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# ANTI-SYMPLECTIC INVOLUTIONS ON MODULI SPACES OF SHEAVES ON K3 SURFACES VIA AUTO-EQUIVALENCES

DANIELE FAENZI, GRÉGOIRE MENET, AND YULIETH PRIETO-MONTAÑEZ

ABSTRACT. We provide new examples of anti-symplectic involutions on moduli spaces of stable sheaves on K3 surfaces. These involutions are constructed through (anti) auto-equivalences of the bounded derived category of coherent sheaves on K3 surfaces arising from spherical bundles. We analyze these induced maps in the moduli space, imposing restrictions on the Mukai vector and considering the preservation of stability conditions. Our construction extends and unifies classical examples, such as the Beauville involutions, Markman-O’Grady reflections and a more recent construction by Beri-Manivel.

## 1. INTRODUCTION

Irreducible holomorphic symplectic manifolds (IHS) are significant objects in algebraic geometry, notably as they play a fundamental part in the Bogomolov decomposition Theorem [Bog75]. When examining these manifolds, a natural question arises regarding their symmetries. Symmetries and, more specifically, involutions have been analyzed recently under several perspectives, for instance derived categories (see [HT24]) and the study of fixed loci, see [FMOS22, FMOS23].

About classification, determining the birational automorphism group of these manifolds is a significant problem. A classical tool for addressing this question is the global Torelli theorem (see [Ver13] and [Mar11, Theorem 1.3]). For instance, Beri and Cattaneo recently classified birational automorphisms on the Hilbert scheme  $X^{[n]}$  of  $n$  points on a K3 surface  $X$  with Picard number one and genus  $g \geq 3$  in terms of minimal solutions of Pell’s equation  $z^2 - (g - 1)(n - 1)y^2 = 1$ . Their result implies that, under certain arithmetic conditions,  $\text{Bir}(X^{[n]}) \simeq \mathbb{Z}/2\mathbb{Z}$ , see [BC22, Theorem 1.1].

However, constructing birational automorphisms explicitly or geometrically remains a challenging problem for Hilbert schemes of points, let alone for more general moduli space  $\mathbf{M}(\mathbf{v})$ .

In this paper, we propose an explicit construction that yields involutions on the moduli spaces of stable sheaves on  $X$  with a given Mukai vector  $\mathbf{v}$ , relying on the twist along a spherical bundle  $\mathcal{S}$ , under certain restrictions on  $\mathbf{v}$  and on the Mukai vector of  $\mathcal{S}$ , see Theorem 1 below. In particular, this gives birational involutions on the Hilbert scheme  $X^{[n]}$  whenever  $g \equiv 2 \pmod{4}$  and  $n = \frac{g+2}{4}$  (see Remark 4.7). The construction recovers and gives a unified framework to several involutions already known in the literature, such as:

- the Beauville involutions [Bea83] (see Example 5.1);
- the Markman-O’Grady involutions [Mar01] (see Section 4.1);
- the Beri-Manivel involution [BM22] (see Example 5.5), see also [BM24].

Let us describe briefly our approach, based on (anti)-auto-equivalences within the bounded derived category  $\mathbf{D}^b(X)$  of coherent sheaves on  $X$ . Let  $\mathcal{S} \in \mathbf{D}^b(X)$  be a spherical object and  $\mathbf{T}_{\mathcal{S}}$  its spherical twist; see Section 3 for more details. Given a line bundle  $\mathcal{L}$  on  $X$ , we introduce the following contravariant endofunctor of  $\mathbf{D}^b(X)$ , actually an anti-auto-equivalence:

$$\mathcal{E} \mapsto \Phi_{\mathcal{S}, \mathcal{L}}^p(\mathcal{E}) = \mathbf{R}\mathcal{H}om(\mathbf{T}_{\mathcal{S}}(\mathcal{E}), \mathcal{L}[p]).$$

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For a fixed  $d \in \mathbb{Z}$ , we set

$$\Phi_{\mathcal{S},d} = \Phi_{\mathcal{S},\mathcal{O}_X(d)}^1.$$

The main observation is that, in some cases of interest,  $\Phi_{\mathcal{S},d}$  induces an involution on moduli spaces of stable sheaves on  $X$  with a well-chosen Mukai vector. Well-chosen here means that only very few choices of the Mukai vector  $\mathbf{s}$  and the twist  $d$  will work and that,  $\mathbf{s}$  and  $d$  being fixed, we have a range of good choices for  $\mathbf{v}$  to get involutions on  $M(\mathbf{v})$ . The two relevant cases for this purpose are the following (see Lemmas 3.4 and 3.8):

- $d = 0$  and  $\mathcal{S} = \mathcal{O}_X$ .
- $d = 1$  and  $\mathcal{S}$  is a stable spherical bundle of Mukai vector  $(2, H, \frac{g}{2})$ .

The first case aligns with Markman-O'Grady involutions. In this case we just recover these well-known involutions and we only point out that their regularity is proved without relying on the study of the ample and movable cones of  $M(\mathbf{v})$ . On the other hand, in the second case our approach introduces new involutions. The new construction obtained in this paper can be summarized in the following result, see Theorem 4.4 and Corollary 5.13.

**Theorem 1.** *Let  $k$  and  $g_0 \geq k(k+1)$  be integers. Let  $(X, H)$  be a polarized K3 surface such that  $\text{Pic}(X) = \mathbb{Z} \cdot H \oplus N$ , with  $H$  ample and  $N$  not containing effective divisors. Let  $\mathcal{S}$  be a spherical bundle with Mukai vector  $\mathbf{v}(\mathcal{S}) = (2, H, 2g_0 + 1)$  and fix  $\mathbf{v} = (v_0, v_1, v_2)$  with*

$$v_0 = 2k + 1, \quad v_1 = (k + 1)H, \quad v_2 = g_0(2k + 3) + k + 1.$$

*Then  $\Phi_{\mathcal{S},1}$  defines an anti-symplectic birational involution on the moduli space of sheaves  $M(\mathbf{v})$ .*

Here,  $(X, H)$  is a polarized K3 surface of genus  $4g_0 + 2$  and the condition  $g_0 \geq k(k+1)$  amounts to  $\mathbf{v}^2 \geq 0$ . The proof that the involution is well-defined on  $M(\mathbf{v})$ , which is to say, that  $\Phi_{\mathcal{S},1}(\mathcal{E})$  is stable for a sufficiently general sheaf  $\mathcal{E}$  in  $M(\mathbf{v})$ , is the point that requires more effort. To achieve this, the first obstacle is to show that, for a generic element  $\mathcal{E}$  of  $M(\mathbf{v})$  one has  $\text{Ext}^1(\mathcal{S}, \mathcal{E}) = 0$ . This holds in the above range, essentially owing to results of Yoshioka, [Yos99]. Once this is done, we show that applying our functor indeed gives a sheaf with the required invariants and check that this sheaf is stable. This forces the above choice of  $v_2$ , which agrees with the requirement that  $\mathbf{v} = (v_0, v_1, v_2)$  is indeed a fixed vector of the cohomological reflection associated with our construction. Finally, to check that the involution is anti-symplectic, we study the action on the cohomology, see Proposition 5.15 for a more detailed statement.

We note that, for  $k = 0$ , our theorem provides involutions on the Hilbert scheme of points  $X^{[n]}$ , with  $n = (g+2)/4$  for all  $g \equiv 2$  modulo 4, generalizing the construction of [BM22].

We contend that any birational involution on  $M_X(\mathbf{v})$  should be induced by an endofunctor of  $\mathbf{D}^b(X)$  obtained as composition of spherical twists, tensor product by line bundles, duality and shifts, together with pull-backs of automorphisms of  $X$  itself. In any concrete case, an obstacle to check that our functor induces a birational (respectively, biregular) involution is to check that  $\text{Ext}^1(\mathcal{S}, \mathcal{E})$  vanishes for sufficiently general (respectively, for all) sheaves  $\mathcal{E}$  in  $M(\mathbf{v})$ , for suitable choices of  $\mathbf{v}$ . This interpolation problem, tightly connected to Brill-Noether theory, has been treated for generic sheaves in recent and less recent times, see for instance [CNY23, Yos01a, Mar01, Yos99]. We don't know much of vanishing result for  $\text{Ext}^1(\mathcal{S}, \mathcal{E})$  for all sheaves in  $M(\mathbf{v})$ , though we prove a small one in Lemma 4.1.

The paper is structured as follows. In Section 2, we introduce the main concepts and results related to the moduli space of sheaves on K3 surfaces, highlighting the role and some properties of (anti) auto-equivalences of derived categories. In Section 3, we delve into the auto-equivalence  $\Phi_{\mathcal{S},d}$  within  $\mathbf{D}^b(X)$  and explore the necessary conditions for obtaining an involution on moduli spaces of stable sheaves. Section 4 establishes the existence of the involutions presented in Theorem 1. Additionally, we obtain the involutions introduced by Markman and studied by O'Grady. Section 5.1 describes several examples; notably, we proved that the Beri–Manivel involution corresponds to  $\Phi_{\mathcal{S},1}$  within the context of its action on the moduli space of stable sheaves  $M(\mathbf{v})$ , involving a suitable vector  $\mathbf{v}$  and  $g$ , see Example 5.5. Finally, the action on cohomology is described in Sections 5.2 and 5.3.

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## 2. PRELIMINARIES

Here we give a short account of the material needed to set up our construction, by recalling some fundamental facts about moduli spaces of semistable sheaves on a projective K3 surface  $X$  and describing the basic equivalences of the derived category of  $X$  and their action on the cohomology of  $X$ .

**2.1. Moduli space of sheaves on K3 surfaces.** The moduli space of semistable sheaves on a projective K3 surface  $X$  stands as a fundamental example in the study of irreducible holomorphic symplectic (IHS) manifolds. Let  $H$  be an ample divisor on  $X$  and set  $g$  for the genus of  $H$ , namely  $H^2 = 2g - 2$ , so  $(X, H)$  is a polarized K3 surface. Let  $H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$  be endowed with the pairing  $\cdot$ , representing the Mukai lattice associated with  $X$ . For a coherent sheaf  $\mathcal{E}$  on  $X$ , its Mukai vector  $\mathbf{v}(\mathcal{E}) = (v_0, c_1, v_2)$  in  $H^*(X, \mathbb{Z})$  satisfies

$$v_0 = \mathrm{rk}(\mathcal{E}), \quad c_1 = c_1(\mathcal{E}), \quad v_2 = \chi(\mathcal{E}) - \mathrm{rk}(\mathcal{E}).$$

This extends to any objects of the derived category  $\mathbf{D}^b(X)$ . Applying Riemann-Roch we obtain  $v_2 = \mathrm{rk}(\mathcal{E}) + c_1(\mathcal{E})^2/2 - c_2(\mathcal{E})$ .

The degree of  $\mathcal{E}$ , denoted by  $\mathrm{deg}_H(\mathcal{E})$ , is defined as the normalized intersection of  $H$  with the first Chern class of  $\mathcal{E}$ , i.e.,  $\mathrm{deg}_H(\mathcal{E}) := \frac{1}{H^2} H \cdot c_1(\mathcal{E})$ . The slope of  $\mathcal{E}$ , denoted by  $\mu_H(\mathcal{E})$ , is defined as

$$\mu_H(\mathcal{E}) := \frac{\mathrm{deg}_H(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}, \text{ if } \mathrm{rk}(\mathcal{E}) > 0.$$

The moduli space  $M_H(\mathbf{v})$  consists of  $H$ -semistable (torsion-free) sheaves on  $X$  with Mukai vector  $\mathbf{v}$ . The dimension of  $M(\mathbf{v})$  is given by  $\mathbf{v}^2 + 2 = c_1^2 - 2v_0v_2 + 2$ . We will omit  $H$  from the notation and simply write  $M_H(\mathbf{v})$ , since we will always work with the polarized surface  $(X, H)$ , the reference to  $H$  being implicit. Thanks to the contributions of various authors, including Mukai, Göttsche-Huybrechts, O'Grady, and Yoshioka (see [Yos01b, Proposition 5.1, Theorem 8.1]), it has been established that under specific restrictions on the Mukai vector  $\mathbf{v}$ , the moduli space  $M(\mathbf{v})$  is an IHS manifold of dimension  $2n$  with the same deformation type as a Hilbert scheme  $X^{[n]}$  of  $n$  points on a K3 surface  $X$ .

**2.2. (Anti)-equivalences and their action on the Mukai lattice.** Let us briefly recall some basic (anti)-auto-equivalences of  $\mathbf{D}^b(X)$  and review their action on the cohomology of the surface  $X$ .

**2.2.1. Duality.** Duality gives an involutive functor  $\mathbf{D}^b(X) \rightarrow \mathbf{D}^b(X)^{\mathrm{op}}$ , i. e. a contravariant functor  $\mathbf{D}^b(X) \rightarrow \mathbf{D}^b(X)$ . This is defined by

$$\mathcal{E} \mapsto \mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{O}_X),$$

where the right-hand side is seen as an object of  $\mathbf{D}^b(X)$ , see [Huy06, Corollary 5.29]. The induced mapping at the level of Mukai vectors is the automorphism  $D$  of  $H^*(X, \mathbb{Z})$  defined as

$$D : (w_0, c_1, w_2) \mapsto (w_0, -c_1, w_2).$$

**2.2.2. Twisting by a line bundle.** Let  $\mathcal{L}$  be a line bundle on  $X$ . Tensoring by  $\mathcal{L}$  gives an auto-equivalence of  $\mathbf{D}^b(X)$  defined by sending an object  $\mathcal{E}$  to  $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}$ . The induced action on Mukai vectors is  $\cdot \mathrm{ch}(\mathcal{L}) : H^*(X, \mathbb{Z}) \rightarrow H^*(X, \mathbb{Z})$ , sending the Mukai vector of a sheaf  $\mathcal{F}$  to the Mukai vector of  $\mathcal{F} \otimes \mathcal{L}$ . For  $\mathcal{L} = \mathcal{O}_X(dH)$ , we have

$$\cdot \mathrm{ch}(\mathcal{O}_X(dH)) : (w_0, c_1, w_2) \mapsto (w_0, c_1 + dw_0H, w_2 + d^2(g-1)w_0 + dc_1 \cdot H).$$

**2.2.3. Shifts.** The shift functor on  $\mathbf{D}^b(X)$  sending an object  $\mathcal{E}$  to  $\mathcal{E}[1]$  has an induced action on cohomology assigning to a Mukai vector  $\mathbf{v}$  its opposite  $-\mathbf{v}$ .

2.2.4. *Spherical twists.* Let  $\mathcal{S}$  be an object of  $\mathbf{D}^b(X)$ . We define  $\mathbf{T}_{\mathcal{S}}$  as the left mutation endofunctor on  $\mathbf{D}^b(X)$  with respect to  $\mathcal{S}$ . This functor applied to an object  $\mathcal{E}$ , is the cone of the natural evaluation morphism  $\mathbf{e}_{\mathcal{S}, \mathcal{E}}$ , hence we have a distinguished triangle

$$\mathbf{RHom}(\mathcal{S}, \mathcal{E}) \otimes \mathcal{S} \xrightarrow{\mathbf{e}_{\mathcal{S}, \mathcal{E}}} \mathcal{E} \rightarrow \mathbf{T}_{\mathcal{S}}(\mathcal{E}) \quad (1)$$

Here, if  $\mathcal{S}$  is not concentrated in a single degree, the tensor product should be intended to be derived as well, however in this paper  $\mathcal{S}$  will always be a vector bundle. If  $\mathcal{S}$  is spherical in the sense of [Huy16, Chapter 16], then  $\mathbf{T}_{\mathcal{S}}$  is an equivalence, known as a spherical twist (we refer to [Huy06, Exercise 8.5 and Proposition 8.6]). The induced action on  $\mathbf{H}^*(X, \mathbb{Z})$  is given by the reflection along the hyperplane orthogonal to the  $(-2)$ -class given by the Mukai vector  $\mathbf{v}(\mathcal{S})$  on  $\mathbf{H}^*(X, \mathbb{Z})$ , see [Huy06, Lemma 8.12]. More explicitly, we set  $R_{\mathcal{S}} : \mathbf{H}^*(X, \mathbb{Z}) \rightarrow \mathbf{H}^*(X, \mathbb{Z}) : x \mapsto x + (x \cdot \mathbf{v}(\mathcal{S})) \mathbf{v}(\mathcal{S})$ . Then, according to [Huy06, Exercise 8.5 and Lemma 8.12], we have:

$$\mathbf{v}(\mathbf{T}_{\mathcal{S}}(\mathcal{E})) = -R_{\mathcal{S}}(\mathbf{v}(\mathcal{E})).$$

### 3. CONSTRUCTION OF THE INVOLUTION

Let  $X$  be a smooth projective K3 surface polarized by  $H$  and let  $\mathcal{S}$  be a spherical object of  $X$  and  $d \in \mathbb{Z}$ . Here we define an anti-auto-equivalence  $\Phi_{\mathcal{S}, d}$  of  $\mathbf{D}^b(X)$  and show that, under suitable conditions on  $\text{Pic}(X)$  and on the Mukai vectors of  $\mathcal{S}$  and  $\mathbf{v}$ , the functor  $\Phi_{\mathcal{S}, d}$  defines a birational or even biregular involution of the moduli space  $\mathbf{M}(\mathbf{v})$  of  $H$ -semistable sheaves on  $X$  with Mukai vector  $\mathbf{v}$ .

3.1. **General construction as an equivalence of derived categories.** Let us introduce the functor we will mostly use. It is the composition of endofunctors of  $\mathbf{D}^b(X)$  described in Section 2.2.

**Definition 3.1.** Let  $\mathcal{L} \in \text{Pic}(X)$ . Set  $\Phi_{\mathcal{S}, \mathcal{L}}^p$  for the contravariant endofunctor of  $\mathbf{D}^b(X)$ :

$$\Phi_{\mathcal{S}, \mathcal{L}}^p : \mathcal{E} \mapsto \Phi_{\mathcal{S}, \mathcal{L}}(\mathcal{E}) = \mathbf{R}\mathcal{H}om(\mathbf{T}_{\mathcal{S}}(\mathcal{E}), \mathcal{L}[p]).$$

For fixed  $H \in \text{Pic}(X)$ , given  $d \in \mathbb{Z}$  we write  $\Phi_{\mathcal{S}, d} = \Phi_{\mathcal{S}, \mathcal{O}_X(dH)}^1$ , so

$$\Phi_{\mathcal{S}, d}(\mathcal{E}) = \mathbf{R}\mathcal{H}om(\mathbf{T}_{\mathcal{S}}(\mathcal{E}), \mathcal{O}_X(dH)[1]).$$

For simplicity, we often abbreviate  $\Phi_{\mathcal{S}, \mathcal{L}}^p$  to  $\Phi_{\mathcal{S}, \mathcal{L}}$  or just  $\Phi$ .

The functor  $\Phi_{\mathcal{S}, \mathcal{L}}^p$  is an anti-auto-equivalence of  $\mathbf{D}^b(X)$  as composition of equivalences and duality (see Section 2.2). Its involutive nature is clear if  $\mathcal{S}$  is self-dual up to a twist. This happens if  $\mathcal{S}$  is locally free of rank one or two.

**Lemma 3.2.** *If  $\mathcal{S} \simeq \mathbf{R}\mathcal{H}om(\mathcal{S}, \mathcal{L}[q])$  for some  $q \in \mathbb{Z}$ , then  $\Phi = \Phi_{\mathcal{S}, \mathcal{L}}^p$  is involutive for all  $p \in \mathbb{Z}$ . If  $\text{Pic}(X) \simeq \mathbb{Z} \cdot H$  and  $\mathcal{S}$  a torsion-free sheaf of rank  $s_0$ , this happens if and only if*

$$s_0 \in \{1, 2\} \quad \text{and} \quad \mathcal{L} \simeq \det(\mathcal{S})^{\frac{2}{s_0}}. \quad (2)$$

**Proof.** Recall the notion of dual spherical twist  $\mathbf{T}'_{\mathcal{S}}$ , sending an object  $\mathcal{E}$  of  $\mathbf{D}^b(X)$  to the cone of the dual evaluation map:

$$\mathcal{E} \rightarrow \mathbf{RHom}(\mathcal{E}, \mathcal{S})^{\vee} \otimes \mathcal{S}.$$

According to [ST01, Proposition 2.10],  $\mathbf{T}_{\mathcal{S}} \circ \mathbf{T}'_{\mathcal{S}}$  and  $\mathbf{T}'_{\mathcal{S}} \circ \mathbf{T}_{\mathcal{S}}$  have natural transformations to the identity functor. Under the condition  $\mathcal{S} \simeq \mathbf{R}\mathcal{H}om(\mathcal{S}, \mathcal{L}[q])$ , for any object  $\mathcal{E}$  of  $\mathbf{D}^b(X)$  we have a natural isomorphism:

$$\mathbf{T}'_{\mathcal{S}}(\mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{L})) \simeq \mathbf{R}\mathcal{H}om(\mathbf{T}_{\mathcal{S}}(\mathcal{E}), \mathcal{L}).$$

This implies that  $\Phi(\Phi(\mathcal{E}))$  is naturally isomorphic to  $\mathcal{E}$ . If  $\mathcal{S}$  is locally free and (2) holds, then  $\mathcal{S} \simeq \mathcal{S}^{\vee} \otimes \mathcal{L} \simeq \mathbf{R}\mathcal{H}om(\mathcal{S}, \mathcal{L})$  so that  $\Phi$  is involutive.

Conversely, assume that  $\mathcal{S}$  is a torsion-free sheaf, suppose  $\text{Pic}(X) \simeq H \cdot \mathbb{Z}$  for some ample divisor  $H$ , with  $H^2 = 2g - 2 \geq 2$ . Write  $\mathbf{s} = (s_0, s_1H, s_2)$  for the Mukai vector of  $\mathcal{S}$ . The

assumption  $\mathcal{S} \simeq \mathbf{R}\mathcal{H}om(\mathcal{S}, \mathcal{L}[q])$  yields  $q = 0$  and we get that  $\mathcal{S}$  is locally free. Moreover, writing  $c_1(\mathcal{S}) = s_1H$  and  $c_1(\mathcal{L}) = dH$ , we get  $s_0d = 2s_1$ . Then, the condition  $\mathbf{s}^2 = -2$  reads

$$s_0^2d^2(g-1) = 4(s_0s_2 - 1).$$

For any prime divisor  $p > 2$  of  $s_0$ , the right-hand-side is non-zero modulo  $p$ , which is a contradiction. Hence  $s_0 = 2^q$  for some  $q \geq 0$  and again the right-hand-side is non-zero modulo  $2^q$  if  $q > 2$ . Similarly one excludes the case  $q = 2$  and we are left with  $s_0 \in \{1, 2\}$ . Now (2) follows from  $s_0d = 2s_1$ .  $\square$

The image of a semistable sheaf by  $\Phi$  may not be a coherent sheaf, and even then, it is not necessarily semistable. Here we provide sufficient conditions for  $\Phi(\mathcal{E})$  to be a coherent sheaf.

**Lemma 3.3.** *We assume that:*

- i)  $\mathcal{E}$  and  $\mathcal{S}$  are torsion-free sheaves;
- ii)  $\text{Ext}^1(\mathcal{S}, \mathcal{E}) = \text{Ext}^2(\mathcal{S}, \mathcal{E}) = 0$ ;
- iii)  $\text{coker}(\mathbf{e}_{\mathcal{S}, \mathcal{E}} : \text{Hom}(\mathcal{S}, \mathcal{E}) \otimes \mathcal{S} \rightarrow \mathcal{E})$  is a torsion sheaf.

Then  $\Phi_{\mathcal{S}, \mathcal{L}}(\mathcal{E})$  is a sheaf that fits in the following exact sequence:

$$0 \rightarrow \mathcal{E}^\vee \otimes \mathcal{L} \rightarrow \text{Hom}(\mathcal{S}, \mathcal{E})^\vee \otimes \mathcal{S}^\vee \otimes \mathcal{L} \rightarrow \Phi_{\mathcal{S}, \mathcal{L}}(\mathcal{E}) \rightarrow \mathcal{E}xt^1(\mathcal{E}, \mathcal{L}) \rightarrow 0. \quad (3)$$

**Proof.** Recally that  $\mathcal{S}$  is locally free. Then, applying  $\mathbf{R}\mathcal{H}om(-, \mathcal{L})$  to the distinguished triangle (1), we obtain the distinguished triangle:

$$\mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{L}) \rightarrow \mathbf{R}\text{Hom}(\mathcal{S}, \mathcal{E})^\vee \otimes \mathcal{S}^\vee \otimes \mathcal{L} \rightarrow \Phi(\mathcal{E}).$$

Taking homology leads to a long exact sequence:

$$\cdots \rightarrow \mathcal{E}xt^k(\mathcal{E}, \mathcal{L}) \rightarrow \text{Ext}^k(\mathcal{S}, \mathcal{E}) \otimes \mathcal{S}^\vee \otimes \mathcal{L} \rightarrow \mathcal{H}^k(\Phi(\mathcal{E})) \rightarrow \mathcal{E}xt^{k+1}(\mathcal{E}, \mathcal{L}) \rightarrow \cdots \quad (4)$$

Since  $\text{coker}(\mathbf{e}_{\mathcal{S}, \mathcal{E}})$  is a torsion sheaf, the transpose of  $\mathbf{e}_{\mathcal{S}, \mathcal{E}}$  is an injective map

$$\mathcal{H}om(\mathcal{E}, \mathcal{L}) \rightarrow \text{Hom}(\mathcal{S}, \mathcal{E}) \otimes \mathcal{S}^\vee \otimes \mathcal{L}.$$

Since the kernel of this map is  $\mathcal{H}^{-1}(\Phi(\mathcal{E}))$ , we get  $\mathcal{H}^{-1}(\Phi(\mathcal{E})) = 0$ . Moreover since  $\mathcal{E}$  is torsion-free,  $\mathcal{E}xt^p(\mathcal{E}, \mathcal{L}) = 0$  for all  $p \geq 2$ . Hence, under the assumption ii),  $\mathcal{H}^p(\Phi(\mathcal{E}))$  vanishes for all  $p \in \mathbb{Z}^*$  and  $\Phi(\mathcal{E}) \simeq \mathcal{H}^0(\Phi(\mathcal{E}))$ . In other words,  $\Phi(\mathcal{E})$  is a coherent sheaf and (4) for  $k = 0$  provides the desired exact sequence.  $\square$

**Lemma 3.4.** *In the assumption of Lemma 3.3, let  $\mathcal{N} \in \text{Pic}(X)$ . Set:*

$$\mathcal{L}' = \mathcal{L} \otimes \mathcal{N}^{\otimes 2}, \quad \mathcal{S}' = \mathcal{S} \otimes \mathcal{N}, \quad \mathbf{v}' = \mathbf{v} \cdot \text{ch}(\mathcal{N})$$

and define  $f : M(\mathbf{v}) \rightarrow M(\mathbf{v}')$  by sending  $\mathcal{E}$  to  $\mathcal{E} \otimes \mathcal{N}$ . If  $\Phi_{\mathcal{S}, \mathcal{L}}$  defines an automorphism on  $M(\mathbf{v})$ , then  $\Phi_{\mathcal{S}', \mathcal{L}'}$  defines an automorphism on  $M(\mathbf{v}')$  and:

$$\Phi_{\mathcal{S}', \mathcal{L}'} = f \circ \Phi_{\mathcal{S}, \mathcal{L}} \circ f^{-1}.$$

**Proof.** For any object  $\mathcal{E}$  of  $\mathbf{D}^b(X)$  we have

$$\mathbf{T}_{\mathcal{S}'}(\mathcal{E} \otimes \mathcal{N}) \simeq \mathbf{T}_{\mathcal{S}}(\mathcal{E}) \otimes \mathcal{N}$$

Therefore we get the following isomorphisms:

$$\Phi_{\mathcal{S}', \mathcal{L}'}(\mathcal{E} \otimes \mathcal{N}) = \mathbf{R}\mathcal{H}om(\mathbf{T}_{\mathcal{S}'}(\mathcal{E} \otimes \mathcal{N}), \mathcal{L}'[1]) \simeq \mathbf{R}\mathcal{H}om(\mathbf{T}_{\mathcal{S}}(\mathcal{E}), \mathcal{L} \otimes \mathcal{N}[1]) = \Phi_{\mathcal{S}, \mathcal{L}}(\mathcal{E}) \otimes \mathcal{N}.$$

This implies the desired formula.  $\square$

**Consequence 3.5.** As Lemma 3.4 shows, while dealing with  $\mathcal{L} = \mathcal{O}_X(dH)$ , modulo conjugation, we can assume that  $d \in \{0, 1\}$ . So without loss of generality, when  $\text{Pic}(X) \simeq \mathbb{Z} \cdot H$ , we assume that  $d \in \{0, 1\}$  for the sequel of this paper. Note that, in view of Lemma 3.2, we deduce:

- either  $\mathcal{S} \simeq \mathcal{O}_X$ ,  $\mathbf{s} = (1, 0, 1)$  and  $d = 0$ ,
- or  $\mathcal{S}$  is a spherical stable bundle of rank 2, the genus  $g$  is even,  $\mathbf{s} = (2, H, \frac{g}{2})$  and  $d = 1$ .

**3.2. Strategy to obtain an involution on the moduli space.** From now, we assume that  $\mathcal{E}$  and  $\mathcal{S}$  are torsion-free sheaves. We set  $\mathbf{v}(\mathcal{E}) = (v_0, v_1H, v_2)$  and  $\mathbf{s} = (s_0, s_1H, s_2) = v(\mathcal{S})$  the Mukai vector of  $\mathcal{E}$  and  $\mathcal{S}$  respectively.

*Objective 3.6.* The objective of the paper is to determine when  $\Phi$  defines an automorphism on the moduli space  $M(\mathbf{v})$ . Therefore, there are two main properties that  $\Phi(\mathcal{E})$  has to verify:

- (i)  $\mathbf{v}(\Phi(\mathcal{E})) = \mathbf{v}$ .
- (ii)  $\Phi(\mathcal{E})$  is torsion-free and semi-stable.

**3.2.1. Determination of Mukai vectors.** In this subsection, we work under the assumption that  $\text{Pic}(X) = \mathbb{Z} \cdot H$ , where  $H$  is an ample divisor with  $H^2 = 2g - 2$ , even though later on we will authorize a slightly more general Picard lattice. The reason of looking at  $\text{Pic}(X) \simeq \mathbb{Z}$  is that we would like to study involutions that do exist for a moduli space  $M(\mathbf{v})$  over  $X$ , with  $X$  general. By the description of the action on cohomology of the functors defining  $\Phi$  given in §2.2, we get:

**Lemma 3.7.** *The action of  $\Phi$  on Mukai vectors satisfies:*

$$\mathbf{v}(\Phi(\mathcal{E})) = -(D \circ R_{\mathcal{S}}(\mathbf{v}(\mathcal{E}))) \cdot \text{ch}(\mathcal{O}_X(dH)). \quad (5)$$

From the previous lemma, we can determine the different possibilities on  $\mathcal{S}$  with Mukai vector  $\mathbf{s}$  to verify condition (i) of Objective 3.6 when  $d = 0$  and when  $d = 1$ .

**Lemma 3.8.** *Let  $\mathbf{v}$  be a Mukai vector with  $\mathbf{s} \cdot \mathbf{v} \neq 0$ . Then  $\mathbf{v}(\Phi(\mathcal{E})) = \mathbf{v}$  if and only if:*

- i)  $(s_0, d) = (1, 0)$ ,  $\mathcal{S} = \mathcal{O}_X$  and  $v_0 = v_2$ , or:
- ii)  $(s_0, d) = (2, 1)$ ,  $\mathcal{S}$  is spherical with  $\mathbf{s} = (2, H, \frac{g}{2})$  and  $2v_2 = (2g - 2)v_1 - v_0(\frac{g}{2} - 1)$ .

**Proof.** In case i), we know that the spherical bundle is  $\mathcal{O}_X$  and that  $d = 0$ , see Consequence 3.5. So  $\mathbf{s} = (1, 0, 1)$ . Moreover, looking at the rank of the involved sheaves, Lemma 3.7 gives  $v_0 = -\mathbf{s} \cdot \mathbf{v} - v_0$ , hence  $v_0 = v_2$ . To finish the proof, it only remains to check that  $\chi(\Phi(\mathcal{E})) = \chi(\mathcal{E})$  when  $\mathbf{s} = (1, 0, 1)$  and  $v_0 = v_2$ . However this follows from the equation on the Euler characteristics obtained from (5).

Let us look at case ii). Set  $a := -\mathbf{s} \cdot \mathbf{v}$ . Lemma 3.7 provides equations on the ranks and the first Chern classes, to the effect that  $2v_0 = as_0$  and  $v_1 - v_0 = -as_1 + v_1$ . We get

$$v_0 = as_1. \quad (6)$$

Recall from Consequence 3.5 that  $\mathbf{s} = (2, H, \frac{g}{2})$ , so that (6) yields  $v_0 = a$ . We obtain

$$2v_2 = (2g - 2)v_1 - v_0 \left( \frac{g}{2} - 1 \right). \quad (7)$$

It remains to verify that we have  $\chi(\Phi(\mathcal{E})) = \chi(\mathcal{E})$ . Lemma 3.7 provides the following equation on the Euler characteristics:

$$\chi((\Phi(\mathcal{E}) \otimes \mathcal{O}_X(-H))) = v_0 \left( \frac{g}{2} + 2 \right) - v_2 - v_0.$$

However:

$$\chi(\mathcal{E}(-H)) = v_2 + v_0 - (g - 1)(2v_1 - v_0).$$

So, we need to verify that:

$$v_2 + v_0 - (g - 1)(2v_1 - v_0) = v_0 \left( \frac{g}{2} + 2 \right) - v_2 - v_0.$$

This is equivalent to relation (7). □

**3.2.2. Preservation of stability.** With the Mukai vectors determined in the previous subsection, we now prove that the functor under consideration preserves the family of semistable sheaves. In this subsection, we make a slightly more general assumption on the Picard group of  $X$ ; we assume that  $\text{Pic}(X) = \mathbb{Z} \cdot H \oplus N$ , where  $H$  is an ample divisor with  $H^2 = 2g - 2$  and  $N$  is a lattice that does not contain any effective divisor.

**Proposition 3.9.** *We assume that  $\text{Ext}^1(\mathcal{S}, \mathcal{E}) = \text{Ext}^2(\mathcal{S}, \mathcal{E}) = 0$ .*

- (i) If  $\dim(\text{coker}(\mathbf{e}_{\mathcal{S}, \mathcal{E}})) \leq 0$ , then  $\Phi(\mathcal{E})$  is a torsion-free sheaf.
- (ii) If in addition  $\text{Ker}(\mathbf{e}_{\mathcal{S}, \mathcal{E}})$  is slope-stable, then  $\Phi(\mathcal{E})$  is slope-stable.

**Proof.** Set  $\mathcal{K} := \text{Ker}(\mathbf{e}_{\mathcal{S}, \mathcal{E}})$  and  $\mathcal{E}_0 := \text{Im}(\mathbf{e}_{\mathcal{S}, \mathcal{E}})$ . We obtain an exact sequence:

$$0 \rightarrow \mathcal{K} \rightarrow \text{Hom}(\mathcal{S}, \mathcal{E}) \otimes \mathcal{S} \rightarrow \mathcal{E}_0 \rightarrow 0. \quad (8)$$

Since  $\text{coker}(\mathbf{e}_{\mathcal{S}, \mathcal{E}})$  has dimension at most 0,  $c_1(\mathcal{E}_0) = c_1(\mathcal{E})$  and  $\mathcal{E}^\vee = \mathcal{E}_0^\vee$ . We can deduce the first Chern class of  $\mathcal{K}$ :

$$c_1(\mathcal{K}) = -(v(\mathcal{S}) \cdot v) c_1(\mathcal{S}) - c_1(\mathcal{E}). \quad (9)$$

Note also that  $\mathcal{K}$  is a reflexive sheaf, hence a vector bundle.

By Lemma 3.3,  $\Phi(\mathcal{E})$  is a sheaf and verifies (3). Consider the image  $\mathcal{G}$  of the middle map in the exact sequence. We have:

$$0 \rightarrow \mathcal{G} \rightarrow \Phi(\mathcal{E}) \rightarrow \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X(dH)) \rightarrow 0. \quad (10)$$

Note that  $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X(dH))$  is supported on isolated points since  $\mathcal{E}$  is torsion free. Taking the dual, tensored by  $\mathcal{O}_X(dH)$ , of (8) we see that  $\mathcal{G}$  is also the image of the middle map of the following sequence:

$$0 \rightarrow \mathcal{E}_0^\vee(dH) \rightarrow \text{Hom}(\mathcal{S}, \mathcal{E})^\vee \otimes \mathcal{S}^\vee(dH) \rightarrow \mathcal{K}^\vee(dH) \rightarrow \mathcal{E}xt^1(\mathcal{E}_0, \mathcal{O}_X(dH)) \rightarrow 0. \quad (11)$$

In particular, we have  $\mathcal{G}^\vee \simeq \Phi(\mathcal{E})^\vee$  and  $\mathcal{G}^\vee \simeq \mathcal{K}(-dH)$ . This shows that  $c_1(\mathcal{G}^\vee) = c_1(\Phi(\mathcal{E})^\vee)$  and  $c_1(\mathcal{G}^\vee) = c_1(\mathcal{K}(-dH))$ . Combined with (9) and (5), we obtain that:

$$c_1(\Phi(\mathcal{E})^\vee) = -c_1(\Phi(\mathcal{E})).$$

This proves that  $\Phi(\mathcal{E})$  cannot have 1-dimensional torsion.

Moreover, we know that  $\Phi$  is an anti-auto-equivalence of  $\mathbf{D}^b(X)$ , which implies

$$\text{Ext}^i(\Phi(\mathcal{E}), \Phi(\mathcal{E})) \simeq \text{Ext}^i(\mathcal{E}, \mathcal{E}), \quad \forall i \in \mathbb{Z}.$$

Therefore, since  $\mathcal{E}$  is stable and hence simple,  $\Phi(\mathcal{E})$  is also simple. This implies that  $\Phi(\mathcal{E})$  has no zero-dimensional torsion. To see this, note that if  $p \in Z$  was a point in the support of a zero-dimensional torsion subsheaf of  $\Phi(\mathcal{E})$ , since  $\Phi(\mathcal{E})$  has positive rank, one could find an epimorphism of  $\Phi(\mathcal{E})$  onto the skyscraper sheaf  $\mathcal{O}_p$ , and since  $\mathcal{O}_p$  maps injectively into  $\Phi(\mathcal{E})$ , we would get an endomorphism of  $\Phi(\mathcal{E})$  factoring through  $\mathcal{O}_p$ , which is absurd. So (i) is proved.

We are ready to prove (ii). Since  $\mathcal{K}$  is a vector bundle. The slope stability of  $\mathcal{K}$  implies the slope stability of  $\mathcal{K}^\vee(dH)$ . However by (11),  $\mathcal{G}$  differs from  $\mathcal{K}^\vee(dH)$  only along a zero-dimensional subset of  $X$ . So  $\mathcal{G}$  is slope-stable. Similarly, by (10)  $\mathcal{G}$  and  $\Phi(\mathcal{E})$  differs along a zero-dimensional subset of  $X$ . We obtain that  $\Phi(\mathcal{E})$  is slope stable.  $\square$

When dealing with twisting about  $\mathcal{O}_X$ , we are able to prove Objective 3.6, (ii) when  $v_1 = 1$ , so the next step would be to establish the following lemma. We omit its proof since the ideas are similar to the ones used by Markman (see [Mar01, P. 682–684] and [O'G05, Lemma 4.10]) and the proof is also similar to the one of Proposition 3.11.

**Lemma 3.10.** *We assume that:*

- (i)  $\text{Ext}^1(\mathcal{O}_X, \mathcal{E}) = \text{Ext}^2(\mathcal{O}_X, \mathcal{E}) = 0$  and  $\text{Hom}(\mathcal{O}_X, \mathcal{E}) \neq 0$ ;
- (ii)  $\mathbf{v}(\Phi(\mathcal{E})) = \mathbf{v}(\mathcal{E})$  and  $c_1(\mathcal{E}) = H$ ;
- (iii)  $\text{Pic}(X) = \mathbb{Z} \cdot H \oplus N$  with  $N$  not containing any effective divisor.

*Then  $\text{coker}(\mathbf{e}_{\mathcal{O}_X, \mathcal{E}})$  is supported on isolated points and  $\text{Ker}(\mathbf{e}_{\mathcal{O}_X, \mathcal{E}})$  is slope-stable.*

When dealing with involutions defined by twisting about a spherical bundle  $\mathcal{S}$  of rank 2, our choice for the Mukai vector is made so that the  $H$ -slope of  $\mathcal{E}$  is just a bit bigger than  $1/2$ . Then, in order to guarantee preservation of stability, we need the following proposition.

**Proposition 3.11.** *Let  $\mathcal{S}$  be a spherical bundle of Mukai vector  $(2, H, \frac{g}{2})$  with  $g$  even. Assume:*

- (i)  $\text{Ext}^1(\mathcal{S}, \mathcal{E}) = \text{Ext}^2(\mathcal{S}, \mathcal{E}) = 0$  and  $\text{Hom}(\mathcal{S}, \mathcal{E}) \neq 0$ ;
- (ii)  $\text{rk}(\mathcal{E}) = 2k + 1$  with  $k \in \mathbb{N}$ ;
- (iii)  $\mathbf{v}(\Phi(\mathcal{E})) = \mathbf{v}(\mathcal{E})$  and  $c_1(\mathcal{E}) = (k + 1)H$ ;

(iv)  $\text{Pic}(X) = \mathbb{Z} \cdot H \oplus N$  with  $N$  not containing any effective divisor.

Then  $\text{coker}(\mathbf{e}_{\mathcal{S}, \mathcal{E}})$  is supported on isolated points and  $\text{Ker}(\mathbf{e}_{\mathcal{S}, \mathcal{E}})$  is slope-stable.

**Proof.** We first prove that  $\text{coker}(\mathbf{e}_{\mathcal{S}, \mathcal{E}})$  is supported on isolated points. We denote by  $\mathcal{E}_0$  the image of  $\mathbf{e}_{\mathcal{S}, \mathcal{E}}$  and by  $v_0$  the rank of  $\mathcal{E}$ . Let  $c \in \mathbb{Z}$  and  $D \in N$  such that  $c_1(\mathcal{E}_0) = cH + D$ . By stability of  $\mathcal{E}$  and  $\mathcal{S}$ , we must have:

$$\frac{1}{2} \leq \frac{c}{\text{rk}(\mathcal{E}_0)} \leq \frac{k+1}{2k+1}.$$

First, we notice that we may not have  $\frac{1}{2} < \frac{c}{\text{rk}(\mathcal{E}_0)} < \frac{k+1}{2k+1}$ . Indeed, this would imply

$$\frac{\text{rk}(\mathcal{E}_0)}{2} < c < \frac{\text{rk}(\mathcal{E}_0)}{2} + \frac{\text{rk}(\mathcal{E}_0)}{4k+2} < \frac{\text{rk}(\mathcal{E}_0)}{2} + \frac{1}{2}.$$

But this is impossible for any choice of the integers  $c$  and  $\text{rk}(\mathcal{E}_0)$ .

Now, we consider the case  $\frac{c}{\text{rk}(\mathcal{E}_0)} = \frac{1}{2}$ . We write  $\mathcal{K} = \text{Ker}(\mathbf{e}_{\mathcal{S}, \mathcal{E}})$ . We have  $\text{rk}(\mathcal{E}_0) = 2c$  and  $\text{rk}(\mathcal{K}) = 2\text{rk}(\mathcal{E}) - \text{rk}(\mathcal{E}_0) = 2(\text{rk}(\mathcal{E}) - c)$ . Moreover  $c_1(\mathcal{K}) = (\text{rk}(\mathcal{E}) - c)H - D$ . Hence the slope of  $\mathcal{K}$  is also  $\frac{1}{2}$ . Therefore by stability of  $\mathcal{S}$ , we must have  $\mathcal{K} \simeq \mathcal{S}^{\oplus(\text{rk}(\mathcal{E})-c)}$ . However, this is impossible because  $\text{Hom}(\mathcal{S}, \mathcal{K}) = 0$ . Indeed, we have the exact sequence:

$$0 \rightarrow \mathcal{K} \rightarrow \text{Hom}(\mathcal{S}, \mathcal{E}) \otimes \mathcal{S} \rightarrow \mathcal{E};$$

if we take the image by the functor  $\text{Hom}(\mathcal{S}, -)$ , since  $\mathcal{S}$  is simple we obtain:

$$0 \rightarrow \text{Hom}(\mathcal{S}, \mathcal{K}) \rightarrow \text{Hom}(\mathcal{S}, \mathcal{E}) \rightarrow \text{Hom}(\mathcal{S}, \mathcal{E}).$$

Note that the last map is induced by the evaluation map and therefore it is just the identity. We obtain  $\text{Hom}(\mathcal{S}, \mathcal{K}) = 0$ .

The only remaining case is  $\frac{c}{\text{rk}(\mathcal{E}_0)} = \frac{k+1}{2k+1}$ . Since  $k+1$  and  $2k+1$  are coprime and  $\text{rk}(\mathcal{E}_0) \leq 2k+1$ , we obtain  $\text{rk}(\mathcal{E}_0) = 2k+1 = \text{rk}(\mathcal{E})$  which is what we wanted to prove. We also get  $c_1(\mathcal{E}_0) = c_1(\mathcal{E})$ , hence  $c_1(\mathcal{E}/\mathcal{E}_0) = 0$  so that  $\text{coker}(\mathbf{e}_{\mathcal{S}, \mathcal{E}}) \simeq \mathcal{E}/\mathcal{E}_0$  is supported on isolated points.

Now, we prove that  $\mathcal{K}$  is slope-stable. We assume that there exists a destabilizing subsheaf  $\mathcal{H} \hookrightarrow \mathcal{K}$  and we will find a contradiction. We have  $\mathcal{H} \hookrightarrow \mathcal{K} \hookrightarrow \text{Hom}(\mathcal{S}, \mathcal{E}) \otimes \mathcal{S}$ . The slope of  $\mathcal{H}$  is given by  $\frac{\text{rk}(\mathcal{H})-c}{\text{rk}(\mathcal{H})} = \frac{k}{2k+1}$ . Hence by stability of  $\mathcal{S}$ , we obtain:

$$\frac{k}{2k+1} \leq \frac{\alpha}{\text{rk}(\mathcal{H})} \leq \frac{1}{2},$$

with  $c_1(\mathcal{H}) = \alpha H + D'$  with  $D' \in N$ . Note that  $\text{rk}(\mathcal{H}) < 2k+1$ . As before, we have:

$$\frac{\text{rk}(\mathcal{H})}{2} - \frac{1}{2} < \frac{\text{rk}(\mathcal{H})}{2} - \frac{\text{rk}(\mathcal{H})}{4k+2} \leq \alpha \leq \frac{\text{rk}(\mathcal{H})}{2}.$$

The only solution is then  $\text{rk}(\mathcal{H}) = 2\alpha$ . As before this implies that  $\mathcal{H} = \mathcal{S}^{\oplus\alpha}$  and it is impossible because we have seen that  $\text{Hom}(\mathcal{S}, \mathcal{K}) = 0$ . Therefore  $\mathcal{K}$  is slope-stable, which finishes the proof.  $\square$

#### 4. EXISTENCE OF THE INVOLUTION

Here we describe the argument to see that the functor  $\Psi_{\mathcal{S}, d}$  indeed induces an involution on the moduli space  $M(\mathbf{v})$  over  $X$ . We first deal with the case  $\mathcal{S} = \mathcal{O}_X$ , where we recover results of Markman and O'Grady, then move to the case  $\text{rk}(\mathcal{S}) = 2$ , where we describe new involutions.

**4.1. Markman-O'Grady's involutions.** In this section, we review Markman's results [Mar01] (see also [O'G05, Section 4]). This corresponds to the spherical twists about  $\mathcal{S} = \mathcal{O}_X$ , with  $d = 0$ , see Consequence 3.5. Our approach provides a direct path to check regularity of Markman-O'Grady's involutions, without any input from the study of divisors on  $M(\mathbf{v})$ .

In this section, for simplicity, we denote  $\Phi_{\mathcal{O}_X, 0} = \Phi$ . We are going to prove the existence of the involution using Proposition 3.9 and Lemma 3.10. First, we propose an alternative proof of the vanishing of  $\text{Ext}^1(\mathcal{O}_X, \mathcal{E}) = H^1(\mathcal{E})$ .

**Lemma 4.1.** *Assume  $r^2 \leq g < (r+1)^2$ . Let  $\mathcal{E}$  be any element of  $M(r, H, r)$ . Then  $H^1(\mathcal{E}) = 0$ .*

**Proof.** Assume  $H^1(\mathcal{E}) \neq 0$  and we will find a contradiction. Note that, by Serre duality, we have  $\text{Ext}^1(\mathcal{E}, \mathcal{O}_X) \simeq H^1(\mathcal{E})^\vee \neq 0$ . Then, we may choose a non-zero element of  $\text{Ext}^1(\mathcal{E}, \mathcal{O}_X)$  and write the corresponding non-trivial extension, which takes the form:

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow 0. \quad (12)$$

The strategy of the proof is the following. The genus  $g$  is chosen in such a way that

$$\mathbf{v}(\mathcal{F})^2 < -2.$$

So if we prove that  $\mathcal{F}$  is slope-stable we will have a contradiction.

Let us prove that  $\mathcal{F}$  is slope-stable. For this, by contradiction we take a saturated destabilizing subsheaf  $\mathcal{K}$  of  $\mathcal{F}$  and set  $\mathcal{Q} = \mathcal{F}/\mathcal{K}$  so that  $\mathcal{Q}$  is a non-trivial torsion-free coherent sheaf on  $X$ . Put  $\mathcal{K}'$  for the intersection of  $\mathcal{K}$  and  $\mathcal{O}_X$  in  $\mathcal{F}$ ,  $\mathcal{K}'' = \mathcal{K}/\mathcal{K}'$ ,  $\mathcal{Q}' = \mathcal{O}_X/\mathcal{K}'$  and  $\mathcal{Q}'' = \mathcal{E}/\mathcal{K}''$ .

Let us summarize the notation in the following commutative exact diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{K}'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{Q}' & \longrightarrow & \mathcal{Q} & \longrightarrow & \mathcal{Q}'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Set  $c = \deg_H(\mathcal{K})$ . Then, since  $\mathcal{K}$  destabilizes  $\mathcal{F}$ , we must have:

$$c \geq 1. \quad (13)$$

Consider the sheaf  $\mathcal{K}'$  and assume  $\mathcal{K}' \neq 0$ . Since  $\mathcal{O}_X$  is locally free of rank 1, the sheaf  $\mathcal{K}'$  must be torsion-free of rank 1. Therefore,  $\mathcal{Q}'$  is a torsion sheaf, which cannot be non-trivial since  $\mathcal{Q}$ , and hence  $\mathcal{Q}'$ , are torsion-free sheaves. This says that  $\mathcal{Q} \simeq \mathcal{Q}''$ . Therefore  $\text{rk}(\mathcal{K}'') < r$  as otherwise  $\mathcal{Q}''$  would be a torsion sheaf. By stability of  $\mathcal{E}$ , we must then have  $c \leq \text{rk}(\mathcal{K}'')/r$ , which is to say  $c \leq 0$ . But this contradicts (13).

We have thus proved that  $\mathcal{K}' = 0$ , which in turn gives  $\mathcal{Q}' \simeq \mathcal{O}_X$  and  $\mathcal{K} \simeq \mathcal{K}''$ . Since  $\mathcal{E}$  is stable, whenever  $\text{rk}(\mathcal{K}'') < r$ , we get  $c \leq 0$ , which again contradicts (13). Therefore  $\text{rk}(\mathcal{K}'') = r$ , so that  $\mathcal{Q}''$  is a torsion sheaf supported on a closed subscheme  $Z \subsetneq X$ . In case  $\dim(Z) = 1$ , then  $Z$  is a divisor of class  $(1-c)H + D$ , with  $D \in N$ . Since  $Z$  is effective we would have  $(1-c) > 0$  which is against (13).

Therefore  $Z$  has finite length. This in turn gives  $c_1(\mathcal{Q}) = 0$ . Now, since the sheaf  $\mathcal{Q}$  is torsion-free of rank 1 and with  $c_1(\mathcal{Q}) = 0$ , the injection  $\mathcal{O}_X \rightarrow \mathcal{Q}$  must be an isomorphism, as one sees by composing with the embedding  $\mathcal{Q} \hookrightarrow \mathcal{Q}^{**} \simeq \mathcal{O}_X$ . In other words, composing the injection  $\mathcal{O}_X \hookrightarrow \mathcal{F}$  with the projection  $\mathcal{F} \rightarrow \mathcal{Q} \simeq \mathcal{O}_X$  we get an isomorphism. This says that the extension (12) splits, a contradiction. We have thus checked that  $\mathcal{F}$  is slope-stable.  $\square$

We deduce the (well-known) regularity if Markman-O'Grady's involutions.

**Proposition 4.2.** *Let  $X$  be a K3 surface with  $\text{Pic}(X) = \mathbb{Z} \cdot H \oplus^\perp N$  with  $H^2 = 2g - 2$  and  $N$  a lattice which does not contain any effective divisor. Let  $r \geq 1$  be an integer with  $r^2 \leq g < (r + 1)^2$ . Then, whenever  $\mathcal{E}$  lies in  $M(r, H, r)$ , also  $\Phi(\mathcal{E})$  lies in  $M(r, H, r)$ . This provides a biregular involution  $\Phi$  of  $M(r, H, r)$ .*

**Proof.** By Lemma 3.8, we already know that  $\mathbf{v}(\Phi(\mathcal{E})) = \mathbf{v}(\mathcal{E})$ . So, it remains to prove that  $\Phi(\mathcal{E})$  is slope-stable and torsion-free. This is a consequence of Proposition 3.9 and Lemma 3.10 if we know that  $\text{Ext}^1(\mathcal{O}_X, \mathcal{E}) = \text{Ext}^2(\mathcal{O}_X, \mathcal{E}) = 0$ . However, the first vanishing is the content of Lemma 4.1 while the second one follows from stability of  $\mathcal{E}$ .  $\square$

*Remark 4.3.* Note that Proposition 4.2 can be generalized when  $g \geq (r + 1)^2$ , in this case we only obtain a birational involution. It follows for instance from Lemma 3.10 and [CNY23, Theorem 1.1] (it is also explained in [O'G05, Section 4.2.2]). The case  $r = 1$  corresponds to the Beauville involutions (see for instance [O'G05, Section 4.1.2] and Example 5.1).

**4.2. Involutions from spherical bundles of rank two.** In this section, we study the involution  $\Phi_{\mathcal{S},1}$  with  $\mathbf{v}(\mathcal{S}) = (2, H, \frac{g}{2})$ , see Lemma 3.8. The main point is to define new involutions from spherical twists about bundles of rank 2, which is the content of our main result, namely Theorem 1. We are in position to prove here, except for the statement on the anti-symplectic nature of the involution, which will be checked in the next section.

**Theorem 4.4.** *Let  $(g, k) \in \mathbb{N}^2$  with  $g \equiv 2$  modulo 4. Let  $(X, H)$  be a polarized K3 surface such that  $\text{Pic}(X) = \mathbb{Z} \cdot H \oplus N$  with  $H^2 = 2g - 2$  and  $N$  not containing any effective divisor. Let  $\mathcal{S}$  be the spherical bundle of Mukai vector  $(2, H, \frac{g}{2})$ . Consider the Mukai vector  $\mathbf{v} = (v_0, v_1, v_2)$  with*

$$v_0 = 2k + 1, \quad v_1 = (k + 1)H, \quad 2v_2 = (2g - 2)(k + 1) - (2k + 1)\left(\frac{g}{2} - 1\right),$$

as given by Lemma 3.8. We assume that  $\dim(M(\mathbf{v})) \geq 2$ , namely  $g \geq (2k + 1)^2 + 1$ . Then  $\Phi_{\mathcal{S},1}$  induces a birational involution on  $M(\mathbf{v})$ .

One of the points we need in order to prove this result is to check that, for  $\mathcal{E}$  sufficiently general in  $M(\mathbf{v})$ , we have  $\text{Ext}^1(\mathcal{S}, \mathcal{E}) = 0$ . However, this essentially follows [Yos99, Lemma 2.6]. We provide here a self-contained proof of this fact.

**Lemma 4.5.** *Let  $g$  be an integer and let  $X$  be a K3 surface having  $\text{Pic}(X) = \mathbb{Z} \cdot H \oplus N$ , with  $N$  which does not contain any effective divisor. We consider two Mukai vectors  $\mathbf{s} = (s_0, s_1H, s_2)$  and  $\mathbf{v} = (v_0, v_1H, v_2)$  which verify the following conditions:*

$$\mathbf{s}^2 = -2, \quad \mathbf{v} \cdot \mathbf{s} \leq 0, \quad s_1v_0 < s_0v_1.$$

Let  $\mathcal{S}$  be the vector bundle of Mukai vector  $\mathbf{s}$ . We assume that  $\text{Ext}^1(\mathcal{S}, \mathcal{E}) \neq 0$  for  $\mathcal{E}$  generic in  $M(\mathbf{v})$ . Then there is a Zariski open dense subset  $U \subset M(\mathbf{v})$  such that for all  $\mathcal{E} \in U$ , a generic nonsplitting extension

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow 0$$

gives a non-simple sheaf  $\mathcal{F}$ .

**Proof.** By assumption, we have  $\text{Ext}^1(\mathcal{S}, \mathcal{E})^\vee \simeq \text{Ext}^1(\mathcal{E}, \mathcal{S}) \neq 0$  for all  $\mathcal{E}$  in  $M(\mathbf{v})$ . We define

$$a = \min\{h^1(\mathcal{S}^\vee \otimes \mathcal{E}) \mid \mathcal{E} \in M(\mathbf{v})\}$$

Then, we consider the open dense subset  $U$  of  $M(\mathbf{v})$  where equality is attained. We get a locally free sheaf  $\mathcal{V}$  over  $U$  and a projective bundle  $\mathbb{P}(\mathcal{V}) \rightarrow U$ , a point of  $\mathbb{P}(\mathcal{V})$  being given by a pair  $(\mathcal{E}, [\xi])$ , where  $[\xi]$  is the proportionality class of  $\xi \in \text{Ext}^1(\mathcal{E}, \mathcal{S}) \setminus \{0\}$ . The element  $\xi$  represents a nonsplitting extension:

$$0 \rightarrow \mathcal{S} \xrightarrow{f} \mathcal{F} \rightarrow \mathcal{E} \rightarrow 0. \quad (14)$$

where the map  $f$  is induced by the extension. We get the following dimension count:

$$\dim(\mathbb{P}(\mathcal{V})) = \dim(M(\mathbf{v})) + h^1(\mathcal{S}^\vee \otimes \mathcal{E}) = \mathbf{v}^2 + a + 1.$$

Note that  $\text{Ext}^2(\mathcal{S}, \mathcal{E})^\vee \simeq \text{Hom}(\mathcal{E}, \mathcal{S}) = 0$  by stability of  $\mathcal{E}$  and  $\mathcal{S}$ . Also, applying  $\text{Hom}(\mathcal{S}, -)$  to (14) we have

$$0 \rightarrow \text{End}(\mathcal{S}) \rightarrow \text{Hom}(\mathcal{S}, \mathcal{F}) \rightarrow \text{Hom}(\mathcal{S}, \mathcal{E}) \rightarrow 0.$$

Note that, using  $\chi(\mathcal{S}^\vee \otimes \mathcal{E}) = -\mathbf{v} \cdot \mathbf{s}$ , we get  $\dim(\text{Hom}(\mathcal{S}, \mathcal{E})) = a - \mathbf{v} \cdot \mathbf{s}$ . Therefore we have

$$\dim(\text{Hom}(\mathcal{S}, \mathcal{F})) = \dim(\text{Hom}(\mathcal{S}, \mathcal{E})) + 1 = a - \mathbf{v} \cdot \mathbf{s} + 1.$$

Assume by contradiction that  $\mathcal{F}$  is simple and consider its class in the moduli space  $\text{Spl}(\mathbf{v} + \mathbf{s})$  of simple sheaves of Mukai vector  $\mathbf{v} + \mathbf{s}$ . Here,  $(\mathcal{E}, [\xi])$  determines the point  $(\mathcal{F}, [f])$ , where  $[f]$  is the proportionality class of  $f \in \text{Hom}(\mathcal{S}, \mathcal{F}) \setminus \{0\}$ . The space of these pairs has dimension

$$\dim(\text{Spl}(\mathbf{v} + \mathbf{s})) + \dim(\mathbb{P}(\text{Hom}(\mathcal{S}, \mathcal{F}))) = (\mathbf{v} + \mathbf{s})^2 + 2 + a - \mathbf{v} \cdot \mathbf{s} = \mathbf{v}^2 + \mathbf{v} \cdot \mathbf{s} + a.$$

However, from the pair  $(\mathcal{F}, [f])$  we recover  $\mathcal{E}$  as  $\mathcal{E} \simeq \text{coker}(f)$  and the class  $[\xi]$  as the extension given by (14), therefore we must have

$$\mathbf{v}^2 + a + 1 \leq \mathbf{v}^2 + \mathbf{v} \cdot \mathbf{s} + a.$$

But this is against the assumption  $\mathbf{v} \cdot \mathbf{s} \leq 0$ .  $\square$

**Lemma 4.6.** *Let  $s_2 \in \mathbb{N}$ . Let  $X$  be a K3 surface having  $\text{Pic}(X) = \mathbb{Z} \cdot H \oplus N$  with  $H^2 = 4s_2 - 2$  and  $N$  containing no effective divisor. We consider  $\mathcal{S}$ , a slope-stable bundle of Mukai vector  $\mathbf{s} = (2, H, s_2)$ . Let  $\mathcal{E}$  be a generic slope-stable sheaf of Mukai vector  $(2k + 1, (k + 1)H, v_2)$  with  $v_2 \in \mathbb{N}$ . Consider a non-trivial extension*

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow 0.$$

*Then  $\mathcal{F}$  is slope-stable.*

**Proof.** Let  $\mathcal{K}$  be a stable destabilizing subsheaf of  $\mathcal{F}$ . We want to check that assuming the existence of  $\mathcal{K}$  we are led to a contradiction. As in Lemma 4.1, write  $\mathcal{K}'$  and  $\mathcal{K}''$  for the sub-sheaves of  $\mathcal{S}$  and  $\mathcal{E}$  induced by the sub-sheaf  $\mathcal{K}$  of  $\mathcal{F}$ .

We start by proving the result for  $k = 0$  which is slightly different. In this case  $\mathcal{E} = \mathcal{I}_Z(H)$  with  $Z$  a generic 0-dimensional scheme of the appropriate length. If  $\mathcal{K}'' = 0$ , then the inclusion of  $\mathcal{K}' \simeq \mathcal{K}$  into  $\mathcal{S}$  would either give a rank-1 destabilizing sub-sheaf, or a subsheaf of rank 2 with  $\deg_H(\mathcal{K}') = k'$  and  $k' \geq 2$ . However both cases are impossible.

Hence  $\mathcal{K}'' \neq 0$ , so actually  $\mathcal{K}''$  must be of the form  $\mathcal{I}_{Z'}(H)$  for some 0-dimensional subscheme  $Z'$  containing  $Z$ , hence  $\mathcal{Q}'' = \mathcal{I}_Z(H)/\mathcal{K}''$  is a 0-dimensional sheaf.

Now if  $\mathcal{K}' = 0$ , since  $\mathcal{Q}''$  is 0-dimensional and  $\mathcal{S}$  is locally free, we get  $\text{Ext}^1(\mathcal{Q}'', \mathcal{S}) = 0$ , hence  $\mathcal{Q}''$  is a direct summand of  $\mathcal{Q} = \mathcal{F}/\mathcal{K}$ , against the assumption that  $\mathcal{K}$  is saturated.

However if  $\mathcal{K}' \neq 0$  then  $\text{rk}(\mathcal{K}') = 1$ , so that in order for  $\mathcal{K}$  to stabilise  $\mathcal{F}$  we would need  $\deg_H(\mathcal{K}') = k'$  with  $k' \geq 2$ , which is impossible by the stability of  $\mathcal{S}$ .

Now, we consider the case  $k \geq 1$ , so  $\text{rk} \mathcal{E} \geq 2$ . Therefore, in this case, for  $\mathcal{E}$  generic, we can assume that  $\mathcal{E}$  is locally free; then  $\mathcal{F}$  and  $\mathcal{K}$  are also locally free. Start by writing

$$\mu(\mathcal{E}) = \frac{k+1}{2k+1} = \frac{1}{2} + \frac{1}{4k+2}, \quad \mu(\mathcal{F}) = \frac{k+2}{2k+3} = \frac{1}{2} + \frac{1}{4k+6}.$$

Since  $\mathcal{K}$  destabilizes  $\mathcal{F}$ , we have:

$$\mu(\mathcal{K}) \geq \mu(\mathcal{F}) = \frac{1}{2} + \frac{1}{4k+6}, \quad \text{rk}(\mathcal{K}) < 2k+3.$$

- Assume  $\mathcal{K}'' = 0$ . Then  $\mathcal{K} = \mathcal{K}'$ , so  $\text{rk}(\mathcal{K}) \in \{1, 2\}$ . If  $\text{rk}(\mathcal{K}) = 1$  then by stability of  $\mathcal{S}$  we have  $\deg_H(\mathcal{K}) = a$  with  $a \leq 0$ , so  $\mathcal{K}$  does not destabilize  $\mathcal{F}$ . If  $\text{rk}(\mathcal{K}) = 2$  then we have  $\deg_H(\mathcal{K}) = a$  with  $a \leq 1$ , so  $\mathcal{K}$  does not destabilize  $\mathcal{F}$ . This case is thus excluded and we assume from now on  $\mathcal{K}'' \neq 0$ .
- Let us prove  $\mu(\mathcal{K}'') < \mu(\mathcal{E})$ . By stability of  $\mathcal{E}$ , we always have  $\mu(\mathcal{K}'') \leq \mu(\mathcal{E})$  and strict inequality holds if  $\text{rk}(\mathcal{K}'') < \text{rk}(\mathcal{E}) = 2k+1$ .

- If  $\mathcal{K}' = 0$ , then  $\mathcal{K} = \mathcal{K}''$  is a locally free subsheaf of  $\mathcal{E}$ . Note that  $\mu(\mathcal{K}'') = \mu(\mathcal{K}) < \mu(\mathcal{E})$  holds unless  $\mathcal{K}$  and  $\mathcal{E}$  have same rank and determinant. But in this case, since  $\mathcal{K}''$  is a locally free subsheaf of  $\mathcal{E}$  with the same slope as  $\mathcal{E}$  and  $\mathcal{E}$  is stable, the inclusion  $\mathcal{K} \rightarrow \mathcal{E}$  is an isomorphism and therefore the sequence defining  $\mathcal{F}$  splits, a contradiction.
- Assume thus  $\mathcal{K}' \neq 0$ . Again,  $\mu(\mathcal{K}'') < \mu(\mathcal{E})$  unless  $\text{rk}(\mathcal{K}'') = \text{rk}(\mathcal{E}) = 2k + 1$ . However in this case  $\mathcal{K}'$  must be of rank 1, since  $\text{rk}(\mathcal{K}) < \text{rk}(\mathcal{F})$ . Then, by stability of  $\mathcal{S}$ , we get  $\deg_H(\mathcal{K}') = a$  with  $a \leq 0$ , hence

$$\mu(\mathcal{K}) \leq \frac{k+1}{2k+2}.$$

But we immediately check

$$\frac{k+1}{2k+2} < \frac{k+2}{2k+3} = \mu(\mathcal{F}).$$

Therefore, we have  $\mu(\mathcal{K}) < \mu(\mathcal{F})$ , a contradiction.

- We proved  $\mu(\mathcal{K}'') < \mu(\mathcal{E})$ . Also,  $\mathcal{K}$  is stable and  $\mathcal{K}''$  is a torsion-free quotient of  $\mathcal{K}$  so  $\mu(\mathcal{K}) \leq \mu(\mathcal{K}'')$ , with strict inequality whenever  $\mathcal{K}' \neq 0$ . Hence

$$\mu(\mathcal{F}) \leq \mu(\mathcal{K}) \leq \mu(\mathcal{K}'') < \mu(\mathcal{E}). \quad (15)$$

In other words

$$\frac{1}{2} + \frac{1}{4k+6} \leq \mu(\mathcal{K}) \leq \mu(\mathcal{K}'') < \frac{1}{2} + \frac{1}{4k+2}$$

Hence:

$$\frac{1}{2} + \frac{1}{4k+6} \leq \frac{c}{\text{rk}(\mathcal{K})} < \frac{1}{2} + \frac{1}{4k+2},$$

with  $\deg_H(\mathcal{K}) = c$ . This gives:

$$\frac{\text{rk}(\mathcal{K})}{2} < c < \frac{\text{rk}(\mathcal{K})}{2} + 1.$$

We obtain that the only solution is  $\text{rk}(\mathcal{K}) = 2r + 1$  and  $c = r + 1$ , with  $0 \leq r \leq k$ . However, the sequence  $\left(\frac{n+1}{2n+1}\right)$  is strictly decreasing. So  $\mu(\mathcal{K}) = \frac{r+1}{2r+1} \geq \frac{k+1}{2k+1}$ , which contradicts (15). Therefore there is no possibility for the slope of  $\mathcal{K}$ , this sheaf cannot exist and  $\mathcal{F}$  is then stable.  $\square$

*Proof of Theorem 4.4.* Note that  $\mathbf{s} \cdot \mathbf{v} = -(2k + 1)$ . So, as follows from [Yos99, Lemma 2.6], or by the two previous lemmas, we have  $\text{Ext}^1(\mathcal{S}, \mathcal{E}) = 0$  for a generic  $\mathcal{E} \in \mathbf{M}(\mathbf{v})$ . Since  $\mathcal{E}$  and  $\mathcal{S}$  are stable, we have  $\text{Ext}^2(\mathcal{S}, \mathcal{E}) = 0$  and by Riemann–Roch  $\dim(\text{Hom}(\mathcal{S}, \mathcal{E})) = 2k + 1 \neq 0$ . Therefore we obtain our result from Lemma 3.8 and Propositions 3.9 and 3.11.  $\square$

*Remark 4.7.* Note that when  $k = 0$ , the involution  $\Phi_{\mathcal{S},1}$  acts on  $X^{[n]}$  with  $n = \frac{g+2}{4}$ . For  $g = 10$ , this agrees with the construction of [BM22]. For all  $g \equiv 2$  modulo 4, one can describe this involution by taking the residual subscheme with respect to a section of  $\mathcal{S}$  vanishing along an element of  $X^{[n]}$ . See the end of §5.1 for more details on this.

*Remark 4.8.* To go further and construct more involutions, we could also consider a construction more complicated than the one exposed in Definition 3.1 and Section 2.2. An idea that could be explored would be to compose several spherical twists.

## 5. PROPERTIES OF THE INVOLUTIONS

Here we discuss some basic properties of the involutions constructed in this paper. After reviewing the connection with several well-known examples, we provide some results about the nature of our involutions and their induced action on the cohomology of the moduli space  $\mathbf{M}(\mathbf{v})$ .

**5.1. Examples.** Let us point out how our construction gives a general framework to many examples of involutions constructed so far on moduli spaces.

5.1.1. *A quick review of involutions defined by  $\mathcal{O}_X$ .* Here we just give a brief account of some well-known involutions and observe that they can be described as  $\Phi = \Phi_{\mathcal{O}_X, 0}$ .

*Example 5.1.* For  $r = 1$ , let us describe the cases  $g = 2$  and  $g = 3$  allowed by the inequalities of Theorem 4.2. The moduli space  $M(\mathbf{v}) = M(1, H, 1)$  is identified with the Hilbert scheme of subschemes  $Z$  of length  $g - 1$  of  $X$ . Note that an element of  $M(\mathbf{v})$  is of the form  $\mathcal{J}_Z(H)$ . Taking double dual we easily see  $\mathcal{E}xt^1(\mathcal{J}_Z(H), \mathcal{O}_X) \simeq \omega_Z$  and for subschemes  $Z \subset X$  of length 1 or 2,  $\omega_Z$  is identified with  $\mathcal{O}_Z$ .

For  $g = 3$ ,  $X \subset \mathbb{P}^3$  is a quartic surface. Let us see that  $\Phi$  agrees with the Beauville involution sending a length-2 subscheme  $Z$  of  $X$  to its residual in  $X$  with respect to the line in  $\mathbb{P}^3$  generated by  $Z$ . For any such  $Z \subset X$ , we have  $h^0(\mathcal{J}_Z(H)) = 2$ . The base locus in  $X$  of the pencil of planes through the line  $L \subset \mathbb{P}^3$  spanned by  $Z$  is the residual subscheme  $Z'$  with respect to  $L \cap X$ . In other words, the cokernel of  $\mathbf{e}_{\mathcal{J}_Z(H)}$  is  $\mathcal{O}_{Z'}$ , while clearly the kernel of  $\mathbf{e}_{\mathcal{J}_Z(H)}$ , being a reflexive sheaf of rank 1, is just  $\mathcal{O}_X(-H)$ . We summarize this in the exact sequence:

$$0 \rightarrow \mathcal{O}_X(-H) \rightarrow \mathcal{O}_X^{\oplus 2} \rightarrow \mathcal{J}_Z(H) \rightarrow \mathcal{O}_{Z'} \rightarrow 0.$$

The image of the middle map in the above sequence is just  $\mathcal{J}_{\bar{Z}}(H)$ , where we put  $\bar{Z} = L \cap X$ . Dualizing the previous display, we see that (3) becomes:

$$0 \rightarrow \mathcal{O}_X(-H) \rightarrow \mathcal{O}_X^{\oplus 2} \rightarrow \Phi(\mathcal{J}_Z(H)) \rightarrow \mathcal{O}_Z \rightarrow 0.$$

It is now clear that  $\Phi(\mathcal{J}_Z(H)) \simeq \mathcal{J}_{Z'}(H)$ .

By the same argument we see that, for  $g = 2$ ,  $\Phi$  is just the involution exchanging the sheets of the double cover  $\pi : X \rightarrow \mathbb{P}^2$ , defined by sending a point  $p$  to  $\mathcal{J}_p(H)$  and then via  $\Phi$  to  $\mathcal{J}_{p'}(H)$ , where  $p'$  is the conjugate point of  $p$ . Indeed,  $\mathcal{O}_{p'}$  appears as cokernel of the map  $\mathcal{O}_X^{\oplus 2} \rightarrow \mathcal{J}_p(H)$  given by the pencil of lines through  $\pi(p)$ .

*Example 5.2.* For  $r = 2$ , we have biregular Markman-O'Grady involutions for  $g = 4, 5, 6, 7, 8$ . For  $g = 5$ , the surface  $X$  is the intersection of three quadrics in  $\mathbb{P}^5$  and the moduli space  $M(\mathbf{v})$  is a K3 surface of genus 2. More specifically,  $M(\mathbf{v})$  can be seen as the double cover  $\pi : M(\mathbf{v}) \rightarrow \mathbb{P}^2$ , where the plane  $\mathbb{P}^2$  parametrizes the web of quadrics whose base locus is  $X$ . The branch locus of  $\pi$  is the discriminant sextic of the web. Pairs of conjugate points  $p, p'$  for this cover correspond to pairs of spinor bundles  $\mathcal{E}, \mathcal{E}'$  defined over a non-singular 4-dimensional quadric  $Q$  of the web. These bundles fit into:

$$0 \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{O}_Q^{\oplus 4} \rightarrow \mathcal{E}' \rightarrow 0.$$

After restriction to  $X$ , the above sequence shows that  $\mathcal{E}'|_X \simeq \Phi(\mathcal{E}|_X)$  so  $\Phi$  is again the involution exchanging sheets of  $\pi : M(\mathbf{v}) \rightarrow \mathbb{P}^2$ .

For  $g = 7$ , the involution  $\Phi$  was described at the level of Fano threefolds in [BF14].

*Example 5.3.* For  $r = 3$ , in case  $g = 10$  we have that  $M(\mathbf{v}) = M(3, H, 3)$  is a K3 surface of genus 2. In this case,  $X$  is a 3-codimensional linear section of the 5-dimensional variety  $\Sigma$  in  $\mathbb{P}^{13}$  homogeneous under the exceptional complex Lie group  $G_2$ . The double cover  $\pi$  of the plane parametrizing the hyperplanes in  $\mathbb{P}^{13}$  whose base locus is  $X$  and ramified along the sextic curve obtained by cutting the dual of  $\Sigma$  along this plane is identified with  $M(\mathbf{v})$ . A point of  $M(\mathbf{v})$  is a rank-3 vector bundle  $\mathcal{E}$  providing an embedding of  $X$  in  $\mathbb{G}(3, 6)$ , cf. [KR13]. The tautological bundles  $\mathcal{E}, \mathcal{E}'$  of rank 3 defined on  $\mathbb{G}(3, 6)$  fit into:

$$0 \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{O}_{\mathbb{G}(3,6)}^{\oplus 6} \rightarrow \mathcal{E}' \rightarrow 0.$$

Again, restricting to  $X$  the above sequence we get that  $\mathcal{E}'|_X \simeq \Phi(\mathcal{E}|_X)$ . Exchanging the sheets of  $\pi$  corresponds to exchanging the restricted tautological bundles on  $\mathbb{G}(3, 6)$ .

5.1.2. *Involutions on Hilbert schemes from spherical bundles of rank 2.* Here we mention that, if  $S$  is a K3 surface of genus  $g$  with  $g \equiv 2$  modulo 4, then, setting  $k = 0$ , the birational involution in Theorem 4.4 acts on  $X^{[n]}$ , with  $n = (g + 2)/4$ . Note that  $X$  carries a spherical bundle  $\mathcal{S}$  of rank 2 with  $c_1(\mathcal{S}) = H$  and  $c_2(\mathcal{S}) = 2n$ . We get the following result. As we have been informed, a similar construction appears also in [BM24].

**Proposition 5.4.** *The involution  $\Phi$  sends a generic subscheme  $Z \in X^{[n]}$  to its residual subscheme with respect to the zero-locus of the only section of  $\mathcal{S}$  vanishing at  $Z$ .*

**Proof.** Note that  $c_2(\mathcal{S}) = 2n$ . Let  $Z \in X^{[n]}$  be a generic element. According to Riemann–Roch and since  $H^1(\mathcal{S} \otimes \mathcal{F}_Z) = 0$  for generic  $Z$  by [Yos99, Lemma 2.6], we know that  $\text{Hom}(\mathcal{S}, \mathcal{F}_Z(H)) = \mathbb{C}$  and  $\text{Ext}^i(\mathcal{S}, \mathcal{F}_Z(H)) = 0$  for all  $i > 0$ . Let  $s \in \text{Hom}(\mathcal{S}, \mathcal{F}_Z(H)) = H^0(\mathcal{S}^\vee \otimes \mathcal{F}_Z(H)) = H^0(\mathcal{S} \otimes \mathcal{F}_Z)$ . So  $Z \subset \mathbb{V}(s)$ . We obtain the following diagram of exact sequences with  $\mathbf{e}$  the evaluation map:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{s} & \mathcal{S} & \longrightarrow & \mathcal{F}_{\mathbb{V}(s)}(H) \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{s} & \mathcal{S} & \xrightarrow{\mathbf{e}} & \mathcal{F}_Z(H) \longrightarrow \text{coker}(\mathbf{e}). \end{array}$$

So  $\text{coker}(\mathbf{e})$  is also the cokernel of the map  $\mathcal{F}_{\mathbb{V}(s)}(H) \hookrightarrow \mathcal{F}_Z(H)$ . Therefore  $\text{coker}(\mathbf{e})$  is a torsion sheaf supported on a subscheme of length  $\ell(\mathbb{V}(s)) - \ell(Z) = n$ . We set  $Z' := \text{Supp}(\text{coker}(\mathbf{e}))$ . Now, we consider the dual of the evaluation map by  $\mathcal{O}_X(H)$  and we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{\mathbf{e}^\vee} & \mathcal{S} & \xrightarrow{s^\vee} & \mathcal{O}_X(H) \longrightarrow \mathcal{E}xt^2(\text{coker}(\mathbf{e}), \mathcal{O}_X) \longrightarrow 0. \\ & & & & \downarrow & \nearrow & \\ & & 0 & \longrightarrow & \Phi_{\mathcal{S},1}(\mathcal{F}_Z(H)) & & \end{array}$$

The sheaf  $\mathcal{E}xt^2(\text{coker}(\mathbf{e}), \mathcal{O}_X)$  is a torsion sheaf supported on  $Z'$ , so we have  $\mathcal{E}xt^2(\text{coker}(\mathbf{e}), \mathcal{O}_X) = \mathcal{O}_{Z'}$ . This shows that  $\Phi_{\mathcal{S},1}(\mathcal{F}_Z(H)) = \mathcal{F}_{Z'}(H)$ . The subscheme  $Z'$  described here is the residual with respect to the only section  $s$  of  $\mathcal{S}$  vanishing at  $Z$ .  $\square$

*Example 5.5.* The involution studied in [BM22, BM24] by Beri and Manivel is the same as the involution of Theorem 4.4 with  $g = 10$  and  $k = 0$ . Indeed, setting  $g = 10$  we get  $n = 3$ , so we consider  $Z = \{p_1, p_2, p_3\} \in X^{[3]}$ , a generic element. The bundle  $\mathcal{S}^\vee$  is the restriction of the tautological rank 2 bundle on  $\mathbb{G}(2, 7)$  via the embedding  $X \hookrightarrow \mathbb{G}(2, 7)$ . Moreover, we have  $\mathcal{S} = \mathcal{S}^\vee(H)$  and  $H^0(\mathcal{S}) = \mathbb{C}^7$ . For all  $s \in H^0(\mathcal{S})$ , we have  $\mathbb{V}(s) = \mathbb{G}(2, 6) \cap X \subset \mathbb{G}(2, 7)$  and  $\mathbb{V}(s)$  is a subscheme of dimension 0 and of length  $c_2(\mathcal{S}) = 6$ . Hence each point in  $X$  can be seen as a plane in  $\mathbb{C}^7$  and, for each  $i \in \{1, 2, 3\}$ , let  $P_i$  be the plane associated to  $p_i$ . The plane  $P_1, P_2, P_3$  genera a sub-vector space  $V_6 \subset \mathbb{C}^7$  of dimension 6. We obtain:  $\mathbb{V}(\tilde{s}) = \mathbb{G}(2, V_6) \subset \mathbb{G}(2, 7)$ , with  $\tilde{s}$  a section of the dual tautological sub-bundle of rank 2 on  $\mathbb{G}(2, 7)$ . We get  $s = \tilde{s}|_X \in H^0(\mathcal{S})$  and  $Z \subset \mathbb{V}(s) = X \cap \mathbb{G}(2, V_6)$  with  $\mathbb{V}(s)$  which is 0-dimensional of length 6. Beri and Manivel in [BM22, Page 8] send  $Z$  to  $Z' = \{q_1, q_2, q_3\}$  with  $\mathbb{V}(s) = \{p_1, p_2, p_3, q_1, q_2, q_3\}$ . This is the same as our involution since  $s \in H^0(\mathcal{S} \otimes \mathcal{F}_Z)$  and  $Z' = \{q_1, q_2, q_3\}$  is the residual of  $Z = \{p_1, p_2, p_3\}$  with respect to the zero-locus of  $s$ .

5.2. **Anti-symplectic involution.** Let  $M(\mathbf{v})$  be a compact moduli space of stable sheaves on  $X$  of dimension  $\geq 4$ . We know from O’Grady [O’G97] that there is a Hodge isometry:

$$\theta : \mathbf{v}^\perp \subset H^*(X, \mathbb{Z}) \rightarrow H^2(M(\mathbf{v}), \mathbb{Z}), \quad (16)$$

with  $H^*(X, \mathbb{Z})$  endowed with the Mukai pairing and  $H^2(M(\mathbf{v}), \mathbb{Z})$  with the Beauville–Bogomolov form. The following result has been proved by O’Grady.

**Proposition 5.6** (See [O’G05, Proposition 4.14]). *Let  $(X, H)$  be a polarized K3 surface such that  $\text{Pic}(X) = \mathbb{Z} \cdot H \oplus N$ , with  $H^2 = 2g - 2$  and  $N$  not containing effective divisors. Let  $\mathbf{v} = (r, H, r)$  be a Mukai vector with  $r \geq 2$  and  $\dim M(\mathbf{v}) \geq 4$ . Let  $h_{\mathbf{v}} := \theta(1, 0, -1)$  and*

$R_{h_{\mathbf{v}}}(\alpha) = -\alpha + B_{M(\mathbf{v})}(h_{\mathbf{v}}, \alpha)h_{\mathbf{v}}$ , with  $B_{M(\mathbf{v})}$  the Beauville–Bogomolov form on  $M(\mathbf{v})$ . The involution  $\Phi_{\mathbb{C}_X, 0}$  on  $M(\mathbf{v})$  is anti-symplectic. Moreover:

$$\Phi_{\mathbb{C}_X, 0}^* = R_{h_{\mathbf{v}}}.$$

*Remark 5.7.* O’Grady also gives the cohomology action of the Beauville involution in [O’G05, Proposition 4.21].

Using the same ideas as O’Grady, we are going to provide a generalization of Proposition 5.6 to the involution  $\Phi_{\mathcal{S}, 1}$ . We consider a Mukai vector  $(v_0, v_1H, v_2)$  with  $v_2 = (g-1)v_1 - \frac{v_0}{2}(\frac{g}{2}-1)$ . Moreover, we assume that  $\dim M(\mathbf{v}) \geq 4$ . As before  $\mathcal{S}$  is the stable bundle on  $X$  with Mukai vector  $(2, H, \frac{g}{2})$ . Let  $U(\mathbf{v}) \subset M(\mathbf{v})$  be the set where  $\Phi_{\mathcal{S}, 1}$  is well defined. Let  $d_{\mathbf{v}} := \theta(2, H, \frac{g}{2}-1)$  and  $R_{d_{\mathbf{v}}}(\alpha) = -\alpha + B_{M(\mathbf{v})}(d_{\mathbf{v}}, \alpha)d_{\mathbf{v}}$ , with  $B_{M(\mathbf{v})}$  the Beauville–Bogomolov form.

**Lemma 5.8.** *We assume that  $U(\mathbf{v})$  is a non-empty Zariski open set in  $M(\mathbf{v})$ . Let  $\iota_{\mathbf{v}} : U(\mathbf{v}) \hookrightarrow M(\mathbf{v})$  be the embedding. We have:*

$$\iota_{\mathbf{v}}^* \circ \Phi_{\mathcal{S}, 1}^* = \iota_{\mathbf{v}}^* \circ R_{d_{\mathbf{v}}}.$$

**Proof.** Consider  $\mathcal{R}$  as a quasi-family of sheaves on  $X$  parametrized by  $M(\mathbf{v})$ , where  $\mathbf{v}$  is the Mukai vector. We also set  $\sigma : X \times M(\mathbf{v}) \rightarrow X$  and  $\pi : X \times M(\mathbf{v}) \rightarrow M(\mathbf{v})$  for the natural projections. We set  $H^4(X, \mathbb{Z}) = \mathbb{Z}\eta$ . We know (see for instance [O’G05, 4.2.8]) that:

$$\theta(\alpha) = \theta_{\mathcal{R}}(\alpha) = \frac{1}{\rho(\mathcal{R})} \pi_* [\text{ch}(\mathcal{R})(1 + \sigma^*(\eta))\sigma^*(\alpha^\vee)]_6,$$

with  $\rho(\mathcal{R})$  an integer such that  $\mathcal{R}|_{X \times \{t\}} = F^{\rho(\mathcal{R})}$  for all  $t \in M(\mathbf{v})$  and with  $F$  a sheaf with Mukai vector  $\mathbf{v}$ . We want to compute:

$$\iota_{\mathbf{v}}^* \circ \Phi_{\mathcal{S}, 1}^* \theta_{\mathcal{R}}(\alpha) = \theta_{(\text{id} \times \Phi)^* \mathcal{R}|_{X \times U(\mathbf{v})}}(\alpha).$$

For simplicity in the notation, we are going to denote  $\mathcal{R}|_{X \times U(\mathbf{v})}$  also by  $\mathcal{R}$ . The first step is to understand  $(\text{id} \times \Phi)^* \mathcal{R}$ . It is given by the complex  $B := \mathbf{R}\mathcal{H}om(A, \sigma^*(\mathbb{C}_X(H)))$ , with

$$A := \pi^*(\pi_*(\mathcal{R} \otimes \sigma^*(\mathcal{S}^\vee))) \otimes \sigma^*\mathcal{S} \rightarrow \mathcal{R}.$$

We have:

$$\begin{aligned} \text{ch}(B) &= \text{ch}(\mathbf{R}\mathcal{H}om(A, \sigma^*(\mathbb{C}_X(H)))) = \text{ch}(\mathbf{R}\mathcal{H}om(A, \sigma^*(\mathbb{C}_X))) \text{ch}(\sigma^*(\mathbb{C}_X(H))) = \\ &= \text{ch}(A)^\vee \text{ch}(\sigma^*(\mathbb{C}_X(H))). \end{aligned}$$

Since  $\Phi_{\mathcal{S}, 1}$  is well defined on  $U(\mathbf{v})$ ,  $B$  is a quasi-family of sheaves and we can consider:

$$\begin{aligned} \theta_B(\alpha) &= \pi_* [\text{ch}(A)^\vee (1 + \sigma^*(\eta))\sigma^*(\alpha^\vee \text{ch}(\mathbb{C}_X(H)))]_6 \\ &= \pi_* [\text{ch}(A)^\vee (1 + \sigma^*(\eta))\sigma^*(\alpha \text{ch}(\mathbb{C}_X(-H)))]_6^\vee. \end{aligned}$$

We refer to Section 2.1 for the computation of  $\alpha \text{ch}(\mathbb{C}_X(-H))$ . We set  $\beta := (\alpha \text{ch}(\mathbb{C}_X(-H)))^\vee$ . We obtain:

$$\begin{aligned} \theta_B(\alpha) &= -\pi_* [\text{ch}(A)(1 + \sigma^*(\eta))\sigma^*(\beta^\vee)]_6 \\ &= \pi_* [\text{ch}(\mathcal{R})(1 + \sigma^*(\eta))\sigma^*(\beta^\vee)]_6 \\ &\quad - \pi_* [\text{ch}(\pi^*(\pi_*(\mathcal{R} \otimes \sigma^*(\mathcal{S}^\vee))) \otimes \sigma^*(\mathcal{S}))(1 + \sigma^*(\eta))\sigma^*(\beta^\vee)]_6. \end{aligned}$$

The first term of the last equality is just  $\theta_{\mathcal{R}}(\beta)$ . It remains to compute the second term; we set:

$$(\heartsuit) := \pi_* [\text{ch}(\pi^*(\pi_*(\mathcal{R} \otimes \sigma^*(\mathcal{S}^\vee))) \otimes \sigma^*(\mathcal{S}))(1 + \sigma^*(\eta))\sigma^*(\beta^\vee)]_6.$$

By projection formula, we have:

$$\begin{aligned} (\heartsuit) &= [\text{ch}(\pi_*(\mathcal{R} \otimes \sigma^*(\mathcal{S}^\vee)))\pi_*(\text{ch}(\sigma^*(\mathcal{S}))(1 + \sigma^*(\eta))\sigma^*(\beta^\vee))]_6 \\ &= c_1(\pi_*(\mathcal{R} \otimes \sigma^*(\mathcal{S}^\vee)))\pi_* [\sigma^*(\text{ch}(\mathcal{S})(1 + \sigma(\eta)))\beta^\vee]_4. \end{aligned}$$

We set  $(\diamond) := c_1(\pi_*(\mathcal{R} \otimes \sigma^*(\mathcal{S}^\vee)))$  and  $(\spadesuit) := \pi_*[\sigma^*(\text{ch}(\mathcal{S})(1 + \sigma(\eta)))\beta^\vee]_4$ . We first compute  $(\diamond)$ ; according to Grothendieck–Riemann–Roch theorem, we have:

$$\begin{aligned} (\diamond) &= \pi_* [\text{ch}(\mathcal{R} \otimes \sigma^*(\mathcal{S}^\vee))\sigma^*(1 + 2\eta)]_6 \\ &= \pi_* [\text{ch}(\mathcal{R})\sigma^*(\text{ch}(\mathcal{S}^\vee)(1 + 2\eta))]_6. \end{aligned}$$

However, we have  $\text{ch}(\mathcal{S}^\vee) = (2, -H, \frac{g}{2} - 2)$ . So:  $\text{ch}(\mathcal{S}^\vee)(1 + 2\eta) = (2, -H, \frac{g}{2} + 2) = (2, -H, \frac{g}{2})(1 + \eta) = \mathbf{v}(\mathcal{S})^\vee(1 + \eta)$ . So:

$$(\diamond) = \theta_{\mathcal{R}}(\mathbf{v}(\mathcal{S})).$$

It remains to compute  $(\spadesuit)$ . We set  $\beta = (\beta_0, \beta_1, \beta_2)$ .

$$\begin{aligned} (\spadesuit) &= \pi_* \left[ \sigma^* \left( \left( 2, H, \frac{g}{2} - 2 \right) (1 + \eta)(\beta_0, -\beta_1, \beta_2) \right) \right]_4 \\ &= \pi_* \left[ \sigma^* \left( \left( 2, H, \frac{g}{2} \right) (\beta_0, -\beta_1, \beta_2) \right) \right]_4 \\ &= 2\beta_2 - (2g - 2)\beta_1 + \frac{g}{2}\beta_0 \\ &= - \left( 2, H, \frac{g}{2} \right) \cdot \beta \\ &= -\mathbf{v}(\mathcal{S}) \cdot \beta. \end{aligned}$$

Combining all the previous computations, we obtain:

$$\iota_{\mathbf{v}}^* \Phi^* \theta_{\mathcal{R}}(\alpha) = \iota_{\mathbf{v}}^* [\theta_{\mathcal{R}}(\beta) + (\mathbf{v}(\mathcal{S}) \cdot \beta) \theta_{\mathcal{R}}(\mathbf{v}(\mathcal{S}))].$$

We set  $R_{\mathcal{S}}(x) = x + (\mathbf{v}(\mathcal{S}) \cdot x) \mathbf{v}(\mathcal{S})$ ,  $D$  the dual and  $T$  the tensorization by  $\mathcal{O}_X(-H)$  of Mukai vectors. We have shown that:

$$\iota_{\mathbf{v}}^* \Phi^* \theta_{\mathcal{R}}(\alpha) = \iota_{\mathbf{v}}^* \circ \theta_{\mathcal{R}} \circ R_{\mathcal{S}} \circ D \circ T(\alpha). \quad (17)$$

Then it remains to show that, for all  $x \in \mathbf{v}^\perp$ ,

$$R_{\mathcal{S}} \circ D \circ T(x) = -x + \left[ \left( 2, H, \frac{g}{2} - 1 \right) \cdot x \right] \left( 2, H, \frac{g}{2} - 1 \right),$$

It is true for all elements in  $(\mathbb{H}^0(X, \mathbb{Z}) \oplus \mathbb{Z}H \oplus \mathbb{H}^4(X, \mathbb{Z}))^\perp$  and we can verify easily that it is also true for a basis of  $(\mathbb{H}^0(X, \mathbb{Z}) \oplus \mathbb{Z}H \oplus \mathbb{H}^4(X, \mathbb{Z})) \cap \mathbf{v}^\perp$ , for instance  $((2, H, \frac{g}{2} - 1), (v_0, 0, -v_2))$ .  $\square$

*Remark 5.9.* A priori the set  $M(\mathbf{v}) \setminus U(\mathbf{v})$  could have codimension 1. To be more precise, the involution  $\Phi$  can always be extended to a bimeromorphism regular in codimension 2, but  $\Phi$  as constructed in Section 3.1 may be defined only in codimension 1. Therefore, if we also denote by  $\Phi$  the extension of  $\Phi$  to a regular involution in codimension 2, the action of  $\Phi^*$  on  $\mathbb{H}^2(M(\mathbf{v}), \mathbb{Z})$  is well defined but could differ from the one found in Lemma 5.8.

*Remark 5.10.* If  $M(\mathbf{v}) \setminus U(\mathbf{v})$  has codimension 2 then according to (17) the action of  $\Phi^*$  on  $\mathbb{H}^2(M(\mathbf{v}), \mathbb{Z})$  is given by :

$$\Phi^* = \theta \circ R_{\mathcal{S}} \circ D \circ T \circ \theta^{-1}.$$

Note that there is another action on  $\mathbb{H}^2(M(\mathbf{v}), \mathbb{Z})$  obtained from the computation on Mukai vector, see Lemma 3.7. It provides the following action :

$$\varphi = -(\theta \circ T^{-1} \circ D \circ R_{\mathcal{S}} \circ \theta^{-1}).$$

This shows that  $\varphi = -\Phi^*$ .

As explained in Remark 5.9, we cannot always determined the action of  $\Phi^*$  on  $\mathbb{H}^2(M(\mathbf{v}), \mathbb{Z})$ , however we can deduce from the previous lemma that  $\Phi_{\mathcal{S},1}$  is anti-symplectic.

**Proposition 5.11.** *Let  $g \geq 2$  even. Let  $(X, H)$  be a K3 surface with  $H^2 = 2g - 2$  and  $\mathcal{S}$  the stable bundle of Mukai vector  $(2, H, \frac{g}{2})$ . Let  $\mathbf{v} = (v_0, v_1H, (g - 1)v_1 - \frac{v_0}{2}(\frac{g}{2} - 1))$  be a Mukai vector (we assume that  $\frac{v_0}{2}(\frac{g}{2} - 1)$  is integral). We set  $U(\mathbf{v})$  the open set where  $\Phi_{\mathcal{S},1}$  is well defined. We assume that:*

- $\dim M(\mathbf{v}) \geq 4$ ;
- $U(\mathbf{v})$  is a non-empty Zariski open set in  $M(\mathbf{v})$ .

Then, the involution  $\Phi_{\mathcal{S},1}$  on  $M(\mathbf{v})$  is anti-symplectic.

**Proof.** Let  $T_{M(\mathbf{v})}$  be the transcendental lattice of  $M(\mathbf{v})$ . From Lemma 5.8, we can prove that  $\Phi_{\mathcal{S},1|T_{M(\mathbf{v})}}^* = -\text{id}_{T_{M(\mathbf{v})}}$ . We have an exact sequence:

$$H^2(M(\mathbf{v}), U(\mathbf{v}), \mathbb{Z}) \xrightarrow{f} H^2(M(\mathbf{v}), \mathbb{Z}) \xrightarrow{\iota_{\mathbf{v}}^*} H^2(U(\mathbf{v}), \mathbb{Z}).$$

According to [Voi03, Section 11.1],  $\text{Im}(f) \subset \text{Pic}(M(\mathbf{v}))$ . Therefore,  $\iota_{\mathbf{v}}^*$  induces an injection  $T_{M(\mathbf{v})} \hookrightarrow H^2(U(\mathbf{v}), \mathbb{Z})$ . So Lemma 5.8 implies our claim.  $\square$

*Remark 5.12.* If  $\dim(M(\mathbf{v})) = 2$ , the involution  $\Phi_{\mathcal{S},1}$  induces a double cover  $M(\mathbf{v}) \rightarrow \mathbb{P}^2$  and is also anti-symplectic.

As a direct consequence, we have the following corollary.

**Corollary 5.13.** *The involutions defined in Theorems 4.4 are anti-symplectic.*

As explained in Remark 5.9, we cannot conclude from Lemma 5.8 the action of  $\Phi_{\mathcal{S},1}^*$  on the cohomology. However, using lattice theory, we can show that  $\Phi_{\mathcal{S},1}^*$  is a reflection through an element of square 2 or  $n - 1$  or  $2(n - 1)$  with  $2n = \dim(M(\mathbf{v}))$ .

**5.3. Action on the cohomology.** In this section, we assume that  $\text{Pic } X = \mathbb{Z} \cdot H$  in order to obtain information on  $H^2(M(\mathbf{v}), \mathbb{Z})^{\Phi_{\mathcal{S},1}}$  from lattice considerations. Let us describe the Picard lattice of  $M(\mathbf{v})$ , based on (16).

**Lemma 5.14.** *Let  $g \geq 2$  even. Let  $X$  be a K3 surface such that  $\text{Pic}(X) = \mathbb{Z} \cdot H$ , with  $H^2 = 2g - 2$ . Let  $\mathbf{v} = (v_0, v_1 H, v_2)$  a Mukai vector with  $v_2 = (g - 1)v_1 - \frac{v_0}{2}(\frac{g}{2} - 1)$  an integer. We assume that  $2n := \dim M(\mathbf{v}) \geq 4$ . We set  $\delta := v_0 \wedge v_2$ ,  $d_{\mathbf{v}} = \theta(2, 1, \frac{g}{2} - 1)$  and  $f_{\mathbf{v}} := \frac{1}{\delta}\theta(v_0, 0, -v_2)$ . Then  $(d_{\mathbf{v}}, f_{\mathbf{v}})$  gives a basis of  $\text{Pic}(M(\mathbf{v}))$  with bilinear form given by the matrix:*

$$\begin{pmatrix} 2 & \frac{2v_2}{\delta} - (\frac{g}{2} - 1)\frac{v_0}{\delta} \\ \frac{2v_2}{\delta} - (\frac{g}{2} - 1)\frac{v_0}{\delta} & 2\frac{v_0 v_2}{\delta^2} \end{pmatrix}.$$

In Particular:

$$\text{discr}(\text{Pic}(M(\mathbf{v}))) = -\frac{4(g-1)}{\delta^2}(n-1). \quad (18)$$

**Proof.** Since  $d_{\mathbf{v}}$  and  $f_{\mathbf{v}}$  are orthogonal to  $\mathbf{v}$ , we have  $\mathbb{Z}d_{\mathbf{v}} \oplus \mathbb{Z}f_{\mathbf{v}} \subset \text{Pic}(X)$ . Moreover,  $\mathbb{Z}d_{\mathbf{v}} \oplus \mathbb{Z}f_{\mathbf{v}}$  is a primitive sublattice of rank two, so  $\text{Pic}(X) = \mathbb{Z}d_{\mathbf{v}} \oplus \mathbb{Z}f_{\mathbf{v}}$ . The bilinear matrix is obtained because  $\theta$  is an isometry. Finally, since

$$n - 1 = (g - 1)v_1^2 - v_0 v_2, \quad (19)$$

we obtain the given discriminant.  $\square$

From the two previous lemmas, we deduce the cohomology action of  $\Phi_{\mathcal{S},1}$ .

**Proposition 5.15.** *Let  $g \in \mathbb{N}$  with  $g \equiv 2 \pmod{4}$  and  $k \in \mathbb{N}$ . Let  $X$  be a K3 surface such that  $\text{Pic}(X) = \mathbb{Z} \cdot H$ , with  $H^2 = 2g - 2$ . Let  $\mathbf{v} = (2k + 1, (k + 1)H, v_2)$  be a Mukai vector with  $v_2 = (g - 1)(k + 1) - \frac{2k+1}{2}(\frac{g}{2} - 1)$ . We set  $\delta = v_0 \wedge v_2$ . We assume that  $2n := \dim(M(\mathbf{v})) \geq 4$ .*

1. For  $n > 2$ , if  $H^2(M(\mathbf{v}), \mathbb{Z})^{\Phi_{\mathcal{S},1}} \simeq (2(n - 1))$  then  $\delta = 1$  and  $\frac{-1}{g-1}$  is a square in  $\mathbb{Z}/(n - 1)\mathbb{Z}$ .
2. For even  $n > 2$ , if  $H^2(M(\mathbf{v}), \mathbb{Z})^{\Phi_{\mathcal{S},1}} \simeq (n - 1)$  then  $\delta = 1$  and  $\frac{-1}{2(g-1)}$  is a square in  $\mathbb{Z}/(n - 1)\mathbb{Z}$ .
3. Otherwise  $H^2(M(\mathbf{v}), \mathbb{Z})^{\Phi_{\mathcal{S},1}} \simeq (2)$ .

About 2., note that  $\frac{-1}{g-1}$  is well defined in  $\mathbb{Z}/(n - 1)\mathbb{Z}$ , because  $\delta = (n - 1) \wedge (g - 1) = 1$ .

**Proof.** Before starting the proof note that  $\delta = (n-1) \wedge (g-1)$ . This is a direct consequence of the two equations  $v_2 = (g-1)v_1 - \frac{v_0}{2}(\frac{g}{2}-1)$  and  $n-1 = (g-1)v_1^2 - v_0v_2$ .

We set  $T := H^2(M(\mathbf{v}), \mathbb{Z})^{\Phi^{s,1}}$ . By Proposition 5.11, the involution  $\Phi$  is anti-symplectic. So:

$$T \subset \text{Pic}(M(\mathbf{v})).$$

Since  $\text{rk}(\text{Pic}(M(\mathbf{v}))) = 2$ , according to [CCC21, Proposition 1.6], there are four possibilities:

- (i)  $A_T = \mathbb{Z}/2\mathbb{Z}$ ;
- (ii)  $A_T = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2(n-1)\mathbb{Z}$ ;
- (iii)  $A_T = \mathbb{Z}/2(n-1)\mathbb{Z}$ ,  $n > 2$ .
- (iv)  $A_T = \mathbb{Z}/(n-1)\mathbb{Z}$ ,  $n > 2$  and  $n$  even.

First note that all these cases imply that  $\text{rk}(T) = 1$ . Indeed if  $\text{rk}(T) = 2$ , it provides  $T = \text{Pic}(M(\mathbf{v}))$  and so by (18):

- (i)  $\delta^2 = 2(n-1)(g-1)$ ;
- (ii)  $\delta^2 = g-1$ ;
- (iii)  $\delta^2 = 2(g-1)$ ;
- (iv)  $\delta^2 = 4(g-1)$ .

However the hypothesis  $\dim(M(\mathbf{v})) \geq 4$  implies that  $g-1 > v_0^2 \geq \delta^2$ . So all the previous cases are impossible. The case (ii) can only occur for  $\text{rk} T = 2$ , so it remains to treat the cases (iii) and (iv) with  $\text{rk} T = 1$ . According to [CCC21, Proposition 1.6], we have the following sub-cases:

- (iii) (a) There exists a primitive element  $x$  of square  $2(n-1)$  and divisibility  $2(n-1)$ .
- (b) There exists a primitive element  $x$  of square  $2(n-1)$  and divisibility  $(n-1)$ .
- (iv) There exists a primitive element  $x$  of square  $(n-1)$  and divisibility  $(n-1)$ .

Let  $y$  be a primitive element such that  $B_{M(\mathbf{v})}(x, y) = 0$ . According to the divisibility of  $x$ , there are two cases, namely  $\mathbb{Z}x \oplus \mathbb{Z}y = \text{Pic}(M(\mathbf{v}))$  or  $\mathbb{Z}x \oplus \mathbb{Z}y$  has index 2 in  $\text{Pic}(M(\mathbf{v}))$  in the sub-case (b) of (iii). So  $\text{discr}(\mathbb{Z}x \oplus \mathbb{Z}y) = -\frac{4(g-1)}{\delta^2}(n-1)$  or  $\text{discr}(\mathbb{Z}x \oplus \mathbb{Z}y) = -\frac{16(g-1)}{\delta^2}(n-1)$  again in the sub-case (b) of (iii). It implies in the different cases that:

- (iii) (a)  $y^2 = \frac{-2(g-1)}{\delta^2}$ ,
- (b)  $y^2 = \frac{-8(g-1)}{\delta^2}$ ,
- (iv)  $y^2 = \frac{-4(g-1)}{\delta^2}$ .

Since  $\delta$  is odd, it means that  $\delta^2 \mid (g-1)$ . Hence by (19),  $\delta^2 \mid (n-1)$  and so  $\delta = 1$ . Moreover, we can write  $d_v = \lambda x + \mu y$  or  $2d_v = \lambda x + \mu y$ , in the sub-case (b) of (iii), that is:

- (iii) (a)  $2 = \lambda^2(2(n-1)) + \mu^2(-2(g-1))$ ;
- (b)  $8 = \lambda^2(2(n-1)) + \mu^2(-8(g-1))$ ;
- (iv)  $2 = \lambda^2(n-1) + \mu^2(-4(g-1))$ .

In the sub-case (b) of (iii),  $\lambda$  has to be divisible by 2; so in case (iii) we obtain that  $\frac{-1}{g-1}$  is a square in  $\mathbb{Z}/(n-1)\mathbb{Z}$ . In case (iv),  $\lambda$  is also divisible by 2 and we obtain that  $\frac{-1}{2(g-1)}$  is a square in  $\mathbb{Z}/(n-1)\mathbb{Z}$ .  $\square$

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