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Optimal Control of Nonsmooth Dynamical Systems using Measure Relaxations

Saroj Prasad Chhatoi

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Didier Henrion

Abstract—We address the problem of optimal control of a nonsmooth dynamical system described by an evolution variational inequality. We consider both the discrete-time and continuous-time versions of the problem and we relax the problem in the space of measures. We show that there is no gap between the original finite-dimensional problem and the relaxed problem. We show the convergence of the relaxed discrete-time optimal control in measures to continuous-time optimal control in measures. This paves the way to a sound implementation of the moment sum-of-squares hierarchy to solve numerically the optimal control of nonsmooth dynamical systems.

I. INTRODUCTION

Optimal control of nonlinear systems with well-posedness properties is a well-studied problem [1]. In contrast, optimal control for nonsmooth dynamical systems with set-valued right-hand side is more complicated and has been an area of active research over the last two decades due to the powerful modeling capabilities of such systems [2].

In the present article, we are concerned with the class of nonsmooth systems described by a normal cone to a convex set as the set-valued mapping along with some perturbation generated by a Lipschitz vector field [3]. In [4], authors consider evolution variational inequality with a closed convex constraint set and obtain first-order conditions on the functional regularized system. They show under certain regularity conditions that the limit adjoint variable exhibits jump behavior. Pontrygin Maximum Principle optimality conditions for discrete approximations of such systems are performed in [5] and the authors use sophisticated tools from variational analysis to derive the limit equation. Recently in [6], the authors use exact penalization-based technique to extend the optimality conditions to more general problems with weaker assumptions on the system data.

Direct methods that discretize the problem and then optimize the discrete problem have fundamental issues and in [7] the authors pointed out that the error in gradients of the simulation results are independent of the step sizes. The authors propose an optimal control strategy based on smooth approximations but rely on the Euler integrator. Finite-element-based discretization along with time-freezing methods have been recently proposed in software packages [8]. For a class of linear complementarity systems (closely related to the formalism adopted in our present work), but where the right-hand side of the differential equation can be

written as a Lipschitz continuous map, the paper [9] presents first-order optimality conditions and some numerical results.

On the other hand, global methods for nonlinear control based on *occupation measures* have gained popularity in the last decade because of their powerful modeling capabilities and the availability of efficient algorithms and semidefinite programming solvers [10]. These methods reformulate the optimal control problem in finite dimensional space to a primal/dual pair of linear problems where the primal problem is expressed in the cone of non-negative Borel measures and a dual problem is expressed in the cone of non-negative continuous functions [11], [12] and then Linear Matrix Inequality (LMI) relaxations of the problem are solved with off-the-shelf semidefinite solvers. Occupation measures were also studied in the context of different system class [13], [14].

In the present work, we address the problem of optimal control of nonsmooth dynamical systems. We use occupation measures to relax both the discrete-time and continuous-time problem into a linear problem defined in the cone of nonnegative Borel measures. Under some mild assumptions, we show that this does not produce any relaxation gap. Further, we use tools from optimal transport to show the convergence of the discrete-time problem to the continuous one when the time step size goes to zero. The overall layout is summarized in Figure 1. The relaxed problems allow us to address the problem of mass transport through controlled nonsmooth dynamical systems, and both the mass transport and the optimal control achieving this task are performed in a single problem. The convergence of the discrete-time problem to the continuous-time problem is of importance for numerical purposes as well because the discrete-time problem in measures has less number of variables and would be a preferable choice for numerical studies using LMI relaxations obtained from the moment-sums-of-squares hierarchy [10]. (Due to space restrictions, we have not included the proofs of Theorem 2, Theorem 3, and Theorem 4 and we rather convey the main ideas; they will be covered in detail along with numerical findings based on LMI relaxations in the extended version of this paper.)

II. OPTIMAL CONTROL PROBLEM FOR NONSMOOTH DYNAMICAL SYSTEMS

We consider a controlled nonsmooth dynamical system modeled as an evolution variational inequality:

$$\dot{x}(t) \in f(x(t), u(t)) - \mathcal{N}_S(x(t)), \quad x(0) = x_0 \quad (1)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a vector field, and $\mathcal{N}_S(x)$ denotes the outward normal cone to the closed convex set

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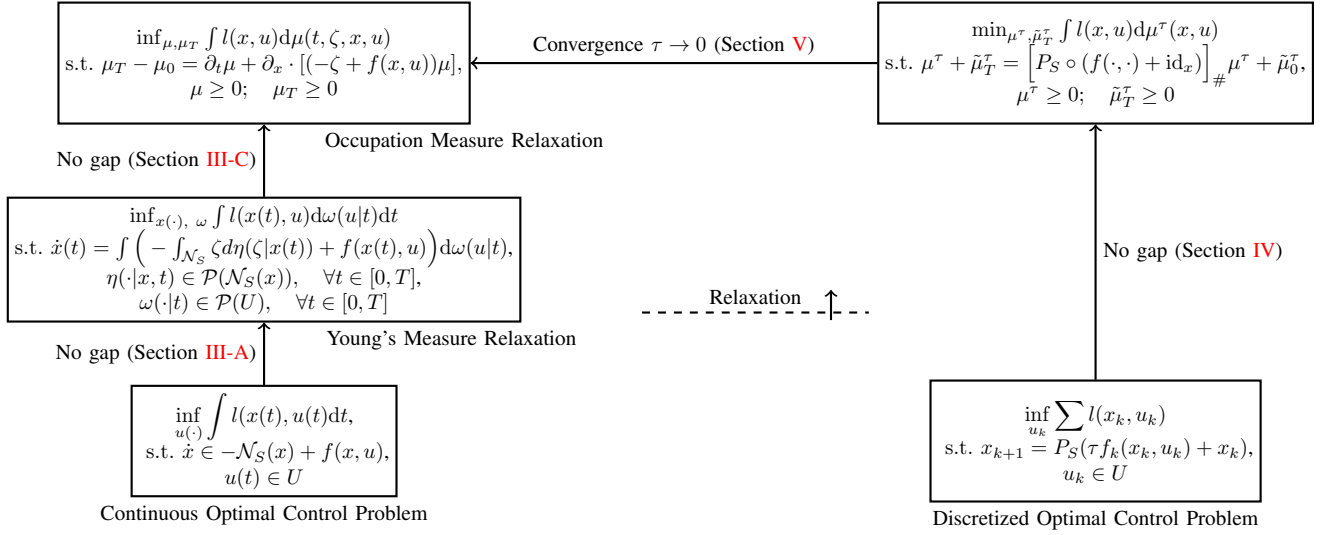


Fig. 1. For the continuous-time problem (left), we first relax the problem using Young measures and then occupation measures. We show that this does not produce any relaxation gap. For the discrete-time problem (right), we lift the problem into the cone of nonnegative Borel measures. We further show the convergence in the limit as $\tau \rightarrow 0$ (no gap).

S at $x \in S$. The control $u(t)$ belongs to a convex compact set U . The initial condition x_0 belongs to S . We make the following assumption for the existence and uniqueness of the absolutely continuous solutions.

Assumption 1. *There exists $L_f > 0$ such that*

$$\begin{aligned} |f(x, u)| &\leq L_f(1 + |x|) \\ |f(x_1, u) - f(x_2, u)| &\leq L_f|x_1 - x_2| \quad \forall u \in U. \end{aligned}$$

An optimal control problem for such a system can be formulated as

$$\begin{aligned} J^*(x_0) &= \inf_{u \in L^\infty[0, T]} \int_0^T l(x(t), u(t)) dt \\ \text{s.t. } \dot{x}(t) &\in -\mathcal{N}_S(x(t)) + f(x(t), u(t)), \\ x(0) &= x_0, \quad u(t) \in U \quad \forall t \in [0, T] \end{aligned} \quad (2)$$

where $l(x, u)$ is a continuous cost function in x, u subject to the following assumption.

Assumption 2. *The cost function satisfies the following upper bound:*

$$l(x, u) \leq h(x)(1 + |u|^2) \quad (3)$$

where h is a given function of $L^\infty(\mathbb{R}^n)$.

Later in Section III-C we will generalize the problem to an optimal control where the initial condition is a random variable whose law is a given probability distribution on S , that is, $x(0) \sim \mu_0$, where $\mu_0 \in \mathcal{P}(S)$.

In order to study the discrete-time version of (2) we first consider discrete-time dynamics defined over a partition of $[0, T] = \{0 = t_0, t_1, \dots, t_i, \dots, t_N = T\}$ with $t_k - t_{k-1} = \tau$, τ be the time step between the two samples. The evolution of states is described by time-stepping scheme [3]. Let x_k, x_{k+1} be the states at time instant $k\tau$ and $(k+1)\tau$, and

u_k be the control at time instant $k\tau$, then these are related by the following,

$$x_{k+1} = P_S(\tau f(x_k, u_k) + x_k) \quad (4)$$

where P_S is the (unique) projection onto the compact convex set S .

The discrete-time optimal control problem can be written as

$$\begin{aligned} J_\tau^*(x_0) &= \inf_{u_k \in \mathbb{R}^m} \sum_{k=0}^N l(x_k, u_k) \\ \text{s.t. } x_{k+1} &= P_S(\tau f(x_k, u_k) + x_k) \\ x_0 &= x_0; \quad u_k \in U \quad \forall 0 \leq k \leq N. \end{aligned} \quad (5)$$

Next, we consider the relaxation of the problems by reformulating these as linear programs in the space of measures.

Notation: Let $X \subset \mathbb{R}^n$, we denote by $C(X)$, respectively $C^1(X)$, the set of continuous, resp. continuously differentiable, functions. We will use $M(X)$ for the set of signed Borel measures. The cone of positive continuous functions is denoted by $C_+(X)$ and $M_+(X)$ is its dual cone of nonnegative Borel measures. We use the following notation

$$\langle l, \mu \rangle := \int_X l(x) d\mu(x) \quad (6)$$

to refer to the duality pairing between $l \in C(X)$, $\mu \in M(X)$. To measure the distance between two probability measures we use the Wasserstein 2 metric defined as follows:

$$W_2(\mu, \nu) = \min_{\gamma} \int |x - y|^2 d\gamma \quad \text{s.t. } \pi_{x\#}\gamma = \mu, \pi_{y\#}\gamma = \nu$$

where $\pi_{x\#}\gamma$ resp. $\pi_{y\#}\gamma$ is the x resp. y marginal of $\gamma \in \mathcal{P}(X \times X)$ and $\mu, \nu \in \mathcal{P}(X)$. W_2 defines a metric in the space of probability measures $(X, \mathcal{B}(X))$ with finite second order moments $\mathcal{B}(X)$ refers to the Borel sigma algebra on X .

III. RELAXED CONTINUOUS-TIME OPTIMAL CONTROL

An important result that will be used later deals with the uniform convergence of trajectories corresponding to weakly converging control sequences (Proof is based on the results in [15]).

Lemma 1. *Let $\{u_j(\cdot)\}_j$ be a weakly converging sequence i.e. $u_j \rightharpoonup u$. Then, for the system (1), the sequence of trajectories $\{x_j(\cdot)\}$ controlled by $\{u_j(\cdot)\}$ converges uniformly to a trajectory $x(\cdot)$ controlled by $u(\cdot)$.*

Next, we will use a foundational result calculus of variation theory and optimal control theory [16, Theorem 6.2], that the weak limit of converging sequence $\{x_j, u_j\}_j$ can be associated with a Young measure $\lambda_t \in \mathcal{P}(S, U)$ for each $t \in [0, T]$.

A. Relaxation using Young measures

We make a selection of vector field from $\mathcal{N}_S(x)$ by choosing a probability measure $\eta(\cdot|t, x) \in \mathcal{P}(\mathcal{N}_S(x(t)))$ (for more details about this selection see [17]). Then (1) can be expressed as

$$\dot{x}(t) = - \int_{\mathcal{N}_S(x(t))} \zeta d\eta(\zeta|x(t)) + f(x(t), u(t)). \quad (7)$$

Let $(x_j(\cdot), u_j(\cdot))$ be a minimizing sequence of state and controls to the problem defined in (2) such that $\lim_j \int l(x_j(t), u_j(t)) dt \rightarrow J^*(x_0)$.

Under the uniform integrability of the sequence $\{f(x_j, u_j)\}_j$, which follows from uniform boundedness of the corresponding $\{\dot{x}_j\}_j$ (Lemma 1) and Lipschitzness of $f(\cdot, u)$, there is an associated Young measure [16, Theorem 6.2] $\lambda_t = \delta_{x(t)} \otimes \omega_t$, where \otimes defines a direct product, such that

$$\dot{x}(t) = - \int_{\mathcal{N}_S(x(t))} \zeta d\eta(\zeta|x(t)) + \int_U f(x(t), u) d\omega(u|t). \quad (8)$$

The uniform bound on $\{|u_j|^2\}_j$ and the upper bound of $l(\cdot, \cdot)$ in Assumption 2 imply that the cost function $\{l(x_j, u_j)\}_j$ is uniformly integrable. So using [16, Theorem 6.2]

$$\lim_{j \rightarrow \infty} \int_0^T l(x_j(t), u_j(t)) dt = \int_0^T \int_U l(x(t), u) d\omega(u|t) dt.$$

We relax the optimal control problem defined in (2) using the Young measure as follows

$$J_r^*(x_0) := \inf_{x(\cdot), \omega(\cdot|\cdot)} \int_0^T \int_U l(x(t), u) d\omega(u|t) dt \quad (9)$$

$$\begin{aligned} \text{s.t. } \dot{x}(t) = & - \int_{\mathcal{N}_S(x(t))} \zeta d\eta(\zeta|x(t)) \\ & + \int_U f(x(t), u) d\omega(u|t), \quad (10) \end{aligned}$$

$$\eta(\cdot|x(t)) \in \mathcal{P}(\mathcal{N}_S(x(t))); \quad \omega(\cdot|t) \in \mathcal{P}(U) \quad \forall t \in [0, T].$$

Remark 1. *The selection of η will be performed in the optimization problem such as to ensure the feasibility of the problem.*

Theorem 1. *The optimal value J^* of (2) and the optimal value J_r^* of (9) are equal, i.e., there is no gap when relaxing to Young measures.*

Proof. (Proof of $J_r^* \leq J^*$): If $(x(\cdot), u(\cdot))$ is feasible for (2) then $(x(\cdot), \delta_{u(\cdot)})$ is feasible for (10), so $J^* \geq J_r^*$.

(Proof of $J^* \leq J_r^*$): $l(x, \cdot)$ is a convex function in u so using Jensen's inequality we get

$$J_r^* = \int_0^T \int_U l(x(t), u) d\omega_t(u|t) dt \geq \int_0^T \int_U l(x(t), u(t)) dt. \quad (11)$$

Now we use the assumption that $g(x, U)$ is convex so the barycenter of ω_t will lead to a valid vector field such that $\dot{x}(t) = g(x(t), u(t))$, $u(t) = \int u d\omega_t(u|t)$. The pair $(x(\cdot), u(\cdot))$ thus obtained is a feasible for (2) and thus $\int_0^T \int_U l(x(t), u(t)) dt \geq J^*$. Using this in (11) we obtain the desired inequality. \square

We observe that problem (9) is still nonlinear in $x(\cdot)$ and we need a linear program in measures. To do so, we further relax into the set of *occupation measures* and we show that these new objects are basically convex combinations of the Young measures. Thus, the set of occupation measures provides a convex closure to the set of Young measures.

B. Relationship between occupation measures and Young measures

In [11], the author associated to each $(t, x(\cdot), \omega(\cdot|\cdot))$ which satisfies (10), a linear functional defined on $C([0, T], S, U)$ such that

$$\langle \mu, h \rangle = \int_0^T \int_U h(t, x(t), u) d\omega(u|t) dt$$

for all $h \in C([0, T], S, U)$. Let us define

$$R := \left\{ (\hat{\mu}, \xi) \in M([0, T], S, U) \times M([0, T], S) \right.$$

$$\left. \langle \hat{\mu}, h \rangle + \langle \xi, g \rangle = \int_0^T \int_U h(t, x(t), u) d\omega(u|t) dt + g(T, x(T)) \right.$$

$$\text{s.t. } (t, x, \omega) \text{ satisfies (10) for some } \eta(\cdot|\cdot, \cdot)$$

$$\left. \text{for all } (h, g) \in C([0, T], S, U) \times C([0, T], S) \right\}. \quad (12)$$

Set R is weak-star sequentially compact by the definition of Young measure [16, Theorem 6.2]. It also satisfies

$$\|\hat{\mu}\|_{C^*} \leq T; \quad \|\xi\|_{C^*} \leq 1; \quad \mu, \xi \geq 0 \quad (13)$$

where bounds are in the dual-norm. It is easy to verify that the elements of the set R satisfy the following equation which is called *continuity equation* or *Liouville equation*

$$\begin{aligned} \int_{[0, T] \times S} \int_U \int_{\mathcal{N}_S(x)} \partial_t \phi(t, x) d\mu(t, x, \zeta, u) + \partial_x \phi(t, x) \cdot (-\zeta \\ + f(x, u)) d\mu(t, x, \zeta, u) = \int_S \phi d\mu_T - \phi(t_0, x_0). \quad (14) \end{aligned}$$

To verify the claim one could substitute $\mu = \delta_{x(t)} d\eta(\zeta|x(t)) d\omega(u|x(t)) dt$ in (14) to obtain (8) in the integrated form. Now define

$$O := \{(\mu, \mu_T) \in M([0, T], S, U) \times M([0, T], S) \mid \mu, \mu_T \text{ satisfies (13) and (14)}\}. \quad (15)$$

Clearly, any element of $\mu \in R$ satisfies (14) [11], and by definition of set O we see that $R \subset O$. Further, set O is convex and weak-star compact where convexity follows from the equation being linear in (μ, μ_T) . The closed balls in $C([0, T], S, U) \times C([0, T], S)$ corresponding to (13) are metrizable, and thus the weak-star sequential compactness of set O imply weak-star compactness. The elements of set O are called *occupation measures*.

C. Occupation Measure Relaxation

We consider an occupation measure based reformulation of the problem (2) as follows

$$J_o^*(x_0) = \inf_{\mu, \mu_T} \int l(x, u) d\mu(t, \zeta, x, u) \quad (16)$$

$$\begin{aligned} \text{s.t. } & \partial_t \mu(t, \zeta, x, u) + \\ & \partial_x \cdot [(\zeta + f(x, u))\mu(t, \zeta, x, u)] = \mu_T - \delta_{x_0, 0}, \end{aligned} \quad (17)$$

$$\mu \geq 0; \quad \mu_T \geq 0$$

where $\partial_x \cdot$ is the divergence operator and (17) has to be interpreted as the weak form of (14) i.e. when integrated against test functions $\phi \in C^1([0, T], S, U)$. Note here the search of the minimizer (μ, μ_T) is over set O which is weak-star compact. Further, the cost function is weakly-star continuous (can be checked using the fact that $l(\cdot, \cdot)$ is continuous) so the existence of a minimizer follows from the direct method of calculus of variations.

Theorem 2. *The optimal value J_o^* of (16) and the optimal value J_r^* of (9) are equal, i.e., there is no relaxation gap when relaxing to occupation measures.*

Note that every feasible solution to (17) is of the form $\mu(dt, d\zeta, dx, du|x_0)$, i.e., it is conditioned w.r.t. to the initial condition $x_0 \in S$. So, the problem of the optimal mass transport through a nonsmooth dynamical system is an easy generalization of problem (16) by integrating the objective function and the dynamics constraints with an initial distribution μ_0 . This will require new notation for the measures solution μ, μ_T to the obtained problem but for the notational convenience we will use the same notation. Thus we have the following linear program in the case where the initial configuration is distributed according to μ_0 :

$$J_o^*(\mu_0) = \inf_{\mu, \mu_T} \int l(x, u) d\mu(t, \zeta, x, u) \quad (18)$$

$$\begin{aligned} \text{s.t. } & \partial_t \mu(t, \zeta, x, u) + \partial_x \cdot [(\zeta + f(x, u))\mu(t, \zeta, x, u)] \\ & = \mu_T - \mu_0 \end{aligned}$$

$$\mu \geq 0, \quad \mu_T \geq 0.$$

IV. RELAXED DISCRETE-TIME OPTIMAL CONTROL

In this section, we consider the relaxation of the discrete-time problem (5) to the space of measures. The arguments presented are inspired by [13]. We consider a partition $\{0 =$

$t_0, t_1, \dots, t_i, \dots, t_N = T\}$ of time interval $[0, T]$ such that $t_{k+1} - t_k = \tau$ and $N = \lceil \frac{T}{\tau} \rceil$. We know from (4) that the states at successive time instants $k\tau$ and $(k+1)\tau$ are related as

$$x_{k+1} = P_S \circ (\tau f(x_k, u_k) + x_k) := G(x_k, u_k)$$

where P_S is the projection mapping on the compact convex set S . To relax the problem in (5) to the space of measures we will rely on probabilistic arguments since we are modeling the initial condition and the control as random variables [13]. For a fixed value of τ , we define the measures $\{\tilde{\mu}_k^\tau\}_{k \in \mathbb{N}}$, such that $\tilde{\mu}_k^\tau \in \mathcal{P}(\mathbb{R}^n)$, at time instances t_k , which are a result of recursive evolution of measures through dynamics (4). We introduce a time-varying stochastic kernel $\omega_k^\tau(u|x) \in \mathcal{P}(U)$ which is the probability measure on controls u at time instant t_k conditioned over the state. Using this stochastic kernel, we define a transition kernel as follows:

$$Q_\omega(A|x) := \int_U I_A(G(x, u)) d\omega(u|x).$$

It is used to describe the probability measure of states at time $k\tau$ given the state was at x_{k-1} at time $(k-1)\tau$, that is,

$$\text{Prob}(x_k \in A|x_{k-1}) = Q_{\omega_{k-1}^\tau}(A|x_{k-1}) \quad (19)$$

so that $Q_{\omega_k^\tau}(\cdot|\cdot)$ captures the effect of feedback $\omega_k^\tau(\cdot|\cdot)$ at each time step. Now, given $\tilde{\mu}_0^\tau$ is the measure at time 0, we can find its successor as

$$\tilde{\mu}_1^\tau(A) = \int_S \int_U I_A(G(x, u)) d\omega_0^\tau(u|x) d\tilde{\mu}_0^\tau(x) \quad (20)$$

where $A \subset \mathbb{R}^n$. We can then use this recursive update to obtain the measures at time $k\tau$

$$\tilde{\mu}_k^\tau(A) = \int_S \int_U I_A(G(x, u)) d\omega_{k-1}^\tau(u|x) d\tilde{\mu}_{k-1}^\tau(x). \quad (21)$$

Next we identify a measure $\mu_k^\tau \in \mathcal{P}(S, U)$

$$\mu_k^\tau(dx, du) = \omega_k^\tau(du|x) \tilde{\mu}_k^\tau(dx) \quad (22)$$

at each time instant. Then we can define an occupation measure $\mu^\tau \in \mathcal{P}(S, U)$ satisfying the following relationship

$$\int_{A, B} \varphi(x) d\mu^\tau(x, u) = \sum_{k=0}^{N-1} \int_{A, B} \varphi(x) d\mu_k^\tau(x, u) \quad (23)$$

for all $\varphi(x)$ bounded measurable function and $A \subset S, B \subset U$. This measure captures the time spent by all possible trajectories in some subset of state and control. Observe also that G defines the push forward between μ_{k-1}^τ and $\tilde{\mu}_k^\tau$ ¹. Thus, we have the following relationship:

$$x_k \xleftarrow{G} (x_{k-1}, u_{k-1}) \quad (24)$$

$$\tilde{\mu}_k^\tau(dx) \xleftarrow{G^\#} \mu_{k-1}^\tau(dx, du). \quad (25)$$

Proposition 1. *Let $\omega_k^\tau(\cdot|x) \in \mathcal{P}(U) \forall x \in S$ be a stochastic kernel and μ_k^τ, μ^τ be as defined in (22), (23). Let $\tilde{\mu}_0^\tau, \tilde{\mu}_T^\tau$*

¹Given two measurable space $(X, \mathcal{B}(X)), (Y, \mathcal{B}(Y))$ and a measurable function $G: X \rightarrow Y$, the pushforward of a measure $\mu \in \mathcal{P}(X)$ is defined as $G_\# \mu(B) = \mu(G^{-1}(B)), \forall B \subset \mathcal{B}(Y)$.

be such that $x_0 \sim \tilde{\mu}_0^\tau$ and $x_N \sim \tilde{\mu}_T^\tau$ respectively. Then, μ^τ , $\tilde{\mu}_0^\tau$, $\tilde{\mu}_T^\tau$ satisfy the following relationship,

$$\begin{aligned} & \int_{S,U} \varphi(x) d\mu^\tau(x, u) + \int_S \varphi(x) d\tilde{\mu}_T^\tau(x) \\ &= \int_{S,U} \varphi(G(x, u)) d\mu^\tau(x, u) + \int_S \varphi(x) d\tilde{\mu}_0^\tau(x) \end{aligned} \quad (26)$$

for all $\varphi(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ bounded, measurable functions.

Proof. Let $\varphi(x)$ be some bounded measurable function, then using (23) and (22),

$$\begin{aligned} \int_{S,U} \varphi(x) d\mu^\tau(x, u) &= \sum_{k=0}^{N-1} \int_{S,U} \varphi(x) d\mu_k^\tau(x, u) \\ &= \sum_{k=0}^{N-1} \int_{S,U} \varphi(x) d\omega_k^\tau(u|x) d\tilde{\mu}_k^\tau(x) = \sum_{k=0}^{N-1} \int_S \varphi(x) d\tilde{\mu}_k^\tau(x) \\ &= \sum_{k=0}^{N-1} \int_S \varphi(x) dG_{\#} \mu_{k-1}^\tau(x, u) \end{aligned}$$

where in the third and fourth equality we have used the fact that $\omega_k^\tau(\cdot|x) \in \mathcal{P}(U)$ and Eq. (25) respectively. Now using the change of variables formula for push-forward measures we get,

$$\begin{aligned} \int_{S,U} \varphi(x) d\mu^\tau(x, u) &= \sum_{k=0}^{N-1} \int_{S,U} \varphi(G(x, u)) d\mu_k^\tau(x, u) \\ &+ \int_S \varphi(x) d\tilde{\mu}_0^\tau(x) - \int_S \varphi(x) d\tilde{\mu}_T^\tau(x). \quad \square \end{aligned}$$

Next, we prove that for any solution of (26), we have a stochastic kernel such that if we propagate this stochastic kernel through (4) we obtain a sequence of measures defining the probability of state at each time instant.

Theorem 3. *Let $\tilde{\mu}_0^\tau, \tilde{\mu}_T^\tau, \mu^\tau$ be the measures which satisfy (26). Then there exists a stochastic kernel $\omega_k^\tau(\cdot|x) \in \mathcal{P}(U)$ at each time $k\tau$ which defines the evolution of measure $\tilde{\mu}_0^\tau$ through (4) into $\tilde{\mu}_T^\tau$. Moreover μ^τ is the corresponding occupation measure satisfying (23).*

Using (26) as the Liouville equation in discrete time for the controlled system (4) we define the following linear program in the cone of non-negative Borel measures:

$$J_d^{*\tau} := \inf_{\mu^\tau} \int l(x, u) d\mu^\tau(x, u) \quad (27)$$

$$\begin{aligned} \text{s.t. } \mu^\tau + \tilde{\mu}_T^\tau &= \left[P_S \circ (f(\cdot, \cdot) + \text{id}_x) \right]_{\#} \mu^\tau + \tilde{\mu}_0^\tau \quad (28) \\ \mu^\tau &\geq 0, \quad \tilde{\mu}_T^\tau \geq 0. \end{aligned}$$

Here, (28) is to be interpreted in weak form like (26) and id_x refers to the partial identity function, i.e., $\text{id}_x(x, u) = x$. So for each fixed τ we obtain $J_d^{*\tau}$ as the optimal value for (27).

Remark 2. *In comparison to the continuous-time problem (16), the selection from the normal cone is taken care of by the projection operator in (27). Thus, when we formulate LMI relaxations one would require lesser number of decision variables.*

V. CONVERGENCE OF THE DISCRETE-TIME OPTIMAL CONTROL PROBLEM TO THE CONTINUOUS-TIME PROBLEM

In this section, we present the underlying ideas to study the pointwise convergence $J_d^{*\tau}(\cdot) \xrightarrow{\tau \rightarrow 0} J_o^*(\cdot)$ where $J_d^{*\tau}$ and J_o^* are defined in (27) and (18) respectively.

Recall from (25) and (21) that $\tilde{\mu}_k^\tau$ is the push-forward of $\mu_{k-1}^\tau(\cdot, \cdot)$, i.e.

$$\tilde{\mu}_k^\tau = \left[P_S \circ (f(\cdot, \cdot) + \text{id}_x) \right]_{\#} \mu_{k-1}^\tau.$$

Using the disintegration of μ_k^τ from (22), we get

$$\tilde{\mu}_k^\tau = \left[P_S \circ \left(\int_U f(\cdot, u) d\omega^\tau(u|k\tau, \cdot) + \text{id}(\cdot) \right) \right]_{\#} \mu_{k-1}^\tau$$

where, $\text{id}(x) = x$. We will denote

$$G_k^\tau(x) := P_S \left(\int_U f(x, u) d\omega^\tau(u|k\tau, x) + x \right).$$

Thus for a partition $\{0 = t_0, t_1, \dots, t_i, \dots, t_N = T\}$ of time interval $[0, T]$ such that $t_{k+1} - t_k = \tau$, we have

$$\tilde{\mu}_{k+1}^\tau = G_{k\#}^\tau \tilde{\mu}_k^\tau \quad (29)$$

$$v_{k+1}^\tau = \frac{G_k^\tau(x) - x}{\tau}. \quad (30)$$

We consider two different interpolations of measures sequence $\{\tilde{\mu}_k^\tau\}_k$ to obtain

(1) Continuous interpolation curves $\tilde{\mu}_t^\tau$ such that

$$\tilde{\mu}_t^\tau = G_{t\#}^\tau \tilde{\mu}_k^\tau \quad \text{for } t \in (k\tau, (k+1)\tau] \quad (31)$$

where, $G_t^\tau = \left(\frac{t-k\tau}{\tau} G^* + \frac{(k+1)\tau-t}{\tau} \text{id} \right)$ and G^* being the optimal transport map between $\tilde{\mu}_{k+1}^\tau$ and $\tilde{\mu}_k^\tau$. The associated velocity is $\tilde{v}_t^\tau = v_{k+1}^\tau \circ (G_t^\tau)^{-1}$.

(2) Piecewise constant interpolation curves $\hat{\mu}_t^\tau$ such that

$$\hat{\mu}_t^\tau = \mu_{k+1}^\tau, \quad \text{for } t \in (k\tau, (k+1)\tau] \quad (32)$$

and velocity $\hat{v}_t^\tau = v_{k+1}^\tau$ for $t \in (k\tau, (k+1)\tau]$.

One important characterization of these curves [18, Chapter 5] is that

$$\|\tilde{v}_t^\tau\|_{L^2(\tilde{\mu}_t^\tau)} = \frac{W_2(\tilde{\mu}_k^\tau, \tilde{\mu}_{k+1}^\tau)}{\tau} = |(\tilde{\mu}^\tau)'(t)| \quad \text{for all } t \in (k\tau, (k+1)\tau]. \quad (33)$$

Remark 3. *The reason for using two interpolations is that the convergence of curve $\tilde{\mu}^\tau$ will imply the convergence to the limit μ_t which is the solution to the Liouville equation and convergence of piecewise constant interpolated curve $\hat{\mu}_t^\tau$ will imply that the limit velocity of Liouville equation is $v_t(x) \in -\mathcal{N}_S(x) + \int_U f(x, u) d\omega(u|t, x)$.*

We make an interpolation of ω^τ s.t. $\omega^\tau(u|t, x) = \omega^\tau(u|(k+1)\tau, x) \forall t \in (k\tau, (k+1)\tau]$. Lets define $g^\tau(k\tau, x) = \int_U f(x, u) d\omega(u|k\tau, x)$. We assume some mild regularity of measure ω^τ which ensure that $g^\tau(t, x) = \int f(x, u) d\omega^\tau(u|t, x)$ is uniformly (w.r.t. τ) Lipschitz continuous. Further using Assumption 1 and the fact that $\omega^\tau \in \mathcal{P}(U)$ we deduce that g^τ is uniformly bounded

w.r.t. τ . We conclude using the Arzelà-Ascoli theorem that $g^\tau(t, x) \rightarrow g(t, x)$. Using the Banach-Alaoglu theorem for a subsequence (without relabelling), $\omega^\tau \rightharpoonup \omega$ in the weak-star sense as $\omega \in \mathcal{P}(U)$, thus

$$g(t, x) = \int_U f(x, u) d\omega(u|t, x), \quad \forall t \in [0, T]. \quad (34)$$

Through the next theorem we formally state the convergence result for the interpolated curves introduced above.

Theorem 4. *For each $t \in [0, T]$, given measure $\mu_0 \in \mathcal{P}(S)$, the curves $\{\tilde{\mu}_t^\tau\}_t \in \mathcal{P}(S)$ defined in (31) converge uniformly in the W_2 metric to $\{\mu_t\}_t \in \mathcal{P}(S)$. The curve $\tilde{\mu}_t^\tau$ (32) converges to $\tilde{\mu}_t^\tau$ in the W_2 metric. Moreover $v_t(x) \in -\mathcal{N}_S(x) + \int_U f(x, u) d\omega(u|t, x)$.*

Theorem 4 states that $\tilde{\mu}_t^\tau$ converges uniformly to μ_t . Next, we would like to show that the objective function for (27) converges to (16) in the limit $\tau \rightarrow 0$. The discrete-time objective function is

$$\sum_{t=0}^N \tau \int_{S \times U} l(x, u) d\omega^\tau(u|t, x) d\tilde{\mu}_t^\tau(x).$$

Using the convergence results proved in Theorem 4 we have

$$\begin{aligned} & \sum_{k=0}^N \tau \int_{S \times U} l(x, u) d\omega^\tau(u|k\tau, x) d\tilde{\mu}_k^\tau(x) \\ & \xrightarrow{\tau \rightarrow 0} \int_0^T \int_{S \times U} l(x, u) d\omega(u|t, x) d\mu_t(x) dt \\ & = \int_0^T \int_{S \times U} \int_{\mathcal{N}_S(x)} l(x, u) d\mu(t, \zeta, x, u). \end{aligned} \quad (35)$$

So, if we take the minimizing sequence μ^τ for the discrete-time problems (27), the sequence converges to a feasible solution of (16). Thus, using (35), we get $\liminf_{\tau \rightarrow 0} J_d^{*\tau}(x) \geq J_o^*(x)$. For the opposite inequality, we observe that for every feasible solution of (16), we disintegrate and then discretize the feedback stochastic kernel. Using (21), we obtain a feasible solution for (27) for each τ , so $J_o^*(x) \geq J_d^{*\tau}(x)$ for each τ and which implies that $J_o^*(x) \geq \limsup_{\tau \rightarrow 0} J_d^{*\tau}(x)$. Combining both we get $\liminf_{\tau \rightarrow 0} J_d^{*\tau}(x) \geq J_o^*(x) \geq \limsup_{\tau \rightarrow 0} J_d^{*\tau}(x)$.

VI. CONCLUSION

We addressed the problem of optimal control for nonsmooth dynamical systems by formulating it as a linear problem in the cone of nonnegative Borel measures. We showed that relaxing the problems from finite dimensional space to infinite dimensional does not produce any relaxation gap. We further showed the convergence of the discrete-time problems to continuous-time problems in the space of measures using tools from optimal transport theory. We defined interpolated curves and then showed that as the time step goes to zero, the interpolated curves converge to the solutions of the Liouville equation where the corresponding velocity vector defines the nonsmooth dynamical system. We then show the pointwise convergence of the value functions.

In future work, we aim to solve numerically these nonsmooth optimal control problems using the moment-SOS hierarchy [10], which produces a family of convex LMI relaxations of the problem. Convex relaxations based on the moment-SOS hierarchy were used in [15] to study the problem of measure evolution of the functional regularized nonsmooth dynamical system. So our current paper can be seen as a first step of the extension to optimal control of the analysis results of [15]. We observe that in comparison to the continuous-time problem, the discrete-time case has fewer variables as there is no extra variable corresponding to the selection of the vector field from the normal cone. So solving the LMI relaxations is expected to be computationally faster. In future work, we will also address the case with nonstationary sets defined on the right-hand side of the dynamics which is expected to add more complexity to the problem.

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