first term of the last equality is

$$A^{(k,\ell),n_1}_{1,1,n,(i_1,i_2,i_3,i_4)}(\vartheta_0)=\left(\frac{2\pi}{n}\right)^3\sum_{j_1,j_2,j_3\in I^3}\exp\left(-i\left(\frac{2\pi j_1}{n}i_1+\frac{2\pi j_2}{n}i_2+\frac{2\pi j_3}{n}i_3-\frac{2\pi\left(j_1+j_2+j_3\right)}{n}i_4\right)\right)$$
$$\times H_n\left(\frac{2\pi\left(j_2+j_3\right)}{n}\right)\frac{\partial^{n_1}}{\partial\vartheta^{n_1}}\left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right)\phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right)\frac{1}{8\pi^3 n}$$
$$\times\left(f_4\left(\frac{2\pi j_1}{n},\frac{2\pi j_2}{n},\frac{2\pi j_3}{n}\right)(n-i_4)+O(1)\right).$$

The support of the kernel H is $[-\pi,\pi]$. Then, it is necessary that $\frac{2\pi\left(j_2+j_3\right)}{n}\in[-k_n\pi,k_n\pi]$ to ensure $H_n\left(\frac{2\pi\left(j_2+j_3\right)}{n}\right)\neq 0$. Therefore, we can write $\frac{2\pi j_3}{n}=-\frac{2\pi j_2}{n}+\frac{2\pi j_4}{n}$ where $j_4\in\widetilde{I}=[-[k_n n],[k_n n]]$. We note that $\frac{2\pi j_4}{n}\longrightarrow 0$ uniformly in $j_4\in\widetilde{I}$ so, by **(A4)**, $f_4\left(\frac{2\pi j_1}{n},\frac{2\pi j_2}{n},\frac{2\pi j_3}{n}\right)=f_4\left(\frac{2\pi j_1}{n},\frac{2\pi j_2}{n},-\frac{2\pi j_2}{n}\right)+O(k_n)$, where the error term $O(k_n)$ is uniform in $(\lambda_1,\lambda_2,\lambda_3)$. Thus, one has

$$\left|A^{(k,\ell),n_1}_{1,1,n,(i_1,i_2,i_3,i_4)}(\vartheta_0)\right|\leqslant\left(\frac{2\pi}{n}\right)^3\sum_{j_1,j_2,j_3\in I^3}\left|H_n\left(\frac{2\pi\left(j_2+j_3\right)}{n}\right)\frac{\partial^{n_1}}{\partial\vartheta^{n_1}}\left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right)\phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right)\right|\frac{1}{8\pi^3 n}$$
$$\times\left|\left(f_4\left(\frac{2\pi j_1}{n},\frac{2\pi j_2}{n},\frac{2\pi j_3}{n}\right)(n-i_4)+O(1)\right)\right|$$
$$\leqslant\left(\frac{2\pi}{n}\right)^3\sum_{j_1,j_2,j_4\in I^2\times\widetilde{I}}\left|H_n\left(\frac{2\pi j_4}{n}\right)\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right)\phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right|\frac{1}{8\pi^3 n}$$
$$\times\left(\left|f_4\left(\frac{2\pi j_1}{n},\frac{2\pi j_2}{n},\frac{-2\pi j_2}{n}\right)\right|(n-i_4)+O\left(k_n(n-i_4)\right)+O(1)\right)$$

and so $A^{(k,\ell)}_{1,1,n,(i_1,i_2,i_3,i_4)}(\vartheta_0) = O(1)$ where the bound is uniform as $(i_1,i_2,i_3,i_4) \in \{0,\dots,n-1\}^4$. The same approach shows that the term $A^{(k,\ell),n_1}_{1,2,n,(i_1,i_2,i_3,i_4)}(\vartheta_0)$ (respectively $A^{(k,\ell),n_1}_{1,3,n,(i_1,i_2,i_3,i_4)}(\vartheta_0)$) which contributes when $j_1 = -j_2$ (respectively $j_1 = -j_3$) are $O(1)$ where the bound is also uniform as $(i_1,i_2,i_3,i_4) \in \{0,\dots,n-1\}^4$. As in **Case 1**, the most delicate term is the last one, namely

$$
\begin{aligned}
A^{(k,\ell),n_1}_{1,4,n,(i_1,i_2,i_3,i_4)}(\vartheta_0) = &\left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3 \in I^3} \exp\left(-i\left(\frac{2\pi j_1}{n} i_1 + \frac{2\pi j_2}{n} i_2 + \frac{2\pi j_3}{n} i_3 - \frac{2\pi(j_1+j_2+j_3)}{n} i_4\right)\right) \\
&\times H_n\left(\frac{2\pi(j_2+j_3)}{n}\right) \frac{\partial^{n_1}}{\partial \vartheta^{n_1}} \left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right) \phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right) \frac{1}{2\pi n} \\
&\times \left(f_2\left(\frac{2\pi j_1}{n}\right) D_{n-i_4}\left(\frac{-2\pi(j_2+j_3)}{n}\right) f_2\left(\frac{2\pi j_2}{n}\right) D_{\min(n-i_2,n-i_3)}\left(\frac{2\pi(j_2+j_3)}{n}\right)\right. \\
&\left. + O\left(D_{\min(n-i_2,n-i_3)}\left(\frac{2\pi(j_2+j_3)}{n}\right)\right)\right)
\end{aligned}
$$

which contributes when $j_3 = -j_2$. This term is unbounded and we have (5.15). Observe now that

$$
\mathbb{V}ar\left(A^{(k,\ell),n_1}_{m,n,(i_1,i_2,i_3,i_4)}(\vartheta_0)\right) = cum\left(A^{(k,\ell),n_1}_{m,n,(i_1,i_2,i_3,i_4)}(\vartheta_0), \overline{A^{(k,\ell),n_1}_{m,n,(i_1,i_2,i_3,i_4)}(\vartheta_0)}\right)
$$

and using Remark 2 of [45] (when $m = 1$),

$$
\begin{aligned}
\mathbb{V}ar\left(A^{(k,\ell),n_1}_{1,n,(i_1,i_2,i_3,i_4)}(\vartheta_0)\right) = &\left(\frac{2\pi}{n}\right)^6 \sum_{j_1,j_2,j_3,j_4,j_5,j_6 \in I^6} \exp\left(-i\left(\frac{2\pi j_1}{n} i_1 + \frac{2\pi j_2}{n} i_2 + \frac{2\pi j_3}{n} i_3 - \frac{2\pi(j_1+j_2+j_3)}{n} i_4\right)\right) \\
&\times \exp\left(i\left(\frac{2\pi j_4}{n} i_1 + \frac{2\pi j_5}{n} i_2 + \frac{2\pi j_6}{n} i_3 - \frac{2\pi(j_4+j_5+j_6)}{n} i_4\right)\right) \\
&\times H_n\left(\frac{2\pi(j_2+j_3)}{n}\right) \frac{\partial^{n_1}}{\partial \vartheta^{n_1}} \left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right) \phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right) \\
&\times H_n\left(\frac{2\pi(j_5+j_6)}{n}\right) \frac{\partial^{n_1}}{\partial \vartheta^{n_1}} \left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_4}{n}\right) \phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_5}{n}\right)\right) \frac{1}{64\pi^6 n^2} \\
&\times \sum_V cum\left(w^0_{n-[r_1]}\left(\frac{2\pi j_{r_1}}{n}\right); r_1 \in V_1\right) \dots cum\left(w^0_{n-[r_p]}\left(\frac{2\pi j_{r_p}}{n}\right); r_p \in V_p\right),
\end{aligned}
$$

where the summation is over all irreductible partitions $V = V_1 \cup \dots \cup V_p$ of the table

$$
\begin{pmatrix} j_1 & j_2 & j_3 & -(j_1+j_2+j_3) \\ -j_4 & -j_5 & -j_6 & (j_4+j_5+j_6) \end{pmatrix},
$$

where in this specific term, we use the notation $\left[r_p\right] = i_1$ if $r_p \in \{j_1, -j_4\}$, ... , $\left[r_p\right] = i_4$ if $r_p \in \{-(j_1 + j_2 + j_3), j_4 + j_5 + j_6\}$. Consequently, we apply Lemma 5.6 to deduce (5.16).

Step 4: The terms of the form $A^{(k,\ell),n_1}_{m,n,(i_1,i_2,i_3,i_4)}\left(\overline{\vartheta}^{n_1}_n\right)$

We assume in this step that $|n_1| = 3$. Using the same expansion as presented in the second step for the discrete Fourier transform, we rewrite the remainder terms as functions of $w^0_n(\cdot)$. Specifically, we maintain the notations introduced in **Step 2** to denote the indices (i_1, i_2, i_3, i_4) involved in the multiple sums. It can be shown using Cauchy-Schwarz inequality together with Lemma 5.6 and **(A1)** that (for each component and in the worst case)

$$\mathbb{E}\left(\left|A^{(k,\ell),n_1}_{m,n,(i_1,i_2,i_3,i_4)}\left(\overline{\vartheta}^{n_1}_n\right)\right|\right) = O\left(\frac{n}{k_n}\right) \tag{5.17}$$

uniformly in $(i_1, i_2, i_3, i_4) \in \{0, 1, \ldots, n-1\}^4$ for any $m \in \{1, 2, 3\}$. Let us be more precise. Observe that

$$\mathbb{E}\left(\left|A^{(k,\ell),n_1}_{1,n,(i_1,i_2,i_3,i_4)}\left(\overline{\vartheta}^{(1)}_n\right)\right|\right) \leqslant \left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3 \in I^3} \frac{1}{8\pi^3 n}\left|H_n\left(\frac{2\pi(j_2+j_3)}{n}\right)\right| \mathbb{E}\left(\left|T^{(j_1,j_2,j_3)}_{1,n,(i_1,i_2,i_3,i_4)}\right|\right),$$

where

$$T^{(j_1,j_2,j_3)}_{1,n,(i_1,i_2,i_3,i_4)}\left(\overline{\vartheta}^{n_1}_n\right) = \frac{\partial^{n_1}}{\partial \vartheta^{n_1}}\left(\phi^{(k)}_{\overline{\vartheta}^{n_1}_n}\left(\frac{2\pi j_1}{n}\right)\phi^{(\ell)}_{\overline{\vartheta}^{n_1}_n}\left(\frac{2\pi j_2}{n}\right)\right) w^0_{n-i_1}\left(\frac{2\pi j_1}{n}\right) w^0_{n-i_2}\left(\frac{2\pi j_2}{n}\right) w^0_{n-i_3}\left(\frac{2\pi j_3}{n}\right) \times w^0_{n-i_4}\left(-\frac{2\pi(j_1+j_2+j_3)}{n}\right).$$

We have, using Cauchy-Schwarz inequality, that

$$\mathbb{E}\left(\left|T^{(j_1,j_2,j_3)}_{1,n,(i_1,i_2,i_3,i_4)}\right|\right) \leqslant \mathbb{E}\left(T^{(j_1,j_2)}_{2,n}\right)^{\frac{1}{2}} \mathbb{E}\left(T^{(j_1,j_2,j_3)}_{3,n,(i_1,i_2,i_3,i_4)}\right)^{\frac{1}{2}},$$

where

$$T^{(j_1,j_2)}_{2,n} = \left(\frac{\partial^{n_1}}{\partial \vartheta^{n_1}}\left(\phi^{(k)}_{\overline{\vartheta}^{n_1}_n}\left(\frac{2\pi j_1}{n}\right)\phi^{(\ell)}_{\overline{\vartheta}^{n_1}_n}\left(\frac{2\pi j_2}{n}\right)\right)\right)^2$$

and

$$T^{(j_1,j_2,j_3)}_{3,n,(i_1,i_2,i_3,i_4)} = \left|w^0_{n-i_1}\left(\frac{2\pi j_1}{n}\right) w^0_{n-i_2}\left(\frac{2\pi j_2}{n}\right) w^0_{n-i_3}\left(\frac{2\pi j_3}{n}\right) w^0_{n-i_4}\left(-\frac{2\pi(j_1+j_2+j_3)}{n}\right)\right|^2.$$

Thanks to **(A1')**, one can obtain that for each component

$$T^{(j_1,j_2)}_{2,n} \leqslant T_{2,n}(j_1, j_2) = K\left|\frac{2\pi j_1}{n}\right|^{-\delta}\left|\frac{2\pi j_2}{n}\right|^{-\delta}$$

for some positive constant K and any $\delta > 0$. We deduce that

$$\mathbb{E}\left(\left|A^{(k,\ell),n_1}_{1,n,(i_1,i_2,i_3,i_4)}\left(\overline{\vartheta}^{n_1}_n\right)\right|\right) \leqslant \left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3\in I^3} \frac{1}{8\pi^3 n}\left|H_n\left(\frac{2\pi\left(j_2+j_3\right)}{n}\right)\right| T_{2,n}\left(j_1,j_2\right)\mathbb{E}\left(\, T^{(j_1,j_2,j_3)}_{3,n,(i_1,i_2,i_3,i_4)}\right)^{\frac{1}{2}}$$

and finally (5.17) from Lemma 5.6.

Step 5: Treatment of the remainder terms

Case 1: when $|n_1| \leqslant 2$

We detail the treatment of certain terms; others are treated in a similar way. We consider one of the remainder terms

$$\begin{aligned} R^{(k,\ell),n_1}_{1,n,(i_1,i_2,i_3,\cdot)}\left(\vartheta_0\right) = &\left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3\in I^3} \exp\left(-i\left(\frac{2\pi j_1}{n}i_1+\frac{2\pi j_2}{n}i_2+\frac{2\pi j_3}{n}i_3-\frac{2\pi\left(j_1+j_2+j_3\right)}{n}i_4\right)\right)\\ &\times H_n\left(\frac{2\pi\left(j_2+j_3\right)}{n}\right)\frac{\partial^{n_1}}{\partial\vartheta^{n_1}}\left(\phi^{(k)}_{\vartheta_0}\left(\frac{2\pi j_1}{n}\right)\phi^{(\ell)}_{\vartheta_0}\left(\frac{2\pi j_2}{n}\right)\right)\frac{1}{8\pi^3 n}w^0_{n-i_1}\left(\frac{2\pi j_1}{n}\right)\\ &\times w^0_{n-i_2}\left(\frac{2\pi j_2}{n}\right)w^0_{n-i_3}\left(\frac{2\pi j_3}{n}\right). \end{aligned}$$

Proceeding as in **Step 3**, one has

$$\mathbb{E}\left(R^{(k,\ell),n_1}_{1,n,(i_1,i_2,i_3,\cdot)}\left(\vartheta_0\right)\right) = O\left(\frac{1}{nk_n}\right)$$

and

$$\mathbb{V}ar\left(R^{(k,\ell),n_1}_{1,n,(i_1,i_2,i_3,\cdot)}\left(\vartheta_0\right)\right) = O\left(\frac{1}{n^2k_n^2}\right).$$

Similarly, one can show that

$$\mathbb{E}\left(R^{(k,\ell),n_1}_{1,n}\left(\vartheta_0\right)\right) = O\left(\frac{1}{nk_n}\right)$$

and

$$\mathbb{V}ar\left(R^{(k,\ell),n_1}_{1,n}\left(\vartheta_0\right)\right) = O\left(\frac{1}{n^2k_n^2}\right).$$

Hence, we deduce that

$$R^{(k,\ell),n_1}_{1,n}\left(\vartheta_0\right) = O_{\mathbb{P}}\left(\frac{1}{nk_n}\right).$$

Case 2: when $|n_1| = 3$

We consider the following remainder term

$$R^{(k,\ell),n_1}_{1,n,(i_1,i_2,i_3,\cdot)}\left(\overline{\vartheta}^{n_1}_n\right) = \left(\frac{2\pi}{n}\right)^3 \sum_{j_1,j_2,j_3\in I^3} \exp\left(-i\left(\frac{2\pi j_1}{n}i_1+\frac{2\pi j_2}{n}i_2+\frac{2\pi j_3}{n}i_3-\frac{2\pi\left(j_1+j_2+j_3\right)}{n}i_4\right)\right)$$

$$\times H_n\left(\frac{2\pi\left(j_2+j_3\right)}{n}\right)\frac{\partial^{n_1}}{\partial\vartheta^{n_1}}\left(\phi^{(k)}_{\overline{\vartheta}_n^{n_1}}\left(\frac{2\pi j_1}{n}\right)\phi^{(\ell)}_{\overline{\vartheta}_n^{n_1}}\left(\frac{2\pi j_2}{n}\right)\right)\frac{1}{8\pi^3 n}w^0_{n-i_1}\left(\frac{2\pi j_1}{n}\right)$$
$$\times w^0_{n-i_2}\left(\frac{2\pi j_2}{n}\right)w^0_{n-i_3}\left(\frac{2\pi j_3}{n}\right).$$

Proceeding as in **Step 4**, we get

$$\mathbb{E}\left(\left|R^{(k,\ell),n_1}_{1,n,(i_1,i_2,i_3,\cdot)}\left(\overline{\vartheta}_n^{n_1}\right)\right|\right)=O\left(\frac{\sqrt{n}}{k_n}\right)$$

Finally, we obtain

$$R^{(k,\ell),n_1}_{1,n}\left(\overline{\vartheta}_n^{n_1}\right)=O_{\mathbb{P}}\left(\frac{\sqrt{n}}{k_n}\right).$$

Step 6: Asymptotic equivalence of the estimator calculated on the residuals and the one on the true noise

***Remark* 5.1.** *In [45], the functions called ϕ used in Theorem 1 and 2 are assumed to be continuous. This is not the case in our situation since the function $\phi^{(k)}$ may have a singularity of magnitude $\lambda^{-\delta}$ at 0 for any small $\delta>0$. Thanks to **(A1)**, this singularity cannot affect the convergence of the Riemann's sums.*

We note that the first term $A^{(k,\ell),0}_{1,n,(0,0,0,0)}\left(\vartheta_0\right)-A^{(k,\ell),0}_{2,n,(0,0,0,0)}\left(\vartheta_0\right)-A^{(k,\ell),0}_{3,n,(0,0,0,0)}\left(\vartheta_0\right)$ is exactly the same estimator presented in Theorem 2 of [45], so we want to show that $V_n^{(k,l)}$ is asymptotically equivalent to $A^{(k,\ell),0}_{1,n,(0,0,0,0)}\left(\vartheta_0\right)-A^{(k,\ell),0}_{2,n,(0,0,0,0)}\left(\vartheta_0\right)-A^{(k,\ell),0}_{3,n,(0,0,0,0)}\left(\vartheta_0\right)$. It is shown in [26] that

$$\xi^{n_2}_j\left(\vartheta\right)=O\left(j^{-1-(d_1-d_0)}\right)\text{ uniformly in }\vartheta\in\Theta_\delta. \tag{5.18}$$

Case 1: when $|n_1|\leqslant 2$

Combining (5.15), (5.16) and (5.18), we have

$$\sum_{i_1=0}^{n-1}\xi^{n_2}_{i_1}\left(\overline{\vartheta}_n^{n_2}\right)A^{(k,\ell),n_1}_{1,n,(i_1,0,0,0)}\left(\vartheta_0\right)=O_{\mathbb{P}}\left(\frac{n^{-(d_1-d_0)}}{k_n}\right).$$

Since $\left(\overline{\vartheta}_n-\vartheta_0\right)=O_{\mathbb{P}}\left(n^{-\frac{1}{2}}\right)$, $d_1-d_0>-\frac{1}{2}$ and $nk_n^2\longrightarrow 0$ we get

$$\left(\overline{\vartheta}_n-\vartheta_0\right)^{n_2}\sum_{i_1=0}^{n-1}\xi^{n_2}_{i_1}\left(\overline{\vartheta}_n^{n_2}\right)A^{(k,\ell),n_1}_{1,n,(i_1,0,0,0)}\left(\vartheta_0\right)=o_{\mathbb{P}}\left(\frac{1}{k_n\sqrt{n}}\right). \tag{5.19}$$

The same treatment applies to the other terms, and the result of **Step 5-Case 1** leads us to

$$\sum_{|n_1|=0}^{2}\frac{1}{n_1!}\left(\overline{\vartheta}_n-\vartheta_0\right)^{n_1}A^{(k,\ell),n_1}_{m,n}\left(\vartheta_0\right)-A^{(k,\ell),0}_{m,n,(0,0,0,0)}\left(\vartheta_0\right)=o_{\mathbb{P}}\left(1\right) \tag{5.20}$$

for any $m \in \{1,2,3\}$.

Case 2: when $|n_1| = 3$

For the last term, one can observe from **Step 4**, together with the same reasoning as for (5.19), that in the worst case, for any $m \in \{1,2,3\}$,

$$\left(\overline{\vartheta}_n - \vartheta_0\right)^{n_1} A^{(k,\ell)}_{m,n,(0,0,0,0)}\left(\overline{\vartheta}_n^{n_1}\right) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}k_n}\right).$$

Similar treatment for the other terms and the results of **Step 5 in Case 2** give

$$\left(\overline{\vartheta}_n - \vartheta_0\right)^{n_1} A^{(k,\ell)}_{m,n}\left(\overline{\vartheta}_n^{n_1}\right) \xrightarrow[n\to\infty]{\mathbb{P}} 0 \tag{5.21}$$

for any $m \in \{1,2,3\}$.

Finally, one can use (5.20)-(5.21) to obtain

$$V_n^{(k,l)} - A^{(k,\ell),0}_{1,n,(0,0,0,0)}(\vartheta_0) + A^{(k,\ell),0}_{2,n,(0,0,0,0)}(\vartheta_0) + A^{(k,\ell),0}_{3,n,(0,0,0,0)}(\vartheta_0) \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Theorem 2 in [45] gives

$$A^{(k,\ell),0}_{1,n,(0,0,0,0)}(\vartheta_0) - A^{(k,\ell),0}_{2,n,(0,0,0,0)}(\vartheta_0) - A^{(k,\ell),0}_{3,n,(0,0,0,0)}(\vartheta_0) \xrightarrow[n\to\infty]{\mathbb{P}} T^{(k,\ell)}$$

and the proof is completed.

5.5. *Proof of Theorem 2.3*

The proof is similar to that of Theorem 2.2. It is obtained by following the same steps as those presented in the subsection 5.4, so it is omitted.

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