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A simple construction of strong solutions to hyperbolic corner problems and an application.

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Abstract

In this article we give a simple construction of strong solutions to hyperbolic corner problems. The main idea of the method is to reduce the analysis to the study of transport equations and to treat the coupling between the transport phenomena as a source term. Then we solve inductively with a loss of one derivative at each step. Such losses being compensated if one considers infinitely regular sources. This simple approach gives strong solutions to hyperbolic corner problems and thus answer a natural question. Indeed in a general setting such existence results are frequently not considered in the literature (we are here thinking to the seminal work of [10]). As an application of the existence of strong solutions, we study the viscous approximation of hyperbolic corner problems and we show that the boundary layers localized along the two sides of the boundary do not interact the one with the other.

Key words: Hyperbolic system of PDE/Boundary value problem/Corner domain/Viscous approximation/Existence of strong solutions

Acknowledgements: This article is directly inspired from a question of Guy Métivier during the PhD defense of the author. This question was essentially "and what about strong solutions?". Some years pass, but the author is now pleased to give a first answer to this pertinent question.

Contents

1	Introduction	2
2	Main results	5
2.1	Some notation and assumptions	5
2.2	Simple construction of strong solutions to hyperbolic corner problems	7
2.2.1	Statement of the main results	7
2.2.2	Consequences	8
2.2.3	Variable coefficients in a nutshell	9
3	Proof of the main results	9
3.1	Proof of Theorem 2.1	9
3.1.1	Reformulation of the equations	9
3.1.2	The iterative scheme	11
3.1.3	Convergence analysis	14
3.2	Proof of Theorem 2.2	16
3.3	Modifications of the proofs for variable coefficients	17

4	Application to viscous perturbations	19
4.1	Half-space viscous perturbation in a nutshell and heuristic for the quarter space	19
4.2	Notation and assumptions	20
4.3	The proof in the quarter-space	20
4.3.1	The leading order term	22
4.3.2	Construction of the correctors	24
4.3.3	Error estimate	26
5	Discussions and examples	27
5.1	Discussion about the functional spaces of resolution	27
5.2	Discussion about Assumption 2.4	27
5.2.1	The wave equation	28
5.2.2	Shallow water equation	28
5.2.3	Linearisation of Euler equation	30

1 Introduction

This article follows [3] and aims to study the strong well-posedness of hyperbolic boundary value problems defined in domains with a corner. We will be particularly interested in strong solutions. We restrict our attention to the problem in the quarter-space. The considered problem in the following reads: for $d \geq 2$ and $T > 0$,

$$\begin{cases} L(\partial)u := \partial_t u + A_1 \partial_1 u + A_2 \partial_2 u + \sum_{j=3}^d A_j \partial_j u + Du = f & \text{for } (t, x) \in]-\infty, T] \times \mathbb{R}_+^2 \times \mathbb{R}^{d-2} := \Omega_T, \\ B_1 u|_{x_1=0} = g_1 & \text{on }]-\infty, T] \times \mathbb{R}_+ \times \mathbb{R}^{d-2} = \partial\Omega_{1,T}, \\ B_2 u|_{x_2=0} = g_2 & \text{on }]-\infty, T] \times \mathbb{R}_+ \times \mathbb{R}^{d-2} = \partial\Omega_{2,T}, \\ u_{t \leq 0} = 0 & \text{on } \mathbb{R}_+^2 \times \mathbb{R}^{d-2} := \Gamma, \end{cases} \quad (1)$$

where the unknown u has its values in \mathbb{R}^N , for some fixed $N \geq 1$ and where f, g_1, g_2 are given source terms in a suitable functional space. The interior coefficients are given matrices $A_j, D \in \mathbf{M}_{N \times N}(\mathbb{R})$, while the boundary matrices $B_1 \in \mathbf{M}_{p_1 \times N}(\mathbb{R})$, $B_2 \in \mathbf{M}_{p_2 \times N}(\mathbb{R})$ encode the good number of boundary conditions. Namely in the above, p_j stands for the number of (strictly) positive eigenvalues of A_j . We will also consider the variable coefficients case and the initial boundary value problem associated to (1) (that is to say the problem defined on $]0, T[\times \Gamma$, $T > 0$, with a non homogeneous initial condition).

Before to describe with more details the results of this article, let us point that even if the study of the well-posedness of problem (1) is a rather old question which appears in the 70's in the litterature with the works of Osher, Sarason and Smoller (we refer to [10] and [14]), and despite the fact that this problem encounters some renewed interest recently with the contributions of Huang-Temam [7], Halpern-Rauch [6], Métivier-Rauch [9] and the author (see [2] and [3]), then a little is known about such a well-posedness theory. Without enter into technical details, let us say that, at present time, a full characterization of the boundary matrices B_1 and B_2 ensuring the strong well-posedness of the problem in the basic hilbert space L^2 seems to be a hard problem and constitutes a widely open question. It does not have a lot of advances since the pioneer work of Osher [10]. However, in some particular cases (we are thinking to the natural subcase of symmetric operators with (strictly) dissipative boundary conditions) we have some (non optimal, but simple) results of well-posedness, see for example [6] or [3].

In [3], we studied the persistence of regularity of the solution to (1). Crudely speaking, we addressed the question "if the sources of the problem are in some (high order) Sobolev space H^s , then what about the solution u ?". We will not describe the results of [3] in all details, let us just indicate that, because corner problems look like characteristic (half-space) problems, we obtained a good persistence of regularity in some tangential Sobolev spaces. These spaces are defined by the differentiation operators $(x_1 \partial_1)^{\alpha_1}$ and $(x_2 \partial_2)^{\alpha_2}$, with $\alpha_1, \alpha_2 \geq 0$, instead of the classical derivatives $\partial_1^{\alpha_1}$ and $\partial_2^{\alpha_2}$. In particular, in such spaces, because of the vanishing weights, we do not have a precise control of the traces of the solution u at the boundary $\partial\Omega_{1,T} \cup \partial\Omega_{2,T}$.

It is fair to say that the proof exposed in [3] is rather long and technical. So one of our goal in the present article was to find some persistence of regularity result, but with a simpler proof. Moreover, it will be motivated below, we also aim to obtain a persistence of regularity result ensuring a good control of the traces of the solution at the boundary.

In the following to simplify the methods of proof, we first choose to restrict our attention to sources with infinite regularity, and we thus want to construct a strong solution u to (1). Let us here point, for future purposes, that because, at the end of the day, the solution $u \in H^\infty$, then the definitions of the traces $u|_{x_1=0}$, $u|_{x_2=0}$ and $u|_{x_1=x_2=0}$ are straightforward.

The main idea of the method of construction of such a strong solution is to look for some decomposition of the space $\mathbb{R}^N := \mathbf{E}_1 \oplus \mathbf{E}_2$, in such a way that if we decompose $u = u_1 + u_2$ with $u_j \in \mathbf{E}_j$, then u_j essentially solves the kind of "transport equation"

$$\partial_t u_j + \lambda_j \partial_j u_j + \tilde{A}_{3-j} \partial_{3-j} u_j + \sum_{k=3}^d \tilde{A}_{j,k} \partial_k u_j + \tilde{D} u_j = f_j - \mathbf{F}_j(\partial u_{3-j}) \text{ in }]-\infty, T] \times \mathbb{R}_+^2 \times \mathbb{R}^{d-2}, \quad (2)$$

where λ_j are some strictly **negative** real numbers, where ∂ denotes the collection of the ∂_j with $j \in \llbracket 1, d \rrbracket$ and finally where the $\tilde{A}_{j,k}$, \tilde{A}_{3-k} stand for some modifications of the coefficients A_k . Similarly, \tilde{D}_j stands for some modification of the zero order term D .

Let us here insist on the fact that the existence of the above decomposition ensuring property (2) does not come from the hyperbolicity assumption. It is a structural assumption made on the interior coefficients of (1). However, the above assumption is satisfied by many examples with physical interest (we refer to Section 5 for a study of some of them).

The keystone remark in the following is that the transport phenomenon in (2) is outgoing for the side $\partial \tilde{\Omega}_{j,T}$. So instead to solve (2) in the quarter-space, we can solve it on the half-space $\tilde{\Omega}_{2,T} :=]-\infty, T] \times \mathbb{R}_{x_1} \times \mathbb{R}_{+,x_2} \times \mathbb{R}_{x'}^{d-2}$ (or $\tilde{\Omega}_{1,T} :=]-\infty, T] \times \mathbb{R}_{+,x_1} \times \mathbb{R}_{x_2,x'}^{1+d-2}$). In particular in such a decomposition, because the transport equation is outgoing, then only one boundary condition needs to be considered. This leads us to solve the couple of transport equations

$$\begin{cases} \partial_t u_1 + \lambda_1 \partial_1 u_1 + \tilde{A}_2 \partial_2 u_1 + \sum_{k=3}^d \tilde{A}_{1,k} \partial_k u_1 = f_1 - \mathbf{F}_1(\partial u_2) & \text{in } \tilde{\Omega}_{2,T}, \\ \text{boundary conditions on } \partial \tilde{\Omega}_{2,T}, \\ u_{1|_{t \leq 0}} = 0, \end{cases} \quad (3)$$

and

$$\begin{cases} \partial_t u_2 + \lambda_2 \partial_2 u_2 + \tilde{A}_1 \partial_1 u_2 + \sum_{k=3}^d \tilde{A}_{2,k} \partial_k u_2 = f_2 - \mathbf{F}_2(\partial u_1) & \text{in } \tilde{\Omega}_{1,T}, \\ \text{boundary conditions on } \partial \tilde{\Omega}_{1,T}, \\ u_{2|_{t \leq 0}} = 0, \end{cases} \quad (4)$$

equations where we treat the coupling $\mathbf{F}_1(\partial u_2)$ and $\mathbf{F}_2(\partial u_1)$ as sources. We then solve (3) and (4) by an iterative scheme which at each step uploads the coupling source term by taking the value obtained at the previous iteration. Because such coupling terms depend on the derivatives of the solution, we loss one derivative at each step. Consequently to make sure that the iterative scheme is well-defined at each step, we use in a non trivial way that the sources are infinitely regular so that the obtained solutions are. Moreover, in order to show that the sequence converges we will also need to ask some "nice" behaviour for the derivatives of the sources. We refer to equations (7) for a precise definition.

Consequently as indicated before our result is not optimal at least in two different directions:

- It necessarily requires infinitely many derivatives on the sources. In particular, it is not a true persistence of regularity result where the sources are expected to be "only" in some H^s for fixed $s \in \mathbb{N}$.
- We can not obtain the whole generality of the space H^∞ because of the requirement imposed on the derivatives of the data. We however justify in Section 5 that with such a restriction we manage to

recover some interesting source terms such as the gaussian. More generically, to be in our goal subspace we essentially require some decrease at infinity but we are rather generic near the boundary. Let us remark here by the way that the application of the existence of strong solution, namely, the viscous perturbation approximation constitute in a study near the boundary. As a consequence the above functional subspace of resolution for the sources is pertinent for such a study.

In the future, to have a totally satisfactory persistence of regularity theory, we will improve the result of [3]. But, in the author's opinion, in spite of its restrictions, the result exposed here has the advantage to show that strong solutions to (1) can effectively be constructed. Moreover, the proof exposed here also has the advantage of simplicity.

Let us emphasize that the question of existence of solutions for the hyperbolic corner problem (1) is not well-understood in the literature. Indeed, the pioneer work of Osher [10] does not give any information about the existence of (weak) solution(s). In the author's knowledge the only existence results in the literature are the ones of [6] and [3]. However these results are restricted to strictly dissipative boundary conditions. Here we do not have such an assumption for existence. It will however be convenient to make it secondarily to recover the uniqueness property, because of the energy estimate (see Corollary 2.2).

To conclude this introduction let us also point that the existence of a strong solution has, in the author's opinion, a very interesting corollary. We can describe this solution as the limit of the solutions to a corner problem perturbed by a small viscosity coefficient, $0 < \varepsilon \ll 1$. More precisely, if we consider the viscous perturbation of (1) given by

$$\begin{cases} L(\partial)u^\varepsilon - \varepsilon\mathcal{E}u^\varepsilon = f & \text{in } \Omega_T, \\ u^\varepsilon|_{x_1=0} = 0 & \text{on } \partial\Omega_{1,T}, \\ u^\varepsilon|_{x_2=0} = 0 & \text{on } \partial\Omega_{2,T}, \\ u^\varepsilon|_{t \leq 0} = 0 & \text{on } \Gamma, \end{cases} \quad (5)$$

where \mathcal{E} is an elliptic operator, so that the system (5) can be seen as a more physically meaningful system of partial differential equation than (1) because it now takes in account viscosity. We will be able to describe, u the solution to (1) (with suitable boundary conditions) as the limit of u^ε when $\varepsilon \downarrow 0$.

This kind of result is not new. It has been deeply studied when the domain of resolution is the half-space (or equivalently if it admits a flat boundary), we refer for instance to the works of Guès [5] or Sueur [16] for such kind of geometric optics expansions. The above works deal with non-linear characteristic problems, so they cover a wider generality than the one we are considering here. Let us also mention the works of Rousset [12] and [11] where the perturbation is authorized to degenerate (question that will not be considered in the following).

To describe what has been our main motivation for studying (5), we should give some more details about the obtained results in the half-space geometry. We refer the interested reader to the articles of [1], [5] or [16]. The main point in the study of such problems is the simple and naive (but crucial) remark that the problem (5) is parabolic while its (expected) limit is hyperbolic. So that, something must come into play in order to correct the (totally prescribed) traces of u^ε into the (partially prescribed) traces of u . A boundary layer localized near the boundary is generated to correct the traces. More precisely, we can show that we have the asymptotic expansion¹:

$$u^\varepsilon(t, x) \sim u(t, x) + \mathcal{U}\left(t, x; \underbrace{\frac{x_d}{\varepsilon}}_{:=X_d}\right) + \text{h.o.t.},$$

where $u \in H^\infty(\Omega_T)$ is the solution to the hyperbolic problem in the half-space with suitable boundary conditions and where $\mathcal{U} \in \mathcal{S}_{X_d}(\mathbb{R}_+; H_{t,x}^\infty(\Omega_T))$ is a boundary layer localized in the strip $\{0 < x_d \lesssim \varepsilon\}$ so that it has fast decay with respect to x_d .

Going back to the geometry of the quarter space it is thus rather natural to postulate for a first approximation:

$$u^\varepsilon(t, x) \sim u(t, x) + \mathcal{U}\left(t, x; \frac{x_1}{\varepsilon}\right) + \mathcal{V}\left(t, x; \frac{x_2}{\varepsilon}\right), \quad (6)$$

¹To fix the ideas, we here work in the half-space indexed by $x_d > 0$.

that is to say that we expect the appearance of two layers, each one localized around one side of the boundary. The question in which we were interested in is: "do these two layers can interact the one with the other ? If yes, do we have to incorporate to (6) a new term $\mathscr{W}(t, x; \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})$ to take this interaction into account, and do this interaction produces some new interesting behaviours ?"

"Unfortunately" the answer of the above question will generically be no. As a consequence, we will show that the two layers do not interact the one with the other and thus that (6) stands for a good approximation of the solution to the corner problem (1). It is an interesting result in itself, but the fact that the two layers do not interact the one with the other is a little disappointing.

The reason is however rather simple. Indeed, because of the fast decays, the two layers can only interact near the corner in the area $\{0 < x_1, x_2 \lesssim \varepsilon\}$. However in this neighbourhood of the corner the layers are determined by the corner value $u|_{x_1=x_2=0}$, but from the boundary conditions and compatibility conditions such a double trace is in fact zero.

The paper is organized as follows: Section 2 describes the several assumptions made on problem (1) and states the main results of the article namely Theorem 2.1 for pure boundary value problems and Theorem 2.2 for initial boundary value problems. The proof of the main results occupies Section 3. This section is divided into three paragraphs. The first one, namely Paragraph 3.1, gives the proof of Theorem 2.1. Then Paragraph 3.2 gives the main ideas in the proof of Theorem 2.2. To facilitate the exposition these detailed proofs are given for constant coefficients. Then Paragraph 3.3 exposes the required modifications to deal with variable coefficients problems. Section 4 deals with the viscous perturbations of (1). The main result of the section establishing the existence of the expansion and its error analysis is Theorem 4.1. Finally, the last section, namely Section 5 discusses the functional spaces where we have shown the existence of a strong solution and gives some examples of physical interest satisfying the structural assumptions that we have done to construct strong solutions and to construct its vanishing viscosity expansion.

2 Main results

2.1 Some notation and assumptions

Let us start by the following generic notation. For $a, b \in \mathbb{R}$, we denote $\llbracket a, b \rrbracket := [a, b] \cap \mathbb{Z}$ and we use the shorthand notation $a \lesssim b$ for "there exists $C > 0$ such that $a \leq Cb$ ". Such a constant C is assumed to be independent of all the considered parameters. If such a dependency occurs and is meaningful we go back to the classical notation. For $x \in \mathbb{R}^d$ we will sometimes use the decomposition $x = (x_1, x_2, x')$ with $x' \in \mathbb{R}^{d-2}$.

For $T > 0$, when dealing with initial boundary value problems, we will use the notations

$$\underline{\Omega}_T := [0, T] \times \Gamma, \partial \underline{\Omega}_{1,T} \sim \partial \underline{\Omega}_{2,T} := [0, T] \times \mathbb{R}_+ \times \mathbb{R}^{d-2}.$$

For $X \subset \Omega_T$ and $s \in \mathbb{N}$, $H^s(X)$ stands for the usual Sobolev space of order s . Classically we will denote $H^\infty(X) := \bigcap_{s \in \mathbb{N}} H^s(X)$.

For fixed $s \in \mathbb{N}$, to state our existence result properly, we shall introduce the following subspaces of $H^s(\Omega_T)$:

$$\begin{aligned} \mathbf{W}_T^s &:= \{u \in \mathcal{C}^0([-\infty, T]; H^s(\Gamma)) \setminus \forall j \in \llbracket 0, s \rrbracket, \partial_t^j u \in \mathcal{C}^0([-\infty, T]; H^{s-j}(\Gamma))\} \text{ and} \\ \mathbf{W}_T^{s,b} &:= \{u \in \mathcal{C}^0([-\infty, T]; H^s(\mathbb{R}_+ \times \mathbb{R}^{d-2})) \setminus \forall j \in \llbracket 0, s \rrbracket, \partial_t^j u \in \mathcal{C}^0([-\infty, T]; H^{s-j}(\mathbb{R}_+ \times \mathbb{R}^{d-2}))\} \end{aligned}$$

equipped with the norms, for $t \leq T$

$$\|u(t)\|_s := \sum_{j=0}^s \|\partial_t^j u(t)\|_{H^{s-j}(\Gamma)} \text{ and } |u(t)|_s := \sum_{j=0}^s \|\partial_t^j u|_{\partial \Omega_{\cdot, T}}(t)\|_{H^{s-j}(\mathbb{R}_+ \times \mathbb{R}^{d-2})}$$

We define $\mathbf{W}_T^\infty := \bigcap_{s \in \mathbb{N}} \mathbf{W}_T^s$ (resp. $\mathbf{W}_T^{\infty, b} := \bigcap_{s \in \mathbb{N}} \mathbf{W}_T^{s, b}$) and we finally introduce $\mathbf{X}_{T, K}^\infty$ (resp. $\mathbf{X}_{T, K}^{\infty, b}$) as the

subspace of \mathbf{W}_T^∞ (resp. $\mathbf{W}_{T,K}^{\infty,b}$) of functions satisfying that

$$\begin{aligned} \mathbf{X}_{T,K}^\infty &:= \{u \in \mathbf{W}_T^\infty \setminus \exists K > 0, \forall s \in \mathbb{N}, \|u(t)\|_s \lesssim K^s s!\} \text{ and} \\ \mathbf{X}_{T,K}^{\infty,b} &:= \{u \in \mathbf{W}_T^\infty \setminus \exists K > 0, \forall s \in \mathbb{N}, |u(t)|_s \lesssim K^s s!\}. \end{aligned} \quad (7)$$

In the above there is no loss of generality by assuming that $K \geq 1$.

Because we are interested in regular solutions to (1), compatibility conditions on the sources f, g_1, g_2 are expected. Because the initial datum vanishes in (1), these sources must be flat at $\{t = 0\}$. Consequently, for some functional space F we denote by F_{\flat} the subspace of F formed by functions $f \in F$ satisfying the flatness condition for all $j \in \mathbb{N}$, $(\partial_t^j f)|_{t \leq 0} = 0$.

The two following assumptions² are classical in the study of hyperbolic boundary value problems.

Assumption 2.1 (Constantly hyperbolic operator) *The operator $L(\partial)$ is assumed to be hyperbolic with constant multiplicity. More precisely, there exist an integer M , homogeneous of degree one with respect to ξ , analytic functions $\lambda_1, \dots, \lambda_M$ on $\Omega_T \times \mathbb{R}^d \setminus \{0\}$ and integers μ_1, \dots, μ_M such that*

$$\forall (t, x) \in \Omega_T, \forall \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \setminus \{0\}, \det \left(\tau I + \sum_{j=1}^d \xi_j A_j(t, x) \right) = \prod_{k=1}^M (\tau - \lambda_k(t, x, \xi))^{\mu_k}.$$

The eigenvalues $\lambda_1, \dots, \lambda_M$ are semi-simple and they satisfy $\lambda_1(\cdot, \xi) < \dots < \lambda_M(\cdot, \xi)$, for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

For simplicity, we assume that the two sides of the boundary are uniformly not characteristic. More precisely, we assume that

Assumption 2.2 (Non characteristic boundary conditions) *For all $(t, x) \in \Omega_T$ the matrices $A_1(t, x)$ and $A_2(t, x)$ corresponding to normal coefficient for the sides $\partial\Omega_1$ and $\partial\Omega_2$ respectively are invertible. Moreover, we assume that p_1 (resp. p_2), the number of positive eigenvalues of A_1 (resp. A_2) does not depend on (t, x) .*

With these assumptions in hand, we have the following decomposition of the space \mathbb{R}^N :

$$\forall (t, x) \in \Omega, \mathbb{R}^N = \mathbf{E}_1^s(t, x) \oplus \mathbf{E}_1^u(t, x) = \mathbf{E}_2^s(t, x) \oplus \mathbf{E}_2^u(t, x),$$

where for $j \in \llbracket 1, 2 \rrbracket$, $\mathbf{E}_j^s(\cdot)$ (resp. $\mathbf{E}_j^u(\cdot)$) stands for the eigenspace associated to negative (resp. positive) eigenvalues of A_j . In the following, we will frequently write $\mathbf{E}_j^{s,u}$ instead of $\mathbf{E}_j^{s,u}(t, x)$.

The following assumption is a reinforcement of the one of Sarason [13]. It has also been used in [3] in the study of persistence of regularity of the solution. We do not know if this assumption is necessary or not, for the existence of a strong solution. But as we will see, it has the advantage to simplify some points of the proof. Let us point here that the counter-example to uniqueness of [9] seems to indicate that a pathological behavior of Sarason's matrix $A_1^{-1}A_2$ implies a pathological behavior of the solution.

Assumption 2.3 *For all $(t, x) \in \Omega_T$, the real eigenvalues of $(A_1^{-1}A_2)(t, x)$ (if they exist) are negative³.*

Our main assumption in this article is the following. It is a structural assumption ensuring that the previous eigenspaces are compatible the one with the other. More precisely:

Assumption 2.4 *We assume that the two following points are satisfied:*

²Our main results are stated for constant coefficients problems, see Theorems 2.1 and 2.2. However, because it can be extended to variable coefficients problems, see Theorem 2.3, we prefer to state the assumptions in the variable framework only in order to save some material.

³In [13], Sarason only requires this sign property for the eigenvalues associated to Jordan blocks of $A_1^{-1}A_2$.

1. For all $(t, x) \in \Omega_T$, we have the decomposition

$$\mathbb{R}^N = \mathbf{E}_1^s(t, x) \oplus \mathbf{E}_2^s(t, x) = \oplus_{\lambda_k < 0} \ker(A_1 - \lambda_k I)(t, x) \oplus_{\lambda_k < 0} \ker(A_2 - \lambda_k I)(t, x). \quad (8)$$

In particular, it implies that the number of lines in the boundary conditions satisfy $N = p_1 + p_2$. We define $P_1^k := P_1^k(t, x)$ (resp. $P_2^k = P_2^k(t, x)$) the projection on $\ker(A_1 - \lambda_k I)$ (resp. $\ker(A_2 - \lambda_k I)$) with respect to decomposition (8).

2. Let $j \in \llbracket 1, 2 \rrbracket$, then for all $(t, x_{3-j}, x') \in \partial\Omega_{j,T}$ we have that

$$\ker B_j(t, x', x_{3-j}) \cap \mathbf{E}_{3-j}^{s,b}(t, x_{3-j}, x') = \{0\}, \quad (9)$$

where $\mathbf{E}_{3-j}^{s,b} := \mathbf{E}_{3-j|_{x_j=0}}^s$.

Let us point here that when $N = 2$, then the first statement of Assumption 2.4 is a consequence of Assumption 2.3. Indeed, in such a setting, in order to have a "true" boundary value problem, we should assume that $p_1 = p_2 = 1$. So we just have to justify that $\mathbf{E}_1^s \cap \mathbf{E}_2^s = \{0\}$. By contradiction, let $u \neq 0$ be in $\mathbf{E}_1^s \cap \mathbf{E}_2^s$. Then, on the one hand, we have $A_1 u = \lambda_1 u$, with $\lambda_1 < 0$ and, on the other hand, we have $A_2 u = \lambda_2 u$, with $\lambda_2 < 0$. Consequently, we have $A_1^{-1} A_2 u = \frac{\lambda_2}{\lambda_1} u$ which contradicts Assumption 2.3.

In the general framework, we do not know if the first statement of Assumption 2.4 can be deduce from Sarason's Assumption 2.3.

To conclude this preliminary paragraph, we recall the following terminology for boundary conditions.

Definition 2.1 (Strictly dissipative boundary conditions) *Let $j \in \llbracket 1, 2 \rrbracket$, we say that the boundary condition B_j is strictly dissipative if for all $(t, x_{3-j}, x') \in \partial\Omega_{j,T}$, we have the inequality*

$$\forall v \in \ker B_j(t, x_{3-j}, x'), \langle A_{j|_{x_j=0}}(t, x_{3-j}, x')v, v \rangle < 0,$$

and if $\ker B_j$ is maximal for the above property. We here mean that $\ker B_j$ is not properly included in any subspace having the above property.

Strictly dissipative boundary conditions have the advantage to be easy to check algebraically. They constitute a particular class of boundary conditions for which the well-posedness of the corner problem (1) can be established (see [6] and [3]). It is known that in the half-space geometry, except when $N = 2$ (see [15]), they do not constitute the most generic possible boundary conditions leading to strong well-posedness. From the result of [8] in the half-space, the most generic conditions in such a geometry are characterized by the so-called uniform Kreiss-Lopatinskii condition.

In the quarter space, the work of Osher [10] indicates that imposing the uniform Kreiss-Lopatinskii condition on each side is not sufficient for strong well-posedness. In [10] a new necessary invertibility condition is identified. It is, however, still not so clear that with this extra invertibility condition in hand, we can show an *a priori* energy estimate without losses of derivatives. Moreover the result of [10] does not give any result about the existence of L^2 solution. Here, by constructing a strong solution we try to fill this lacuna (at least partially).

2.2 Simple construction of strong solutions to hyperbolic corner problems

The main results of this article state the existence of regular strong solutions to the hyperbolic corner problem reading under the form (1) or to its associated initial boundary value problem. They occupy the following paragraph.

2.2.1 Statement of the main results

Let us first consider the boundary value problem, for $T > 0$

$$\begin{cases} L(\partial)u = f & \text{in } \Omega_T, \\ B_1 u|_{x_1=0} = g_1 & \text{on } \partial\Omega_{1,T}, \\ B_2 u|_{x_2=0} = g_2 & \text{on } \partial\Omega_{2,T}, \\ u|_{t \leq 0} = 0 & \text{on } \Gamma, \end{cases} \quad (10)$$

then we have the following result:

Theorem 2.1 *Assume that the coefficients of (10) are constant and satisfy Assumptions 2.1, 2.2, 2.3 and 2.4. Assume moreover that there exist $K \geq 1$ and $T > 0$ such that the sources $f \in \mathbf{X}_{K,T,\natural}^\infty$ and $(g_1, g_2) \in (\mathbf{X}_{K,T,\natural}^{\infty,b})^2$. Then for K' large enough and $0 < T' \leq T$ small enough, there exists $u \in \mathbf{X}_{K',T',\natural}^\infty$ solution to (10).*

So that, if the sources live in the spaces $\mathbf{X}_{T,K,\natural}^\infty$ and $\mathbf{X}_{T,K,\natural}^{\infty,b}$ that is to say that we have a control on the growth of the derivatives, then we have the existence of a regular strong solution and we also have the same kind of control on the derivatives of the solution.

Let us stress that the above result holds locally in time. The maximal time of resolution depends on the coefficients of $L(\partial)$ and on the control parameter K . Such a maximal time of resolution already appears for any operator $L(\partial)$ with $D \neq 0$, so that we recover the same behaviour of the solutions. The maximal time of resolution can be decreasing compared to the problem in the half-space, but the behaviour of the solution is qualitatively the same.

Our second result deals with the initial boundary value problem. By linearity there is no loss of generality by assuming that $f, g_1, g_2 \equiv 0$. We thus consider for $T > 0$:

$$\begin{cases} L(\partial)u = 0 & \text{in } \underline{\Omega}_T, \\ B_1 u|_{x_1=0} = 0 & \text{on } \partial\underline{\Omega}_{1,T}, \\ B_2 u|_{x_2=0} = 0 & \text{on } \partial\underline{\Omega}_{2,T}, \\ u|_{t=0} = u_0 & \text{on } \Gamma. \end{cases} \quad (11)$$

We introduce the following functional space for the initial datum:

$$\mathbf{Y}_K^\infty := \{u \in H^\infty(\Gamma) \mid \exists K \geq 1, \forall s \in \mathbb{N}, \|u\|_{H^s(\Gamma)} \lesssim K^s s!\}.$$

Then the result is the following:

Theorem 2.2 *Under the assumptions of Theorem 2.1, let the initial datum $u_0 \in \mathbf{Y}_K^\infty$ satisfies the compatibility conditions (31) at any order, then there exists $K' > 0$ large enough and $0 < T' \leq T$ small enough such that $u \in \mathbf{X}_{K',T'}^\infty$ is a solution to (11).*

We would like to insist on the fact that in Theorems 2.1 and 2.2, we do not made any assumptions on the type of the boundary conditions. We do not require the boundary conditions to be dissipative or to satisfy the uniform Kreiss-Lopatinskii condition. It is also the case in the half-space geometry where the construction of a solution can be made if one has some dual energy estimate. Such a dual energy estimate is of course a consequence of a good choice for the (primal) boundary condition. But it could also be stated independently of the boundary condition. Here the only point which is required about these boundary conditions is that they satisfy the second point of the structure Assumption 2.4.

2.2.2 Consequences

If we ask a little more about the boundary conditions, more precisely that they gives rise to a well-posed problem having an *a priori* energy estimate in L^2 , then the constructed strong solution of Theorems 2.1 and 2.2 becomes necessarily unique. We thus have the following corollary

Corollary 2.1 *Under assumptions of Theorem 2.1, if the boundary value problem (10) (resp. (11)) comes with the *a priori* energy estimate, there exists $0 < T' \leq T$ such that for all $0 < t \leq T'$,*

$$\begin{aligned} & \|u(t, \cdot)\|_{L^2(\Gamma)}^2 + \|u|_{x_1=0}\|_{L^2(\partial\Omega_{1,t})}^2 + \|u|_{x_2=0}\|_{L^2(\partial\Omega_{2,t})}^2 \\ & \lesssim \int_0^t \|f(s, \cdot)\|_{L^2(\Gamma)}^2 ds + \|g_1\|_{L^2(\partial\Omega_{1,t})}^2 + \|g_2\|_{L^2(\partial\Omega_{2,t})}^2 \\ & \text{(resp. the l.h.s of (12) } \lesssim \|u_0\|_{L^2(\Gamma)} \text{)}, \end{aligned} \quad (12)$$

then the strong solution to (10) (resp. (11)) given by Theorem 2.1 (resp. 2.2) is unique.

A complete characterization of the boundary conditions leading to L^2 -well-posedness is, in the author's knowledge, a widely open question. However, in the particular framework of symmetric operators with strictly dissipative boundary conditions, then such an estimate can be shown. We refer for instance to [2] or [3]. As a consequence, in this particular interesting setting we can show that

Corollary 2.2 *Under the assumptions of Theorem 2.1, if we also assume that the coefficients of $L(\partial)$ are symmetric and that the boundary conditions are strictly dissipatives in the sense of Definition 2.1, then from [3] the problem is L^2 -well-posed. Consequently, the strong solution to (10) (resp. (11)) given by Theorem 2.1 (resp. Theorem 2.2) is unique.*

2.2.3 Variable coefficients in a nutshell

To conclude our statement of the results of this article we give some extensions to variable coefficients boundary value problems or initial boundary value problems. Classically for $s \in \mathbb{N}$, we consider $W^{s,\infty}(X)$ the usual Sobolev space of functions which have bounded derivatives up to the order s , and we denote $W^{\infty,\infty}(X) := \bigcap_{s \in \mathbb{N}} W^{s,\infty}(X)$. Then the result is the following:

Theorem 2.3 *Assume that the coefficients of the problem (10) (resp. (11)), namely A_j , D , B_1 and B_2 admit the regularity*

$$A_j \in \mathcal{C}(\Omega_T) \cap W^{\infty,\infty}(\Omega_T) \text{ (resp. } A_j \in \mathcal{C}(\underline{\Omega}_T) \cap W^{\infty,\infty}(\underline{\Omega}_T)), D \in W^{\infty,\infty}(\Omega_T) \text{ (resp. } W^{\infty,\infty}(\underline{\Omega}_T))$$

$$\text{and for } k = 1, 2, B_k \in W^{\infty,\infty}(\partial\Omega_{k,T}) \text{ (resp. } W^{\infty,\infty}(\partial\underline{\Omega}_{k,T})).$$

To simplify the following⁴, we assume that there exists a universal constant $C > 0$ such that

$$\forall j \in \llbracket 1, d \rrbracket, \forall \alpha \in \mathbb{N}^{1+d}, \|\partial^\alpha A_j\|_{L^\infty(X)}, \|\partial^\alpha D\|_{L^\infty(X)} \leq C \text{ and for } k \in \{1, 2\}, \forall \beta_k \in \mathbb{N}^{1+d-1}, \|\partial^{\beta_k} B_k\|_{L^\infty(Y_k)}$$

where $X = \Omega_T$ or $X = \underline{\Omega}_T$, $Y_k = \partial\Omega_{k,T}$ or $\partial\underline{\Omega}_{k,T}$ and where the derivatives are defined by $\partial^\alpha := \partial_t^{\alpha_0} \dots \partial_d^{\alpha_d}$, $\partial^{\beta_1} := \partial_t^{\beta_1,0} \partial_2^{\beta_1,2} \dots \partial_d^{\beta_1,d}$ and $\partial^{\beta_2} := \partial_t^{\beta_2,0} \partial_1^{\beta_2,1} \partial_3^{\beta_2,3} \dots \partial_d^{\beta_2,d}$. Finally, we assume that the coefficients satisfy the assumptions of Theorem 2.1 (resp. Theorem 2.2). Then the conclusion of Theorem 2.1 (resp. Theorem 2.2) still holds.

3 Proof of the main results

This section gives a detailed proof of Theorem 2.1 (see Paragraph 3.1). The proof of Theorem 2.2 follows essentially the same lines, or it can be deduced from Theorem 2.1 and the use of some Borel extension lemma. We refer to Paragraph 3.2 for more details. Finally Paragraph 3.3 contains the required modifications to apply the previously described methods in the variable coefficients framework.

3.1 Proof of Theorem 2.1

The proof of Theorem 2.1 is made in three steps, we reformulate the problem in terms of outgoing transport equations by using Assumption 2.4, see Paragraph 3.1.1. We then treat the coupling between these transport phenomena by an iterative scheme. The well-posedness of such a scheme is studied in Paragraph 3.1.2, while its convergence is investigated in Paragraph 3.1.3.

3.1.1 Reformulation of the equations

First let us stress that in this paragraph the coefficients A_j , $j \in \llbracket 1, d \rrbracket$, D and B_1, B_2 are assumed to be constant. We recall that from Assumption 2.4 we can decompose any $u \in \mathbb{R}^N$ as

$$u = \sum_{k=1}^{M_1} P_1^k u + \sum_{\ell=1}^{M_2} P_2^\ell u \tag{13}$$

$$\text{where } \forall k \in \llbracket 1, M_1 \rrbracket, A_1 P_1^k u = \lambda_{1,k} P_1^k u \text{ and } \forall \ell \in \llbracket 1, M_2 \rrbracket, A_2 P_2^\ell u = \lambda_{2,\ell} P_2^\ell u, \tag{14}$$

⁴We have good reasons to believe that such a uniform bound for the derivatives can be weakened in estimates reading $\cdot \leq C_s$, with $s = |\alpha|$. The main obstructions in the present proofs are however pointed in the core of the proofs.

with $\lambda_{1,\cdot}, \lambda_{2,\cdot} < 0$.

We simply denote for $k \in \llbracket 1, M_1 \rrbracket$ and $\ell \in \llbracket 1, M_2 \rrbracket$, $u_{1,k} := P_1^k u$ and $u_{2,\ell} := P_2^\ell u$. We also introduce for given k and ℓ

$$\bar{u}_{1,k}^* := \sum_{k' \neq k} u_{1,k'}, \bar{u}_{2,\ell}^* := \sum_{\ell' \neq \ell} u_{2,\ell'} \text{ and } \bar{u}_1 := \sum_{k=1}^{M_1} u_{1,k}, \bar{u}_2 := \sum_{\ell=1}^{M_2} u_{2,\ell}.$$

If we use the decomposition (13) in the interior equation of (1), then we obtain the equations

$$\partial_t(\bar{u}_1 + \bar{u}_2) + \sum_{k=1}^{M_1} \lambda_{1,k} \partial_1 u_{1,k} + \sum_{\ell=1}^{M_2} \lambda_{2,\ell} \partial_2 u_{2,\ell} + A_1 \partial_1 \bar{u}_2 + A_2 \partial_2 \bar{u}_1 + \sum_{j=3}^d A_j \partial_j (\bar{u}_1 + \bar{u}_2) + D(\bar{u}_1 + \bar{u}_2) = f.$$

We apply P_1^k (resp. P_2^ℓ) for all $k \in \llbracket 1, M_1 \rrbracket$ (resp. $\ell \in \llbracket 1, M_2 \rrbracket$) to obtain the equations

$$\begin{aligned} \forall k \in \llbracket 1, M_1 \rrbracket, \partial_t u_{1,k} + \lambda_{1,k} \partial_1 u_{1,k} + P_1^k A_2 \partial_2 u_{1,k} + P_1^k \sum_{j=3}^d A_j \partial_j u_{1,k} + P_1^k D u_{1,k} \\ = f_{1,k} - P_1^k A_1 \partial_1 \bar{u}_2 - P_1^k A_2 \partial_2 \bar{u}_{1,k}^* - \sum_{j=3}^d P_1^k A_j \partial_j (\bar{u}_2 + \bar{u}_{1,k}^*) - P_1^k D(\bar{u}_{1,k}^* + \bar{u}_2) \end{aligned}$$

and

$$\begin{aligned} \forall \ell \in \llbracket 1, M_2 \rrbracket, \partial_t u_{2,\ell} + \lambda_{2,\ell} \partial_2 u_{2,\ell} + P_2^\ell A_1 \partial_1 u_{2,\ell} + P_2^\ell \sum_{j=3}^d A_j \partial_j u_{2,\ell} + P_2^\ell D u_{2,\ell} \\ = f_{2,\ell} - P_2^\ell A_1 \partial_2 \bar{u}_1 - P_2^\ell A_1 \partial_1 \bar{u}_{2,\ell}^* - \sum_{j=3}^d P_2^\ell A_j \partial_j (\bar{u}_1 + \bar{u}_{2,\ell}^*) - P_2^\ell D(\bar{u}_{2,\ell}^* + \bar{u}_1) \end{aligned}$$

where we defined $f_{1,k} := P_1^k f$ and $f_{2,\ell} := P_2^\ell f$.

In the following we will thus consider equations reading under the form (we write $u_k = u_{1,k}$ and $u_\ell = u_{2,\ell}$ to save some notation)

$$\begin{aligned} \partial_t u_k + \lambda_{1,k} \partial_1 u_k + \tilde{A}_{2,k} \partial_2 u_k + \sum_{j=3}^d \tilde{A}_{j,k} \partial_j u_k + D_k u_k & \quad (15) \\ = f_{1,k} - F_k(\partial_1 \bar{u}_2) - G_k(\partial_2 \bar{u}_{1,k}^*) - H_k(\partial_j (\bar{u}_2 + \bar{u}_{1,k}^*)) - D_k(\bar{u}_{1,k}^* + \bar{u}_2) & \quad \text{in } \Omega_T, \\ \partial_t u_\ell + \lambda_{2,\ell} \partial_2 u_\ell + \tilde{A}_{1,\ell} \partial_1 u_\ell + \sum_{j=3}^d \tilde{A}_{j,\ell} \partial_j u_\ell + D_\ell u_\ell & \\ = f_{2,\ell} - F_\ell(\partial_2 \bar{u}_1) - G_\ell(\partial_1 \bar{u}_{2,\ell}^*) - H_\ell(\partial_j (\bar{u}_1 + \bar{u}_{2,\ell}^*)) - D_\ell(\bar{u}_{2,\ell}^* + \bar{u}_1) & \quad \text{in } \Omega_T, \end{aligned}$$

where we defined $\tilde{A}_{j,\cdot} = P A_j P$ and $D_\cdot = P D$ and where the terms in the right-hand sides can be made explicit. Note that in the right-hand side we make the slight abuse of notation ∂_j for the collection of the ∂_j for $j \in \llbracket 1, d \rrbracket$. In the following we will just have to keep in mind that they depend linearly on their variables. We thus obtain transport equations where the transport are coupled the one to the others *via* the terms in the right-hand sides of (15).

Then we make the simple (but crucial) remark that because we have $\lambda_{1,k} < 0$ (resp. $\lambda_{2,\ell} < 0$), the transport in the first (resp. second) equation of (15) is outgoing for the side $\partial\Omega_{1,T}$ (resp. $\partial\Omega_{2,T}$). As a matter of fact, we can then solve the first (resp. second) equation for $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+$ (resp. $\mathbb{R}_+ \times \mathbb{R}$) instead of \mathbb{R}_+^2 . Moreover no boundary condition on $\partial\Omega_{1,T}$ (resp. $\partial\Omega_{2,T}$) is required because of the fact that the transport is outgoing.

Consequently, we now have to solve two boundary value problems in the upper and right half-spaces. Let us have a look to the boundary conditions, we have using the decomposition (13):

$$\begin{cases} B_1 \bar{u}_2|_{x_1=0} = g_1 - B_1 \bar{u}_1|_{x_1=0} & \text{on } \partial\Omega_{1,T}, \\ B_2 \bar{u}_1|_{x_2=0} = g_2 - B_2 \bar{u}_2|_{x_2=0} & \text{on } \partial\Omega_{2,T}, \end{cases} \quad (16)$$

and in order to solve these conditions we shall be more precise about the eigenvalues of the coefficients $\tilde{A}_{1,\ell}$ and $\tilde{A}_{2,k}$. Firstly, let us point that from [[4]-Theorem 1.6] the subsystems (15) inherit the hyperbolicity of the one of $L(\partial)$. In particular, the eigenvalues of $\tilde{A}_{1,\ell}$ and $\tilde{A}_{2,k}$ are real. The following lemma precises the sign of the eigenvalues:

Lemma 3.1 *Under Assumption 2.3, for any $k \in \llbracket 1, M_1 \rrbracket$ and any $\ell \in \llbracket 1, M_2 \rrbracket$, let $\mu \in \mathbb{R}$ be such that there exists $v \in \mathbb{R}^N \setminus \{0\}$ satisfying $\tilde{A}_{1,\ell} v = \mu v$ or $\tilde{A}_{2,k} v = \mu v$, then $\mu > 0$.*

Proof of Lemma 3.1 We show the result for $\tilde{A}_{1,\ell}$. The proof being analogous for $\tilde{A}_{2,k}$. Let v be as specified. By definition of $\tilde{A}_{1,\ell}$ we have $P_2^\ell A_1 P_2^\ell v = \mu v$. We apply P_2^ℓ on the left-hand side and we use that fact that $\text{Ran } P_2^\ell = \ker(A_2 - \lambda_{2,\ell} I)$. Consequently, we have $P_2^\ell (A_1 - \frac{\mu}{\lambda_{2,\ell}} A_2) P_2^\ell v = 0$. As a consequence, $\frac{\mu}{\lambda_{2,\ell}}$ is a (real) eigenvalue of $A_2^{-1} A_1$, but $\lambda_{2,\ell} < 0$, so that from Assumption 2.3, we necessarily have $\mu > 0$. \square

Consequently, all the eigenvalues of the normal coefficients in the systems (15) are positive, we thus have to prescribe all the components of the trace of $u_{1,k}$ (resp. $u_{2,\ell}$) on $\partial\Omega_{2,T}$ (resp. $\partial\Omega_{1,T}$). But because of the second statement of Assumption 2.4, we have $\ker B_1 \cap \mathbf{E}_2^s = \{0\}$, so that we can invert the matrix B_1 in the first equation of (16). We proceed similarly for the second equation and we thus obtain the couple of boundary conditions:

$$\begin{cases} u_{1,k}|_{x_2=0} = P_1^k \phi_2 \left(g_2 - B_2 \bar{u}_2|_{x_2=0} \right) := g_{2,k} - \mathbf{g}_{2,k}(\bar{u}_2|_{x_2=0}) & \text{on } \partial\Omega_{2,T}, \\ u_{2,\ell}|_{x_1=0} = P_2^\ell \phi_1 \left(g_1 - B_1 \bar{u}_1|_{x_1=0} \right) := g_{1,\ell} - \mathbf{g}_{1,\ell}(\bar{u}_1|_{x_1=0}) & \text{on } \partial\Omega_{1,T}, \end{cases}$$

where $\phi_j := B_{j|\mathbf{E}_{3-j}^s}^{-1}$.

So that, finally we have to solve the couple of systems

$$\begin{cases} \partial_t u_k + \lambda_{1,k} \partial_1 u_k + \tilde{A}_{2,k} \partial_2 u_k + \sum_{j=3}^d \tilde{A}_{j,k} \partial_j u_k + D_k u_k = f_{1,k} - \mathbf{F}_k(u_k^*) & \text{for } (x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+, \\ u_k|_{x_2=0} = g_{2,k} - \mathbf{g}_{2,k}(\bar{u}_2|_{x_2=0}), \\ u_k|_{t \leq 0} = 0, \end{cases} \quad (17)$$

and

$$\begin{cases} \partial_t u_\ell + \lambda_{2,\ell} \partial_2 u_\ell + \tilde{A}_{1,\ell} \partial_1 u_\ell + \sum_{j=3}^d \tilde{A}_{j,\ell} \partial_j u_\ell + D_\ell u_\ell = f_{2,\ell} - \mathbf{F}_\ell(u_\ell^*) & \text{for } (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}, \\ u_\ell|_{x_1=0} = g_{1,\ell} - \mathbf{g}_{1,\ell}(\bar{u}_1|_{x_1=0}), \\ u_\ell|_{t \leq 0} = 0, \end{cases} \quad (18)$$

where u^* stands for the collection of the $(u_{1,a}, u_{2,b})$ with $a, b \neq \cdot$, and where we introduced

$$\begin{aligned} \mathbf{F}_k(\bar{u}_k^*) &:= F_k(\partial_1 \bar{u}_2) - G_k(\partial_2 \bar{u}_{1,k}^*) - H_k(\partial_j(\bar{u}_2 + \bar{u}_{1,k}^*)) - D_k(\bar{u}_2 + \bar{u}_{1,k}^*), \\ \mathbf{F}_\ell(\bar{u}_\ell^*) &:= F_\ell(\partial_2 \bar{u}_1) - G_\ell(\partial_1 \bar{u}_{2,\ell}^*) - H_\ell(\partial_j(\bar{u}_1 + \bar{u}_{2,\ell}^*)) - D_\ell(\bar{u}_1 + \bar{u}_{2,\ell}^*). \end{aligned}$$

3.1.2 The iterative scheme

In order to solve systems (17) and (18) we treat the coupling in the right-hand sides inductively. We thus consider $(u^n)_{n \in \mathbb{N}} := (u_k^n, u_\ell^n)_{n \in \mathbb{N}}$, the sequence of functions defined by:

$$\begin{cases} \partial u_k^0 + \lambda_{1,k} \partial_1 u_k^0 + \tilde{A}_{2,k} \partial_2 u_k^0 + \sum_{j=3}^d \tilde{A}_{j,k} \partial_j u_k^0 + D_k u_k^0 = f_{1,k} & \text{in } \tilde{\Omega}_{2,T}, \\ u_k^0|_{x_2=0} = g_{2,k} & \text{for } x_2 = 0, \\ u_k^0|_{t \leq 0} = 0 & \text{for } t \leq 0, \end{cases} \quad (19)$$

and

$$\begin{cases} \partial_t u_\ell^0 + \lambda_{2,\ell} \partial_2 u_\ell^0 + \tilde{A}_{1,\ell} \partial_1 u_\ell^0 + \sum_{j=3}^d \tilde{A}_{j,\ell} \partial_j u_\ell^0 + D_\ell u_\ell^0 = f_{2,\ell} & \text{in } \tilde{\Omega}_{1,T}, \\ u_{\ell|_{x_1=0}}^0 = g_{1,\ell} & \text{for } x_1 = 0, \\ u_{\ell|_{t \leq 0}}^0 = 0 & \text{for } t \leq 0, \end{cases} \quad (20)$$

and then for $n \in \mathbb{N}$, we define $(u^{n+1})_{n \in \mathbb{N}} := (u_k^{n+1}, u_\ell^{n+1})_{n \in \mathbb{N}}$ as the solutions to the problems

$$\begin{cases} \partial_t u_k^{n+1} + \lambda_{1,k} \partial_1 u_k^{n+1} + \tilde{A}_{2,k} \partial_2 u_k^{n+1} + \sum_{j=3}^d \tilde{A}_{j,k} \partial_j u_k^{n+1} = f_{1,k} - \mathbf{F}_k(\bar{u}_k^{n,*}) & \text{in } \tilde{\Omega}_{2,T}, \\ u_{k|_{x_2=0}}^{n+1} = g_{2,k} - \mathbf{g}_{2,k}(\bar{u}_{2|x_2=0}^n) & \text{on } x_2 = 0, \\ u_{k|_{t \leq 0}}^{n+1} = 0 & \text{for } t \leq 0, \end{cases} \quad (21)$$

and

$$\begin{cases} \partial_t u_\ell^{n+1} + \lambda_{2,\ell} \partial_2 u_\ell^{n+1} + \tilde{A}_{1,\ell} \partial_1 u_\ell^{n+1} + \sum_{j=3}^d \tilde{A}_{j,\ell} \partial_j u_\ell^{n+1} = f_{2,\ell} - \mathbf{F}_\ell(\bar{u}_\ell^{n,*}) & \text{in } \tilde{\Omega}_{1,T}, \\ u_{\ell|_{x_1=0}}^{n+1} = g_{1,\ell} - \mathbf{g}_{1,\ell}(\bar{u}_{1|x_1=0}^n) & \text{on } x_1 = 0, \\ u_{\ell|_{t \leq 0}}^{n+1} = 0 & \text{for } t \leq 0. \end{cases} \quad (22)$$

The following lemma gives the necessary well-posedness result and the energy estimates for the problems (21). The result for the problem (22) being analogous.

Lemma 3.2 *Let $s \in \mathbb{N}$, $T > 0$ and $\lambda < 0$ be given, let two sources $f \in \mathbf{W}_{T,\natural}^s$ and $g \in \mathbf{W}_{T,\natural}^{s,b}$ then the boundary value problem:*

$$\begin{cases} \mathbf{L}u := \partial_t u + \lambda \partial_1 u + \tilde{A}_2 \partial_2 u + \sum_{j=3}^d \tilde{A}_j \partial_j u + Du = f & \text{in } \tilde{\Omega}_{2,T}, \\ u_{|_{x_2=0}} = g & \text{on } \partial \tilde{\Omega}_{2,T}, \\ u_{|_{t \leq 0}} = 0 & \text{on } \mathbb{R}^{1+d-2}, \end{cases} \quad (23)$$

admits a unique solution $u \in \mathbf{W}_{T,\natural}^s$ satisfying the energy estimate: there exist $0 < T' := T'(D) \leq T$ such that for all $0 < t \leq T'$ we have:

$$\|u(t)\|_s + |u_{|_{x_1=0}}(t)|_s \lesssim \int_0^t \|f(t')\|_s dt' + \|g\|_{H^s(\partial \tilde{\Omega}_{2,t})}. \quad (24)$$

In fact, to have a more convenient energy estimate for our convergence analysis, it will be useful to modify a little the energy estimate (24) in order to express the right-hand side as an integral term. In order to do so, we remark that we can bound the term $\|g\|_{H^s(\partial \tilde{\Omega}_{2,t})}$ by $|g(t)|_s$. Then we write

$$|g(t)| \leq \underbrace{|g(0)|_s}_{=0} + \int_0^t |\partial_t g(t')|_s dt' \leq \int_0^t |g(t')|_{s+1} dt',$$

if we restrict our attention to sources in $\mathbf{W}_{T,\natural}^{s,b}$. As a consequence, we pay a derivative on the boundary in order to recover the integral factor. However as we will see, because of the definitions (21) and (22) such a loss in the interior is automatic. The advantage to have an integral term in the right-hand side of (24) will be clarified in the convergence analysis. We end up with the following well-posedness result:

Corollary 3.1 *Let $s \in \mathbb{N}$ and $T > 0$ be given, let two sources $f \in \mathbf{W}_{T,\natural}^s$ and $g \in \mathbf{W}_{T,\natural}^{s+1,b}$ then the boundary value problem (23) admits a unique solution $u \in \mathbf{W}_T^s$ satisfying the energy estimate: there exists $0 < T_0 < T$ such that for all $0 < t \leq T_0$ we have:*

$$\|u(t)\|_s + |u_{|_{x_1=0}}(t)|_s \lesssim \int_0^t \|f(t')\|_s + |g(t')|_{s+1} dt'. \quad (25)$$

Proof of Lemma 3.2 We just give here the main lines in the proof of the *a priori* energy estimate (24). The construction of a solution can be made classically by duality (see for instance [[4]-Section 4.4 and Paragraph 4.5.3]). Finally, the fact that this solution satisfies (24) follows by a regularization procedure by some

mollifier depending on the tangential variables (t, x_1, x') .

We first show (24) for $s = 0$. Let $t < T$ we test the evolution equation of (23) against u and we integrate over Ω_t . We thus have

$$\begin{aligned} \|u(t, \cdot)\|_{L^2(\Gamma)}^2 - \underbrace{\lambda}_{<0} \|u|_{x_1=0}\|_{L^2(\partial\Omega_{1,t})}^2 - t\|D\| \times \sup_{t' \leq t} \|u(t', \cdot)\|_{L^2(\Gamma)}^2 \\ \leq \int_0^t \int_{\Gamma} \langle f, u \rangle dx dt + \int_{\partial\Omega_{2,t}} \langle (A_2 u)|_{x_2=0}, u|_{x_2=0} \rangle dt dx_1 dx'. \end{aligned}$$

We conclude to (24) by using the fact that the full trace $u|_{x_2=0}$ is prescribed, it equals g . So that the second term in the right-hand side is bounded by $C\|g\|_{L^2_\gamma(\partial\tilde{\Omega}_{2,t})}$. Let $n(t) := \sup_{t' \leq t} \|u(t', \cdot)\|_{L^2(\Gamma)}$, if we take the supremum for $t' \leq t$ we can replace the term $\|u(t, \cdot)\|_{L^2(\Gamma)}^2$ appearing in the left-hand side by $n^2(t)$.

We use Young inequality for the first term in the right-hand side

$$\int_0^t \int_{\Gamma} \langle f, u \rangle dx dt \leq n(t) \int_0^t \|f(t', \cdot)\|_{L^2(\Gamma)} dt' \leq \frac{1}{2} n^2(t) + \frac{1}{2} \left(\int_0^t \|f(t', \cdot)\|_{L^2(\Gamma)} dt' \right)^2$$

and we absorb the term $\frac{1}{2} n^2(t)$ by the one appearing in the left-hand side. Finally we choose T' small enough such that (for instance) $T' \leq \frac{1}{4\|D\|}$ to ensure that the left-hand side is bounded by below by $\frac{1}{4} n^2(t)$. Estimate (24) for $s = 0$ follows.

We now turn to the proof for $s \in \mathbb{N}$. Clearly, the tangential derivatives (x_1, x') commute with the boundary condition of (23). Similarly, the time derivatives commute with the boundary condition. For the initial condition we use the interior equation to write

$$(\partial_t u)|_{t \leq 0} = - \left(D + \sum_{j=1}^d A_j \partial_j \right) \underbrace{u|_{t \leq 0}}_{=0} + \underbrace{f|_{t \leq 0}}_{=0} = 0,$$

because f is flat at the origin. A simple induction shows that we have for all $0 \leq \alpha \leq s$, $(\partial_t^{\alpha_0} u)|_{t \leq 0} = 0$. So that if $\partial^\alpha := \partial_t^{\alpha_0} \partial_1^{\alpha_1} \partial_3^{\alpha_3} \dots \partial_d^{\alpha_d}$, with $\alpha \in \mathbb{N}^{1+d-2}$ satisfying $|\alpha| = s$, then we have the system of equations:

$$\begin{cases} \mathbf{L} \partial^\alpha u = \partial^\alpha f & \text{in } \tilde{\Omega}_{2,T}, \\ (\partial^\alpha u)|_{x_2=0} = \partial^\alpha g & \text{on } \partial\tilde{\Omega}_{2,T}, \\ (\partial^\alpha u)|_{t \leq 0} = 0 & \text{on } \Gamma, \end{cases}$$

and the study of the case $s = 0$ applies. It gives the *a priori* energy estimate (24) up to the fact that we have to control the normal derivatives with respect to the normal variable x_2 . More precisely if we denote by $\mathbf{W}_{T,tan}^s$ the set of functions u defined by the condition $\|u(t)\|'_s < \infty$ with $\|u(t)\|'_s := \sum_{j=0}^s \|\partial_t^j u(t)\|_{H_{tan}^{s-j}(\Gamma)}$, where the tangential Sobolev space H_{tan}^s is the Sobolev space of order s generated by the tangential derivatives, that is to say the ones with respect to t, x_1, x_3, \dots, x_d . Then we have the estimate:

$$\|u(t)\|'_s + |u(t)|_s \leq \int_0^t \|f(t')\|'_s dt' + \|g\|_{H^s(\partial\tilde{\Omega}_{2,t})}.$$

We are in a non characteristic framework so that we can use the interior equation to isolate $\partial_2 u$ and express this derivative in terms of the tangential derivatives. More precisely we write

$$\partial_2 u = \tilde{A}_2^{-1} \left[f - \partial_t u - \sum_{j=3}^d \tilde{A}_j \partial_j u - \lambda \partial_1 u - Du \right],$$

so that by induction we obtain:

$$\|u(t)\|_s \lesssim \|u(t)\|'_s + \|f(t)\|_{s-1} \lesssim \|u(t)\|'_s + \underbrace{\|f(0)\|_{s-1}}_{=0} + \int_0^t \underbrace{\|\partial_t f(t')\|_{s-1}}_{\leq \|f(t')\|_s} dt',$$

where we use the flatness of f at the origin. This completes the proof of the energy estimate (24).

□

As a consequence, from Corollary 3.1 we can justify that the sequence $(u^n)_{n \in \mathbb{N}}$ is well-defined with values in $\mathbf{W}_{T_0}^s$ if $0 < T_0 \leq T$ is sufficiently small compared to $\|D\|$. Indeed we have the following proposition:

Proposition 3.1 *For all $n \in \mathbb{N}$, for all $s \in \mathbb{N}$, there exists $0 < T'(D) \leq T$ such that we have the energy estimates: for all $0 \leq t_n \leq T$, for all $k \in \llbracket 1, M_1 \rrbracket$, $\ell \in \llbracket 1, M_2 \rrbracket$*

$$\left\| \|u_{1,k}^{n+1}(t_{n+1})\|_s + |u_{1,k}|_{x_1=0}^{n+1}(t_{n+1})|_s \right\| \lesssim \int_0^{t_n} \cdots \int_0^{t_1} (\|f(t_0)\|_{s+n} + |g(t_0)|_{s+n}) dt_0 \cdots dt_{n-1}, \quad (26)$$

$$\left\| \|u_{2,\ell}^n(t_n)\|_s + |u_{2,\ell}|_{x_2=0}^n(t_n)|_s \right\| \lesssim \int_0^{t_n} \cdots \int_0^{t_1} (\|f(t_0)\|_{s+n} + |g(t_0)|_{s+n}) dt_0 \cdots dt_{n-1}. \quad (27)$$

In particular we note that to define the sequence $(u^n)_{n \in \mathbb{N}}$ for all $n \in \mathbb{N}$ we use in a non trivial way the fact that the initial data f , g_1 and g_2 are infinitely regular. Our analysis clearly breaks down if the sources are chosen with a finite (but large) regularity H^s .

Proof of Proposition 3.1 We proceed by induction. For $n = 0$, from the definitions (19) and (20), the estimates (26) and (27) follow from Corollary 3.1.

Assume that the estimates (26) and (27) hold for some $n \in \mathbb{N}$. We fix T' small enough to apply Corollary 3.1, we thus have the inequalities: for all $k \in \llbracket 1, M_1 \rrbracket$, for all $\ell \in \llbracket 1, M_2 \rrbracket$ and for all $t_{n+1} \leq T_0$

$$\begin{aligned} & \left\| \|u_{1,k}^{n+1}(t_{n+1})\|_s + |u_{1,k}|_{x_1=0}^{n+1}(t_{n+1})|_s \right\| \\ & \lesssim \int_0^t \|f(t_n)\|_s + |g(t_n)|_{s+1} + \|\mathbf{F}_k(\bar{u}_k^{n,*})(t_n)\|_s + |\mathbf{g}_{2,k}(\bar{u}_{2|x_2=0}^n)(t_n)|_{s+1} dt_n, \end{aligned}$$

and

$$\begin{aligned} & \left\| \|u_{2,\ell}^{n+1}(t_{n+1})\|_s + |u_{2,\ell}|_{x_2=0}^{n+1}(t_{n+1})|_s \right\| \\ & \lesssim \int_0^t \|f(t_n)\|_s + |g(t_n)|_{s+1} + \|\mathbf{F}_\ell(\bar{u}_\ell^{n,*})(t_n)\|_s + |\mathbf{g}_{1,\ell}(\bar{u}_{1|x_1=0}^n)(t_n)|_{s+1} dt_n. \end{aligned}$$

Because the \mathbf{F}_\cdot , $\mathbf{g}_{1,\cdot}$ and $\mathbf{g}_{2,\cdot}$ are linear⁵ we have:

$$\begin{aligned} & \left\| \|u_{1,k}^{n+1}(t_{n+1})\|_s + |u_{1,k}|_{x_1=0}^{n+1}(t_{n+1})|_s \right\| \lesssim \int_0^t \|f(t_n)\|_s + |g(t_n)|_{s+1} + \|\bar{u}_k^{n,*}(t_n)\|_{s+1} + |\bar{u}_{2|x_2=0}^n(t_n)|_{s+1} dt_n, \\ & \left\| \|u_{2,\ell}^{n+1}(t_{n+1})\|_s + |u_{2,\ell}|_{x_2=0}^{n+1}(t_{n+1})|_s \right\| \lesssim \int_0^t \|f(t_n)\|_s + |g(t_n)|_{s+1} + \|\bar{u}_\ell^{n,*}(t_n)\|_{s+1} + |\bar{u}_{1|x_1=0}^n(t_n)|_{s+1} dt_n, \end{aligned}$$

we conclude by using (26) and (27) to estimate the last terms in the right-hand sides of the previous equations. □

3.1.3 Convergence analysis

The aim of the following paragraph is to show that the sequence $(u^n)_{n \in \mathbb{N}}$ defined in Paragraph 3.1.2 is a Cauchy sequence in $\mathbf{W}_{T',\mathfrak{h}}^s$ for all $s \in \mathbb{N}$ when we choose T' small enough. This can be ensured if we restrict our study to source terms in the functional spaces $\mathbf{X}_{K,T,\mathfrak{h}}^{\infty}$ for the interior source term and $\mathbf{X}_{K,T,\mathfrak{h}}^{\infty,b}$ for the boundary ones. We recall the definition for convenience:

$$\mathbf{X}_{K,T}^{\infty} := \{u \in H^\infty \setminus \exists K \geq 1, \forall s \in \mathbb{N}, \|u(t)\|_s \lesssim K^s s!\},$$

more comments about this space are given in Paragraph 5.1.

In order to show that the sequence $(u^n)_{n \in \mathbb{N}}$ is a Cauchy sequence we will use the following propositions⁶:

⁵We recall that the \mathbf{F}_\cdot are evaluated in $\partial_j u$, $j \in \llbracket 1, d \rrbracket$.

⁶It is here where it is convenient to work with the energy estimate of Corollary 3.1 instead of the one of Lemma 3.2. If one uses (24), then the basic energy estimate admits a right-hand side reading $t\mathbf{F} + \mathbf{G}$, where \mathbf{F} (resp. \mathbf{G}) depends on the norm of the source f (resp. g). When we iterate such a bound in the induction some binomial coefficients appear and we can not absorb them at the end of the day because we have one extra factorial (see (28))

Proposition 3.2 For all $s \in \mathbb{N}$, for all $n \in \mathbb{N}$ and for all $T' \leq \min(1, T_0)$ small enough, we have the inequalities: for all $k \in \llbracket 1, M_1 \rrbracket$ and for all $\ell \in \llbracket 1, M_2 \rrbracket$,

$$\begin{aligned} \left\| (u_{1,k}^{n+1} - u_{1,k}^n)(t_{n+1}) \right\|_s &\leq C^{n+1} \int_0^{t_{n+1}} \mathbf{E}^{s+n+1}(t_n) dt_n, \\ \left\| (u_{2,\ell}^{n+1} - u_{2,\ell}^n)(t_{n+1}) \right\|_s &\leq C^{n+1} \int_0^{t_{n+1}} \mathbf{E}^{s+n+1}(t_n) dt_n \end{aligned}$$

where we defined for $p \in \mathbb{N}$:

$$\mathbf{E}^p(t_n) := \int_0^{t_n} \cdots \int_0^{t_2} \int_0^{t_1} (\|f(t_0)\|_p + |g(t_0)|_p) dt_0 dt_1 \cdots dt_{n-1}.$$

As a direct consequence we obtain:

Proposition 3.3 For all $s, n, p \in \mathbb{N}$, for all $T' \leq \min(1, T_0)$ small enough, we have the inequalities: for all $k \in \llbracket 1, M_1 \rrbracket$ and for all $\ell \in \llbracket 1, M_2 \rrbracket$,

$$\left\| (u_{1,k}^{n+p} - u_{1,k}^n)(t) \right\|_s \leq \sum_{q=1}^p C^{n+q} \int_0^t \mathbf{E}^{s+n+q}(t') dt' \text{ and } \left\| (u_{2,\ell}^{n+p} - u_{2,\ell}^n)(t) \right\|_s \leq \sum_{q=1}^p C^{n+q} \int_0^t \mathbf{E}^{s+n+q}(t') dt'.$$

With these propositions in hand, we can easily show that $(u^n)_{n \in \mathbb{N}}$ is a Cauchy sequence when we choose T' small enough, specified below and when the sources are restricted to the space $\mathbf{X}_{K,T,\natural}^\infty$ i.e. $f \in \mathbf{X}_{K,T,\natural}^\infty$, $g_1, g_2 \in \mathbf{X}_{K,T,\natural}^{\infty,b}$. We show the result for the terms $(u_k^n)_{n \in \mathbb{N}, k \in \llbracket 1, M_1 \rrbracket}$, the arguments being analogous for the $(u_\ell^n)_{n \in \mathbb{N}, \ell \in \llbracket 1, M_2 \rrbracket}$.

Using Proposition 3.3, we have

$$\left\| (u_{1,k}^{n+p} - u_{1,k}^n)(t_{n+p}) \right\|_s \leq C_p C^n \int_0^{t_{n+p}} \mathbf{E}^{s+n+p}(t_{n+p-1}) dt_{n+p-1},$$

where we used the bounds, for $q \leq p$, $\mathbf{E}^{s+n+q} \leq \mathbf{E}^{s+n+p}$, and bound in the first integral $\|f(t_0)\|_{s+n+p} + |g(t_0)|_{s+n+p} \leq \mathbf{M}^{s+n+p} := \sup_{t' \in [0, T']}$ $\|f(t')\|_{s+n+p} + |g(t')|_{s+n+p}$.

However if the sources $f \in \mathbf{W}_{K,T,\natural}^\infty$, $g_1, g_2 \in \mathbf{W}_{K,T,\natural}^{\infty,b}$ then we have

$$\mathbf{M}^{s+n+p} \lesssim K^{s+n+p} (n+s+p)!.$$

As a consequence, we end up with

$$\left\| (u_{1,k}^{n+p} - u_{1,k}^n)(t_{n+p}) \right\|_s \leq C_{s,p} C^n \frac{(t_{n+p})^n}{(n+p)!} K^n (n+s+p)! \leq C_{s,p} C^n (t_{n+p})^n K^n (K')^n, \quad (28)$$

where $K' > 0$ is chosen large enough to have $(n+s+p)^s \lesssim (K')^n$. We restrict the final time variable $t_{n+p} \leq T' := \frac{1}{2CK'}$, so that the right-hand side goes to zero when $n \uparrow \infty$. The space $\mathbf{W}_{T'}^s$ being a Banach space we obtain that $(u_{1,k}^n)$ converges to some $\mathbf{u}_{1,k} \in \mathbf{W}_{T'}^s$. The result being true for all $s \geq 0$, we deduce that $\mathbf{u}_{1,k} \in \mathbf{W}_{T'}^\infty$. Moreover, passing to the limit in (17) we obtain that $\mathbf{u}_{1,k}$ is a solution to (17). We proceed similarly for the sequence $(u_{2,\ell}^n)_{n \in \mathbb{N}}$ and we show that it converges to some $\mathbf{u}_{2,\ell}$ a solution to (18). Finally, we define $u := \sum_{k=1}^{M_1} \mathbf{u}_{1,k} + \sum_{\ell=1}^{M_2} \mathbf{u}_{2,\ell}$ which thus define a solution to (10) in $\mathbf{W}_{T'}^\infty$.

To conclude the proof of Theorem 2.1, we have to justify that $u \in \mathbf{W}_{K',T'}^\infty$. It is sufficient to show the result for one of the $\mathbf{u}_{1,k}$. Recall that we assumed $K \geq 1$, we fix $n_0 \in \mathbb{N}$ such that $\left\| (u_{1,k}^{n_0} - \mathbf{u}_{1,k})(t) \right\|_s \leq \frac{1}{2}$, the triangle inequality combined with the estimate (26) and the definition of the space $\mathbf{X}_{K,T}^\infty$ gives

$$\|\mathbf{u}_{1,k}(t)\|_s \leq \frac{1}{2} + CK^{n_0} K^s s! \lesssim (K')^s s!.$$

The proof of Theorem 2.1 is complete.

We end up with the proof of Proposition 3.2.

Proof of Proposition 3.2 By linearity of the systems (21) and (22) we have that for all $n \in \mathbb{N}$, $v_k^n := u_{1,k}^{n+1} - u_{1,k}^n$ (resp. $v_\ell^n := u_{2,\ell}^{n+1} - u_{2,\ell}^n$), verifies the equation

$$\begin{cases} \partial_t v_k^n + \lambda_{1,k} \partial_1 v_k^n + \tilde{A}_{2,k} \partial_2 v_k^n + \sum_{j=3}^d \tilde{A}_{j,k} \partial_j v_k^n + D_k v_k^n = -\mathbf{F}_k(v_k^{n-1,*}) & \text{in } \tilde{\Omega}_{2,T}, \\ v_{k|_{x_2=0}}^n = -\mathbf{g}_{2,k}(\bar{v}_{2|x_2=0}^{n-1}) & \text{on } \partial\tilde{\Omega}_{2,T}, \\ v_{k|_{t \leq 0}}^n = 0, \end{cases}$$

$$\left(\text{resp. } \begin{cases} \partial_t v_\ell^n + \lambda_{2,\ell} \partial_2 v_\ell^n + \tilde{A}_{1,\ell} \partial_1 v_\ell^n + \sum_{j=3}^d \tilde{A}_{j,\ell} \partial_j v_\ell^n + D_\ell v_\ell^n = -\mathbf{F}_\ell(v_\ell^{n-1,*}) & \text{in } \tilde{\Omega}_{1,T}, \\ v_{\ell|_{x_1=0}}^n = -\mathbf{g}_{1,\ell}(\bar{v}_{1|x_1=0}^{n-1}) & \text{on } \tilde{\Omega}_{1,T}, \\ v_{\ell|_{t \leq 0}}^n = 0. \end{cases} \right) \quad (29)$$

We choose T_0 small enough to apply Corollary 3.1. The energy estimate gives that for all $n \geq 1$ and for all $0 \leq t_n \leq T_0$:

$$\begin{aligned} \|v_k^n(t_n)\|_s + |v_{k|_{x_1=0}}^n(t_n)|_s &\lesssim \int_0^{t_n} \left\| \bar{v}_k^{n-1,*}(t_{n-1}) \right\|_{s+1} + |\bar{v}_{2|x_2=0}^{n-1}(t_{n-1})|_{s+1} dt_{n-1}, \\ \|v_\ell^n(t_n)\|_s + |v_{\ell|_{x_2=0}}^n(t_n)|_s &\lesssim \int_0^{t_n} \left\| \bar{v}_\ell^{n-1,*}(t_{n-1}) \right\|_{s+1} + |\bar{v}_{1|x_1=0}^{n-1}(t_{n-1})|_{s+1} dt_{n-1}, \end{aligned}$$

and for $n = 0$,

$$\|v_k^1(t_1)\|_s + |v_{k|_{x_1=0}}^1(t_1)|_s \lesssim \int_0^{t_1} \mathbf{E}^{s+1}(t_0) dt_0 \quad \text{and} \quad \|v_\ell^1(t_1)\|_s + |v_{\ell|_{x_2=0}}^1(t_1)|_s \lesssim \int_0^{t_1} \mathbf{E}^{s+1}(t_0) dt_0.$$

A direct induction then gives the desired result. \square

3.2 Proof of Theorem 2.2

Because we authorize losses of derivatives in our estimates, a simple way to treat non homogeneous initial condition is to extend the initial condition u_0 on the whole time line by some function let us say $U \in \mathbf{X}_{K,T}^\infty$ such that $U|_{t=0} = u_0$, from Borel lemma, and then to consider the new unknown $\tilde{u} := u - U$ which solves a boundary problem reading under the form (1), so that if the new sources (depending on the extension U) are regular we can apply Theorem 2.1.

In order to have regular sources, compatibility conditions are expected and required. Let us assume that the solution to

$$\begin{cases} L(\partial)u := \partial_t u + \mathbf{A}u = 0 & \text{in } \Omega_T, \\ B_1 u|_{x_1=0} = 0 & \text{on } \partial\Omega_{1,T}, \\ B_2 u|_{x_2=0} = 0 & \text{on } \partial\Omega_{2,T}, \\ u|_{t=0} = u_0 & \text{on } \Gamma, \end{cases} \quad (30)$$

is regular enough so that all the following traces make sense. We should then first have

$$(B_1 u|_{x_1=0})|_{t=0} = 0 = (B_2 u|_{x_2=0})|_{t=0} \quad \text{so that} \quad \begin{cases} u_{0|x_1=0} \in \ker B_{1|_{t=0}}, \\ u_{0|x_2=0} \in \ker B_{2|_{t=0}}, \end{cases}$$

the so-called compatibility condition of order zero.

Then, we introduce the sequence $(u^n)_{n \in \mathbb{N}}$ defined for all $n \geq 1$ by $u^n := (\partial_t^n u)|_{t=0}$ and $u^0 = u_0$. We isolate the time derivative in the evolution equation of (30), we apply ∂_t^n and we use Leibniz formula to obtain the induction relation

$$u^{n+1} = - \sum_{k=0}^n \binom{n}{k} (\partial_t^{n-k} \mathbf{A})|_{t=0} u^k = -F_n(u^0, u^1, \dots, u^n).$$

So that we have by a straightforward induction $u^n := -\mathbf{F}_{n-1}(u^0)$ for all $n \geq 1$. We then apply ∂_t^n to the boundary conditions of (30), we use Leibniz formula and evaluate at $t = 0$ to obtain

$$\begin{cases} \mathbf{G}_{1,n}(u_0) := (\partial_t^n B_1)|_{t=0} u_0|_{x_1=0} - \sum_{k=1}^n \binom{n}{k} (\partial_t^{n-k} B_1)|_{t=0} (\mathbf{F}_{k-1}(u_0))|_{x_1=0} = 0, \\ \mathbf{G}_{1,n}(u_0) := (\partial_t^n B_2)|_{t=0} u_0|_{x_2=0} - \sum_{k=1}^n \binom{n}{k} (\partial_t^{n-k} B_2)|_{t=0} (\mathbf{F}_{k-1}(u_0))|_{x_2=0} = 0. \end{cases} \quad (31)$$

The above relations being the so-called compatibility condition of order $n \geq 1$.

As pointed before with these conditions in hand then the proof of Theorem 2.2 can be deduced from Theorem 2.1 by using some Borel lemma.

An other proof, which is more in the spirit of the present article is to reiterate the proof exposed in Paragraph 3.1, but within the space $\underline{\Omega}_T$. Clearly, the reformulation part of Paragraph 3.1.1 still holds and the convergence analysis of Paragraph 3.1.3 operates if the energy estimates of Paragraph 3.1.2 are suitably modified.

Clearly when we have initial sources the energy estimate Corollary 3.1 becomes

$$\|u(t)\|_s + |u|_{x_1=0}(t)|_s \lesssim \|u_0\|_{H^s(\Gamma)} + \int_0^t (\|f(t')\|_s + |g(t')|_s) dt'. \quad (32)$$

One can worry that in the above estimate we can not express (in a nature way) the term $\|u_0\|_{H^s(\Gamma)}$ as a time integral form. Let us however point that in such a setting we require the iterative scheme to be exact one initial conditions. More precisely, we will consider the solutions to (19), (21) (resp. (20) and (22)) with the initial condition $u_{k|t=0}^n = P_1^k u_0$ (resp. $u_{\ell|t=0}^n = P_1^\ell u_0$) for all $n \geq 0$. As a consequence, the extra term appearing in the right-hand side of (32) vanishes in the proof of Proposition 3.2 and the rest of the analysis is unchanged.

To be totally precise, let us indicate that the proof of the energy estimate (32) requires the compatibility conditions (31). Indeed, the proof of the energy estimate requires the fact that, when the sources are flat, we have $\|f(0)\|_s = |g(0)|_s = 0$. This is not true any more but the compatibility conditions (31) ensure that these norms can be estimated in terms of $\|u_0\|_{H^s(\Gamma)}$.

3.3 Modifications of the proofs for variable coefficients

In the following paragraph we end up our construction of strong solutions by considering the case of variable coefficients. We thus consider from now on that the interior coefficients A_j , D depend on the variable (t, x) and that the boundary matrix B_1 (resp. B_2) depends on (t, x_2, x') (resp. (t, x_1, x')).

Such a modification of the framework of resolution does not modify the proof exposed in Paragraph 3.1 for constant coefficients and instead of reproduce the whole proof let us indicate the main modifications to take into account.

To modify the proof of Paragraph 3.1 to variable coefficients we have to be careful about the following new facts:

1. First, let us mention that the reformulation part of system (1) into the family of transport equations (21) and (22) still holds with variable coefficients because the spectral properties of Assumption 2.4 are now assumed to hold pointwise in (t, x) . Let us however mention that in the reformulation procedure of Paragraph 3.1.1, all the objects (in particular the projection) inherit the dependence with respect to (t, x) , so that some commutators (typically the derivatives of the projections) appear and act as zero order terms. We here ask regularity upon the coefficients A_j to make sure that such derivatives are understood in a strong sense. Let us however point that because (1) and consequently, the reformulated systems (21) and (22), already include such a zero order term, the above reformulation hold up to a suitable modification of D . So that at this step of the proof we have to solve (21) and (22) with variables coefficients.

2. In order to do so, we have to modify Lemma 3.2 and Corollary 3.1 suitably. We point that the basic energy estimates (24) and (25) (for $s = 0$) still hold in a variable coefficient setting because the sign property on the eigenvalue $\lambda(t, x) < 0$ holds for all $(t, x) \in \Omega_T$. The only modifications appear in order to obtain the persistence of regularity result (that is to say when $s \geq 1$). Once again because the coefficients of \mathbf{L} now depend on (t, x) some commutators appear. Let us precise how we can now recover the regularity of the first order derivatives, then the persistence of regularity follows by some induction procedure. Let ∂_k , $k \in \llbracket 3, d \rrbracket \cup \{1, t\}$ denote a tangential derivative. We have that $\partial_k u$ solves the transport equation

$$\begin{cases} \mathbf{L}\partial_k u + (\partial_k \lambda)\partial_1 u + \underbrace{\sum_{j=2}^d (\partial_k \tilde{A}_j)\partial_j u}_{\mathbf{M}u} + (\partial_k D)u = \partial_k f, \\ (\partial_k u)|_{x_2=0} = \partial_k g, \\ (\partial_k u)|_{t \leq 0} = 0, \end{cases} \quad (33)$$

where we used the flatness of the interior source in the initial condition. Note that because $\mathbf{M}u$ contains the normal $\partial_2 u$, we can not express it as a zero order operator with respect to the collection of tangential derivatives. We can however reproduce the analysis performed in the constant coefficient setting and show that up to choose t small with respect to $\|\partial_k \tilde{A}_k\|_{L^\infty(\Omega_T)}$ and $\|\partial_k D\|_{L^\infty(\Omega_T)}$ we have the energy estimate⁷⁸:

$$\begin{aligned} & \|\partial_k u(t, \cdot)\|_{L^2(\Gamma)}^2 - t\|\partial_k \lambda\|_{L^\infty(\Omega_T)} \sup_{t \leq T} \|\partial_1 u(t, \cdot)\|_{L^2(\Gamma)}^2 \\ & - t\|\partial_k \tilde{A}_2\|_{L^\infty(\Omega_T)} \sup_{t \leq T} \|\partial_2 u(t, \cdot)\|_{L^2(\Gamma)}^2 - t \sum_{j \neq k} \|\partial_k \tilde{A}_j\|_{L^\infty(\Omega_T)} \sup_{t \leq T} \|\partial_j u(t, \cdot)\|_{L^2(\Gamma)}^2 \\ & \lesssim \int_0^t \|\partial_k f(s, \cdot)\|_{L^2(\Gamma)}^2 ds + \|\partial_k g\|_{L^2(\partial\tilde{\Omega}_{2,T})}^2 + t\|\partial_k D\|_{L^\infty(\Omega_T)} \sup_{t \leq T} \|u(t, \cdot)\|_{L^2(\Gamma)}^2. \end{aligned} \quad (34)$$

We use the preceding energy inequality to estimate the last term in the right-hand side in terms of the L^2 -norm. We sum the above estimates for all $k \in \llbracket 3, d \rrbracket \cup \{1, t\}$ for the lowest possible time of resolution we thus have

$$\|u(t, \cdot)\|_{H_{tan}^1(\Gamma)}^2 - t\|\partial_k \tilde{A}_2\|_{L^\infty(\Omega_T)} \sup_{t \leq T} \|\partial_2 u(t, \cdot)\|_{L^2(\Gamma)}^2 \lesssim \int_0^t \|f(s, \cdot)\|_{H_{tan}^1(\Gamma)}^2 ds + \|g\|_{H^1(\partial\tilde{\Omega}_{2,t})}^2, \quad (35)$$

where we recall that H_{tan}^1 denotes the Sobolev space of order one generated by the tangential derivatives only. So that it remains to deal with the term depending on the normal derivative in the right-hand side in order to do so we express

$$\partial_2 u = \tilde{A}_2^{-1} \left[f - \partial_t u - \lambda \partial_1 u - \sum_{j=3}^d \tilde{A}_j \partial_j u \right], \quad (36)$$

so that $\|\partial_2 u\|$ can be expressed in terms of the tangential derivatives and up to choose t small enough the second term in the left-hand side of (35) can be absorbed by the first one. We thus have the tangential estimate and (36) gives the missing normal estimate.

3. Because the convergence analysis only required the energy estimate (24) which has been shown to hold. We can reproduce the analysis of Paragraph 3.1.3 to conclude the proof of Theorem 2.3.

⁷Note that the regularity of the traces is here again obvious so that we will not incorporate it in the following estimates.

⁸It is here that a uniform bound on the derivatives of the coefficients is convenient to ensure that such a t can be made independent of the order of the derivation s . This point is convenient because at the end of the day the above energy estimate will be used at order $s + n$, so that we have to make sure that the maximal time of resolution does not go to zero when n goes to infinity.

4 Application to viscous perturbations

In the following paragraph we give a direct consequence of the existence of strong solutions to hyperbolic corner problems. More precisely, we will consider a viscous approximation of the hyperbolic problem (1): for $T > 0$

$$\begin{cases} L(\partial)u^\varepsilon - \varepsilon\mathcal{E}u^\varepsilon = f & \text{for } (t, x) \in \Omega_T, \\ u^\varepsilon|_{x_1=0} = 0 & \text{on } \partial\Omega_{1,T}, \\ u^\varepsilon|_{x_2=0} = 0 & \text{on } \partial\Omega_{2,T}, \\ u^\varepsilon|_{t \leq 0} = 0 & \text{on } \Gamma, \end{cases} \quad (37)$$

where \mathcal{E} is some given elliptic operator. The aim of the following is to describe the limit of the solution u^ε when the viscosity parameter ε goes to zero.

Such problems have been intensively studied in the geometry of the half-space (we refer to the consequent works of [1], [5] and [16]). Because, ε goes to zero, the considered problem changes of type (it is parabolic for $\varepsilon > 0$ and hyperbolic for $\varepsilon = 0$). So some boundary layers localized around the boundary are expected in order to correct the modification of the prescribed boundary conditions.

Consequently in a quarter space geometry, two boundary layers, localized around the border $\partial\Omega_1$ and $\partial\Omega_2$ are expected and the main question that we are addressing here is : "do the two boundary layers interact together ?".

To answer this question, let us be more precise about the functional space used to construct the boundary layer in the half-space.

4.1 Half-space viscous perturbation in a nutshell and heuristic for the quarter space

If we consider the half-space viscous approximation

$$\begin{cases} L(\partial)u^\varepsilon - \varepsilon\mathcal{E}u^\varepsilon = f & \text{in }]0, T] \times \mathbb{R}^{d-1} \times \mathbb{R}_+, \\ u^\varepsilon|_{x_d=0} = 0 & \text{on }]0, T] \times \mathbb{R}^{d-1}, \\ u^\varepsilon|_{t \leq 0} = 0 & \text{on } \mathbb{R}^{d-1} \times \mathbb{R}_+, \end{cases} \quad (38)$$

then the classical analysis (see for instance [1]) shows the existence of some elements $\mathcal{U} \in \mathcal{S}_{X_d}(\mathbb{R}_+; H_{t,x}^\infty(]0, T] \times \mathbb{R}_+^d))$, $u^0 \in H^\infty(]0, T] \times \mathbb{R}_+^d)$ such that the function

$$u_{app}^\varepsilon(t, x) := u^0(t, x) + \varepsilon\mathcal{U}(t, x; \frac{x_d}{\varepsilon}), \quad (39)$$

is an approximation of the solution u^ε in the sense that $\|u^\varepsilon - u_{app}^\varepsilon\|_{L^2(\Omega_T)} \lesssim \varepsilon^\alpha$, for some $\alpha \in \mathbb{N}$. In the previous formula, let us point that the terms \mathcal{U} and u^0 are explicitly constructive. In particular, u^0 is the solution to a hyperbolic boundary value problem with suitable boundary conditions.

As a consequence, for the quarter space problem (37) we expect two boundary layers: \mathcal{U} localized in the strip $\{0 < x_1 \lesssim \varepsilon\}$ and \mathcal{V} localized in the strip $\{0 < x_2 \lesssim \varepsilon\}$. At first glance, we can not exclude that the two layers interact the one with the other near the corner in the square $\{0 < x_1, x_2 \lesssim \varepsilon\}$. So that we shall *a priori* look for an approximation of the exact solution u^ε reading under the form

$$u_{app}^\varepsilon(t, x) := u^0(t, x) + \varepsilon \left(\mathcal{U}(t, x; \frac{x_1}{\varepsilon}) + \mathcal{V}(t, x; \frac{x_2}{\varepsilon}) + \mathcal{W}(t, x; \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \right),$$

where \mathcal{W} is localized in $\{0 < x_1, x_2 \lesssim \varepsilon\}$ and takes into account such an interaction.

The aim of the following is to justify rigorously that such an interaction corrector is not necessary if the elliptic perturbation is suitably chosen. The reason is that near the corner, the boundary layers \mathcal{U} and \mathcal{V} behave like the corner value $u|_{x_1=x_2=0}$ which is zero for compatibility reasons on the boundary conditions of the limiting hyperbolic problem.

4.2 Notation and assumptions

First, let us be more precise about the elliptic perturbation. It reads $\mathcal{E} := \mathcal{E}(x) := \sum_{i,j=1}^d \partial_i(E_{i,j}(x)\partial_j)$, and we require that for all $x \in \Gamma$, the coefficients $E_{i,j}(x) \in \mathbf{M}_{N \times N}(\mathbb{R})$ satisfy the following assumption:

Assumption 4.1 *We assume that for all $i, j \in \llbracket 1, d \rrbracket^2$, for all $x \in \Gamma$, the matrix $E_{i,j}(x) \in \mathbf{M}_{N \times N}(\mathbb{R})$ is symmetric. The map $x \mapsto E_{i,j}(x)$ is regular. Moreover the operator \mathcal{E} is assumed to be coercive in the sense that there exists $\alpha > 0$ such that*

$$\forall x \in \Gamma, \forall \xi \in \mathbb{S}^{d-1}, \sum_{i,j=1}^d \xi_i \xi_j E_{i,j}(x) \geq \alpha I.$$

Finally to simplify the exposition we require a universal bound for the derivatives of these coefficients. More precisely, we assume that there exists $C > 0$ such that for all $\gamma \in \mathbb{N}^d$, we have for all $i, j \in \llbracket 1, d \rrbracket$

$$\|\partial^\gamma E_{i,j}\|_{L^\infty(\Gamma)} \leq C, \text{ where } \partial^\gamma := \partial_1^{\gamma_1} \dots \partial_d^{\gamma_d}.$$

Following the analysis in the half-space geometry, it will be convenient to consider some matrices $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ which are well-adapted to describe the boundary layers. We define $\mathbf{\Lambda}_1 := \mathbf{\Lambda}_1(t, x) = E_{1,1}^{-1/2} A_1 E_{1,1}^{-1/2}$ and $\mathbf{\Lambda}_2 := \mathbf{\Lambda}_2(t, x) = E_{2,2}^{-1/2} A_2 E_{2,2}^{-1/2}$. For $j \in \{1, 2\}$, also introduce $\tilde{\mathbf{E}}_j^s$ (resp. $\tilde{\mathbf{E}}_j^u$) the eigenspace associated to the negative (resp. positive) eigenvalues of $\mathbf{\Lambda}_j$. Finally, we define $\tilde{\Pi}_j^s$ (resp. $\tilde{\Pi}_j^u$) the projection upon $\tilde{\mathbf{E}}_j^s$ (resp. $\tilde{\mathbf{E}}_j^u$) with respect to the decomposition $\mathbb{R}^N = \tilde{\mathbf{E}}_j^s \oplus \tilde{\mathbf{E}}_j^u$.

Our main assumption about the "admissible" perturbation coefficients is the following. It requires some structural assumptions which are made to ensure that Theorem 2.1 applies to determine the slow amplitude u^0 and that the obtained amplitude vanishes at the corner.

Assumption 4.2 *We assume that the perturbation coefficients $E_{1,1}$ and $E_{2,2}$ satisfy the following structural assumptions:*

1. For $j \in \{1, 2\}$, for all $(t, x_{3-j}, x') \in \partial\Omega_{j,T}$ we have the non intersecting conditions:

$$\left[(E_{j,j}^{-1/2} \tilde{\mathbf{E}}_j^s)|_{x_j=0} \cap (\mathbf{E}_{3-j}^s)|_{x_j=0} \right] (t, x_{3-j}, x') = \{0\}.$$

2. We have

$$(E_{1,1}^{-1/2} \tilde{\mathbf{E}}_1^s)|_{x_1=x_2=0} \cap (E_{2,2}^{-1/2} \tilde{\mathbf{E}}_2^s)|_{x_1=x_2=0} = \{0\}.$$

3. Finally we have

$$(A_2^{-1} A_1 E_{2,2}^{-1/2} \tilde{\mathbf{E}}_2^s)|_{x_1=x_2=0} \cap (E_{1,1}^{-1/2} \tilde{\mathbf{E}}_1^s)|_{x_1=x_2=0} = (A_1^{-1} A_2 E_{1,1}^{-1/2} \tilde{\mathbf{E}}_1^s)|_{x_1=x_2=0} \cap (E_{2,2}^{-1/2} \tilde{\mathbf{E}}_2^s)|_{x_1=x_2=0} = \{0\}.$$

Let us point that in the particular case where $E_{1,1} = E_{2,2} = I$, then Assumption 4.2 is a trivial consequence of Assumption 2.1. In particular, Assumption 4.2 is automatically satisfied in the (physically) interesting particular case where the viscous perturbation $\mathcal{E} = \Delta$.

4.3 The proof in the quarter-space

We look for an approximate solution of u^ε the solution to (37) under the form:

$$u^\varepsilon(t, x) \sim \sum_{n \geq 0} \varepsilon^n U^n \left(t, x; \underbrace{\frac{x_1}{\varepsilon}}_{:=X_1}, \underbrace{\frac{x_2}{\varepsilon}}_{:=X_2} \right) \quad (40)$$

where the amplitudes U^n will be constructed in the following set of profiles:

Definition 4.1 (Set of profiles) Let $T > 0$ and $K \geq 1$, a function U is said to be in the set of profiles $\mathcal{P}_{K,T}$ if $U \in \mathbf{X}_{K,T}^\infty \oplus \mathcal{S}_{X_1}(\mathbb{R}_+; \mathbf{X}_{K,T}^\infty) \oplus \mathcal{S}_{X_2}(\mathbb{R}_+; \mathbf{X}_{K,T}^\infty)$. For such a function U , we use the decomposition:

$$U(t, x; X_1, X_2) := u(t, x) + \mathcal{U}(t, x; X_1) + \mathcal{V}(t, x; X_2).$$

Inject the ansatz (40) in the interior equation of (37) gives the cascade of equations:

$$\underbrace{(A_1 - E_{1,1}\partial_{X_1})}_{:=\mathbf{X}_1} \partial_{X_1} \mathcal{U}^0 + \underbrace{(A_2 - E_{2,2}\partial_{X_2})}_{:=\mathbf{X}_2} \partial_{X_2} \mathcal{V}^0 = 0 \text{ for the coefficient } \varepsilon^{-1}, \quad (41)$$

then

$$\mathbf{X}_1 \mathcal{U}^1 + \mathbf{X}_2 \mathcal{V}^1 + L(\partial)U^0 - \underbrace{\sum_{j=1}^d (E_{1,j} + E_{j,1}) \partial_j \partial_{X_1} \mathcal{U}^0 - \sum_{j=1}^d \partial_j (E_{j,1}) \partial_{X_1} \mathcal{U}^0}_{:=\mathbf{E}_1 \mathcal{U}^0} \quad (42)$$

$$- \underbrace{\sum_{j=1}^d (E_{2,j} + E_{j,2}) \partial_j \partial_{X_2} \mathcal{V}^0 - \sum_{j=1}^d \partial_j (E_{j,2}) \partial_{X_2} \mathcal{V}^0}_{:=\mathbf{E}_2 \mathcal{V}^0} = f \quad \text{for the coefficient } \varepsilon^0, \quad (43)$$

and, finally, for all $n \geq 1$,

$$\mathbf{X}_1 \mathcal{U}^{n+1} + \mathbf{X}_2 \mathcal{V}^{n+1} + L(\partial)U^n - \mathbf{E}_1 \mathcal{U}^n - \mathbf{E}_2 \mathcal{V}^n - \mathcal{E}U^{n-1} = 0 \quad \text{for the coefficient } \varepsilon^n. \quad (44)$$

When we inject the ansatz (40) in the boundary conditions of (37), we obtain the couple of conditions (to save some notation, we only keep the dependence with respect to the couples (x_1, X_1) and (x_2, X_2)):

$$\forall n \geq 0, \begin{cases} u_{|x_1=0}^n(x_2) + \mathcal{U}_{|x_1=X_1=0}^n(x_2) + \mathcal{V}_{|x_1=0}^n(x_2, \frac{x_2}{\varepsilon}) = 0, \\ u_{|x_2=0}^n(x_1) + \mathcal{U}_{|x_2=0}^n(x_1, \frac{x_1}{\varepsilon}) + \mathcal{V}_{|x_2=0}^n(x_1) = 0. \end{cases} \quad (45)$$

Finally, injecting the ansatz (40) in the initial condition of (37) give us after decoupling the variables

$$\forall n \geq 0, u_{|t \leq 0}^n = \mathcal{U}_{|t \leq 0}^n = \mathcal{V}_{|t \leq 0}^n = 0. \quad (46)$$

The main result of this current section is the following theorem describing the asymptotic of the solutions to (37) when $\varepsilon \downarrow 0$.

Theorem 4.1 Under Assumptions 2.1, 2.2, 2.3, 4.1 and 4.2, we moreover assume that the coefficients of $L(\partial)$ are symmetric. Then we can choose γ large enough, such that if the source $f \in H_{K, \gamma, \frac{1}{\varepsilon}}^\infty(\Omega)$ then for all $n \in \mathbb{N}$ there exist T' small enough, K' large enough and $U^n \in \mathcal{P}_{K', T'}$ a sequence of profiles satisfying the cascade of equations (41)-(44), (45) and (46). For given $N_0 \in \mathbb{N}$, if we define the truncated asymptotic expansion:

$$u_{app, N_0}^\varepsilon := \sum_{n=0}^{N_0} \varepsilon^n U^n, \quad (47)$$

and if u^ε stands for the solution to (37), then we have the error estimate

$$\|u^\varepsilon - u_{app, N_0}^\varepsilon\|_{L^2(\Omega_{T'})} \lesssim \varepsilon^{N_0+1}.$$

In particular, because the first layers $\mathcal{U}^0(t, x; \frac{x_1}{\varepsilon})$ and $\mathcal{V}^0(t, x; \frac{x_2}{\varepsilon})$ are $O(\sqrt{\varepsilon})$ in $L^2(\Omega_{T'})$ we have the estimate

$$\|u^\varepsilon - u^0\|_{L^2(\Omega_{T'})} \lesssim \sqrt{\varepsilon}.$$

In the following paragraph, we describe the construction of the leading order term. A sketch of proof for the construction of the higher orders correctors is given in Paragraph 4.3.2. It is illustrated on the first correctors. Once all the amplitudes appearing in the ansatz (40) are constructed, the error analysis used to conclude to Theorem 4.1 is given in Paragraph 4.3.3.

4.3.1 The leading order term

It will be more interesting to study the construction of the boundary layers \mathcal{U}^0 and \mathcal{V}^0 before the one of the leading order slow variables amplitude u^0 . However, let us stress that at the end of the day, the slow amplitude u^0 will be determined before the boundary layers \mathcal{U}^0 and \mathcal{V}^0 .

First, we decouple the fast variables in (41). Then, we integrate each equation with respect to its fast variable X_j and use the condition $\lim_{X_1 \rightarrow \infty} \mathcal{U}^0 = \lim_{X_2 \rightarrow \infty} \mathcal{V}^0 = 0$ to transform (41) into a first order ordinary differential equation. Consequently, to determine the first boundary layers, we have to solve

$$\begin{cases} \mathbf{X}_1 \mathcal{U}^0 = 0 & \text{in } \Omega_T \times \mathbb{R}_+, \\ \text{boundary condition on } \{X_1 = 0\}, \\ \mathcal{U}^0|_{t \leq 0} = 0, \end{cases} \quad (48)$$

and

$$\begin{cases} \mathbf{X}_2 \mathcal{V}^0 = 0 & \text{in } \Omega_T \times \mathbb{R}_+, \\ \text{boundary condition on } \{X_2 = 0\}, \\ \mathcal{V}^0|_{t \leq 0} = 0, \end{cases} \quad (49)$$

In the following we will only describe the resolution of (48), the construction of the solution to (49) follows the same lines. In order to do so, we first look for a solution to the interior equation and then we aim to determine the suitable boundary value. Define $\mathscr{W}^0 := E_{1,1}^{1/2} \mathcal{U}^0$, then the interior equation of (48) reads:

$$-\partial_{X_1} \mathscr{W}^0 + \mathbf{\Lambda}_1 \mathscr{W}^0 = 0, \quad (50)$$

where we recall that $\mathbf{\Lambda}_1 := E_{1,1}^{-1/2} A_1 E_{1,1}^{-1/2}$ and where the slow variables (t, x) act as parameters. The L^2 solution to (50) is explicitly given, using Duhamel formula, by

$$\mathscr{W}^0(t, x; X_1) := e^{X_1 \mathbf{\Lambda}_1(t, x)} \tilde{\Pi}_1^s(t, x) \mathscr{W}^0|_{X_1=0}, \quad (51)$$

where we recall that $\tilde{\Pi}_1^s$ stands for the projection upon the eigenspace associated to the negative real eigenvalues of $\mathbf{\Lambda}_1$. We note that from the explicit formula (51), it is clear that $\mathscr{W}^0 \in \mathcal{S}_{X_1}(\mathbb{R}_+; \mathbf{X}_{K,T}^\infty)$ if the trace $\mathscr{W}^0|_{X_1=0} \in \mathbf{X}_{K,T}^\infty$. Indeed \mathscr{W}_0 reads under the form $\mathscr{W}^0 = \mathcal{F}(t, x) \mathscr{W}^0|_{x_1=0}$, where \mathcal{F} is a product function so that from Leibniz formula combined with the uniform bounds for the derivatives of the coefficients, we have

$$\|\partial^\alpha \mathcal{F}\|_{L^\infty(\Omega_T)} \lesssim (K')^{|\alpha|} \alpha! \text{ for some large } K'.$$

So that using once again Leibniz formula we obtain

$$\begin{aligned} \|\partial^\alpha \mathscr{W}^0(t)\|_{L^2(\Gamma)} &\leq \sum_{n \leq \alpha} \binom{\alpha}{n} \|\partial^n \mathcal{F}\|_{L^\infty(\Omega_T)} \|\partial^{\alpha-n} \mathscr{W}^0|_{X_1=0}(t)\|_{L^2(\Gamma)} \\ &\lesssim K^{|\alpha|} K'^{|\alpha|} \sum_{n \leq \alpha} \frac{\alpha!}{(\alpha-n)! \cdot n!} n! (\alpha-n)! \lesssim (K'')^{|\alpha|} \alpha!, \end{aligned}$$

as desired for some large K'' , if $\mathscr{W}^0|_{X_1} \in \mathbf{X}_{K,T}^\infty$.

Finally, let us note the important point that from (51), the leading order boundary layer \mathscr{W}^0 satisfies the polarization type condition

$$\mathscr{W}^0 = \tilde{\Pi}_1^s \mathscr{W}^0. \quad (52)$$

Consequently to determine the whole boundary layer, it is sufficient to determine its trace upon $\{X_1 = 0\}$. In order to do so, we consider the first equation of (45) written for $n = 0$. We recall this equation for convenience:

$$u^0|_{x_1=0}(x_2) + \mathcal{U}^0|_{x_1=X_1=0}(x_2) + \mathcal{V}^0|_{x_1=0}(x_2; \frac{x_2}{\varepsilon}) = 0. \quad (53)$$

To determine the value of the required trace from (53), the first idea may be to say that because of an analogous explicit formula (51) for the second layer \mathcal{V}^0 , the last term in (53) is $O(\varepsilon^\infty)$ so that it should be

neglected and thus (53) should give the desired trace in terms of $u^0_{|x_1=0}$. Let us however stress that this argument breaks down in the area $\{0 < x_2 \lesssim \varepsilon\}$ that is to say near the corner, where $\mathcal{V}^0_{|x_1=0}(x_2; \frac{x_2}{\varepsilon})$ is $O(1)$. Consequently, a good way to consider (53) is to decouple once again the variables. As a consequence, we will need to solve the couple of equations:

$$\begin{cases} \mathcal{W}^0_{|x_1=X_1=0} = -E_{1,1}^{-1/2} u^0_{|x_1=0}, \\ \mathcal{V}^0_{|x_1=0} = 0. \end{cases} \quad (54)$$

We proceed similarly for the second equation of (45), written for $n = 0$, to obtain a second couple of boundary conditions

$$\begin{cases} \mathcal{X}^0_{|x_2=X_2=0} = -E_{2,2}^{-1/2} u^0_{|x_2=0}, \\ \mathcal{Z}^0_{|x_2=0} = 0, \end{cases} \quad (55)$$

where we introduced $\mathcal{X}^0 := E_{2,2}^{1/2} \mathcal{V}^0$. The rest of the construction is rather clear: to construct $\mathcal{W}^0_{|X_1=0}$ (resp. $\mathcal{X}^0_{|X_2=0}$), we use the first boundary condition in (54) (resp. (55)). We then use the explicit formula (51) to show that the second boundary condition in (54) (resp. (55)) is satisfied under Assumption 4.1.

From the polarization condition (52), the first equation of (54) amounts to solve:

$$(\tilde{\Pi}_1^s \mathcal{W}^0)_{|x_1=X_1=0} = -(\tilde{\Pi}_1^s E_{1,1}^{1/2} u^0)_{|x_1=0} \text{ and } (\tilde{\Pi}_1^u E_{1,1}^{1/2} u^0)_{|x_1=0} = 0. \quad (56)$$

Because x_1 acts as a parameter in (50) it is sufficient to consider a lifted trace by introducing some localization function $\chi \in \mathcal{D}(-1, \infty[)$ satisfying $\chi(0) = 1$. We thus define

$$\mathcal{W}^0(t, x; X_1) := -\chi(x_1) E_{1,1}^{1/2} e^{X_1 \Lambda_1} (\tilde{\Pi}_1^s E_{1,1}^{1/2} u^0)_{|x_1=0}(t, x_2). \quad (57)$$

We proceed similarly for the first equation of (55) to obtain the boundary condition on u^0 :

$$(\tilde{\Pi}_2^u E_{2,2}^{1/2} u^0)_{|x_2=0} = 0, \quad (58)$$

and the explicit formula

$$\mathcal{V}^0(t, x; X_2) = -\chi(x_2) E_{2,2}^{1/2} e^{X_2 \Lambda_2} (\tilde{\Pi}_2^s E_{2,2}^{1/2} u^0)_{|x_2=0}(t, x_1). \quad (59)$$

Consequently, to complete the construction we have to determine u^0 in a unique way, then verify that the second boundary condition in (54) and (55) are verified by the expressions given by (56) and (59), and conclude by checking that the amplitudes satisfy the initial condition 46 at order zero.

To determine the evolution equation satisfied by u^0 , we consider the second equation in the interior, namely (42). When we decouple the variables, we obtain that u^0 solves $L(\partial)u^0 = f$ in Ω . Using the boundary conditions (56) and (58) we thus have to solve the following hyperbolic corner problem in order to determine u^0 :

$$\begin{cases} L(\partial)u^0 = f & \text{in } \Omega_T, \\ B_1 u^0_{|x_1=0} = 0 & \text{on } \partial\Omega_{1,T}, \\ B_2 u^0_{|x_2=0} = 0 & \text{on } \partial\Omega_{2,T}, \\ u^0_{|t \leq 0} = 0 & \text{on } \Gamma, \end{cases} \quad (60)$$

where we defined $B_1 := (\tilde{\Pi}_1^u E_{1,1}^{1/2})_{|x_1=0}$ and $B_2 := (\tilde{\Pi}_2^u E_{2,2}^{1/2})_{|x_2=0}$.

The next lemma justifies that we can apply Theorem 2.1 in order to obtain a unique solution $u^0 \in \mathbf{X}_{K', T', \natural}^\infty$ to (60).

Lemma 4.1 *Under Assumption 4.2, then Assumption 2.4 for the system (60) is satisfied. Assume that $f \in \mathbf{X}_{K, T, \natural}^\infty$, then Theorem 2.1 applies and (60) admits a solution $u^0 \in \mathbf{X}_{K', T', \natural}^\infty$. Moreover, the boundary conditions of the boundary value problems (60) are strictly dissipative and the coefficients are symmetric, so that Corollary 2.2 applies and we have a unique solution $u^0 \in \mathbf{X}_{K', T', \natural}^\infty$.*

In particular, the corner value $u^0_{|x_1=x_2=0}$ makes sense.

Proof : The first condition of Assumption 2.4 is readable. For the second one, let us point that by definition $\ker B_j = (E_{j,j}^{-1/2} \tilde{\mathbf{E}}_j^s)|_{x_j=0}$, so that the second requirement of Assumption 2.4 is equivalent to Assumption 4.2.

The fact that the boundary conditions of problem (60) are strictly dissipative is a well-known fact from the literature in the half-space. We reproduce the proof here for a sake of completeness. Let $u \in \ker B_1$, we write $u = E_{1,1}^{-1/2} v$, with $v \in \tilde{\mathbf{E}}_1^s$. We have, using the symmetry of the matrix $E_{1,1}$,

$$\langle A_1 u, u \rangle = \langle \mathbf{A}_1 v, v \rangle < 0,$$

because of the definition of $\tilde{\mathbf{E}}_1^s$. □

Now that the slow amplitude u^0 is constructed in a unique way, to conclude the construction of the leading order amplitude, it remains to justify that with the definitions (57) and (59), the boundary layers satisfy the second condition of (54) and (55). From the explicit formula (57) we have

$$\mathcal{U}_{|x_2=0}^0(x_1; X_1) = \chi(x_1) (E_{1,1}^{1/2} e^{X_1 \mathbf{A}_1})_{|x_2=0} (\tilde{\Pi}_1^s E_{1,1}^{1/2})_{|x_1=x_2=0} u_{|x_1=x_2=0}^0.$$

Consequently it is sufficient to determine the value of the double trace $u_{|x_1=x_2=0}^0$. To conclude, we consider the compatibility conditions of (60). We have

$$B_1 u_{|x_1=0}^0 = 0 = B_2 u_{|x_2=0}^0 \implies u_{|x_1=x_2=0}^0 \in \ker B_{1|_{x_2=0}} \cap \ker B_{2|_{x_1=0}}.$$

We note that under the second structural requirement in Assumption 4.2 the last intersection reduces to the null vector. At last from the explicit formulas (57) and (59) it is readable that $\mathcal{U}_{|t \leq 0}^0 = \mathcal{V}_{|t \leq 0}^0 = 0$, if $u_{|t \leq 0}^0 = 0$. It concludes the construction of the leading order term.

For later purposes, let us here indicate that the traces $\mathcal{U}_{|x_1=X_1=0}^0$ and $\mathcal{V}_{|x_2=X_2=0}^0$ inherit the regularity $\mathbf{X}_{K,T}^\infty$. Indeed, it is readable on the explicit formulas (57) and (59).

4.3.2 Construction of the correctors

For convenience, we describe the construction of the first correctors u^1 , \mathcal{U}^1 and \mathcal{V}^1 . The construction of the higher orders correctors follows exactly the same lines.

To obtain the interior equations, we decouple the fast variables in (42) and isolate the slow variable in (44) (written for $n = 1$). We thus have after the change of unknowns, $\mathcal{W}^1 := E_{1,1}^{-1/2} \mathscr{W}^1$ and $\mathcal{X}^1 := E_{2,2}^{-1/2} \mathscr{V}^1$:

$$\begin{cases} -\partial_{X_1} \mathscr{W}^1 + \mathbf{A}_1 \mathscr{W}^1 = E_{1,1}^{-1/2} F_1 & \text{for } X_1 > 0, \\ -\partial_{X_2} \mathscr{X}^1 + \mathbf{A}_2 \mathscr{X}^1 = E_{2,2}^{-1/2} F_2 & \text{for } X_2 > 0, \\ L(\partial) u^1 = \mathcal{E} u^0 & \text{for } (t, x) \in \Omega_T, \end{cases} \quad (61)$$

where we defined

$$F_1(X_1) := F_1(t, x; X_1) = - \int_{X_1}^{\infty} (\mathbf{E}_1 \mathcal{W}^0)(y_1) dy_1 \text{ and } F_2(X_2) := F_2(t, x; X_2) = - \int_{X_2}^{\infty} (\mathbf{E}_2 \mathcal{V}^0)(y_2) dy_2. \quad (62)$$

To solve the two firsts equations of (61) we use Duhamel formula (with a non vanishing source term compared to Paragraph 4.3.1). The solution of the last equation of (61) is given as the solution of a hyperbolic corner problem with suitable boundary conditions.

From Duhamel formula we can decompose $\mathscr{W}^1 = \tilde{\Pi}_1^s \mathscr{W}^1 + \tilde{\Pi}_1^u \mathscr{W}^1$, where the terms in the right-hand side are explicitly given by:

$$\begin{aligned} \tilde{\Pi}_1^s \mathscr{W}^1(X_1) &:= e^{X_1 \mathbf{A}_1} \tilde{\Pi}_1^s \mathscr{W}^1|_{X_1=0} + \int_0^{X_1} e^{(X_1-y_1) \mathbf{A}_1} \tilde{\Pi}_1^s E_{1,1}^{-1/2} F_1(y_1) dy_1, \\ \tilde{\Pi}_1^u \mathscr{W}^1(X_1) &:= - \int_{X_1}^{\infty} e^{(X_1-y_1) \mathbf{A}_1} \tilde{\Pi}_1^u E_{1,1}^{-1/2} F_1(y_1) dy_1. \end{aligned} \quad (63)$$

We have analogous formulas for \mathcal{X}^1 up to a change in the index. The important thing to remark here is that because F_1 is an explicit function of the constructed amplitude \mathcal{U}^0 , then to determine \mathcal{U}^1 it is sufficient to determine the trace $\mathcal{W}^1_{|X_1=0}$. For later purposes, let us keep in mind that the unstable part of the trace, namely $\tilde{\Pi}_1^u \mathcal{W}^1_{|X_1=0}$, can be determined in a unique way. This can be done before to determine the stable part, just by evaluating (63) at $X_1 = 0$.

Reiterating the same kind of decoupling as in the previous paragraph, the boundary condition (45) written for $n = 1$, gives

$$\begin{cases} \mathcal{W}^1_{|x_1=X_1=0} = -(E_{1,1}^{1/2} u^1)_{|x_1=0}, \\ \mathcal{V}^1_{|x_1=0} = 0, \\ \mathcal{X}^1_{|x_2=X_2=0} = -(E_{2,2}^{1/2} u^1)_{|x_2=0}, \\ \mathcal{Q}^1_{|x_2=0} = 0. \end{cases} \quad (64)$$

Because the layers \mathcal{W}^1 and \mathcal{X}^1 are not "polarized" anymore, a decomposition in the first and third lines of (64) gives

$$\begin{cases} (\tilde{\Pi}_1^s \mathcal{W}^1)_{|x_1=X_1=0} = -(\tilde{\Pi}_1^s E_{1,1}^{1/2} u^1)_{|x_1=0}, \\ (\tilde{\Pi}_2^s \mathcal{X}^1)_{|x_2=X_2=0} = -(\tilde{\Pi}_2^s E_{2,2}^{1/2} u^1)_{|x_2=0}, \\ (\tilde{\Pi}_1^u E_{1,1}^{1/2} u^1)_{|x_1=0} = -(\tilde{\Pi}_1^u \mathcal{W}^1)_{|x_1=X_1=0} := g_1, \\ (\tilde{\Pi}_2^u E_{2,2}^{1/2} u^1)_{|x_2=0} = -(\tilde{\Pi}_2^u \mathcal{X}^1)_{|x_2=X_2=0} := g_2, \end{cases} \quad (65)$$

where we recall that the right-hand sides in the third and fourth lines are explicitly determined from \mathcal{U}^0 and \mathcal{V}^0 by using (63). Consequently, we define u^1 as the unique solution to the corner problem:

$$\begin{cases} L(\partial)u^1 = \mathcal{E}u^0 & \text{in } \Omega_T, \\ B_1 u^1_{|x_1=0} = g_1 & \text{on } \partial\Omega_{1,T}, \\ B_2 u^1_{|x_2=0} = g_2 & \text{on } \partial\Omega_{2,T}, \\ u^1_{|t \leq 0} = 0 & \text{on } \Gamma, \end{cases} \quad (66)$$

where we recall that we defined $B_1 := (\tilde{\Pi}_1^u E_{1,1}^{1/2})_{|x_1=0}$ and $B_2 := (\tilde{\Pi}_2^u E_{2,2}^{1/2})_{|x_2=0}$.

To justify the existence of such a u^1 , we need to apply Theorem 2.1. In order to do this, we have to justify that the sources lie in the spaces $\mathbf{X}_{K',T',\natural}^\infty$ and $\mathbf{X}_{K',T',\natural}^{\infty,b}$. For example for the interior source, it is clear that it is flat at the origin because u^0 is. Then, with a slight abuse of notation,

$$\sum_{j=0}^d \|\partial_t^j \partial^\gamma (\mathcal{E}u^0)(t)\|_{L^2(\Gamma)} \lesssim \sum_{j=0}^d \|\partial_t^j \partial^{\gamma+2} u^0(t)\|_{L^2(\Gamma)} \lesssim K'(K')^s (s+2)! \lesssim (K'')^s s!,$$

for K'' large enough.

Similarly the source terms are flat at $t = 0$ and the behaviour of the derivatives follows the same lines as the one described for the leading order term \mathcal{W}^0 (see equation (??)) but applied to the unstable explicit formula (63). When this problem is solved, it determines the right-hand sides in the first and second equation of (64). Finally, we lift the double trace and define:

$$\begin{aligned} \mathcal{U}^1(t, xX_1) &= -\chi(x_1) E_{1,1}^{1/2} e^{X_1 \Lambda_1} \tilde{\Pi}_1^s E_{1,1}^{1/2} u^1_{|x_1=0} + \int_0^{X_1} E_{1,1}^{1/2} e^{(X_1-y_1)\Lambda_1} \tilde{\Pi}_1^s E_{1,1}^{1/2} F_1(y_1) dy_1 \\ &\quad - \int_{X_1}^\infty E_{1,1}^{1/2} e^{(X_1-y_1)\Lambda_1} \tilde{\Pi}_1^u E_{1,1}^{1/2} F_1(y_1) dy_1, \end{aligned} \quad (67)$$

and we have a totally similar expression for \mathcal{V}^1 up to change the index one into two.

To conclude the construction, we have to justify that the second and the fourth lines of (64) are verified. We give the proof for the second line that is to say that we want to verify that $\mathcal{Q}^1_{|x_2=0} = 0$. Using the explicit expression (67), where we can not expect some cancellation between the different terms, it amounts to verify that $u^1_{|x_1=x_2=0} = 0$ and $F_1(t, x_1, 0; X_1) = 0$. We treat each term separately:

- To show that $u^1|_{x_1=x_2=0} = 0$, using the boundary conditions of (66) we obtain that

$$B_1 u|_{x_1=x_2=0} = g_1|_{x_2=0} \quad \text{and} \quad B_2 u|_{x_1=x_2=0} = g_2|_{x_1=0}.$$

So that if we can justify that $g_1|_{x_2=0} = g_2|_{x_1=0} = 0$, we end up with $u^1|_{x_1=x_2=0} \in \ker B_1 \cap \ker B_2 = \{0\}$ thanks to Assumption 4.2. From the definition of the boundary sources, to show that $g_1|_{x_2=0} = 0$ we only have to justify that $F_1(t, x_1, 0; X_1) = 0$ which is performed in the following step.

- To show that $F_1(t, x_1, 0; X_1) = 0$, we recall that F_1 is the integral with respect to the fast variable X_1 of $\mathbf{E}_1 \mathcal{U}^0$. For convenience let us also recall that by definition

$$\mathbf{E}_1 \mathcal{U} := \sum_{j=1}^d (E_{1,j} + E_{j,1}) \partial_j \partial_{X_1} \mathcal{U} - \sum_{j=1}^d \partial_j (E_{j,1}) \partial_{X_1} \mathcal{U},$$

so that the trace on $\{x_2 = 0\}$ commutes (compared to \mathcal{U}) with all the terms except the term for $j = 2$ in the first sum. Because we have already justified that $\mathcal{U}^0|_{x_2=0} = 0$ it is sufficient to show that $(\partial_2 \mathcal{U}^0)|_{x_1=x_2=0} = 0$.

From its definition (see (57)), \mathcal{U}^0 reads under the form $\mathcal{U}^0(t, x, X_1) := \mathcal{F}(t, x; X_1) u^0|_{x_1=0}(x_2)$ so that from Leibniz rule, using that $u^0|_{x_1=x_2=0} = 0$, it is sufficient to show that $(\partial_2 u^0)|_{x_1=x_2=0} = 0$. To do so we differentiate the boundary condition on $\partial\Omega_{1,T}$ of (60) with respect to x_2 , and we evaluate at $x_2 = 0$. We obtain

$$(\partial_2 B_1)|_{x_2=0} \underbrace{u^0|_{x_1=x_2=0}}_{=0} + B_1 (\partial_2 u^0)|_{x_1=x_2=0} = 0 \quad \text{so that} \quad (\partial_2 u^0)|_{x_1=x_2=0} \in (\ker B_1)|_{x_2=0} = (E_{1,1}^{-1/2} \tilde{\mathbf{E}}_1^s)|_{x_1=x_2=0}.$$

Proceeding similarly for the other boundary condition gives $(\partial_1 u^0)|_{x_1=x_2=0} \in (\ker B_2)|_{x_1=0} = (E_{2,2}^{-1/2} \tilde{\mathbf{E}}_2^s)|_{x_1=x_2=0}$.

However because we are in a non-characteristic framework we can isolate the ∂_2 derivative in the evolution equation of u^0 . We thus have

$$\partial_2 u^0 = -A_2^{-1} \left(\partial_t u^0 + A_1 \partial_1 u^0 + \sum_{j=3}^d A_j \partial_j u^0 \right),$$

so that if we evaluate at the corner, we obtain

$$(\partial_2 u^0)|_{x_1=x_2=0} = -(A_2^{-1} A_1)|_{x_1=x_2=0} (\partial_1 u^0)|_{x_1=x_2=0} \in (A_2^{-1} A_1)|_{x_1=x_2=0} \ker B_1.$$

Consequently we obtain at the end of the day $(\partial_2 u^0)|_{x_1=x_2=0} \in (\ker B_2)|_{x_1=0} \cap (A_2^{-1} A_1)|_{x_1=x_2=0} (\ker B_1)|_{x_2=0} = \{0\}$ from the last requirement of Assumption 4.2.

This concludes the construction of the first correctors u^1 , \mathcal{U}^1 and \mathcal{V}^1 . To justify the regularity we insist on the fact that the one for u^1 comes from Theorem 2.1 and then the ones for the layers are readable from the explicit formula (63). At last the amplitudes \mathcal{U}^1 and \mathcal{V}^1 vanish at initial time because u^0 and u^1 do.

4.3.3 Error estimate

In this paragraph we will use the following well-posedness result for the solution to (37). The proof of this result is admitted here

Lemma 4.2 *Let $\varepsilon > 0$ and $T > 0$ be given and $f \in L^2(\Omega_T)$ be given. We assume that the coefficients of $L(\partial)$ are symmetric and that the perturbation \mathcal{E} satisfies Assumption 4.1, then there exists $0 < T_0 := T_0(D) \leq T$ such that (37) admits a unique solution $u^\varepsilon \in \mathcal{C}^0([-\infty, T_0]; L^2(\Gamma))$. It satisfies the energy estimate: for all $0 \leq t \leq T_0$ we have the inequality*

$$\|u^\varepsilon(t, \cdot)\|_{L^2(\Omega_T)}^2 + \alpha \varepsilon \|\nabla_x u^\varepsilon\|_{L^2(\Omega_T)}^2 \lesssim \int_0^t \|f(t', \cdot)\|_{L^2(\Omega_T)}^2 dt'. \quad (68)$$

Let $N_0 \in \mathbb{N}$ be given, for T' small enough, we consider the amplitudes $(U_n)_{n \in \mathbb{N}}$ defined in Paragraphs 4.3.1 and 4.3.2. We consider the truncated expansion

$$u_{app, N_0+1}^\varepsilon := \sum_{n=0}^{N_0+1} \varepsilon^n U_n.$$

By construction $u^\varepsilon - u_{app, N_0+1}^\varepsilon$ satisfies the parabolic problem:

$$\begin{cases} (L(\partial) - \varepsilon \mathcal{E})(u^\varepsilon - u_{app, N_0+1}^\varepsilon) = -\varepsilon^{N_0+1} \mathbf{f}_{err} & \text{in } \Omega_T, \\ (u^\varepsilon - u_{app, N_0+1}^\varepsilon)|_{x_1=0} = 0 & \text{on } \partial\Omega_{1,T}, \\ (u^\varepsilon - u_{app, N_0+1}^\varepsilon)|_{x_2=0} = 0 & \text{on } \partial\Omega_{2,T}, \\ (u^\varepsilon - u_{app, N_0+1}^\varepsilon)|_{t \leq 0} = 0 & \text{on } \Gamma, \end{cases}$$

where $\mathbf{f}_{err}^\varepsilon$ is explicitly given by

$$\mathbf{f}_{err}^\varepsilon := L(\partial)u^{N_0+1} - \mathcal{E}u^{N_0+1} + (L(\partial) - \mathcal{E})(\mathcal{U}^{N_0+1} + \mathcal{V}^{N_0+1}) - \mathbf{E}_1 \mathcal{U}^{N_0+1} - \mathbf{E}_2 \mathcal{V}^{N_0+1} + \varepsilon \mathcal{E}U^{N_0+2}. \quad (69)$$

In particular we have $\mathbf{f}_{err} = O(1)$ in $L^2(\Omega_{T'})$ (the limiting terms being the two first ones in the right-hand side of (69)). So that, using the energy estimate (68) we obtain $\|u^\varepsilon - u_{app, N_0+1}^\varepsilon\|_{L^2(\Omega_{T'})} \lesssim \varepsilon^{N_0+1}$. The triangle inequality ends the proof of Theorem 4.1.

5 Discussions and examples

In this concluding section, we show that Assumption 2.4 holds for some examples coming from the Physics, see Paragraph 5.2. We also give some examples of functions lying in the space $X_{K,T}^\infty$ in Paragraph 5.1.

5.1 Discussion about the functional spaces of resolution

To simplify a little the exposition let us assume without loss of generality that $d = 2$. Clearly the set $\mathbf{X}_{T,K}^\infty$ is not empty, because $\mathbf{f}(t, x) := e^{-t} e^{-(x_1+x_2)}$ is in $\mathbf{X}_{T,0}^\infty$. Similarly, we can show that any function reading $\mathbf{f}(t, x_1, x_2) := P(t, x_1, x_2) e^{-(t+x_1+x_2)(\nu+i\theta)}$, with $\nu > 0$, $\theta \in \mathbb{R}$ and where P if some polynomial is also in $\mathbf{X}_{T,K}^\infty$. Because such functions behave like P in the neighbourhood of the corner, it give rather generic examples of sources for which we can justify that the two layers studied in Section 4 do not interact the one with the other.

At last let us mention that the gaussian lies in $\mathbf{W}_{2,T}^\infty$. Indeed for $n \geq 0$ and $y \geq 0$, let $H_n(y) := (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}$ denotes the Hermite polynomial. Then it is a well-known result that such polynomial form an orthogonal basis for $L^2(\mathbb{R}, \frac{e^{-x^2}}{\sqrt{\pi}} dy)$. So that we deduce

$$\int_0^\infty \left(\frac{d^n}{dy^n} e^{-y^2} \right)^2 dy \leq \int_0^\infty e^{-y^2} (H_n(y))^2 dy = \sqrt{\pi} 2^{n-1} n!,$$

which shows the claim.

5.2 Discussion about Assumption 2.4

In this last paragraph we discuss the structure assumptions (Assumptions 2.4 and 4.2) for some examples coming from the Physics. We here recover the examples of [7], so that Assumption 2.4 seems to be rather generic, at least from a physical point of view.

5.2.1 The wave equation

We consider the 2d wave equation in the quarter space \mathbb{R}_+^2 . This equation can classically be written under the form of a first order system. We thus consider for $T > 0$

$$\begin{cases} \partial_t u + A_1 \partial_1 u + A_2 \partial_2 u = f & \text{in }]-\infty, T] \times \mathbb{R}_+^2, \\ B_1 u|_{x_1=0} = 0 & \text{on }]-\infty, T] \times \mathbb{R}_+, \\ B_2 u|_{x_2=0} = 0 & \text{on }]-\infty, T] \times \mathbb{R}_+, \\ u|_{t \leq 0} = 0 & \text{on } \mathbb{R}_+^2, \end{cases} \quad (70)$$

where the interior coefficients A_1 and A_2 are given by

$$A_1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } A_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We thus have $\mathbf{E}_1^s := \text{vect}\{(0, 1)^T\}$ and $\mathbf{E}_2^s := \text{vect}\{(1, 1)^T\}$. Clearly $\mathbb{R}^2 := \mathbf{E}_1^s \oplus \mathbf{E}_2^s$. To study the second requirement of Assumption 2.4, we need to specify the boundary conditions in (70). They can be parametrized by two real numbers α, β by defining

$$B_1 := [1 \quad -\alpha] \text{ and } B_2 := [1 \quad -\beta].$$

The parameters leading to strictly dissipative boundary conditions (and thus to L^2 well-posed problems in the two half-spaces, from the result of [15]) are given by:

$$|\alpha| < 1 \text{ and } \beta < 0.$$

However, in order that $\ker B_1 \cap \mathbf{E}_2^s = \{0\} = \ker B_2 \cap \mathbf{E}_1^s$ we will have to avoid the exceptional case $\alpha = 1$ and $\beta = 0$. So that in particular, Assumption 2.4 is satisfied by all parameters leading to strictly dissipative boundary conditions.

As a consequence, Theorem 2.1 applies and we have existence and uniqueness of a strong solution to (70).

We now want to study Assumption 4.2 characterizing admissible viscous perturbations. In the following to please the reading, we consider a viscous perturbation \mathcal{E} whose diagonal coefficients are given by

$$\text{for } j \in \{1, 2\}, E_{j,j} := \begin{bmatrix} \alpha_j & 0 \\ 0 & \beta_j \end{bmatrix} \text{ with } \alpha_j, \beta_j > 0.$$

We thus have

$$\mathbf{\Lambda}_1 = \begin{bmatrix} \frac{1}{\alpha_1} & 0 \\ 0 & -\frac{1}{\beta_1} \end{bmatrix} \text{ and } \mathbf{\Lambda}_2 = \frac{1}{\sqrt{\alpha_2 \beta_2}} A_2.$$

So that independently of the coefficients α_j, β_j , we have $\tilde{\mathbf{E}}_1^s = \mathbf{E}_1^s$ and $\tilde{\mathbf{E}}_2^s = \mathbf{E}_2^s$. It is then a simple exercise to show that Assumption 4.2 is satisfied unconditionally on the parameters α_j and β_j . As a consequence, Theorem 4.1 applies.

5.2.2 Shallow water equation

For this second example we consider the inviscid shallow water equation. For $g > 0$ the acceleration, $\varphi_0 > 0$ the typical height and a state $(u_0, v_0) \in \mathbb{R}^2$, we consider the linearization⁹

$$\begin{cases} \partial_t u + A_1 \partial_1 u + A_2 \partial_2 u + C = F, & \text{in }]-\infty, T] \times \mathbb{R}_+^2, \\ B_1 u|_{x_1=0} = 0 & \text{on }]-\infty, T] \times \mathbb{R}_+, \\ B_2 u|_{x_2=0} = 0 & \text{on }]-\infty, T] \times \mathbb{R}_+, \\ u|_{t \leq 0} = 0 & \text{on } \mathbb{R}_+^2, \end{cases} \quad (71)$$

⁹Let us point here that in this formulation, the shallow water equation (71) does not constitute a symmetric corner problem. It is however symmetrizable, moreover the study of Assumptions 2.4 and 4.2 does not require the fact that the system is symmetric.

where the interior coefficient are given by:

$$A_1 := \begin{bmatrix} u_0 & 0 & g \\ 0 & u_0 & 0 \\ \phi_0 & 0 & u_0 \end{bmatrix}, A_2 := \begin{bmatrix} v_0 & 0 & 0 \\ 0 & v_0 & g \\ 0 & \phi_0 & v_0 \end{bmatrix} \text{ and } C := \begin{bmatrix} 0 & f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$f \in \mathbb{R}$ being the Coriolis parameter.

The eigenvalues of A_1 and A_2 are respectively given by

$$\lambda_{1,\pm} := u_0 \pm \sqrt{g\phi_0}, \lambda_{1,0} := u_0 \text{ and } \lambda_{2,\pm} := v_0 \pm \sqrt{g\phi_0}, \lambda_{2,0} := v_0.$$

In the following, we consider the case $\dim \mathbf{E}_1^s = 1$ and $\dim \mathbf{E}_2^s = 2$ (the other one being analogous), so that we impose the following restriction on the state of linearization (u_0, v_0) :

$$u_0 > 0, u_0 < \sqrt{g\phi_0} \text{ and } v_0 < 0, v_0 > -\sqrt{g\phi_0}. \quad (72)$$

We then compute easily $\mathbf{E}_1^s = \text{vect}\{(-\sqrt{g}, 0, \sqrt{\phi_0})^T\}$ and $\mathbf{E}_2^s := \text{vect}\{(0, 1, 0)^T; (0, -\sqrt{g}, -\sqrt{\phi_0})^T\}$, so that it is clear that the first point of Assumption 2.4 holds.

About the second point of Assumption 2.4, we have $B_1 \in \mathbf{M}_{2 \times 3}(\mathbb{R})$ so that we can characterize $\ker B_1 = \text{vect}\{(\alpha, \beta, \gamma)^T\}$. Thus

$$\begin{vmatrix} \alpha & 0 & 0 \\ \beta & 1 & -\sqrt{g} \\ \gamma & 0 & \sqrt{\phi_0} \end{vmatrix} = \alpha\sqrt{\phi_0} \neq 0 \text{ if and only if } \alpha \neq 0.$$

So that the condition $\ker B_1 \cap \mathbf{E}_2^s$ holds for all $\alpha \neq 0$.

For the second requirement, we parametrize $\ker B_2 := \text{vect}\{(\alpha_1, \beta_1, \gamma_1)^T; (\alpha_2, \beta_2, \gamma_2)^T\}$. We see that $\ker B_2 \cap \mathbf{E}_1^s \neq \{0\}$ if $\beta_1 = \beta_2 = 0$. If $\beta_1 \neq 0$ and $\beta_2 = 0$, to have $\ker B_2 \cap \mathbf{E}_1^s = \{0\}$ it is sufficient to impose $\sqrt{g}\gamma_1 - \sqrt{\phi_0}\alpha_1 \neq 0$. We have an analogous characterization for $\beta_1 = 0$ and $\beta_2 \neq 0$. Finally if $\beta_1 = \beta_2 = 1$, it is sufficient to have $-\sqrt{g}(\gamma_2 - \gamma_1) - \sqrt{\phi_0}(\alpha_1 - \alpha_2) \neq 0$.

Consequently Assumption 2.4 is satisfied for a "large number" of boundary conditions.

Assumption 2.3 has been considered in [3]. The obtained result is that it holds if and only if $|(u_0, v_0)|^2 < g\phi_0$ or if $|(u_0, v_0)| \geq \sqrt{g\phi_0}$ but with $u_0^2 v_0^2 > \phi_0 g(u_0^2 + v_0^2 - \phi_0 g)$.

To conclude we consider Assumption 4.2 in the particular case where $E_{1,1} = E_{2,2} = I$, consequently the two first points of Assumption 4.2 follow from Assumption 2.4. For the last point we compute:

$$A_2^{-1}A_1 = \frac{1}{\det A_2} \begin{pmatrix} \frac{\det A_2}{v_0} & 0 & \det A_2 \frac{g}{v_0} \\ -g\phi_0 & u_0 v_0 & -g u_0 \\ v_0 \phi_0 & -\phi_0 u_0 & u_0 v_0 \end{pmatrix}.$$

Let $\alpha, \beta_1, \beta_2 \in \mathbb{R}$ be such that

$$\alpha \begin{bmatrix} -\sqrt{g} \\ 0 \\ -\sqrt{\phi_0} \end{bmatrix} = \beta_1 A_2^{-1} A_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \beta_2 A_2^{-1} A_1 \begin{bmatrix} 0 \\ -\sqrt{g} \\ \sqrt{\phi_0} \end{bmatrix}$$

then solving this system in terms of α, β_1, β_2 gives

$$\begin{cases} \alpha = -\frac{\det A_2}{v_0} \sqrt{g\phi_0} \beta_2, \\ \beta_1 = (\sqrt{g}u_0 v_0 + g u_0 \sqrt{\phi_0}) \frac{\beta_2}{v_0}, \\ \beta_2 (\underbrace{\sqrt{g}u_0^2 + g \frac{u_0^2}{v_0} \sqrt{\phi_0} - \sqrt{g}\phi_0 u_0 + u_0 v_0 \sqrt{\phi_0} + \frac{\det A_2}{v_0} \sqrt{g\phi_0}}_{<0}) = 0. \end{cases}$$

But from (72) we have $\sqrt{g}u_0^2 + g \frac{u_0^2}{v_0} \sqrt{\phi_0} < 0$, so that $\alpha = \beta_1 = \beta_2 = 0$. Consequently we have $A_2^{-1}A_1 \mathbf{E}_2^s \cap \mathbf{E}_1^s = \{0\}$ unconditionally. We proceed similarly to show that $A_1^{-1}A_2 \mathbf{E}_1^s \cap \mathbf{E}_2^s = \{0\}$ unconditionally. Here again Theorem 4.1 applies (to the symmetrized formulation of (71)).

5.2.3 Linearisation of Euler equation

As a last example, we will give a partial study of Assumptions 2.4 and 4.2 for the linearisation of Euler equations. Let us consider the classical Euler equation on the quadruple $(\rho, \mathbf{u} = (u, v), e) \in \mathbb{R}^4$, describing the density ρ , the velocity \mathbf{u} and the internal energy e . After linearisation around a constant state $(\rho_0, \mathbf{u}_0, e_0) \in \mathbb{R}^4$, we will consider the system of equations (1) with coefficients

$$A_1 := \begin{bmatrix} u_0 & 0 & \frac{1}{\rho_0} \partial_\rho p_0 & \frac{1}{\rho_0} \partial_e p_0 \\ 0 & u_0 & 0 & 0 \\ \rho_0 & 0 & u_0 & 0 \\ \frac{1}{\rho_0} p_0 & 0 & 0 & u_0 \end{bmatrix} \quad \text{and} \quad A_2 := \begin{bmatrix} v_0 & 0 & 0 & 0 \\ 0 & v_0 & \frac{1}{\rho_0} \partial_\rho p_0 & \frac{1}{\rho_0} \partial_e p_0 \\ 0 & \rho_0 & v_0 & 0 \\ 0 & \frac{1}{\rho_0} p_0 & 0 & v_0 \end{bmatrix},$$

where $p = p(\rho, e)$ stands for the pressure and where $\partial_\cdot p_0$ is a short-hand notation for the evaluation of the derivative of p with respect to \cdot evaluated at the linearisation state. To have a hyperbolic system we make the classical assumption that $\partial_\rho p_0 + \frac{1}{\rho_0^2} p_0 \partial_e p_0 > 0$ and we thus will denote by $c := \sqrt{\partial_\rho p_0 + \frac{1}{\rho_0^2} p_0 \partial_e p_0}$ the local speed of sound.

The eigenvalues $\lambda_{1,\cdot}$ (resp. $\lambda_{2,\cdot}$) of A_1 (resp. A_2) are given by:

$$\lambda_{1,0} = u_0, \lambda_{1,\pm} = u_0 \pm c \text{ (resp. } \lambda_{2,0} = v_0, \lambda_{2,\pm} = v_0 \pm c \text{)}.$$

Moreover, in order to have Assumption 2.3, see [3], we need to assume that $u_0 v_0 < 0$ and that the linearisation state is subsonic $|\mathbf{u}_0| < c$. To fix the ideas, we assume that $u_0 > 0$ and $v_0 < 0$, the analysis being similar in for the opposite signs. In order that we have the suitable dimensions of the subspaces constituting Assumption 2.4, we assume that $u_0 < c$ and $v_0 > -c$. Consequently A_1 has a single negative eigenvalue, namely $\lambda_{1,-}$ and A_2 has three negative eigenvalues, namely $\lambda_{2,0}$ (with multiplicity two) and $\lambda_{2,-}$.

It is then a simple exercise to parametrize the stable subspaces \mathbf{E}_1^s and \mathbf{E}_2^s . We have

$$\mathbf{E}_1^s := \text{vect}\left\{ \left(c, 0, -\rho_0, -\frac{p_0}{\rho_0} \right)^T \right\} \quad \text{and} \quad \mathbf{E}_2^s := \text{vect}\left\{ (1, 0, 0, 0)^T; (0, 0, -\partial_e p_0, \partial_\rho p_0)^T; (0, c, -\rho_0, \frac{-p_0}{\rho_0})^T \right\},$$

Thanks to these explicit formulas it is easy to verify that the first requirement of Assumption 2.4 holds. It seems to indicate that such decomposition is rather generic, at least for systems with a physical meaning.

It is also simple to see that generically $\ker B_1 \cap \mathbf{E}_2^s = \{0\}$. However, because $\ker B_2$ is three dimensional it is complicated to verify for all possible choices of matrix B_2 that the last requirement of Assumption 2.4, namely $\ker B_2 \cap \mathbf{E}_1^s = \{0\}$, holds generically. However for a given boundary condition B_2 check if $\ker B_2 \cap \mathbf{E}_1^s = \{0\}$ or not is just a straightforward algebraic computation.

Similarly, because the dimension of one of the subspaces constituting Assumption 4.2 is now of dimension three, having a generic verification of the viscous perturbations fulfilling Assumption 4.2 is not so clear. However, once again verify that a given viscous perturbation checks or not Assumption 4.2 can be made by an algebraic computation.

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