Asymptotic blow-up behavior for the semilinear heat equation with non scale invariant nonlinearity

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Abstract

We characterize the asymptotic behavior near blowup points for positive solutions of the semilinear heat equation

$$\partial_t u - \Delta u = f(u),$$

for nonlinearities which are genuinely non scale invariant, unlike in the standard case $f(u) = u^p$. Indeed, our results apply to a large class of nonlinearities of the form $f(u) = u^p L(u)$, where $p > 1$ is Sobolev subcritical and L is a slowly varying function at infinity (which includes for instance logarithms and their powers and iterates, as well as some strongly oscillating functions).

More precisely, denoting by ψ the unique positive solution of the corresponding ODE $y'(t) = f(y(t))$ which blows up at the same time T, we show that if $a \in \Omega$ is a blowup point of u, then

$$\lim_{t\to T} \frac{u(a + y\sqrt{T-t}, t)}{\psi(t)} = 1, \quad \text{uniformly for } y \text{ bounded.}$$

Additional blow-up properties are obtained, including the compactness of the blow-up set for the Cauchy problem with decaying initial data.

Key words: Semilinear heat equation, asymptotic blowup behavior, blow-up set, regular variation, weighted energy.

1 Introduction

We consider the semilinear heat equation

$$\begin{cases} u_t - \Delta u = f(u), & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Throughout this article, Ω is a, possibly unbounded, uniformly smooth domain of $\mathbb{R}^n$ ($n \geq 1$) and $f \in C^1([0,\infty))$ satisfies $f(0) \geq 0$ and $f(s) > 0$ for s large. It is well known that, for $u_0 \in L^\infty(\Omega)$, problem (1.1) has a unique nonnegative classical solution. Throughout this article we will denote by u this solution and by $T = T(u_0) \in (0,\infty]$ its maximal existence time. If f has superlinear growth in the sense that $1/f$ is integrable at infinity then (see, e.g., [17, Section 17]), under suitable largeness condition on the initial data, u blows up in finite time, i.e $T < \infty$ and

$$\lim_{t\to T} ||u(t)||_\infty = \infty.$$

In this case, T is called the blowup time of u. Given $a \in \overline{\Omega}$, we say that a is a blowup point of u if there exists $(a_j, t_j) \to (a, T)$ such that $|u(a_j, t_j)| \to \infty$ as $j \to \infty$.

The asymptotic behavior of blowup solutions for problem (1.1) has been studied in great detail in the special case $f(u) = |u|^{p-1}u$, especially in the Sobolev subcritical range $p \in (1, p_S)$ with

$$p_S = \begin{cases} \frac{n+2}{n-2}, & \text{if } n \geq 3 \\ \infty, & \text{if } n \leq 2. \end{cases}$$

In their fundamental work [9, 10], Giga and Kohn have studied the local behavior of solutions near blow-up points for this range of p and have discovered that, in backward self-similar parabolas, the solution behaves like the solution of the corresponding ODE, namely:

$$\lim_{t\to T} (p-1)^{\frac{1}{p-1}} (T-t)^{\frac{1}{p-1}} u(a + y\sqrt{T-t}, t) = 1. \tag{1.2}$$

Later on, building on the result in [9, 10], the sharp final blowup profiles and the corresponding refined space-time behaviors have been completely classified in the Sobolev subcritical range (cf. [7, 13, 14, 21, 22, 3, 15, 16, 19]). The complementary range $p \geq p_S$ has also been the subject of a number of investigations, but this range exhibits more complicated behaviors and is less understood (for instance other self-similar or non self-similar behaviors are possible; see, e.g., [17, Section 25] and the references therein for details).

On the other hand, the above mentioned analysis for the pure power nonlinearity in the Sobolev subcritical range heavily depends on the scale invariance properties of the equation, namely the fact that the equation is invariant by the transformation

$$u \mapsto \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t), \quad \lambda > 0.$$

As already noted in [1, Section 5.3], the precise asymptotic blow-up behavior for general nonlinearities is still a widely open problem. The main goal of the present work and of the companion paper [4] is to partially fill this gap and to provide a precise description of the blow-up behavior of solutions for a large class of non scale invariant nonlinearities. The present paper is devoted to the local behavior of general solutions near arbitrary blow-up points, whereas [4] will concentrate on radial decreasing solutions and describe the sharp final blowup profile and the refined space-time behavior (note that [4] will make essential use of the results of the present paper).

As far as we know, the only previous study of local blow-up asymptotics for problem (1.1) with a genuinely non-scale invariant nonlinearity [1] was recently carried out in [6] where, for the special case of the logarithmic nonlinearity $|u|^{p-1}u \log^q(2+u^2)$, the authors construct a special, single-point blow-up solution with a prescribed final and space-time blow-up profile. A starting point in the approach of [6] is to rescale the problem by similarity variables and ODE renormalization, using as normalization factor the positive solution of the ODE $y' = f(y)$ blowing up at the same time T (instead of $((p-1)(T-t))^{-1/(p-1)}$). We will here use this idea with a different goal, and in a more systematic way as regards the nonlinearity. Namely, for a large class of nonlinearities satisfying a suitable regular variation assumption at infinity, we will prove results that can be seen as a counterpart of [9, 10], describing the local behavior of any solution near an arbitrary blow-up point. We will also show the compactness of the blow-up set in the case $\Omega = \mathbb{R}^n$ with decaying initial data.

2 Main results

2.1 Statements of main results

For $p \in \mathbb{R}$, we say that the function f has regular variation at ∞ of index p if the function L defined by $L(s) := s^{-p} f(s)$ satisfies

$$\lim_{\lambda \to \infty} \frac{L(\lambda s)}{L(\lambda)} = 1 \quad \text{for each} \quad s > 0. \tag{2.1}$$

A function L with the property (2.1) is called a function with slow variation at ∞. When L is C^1 near infinity, a well-known sufficient condition for (2.1) is $\lim_{s\to\infty} s\frac{L'(s)}{L(s)} = 0$. We shall consider the following subclass of functions with regular variation, with index $p > 1$:

$$f \in C^1([0,\infty)), \quad f(0) \geq 0, \quad f \text{ is positive and } C^2 \text{ for large } s \tag{2.2}$$

$$L(s) := \frac{f(s)}{s^p} \text{ satisfies } \frac{sL'(s)}{L(s)} = O\big(\log^{-\alpha}(s)\big) \text{ and } \frac{s^2 L''(s)}{L(s)} = o(1) \text{ as } s \to \infty, \text{ for some } \alpha > \tfrac{1}{2}. \tag{2.3}$$

Some examples of function L with slow variation that satisfy (2.3) are given in Remark 2.1 below (see [18] for a general reference on regularly varying functions and, e.g, [2] and [5, 20]). Throughout this paper, we shall denote by $\psi(\cdot)$ the unique positive increasing solution of $y' = f(y)$ which blows up at $T < \infty$ (see at the end of this section for details). We also denote $\beta = \frac{1}{p-1}$ and $\kappa = \beta^\beta$.

[1] Nonlinearities with asymptotic scale invariance as $s \to \infty$ can be treated by similar methods as for the case $f(s) = s^p$, see [10, Section 6A] and [3].

Theorem 2.1. *Let $1 < p < p_S$ and $u_0 \in L^\infty(\Omega)$ with $u_0 \geq 0$. Assume* (2.2)-(2.3) *and $T := T_{\max}(u_0) < \infty$. If $a \in \Omega$ is a blow-up point of u, then*

$$\lim_{t\to T} \frac{u(a + y\sqrt{T-t}, t)}{\psi(t)} = 1, \tag{2.4}$$

uniformly on compact sets $|y| \leq C$.

Remark 2.1.

- *The conclusion of Theorem 2.1, with limit ± 1, remains valid for sign changing solutions (i.e., without the assumption $u_0 \geq 0$ and, for instance, extending f as as odd function for $s < 0$) provided the solution is of type I, namely:*

$$\|u(t)\|_\infty \leq M\psi(t), \quad T - \delta < t < T, \tag{2.5}$$

for some $M, \delta > 0$. See Theorem 3.1 below.

- *For nonnegative initial data, property* (2.5) *follows from the recent result [20, Theorem 3.1], whenever $p \in (1, p_S)$ and L has slow variation at ∞, hence in particular under the assumptions of Theorem 2.1 (see Theorem 5.1 in appendix below). However, for sign-changing solutions and $p \in (1, p_S)$, property* (2.5) *so far is known only for $f(u) = |u|^{p-1}u$ [9, 10] or for $f(u) = |u|^{p-1}u \log^q(2 + u^2)$ [12]. The result also remains true for $p = p_S$ under assumption* (2.5) *(but* (2.5) *is not true in general when $p = p_S$ (see [17, Section 25]).*

- *As examples of nonlinearities such that assumptions* (2.2)-(2.3) *are satisfied, so that Theorem 2.1 applies, we have $f(s) = s^p L(s)$ with L given by:*

$$\begin{cases} \star\ \log^a(K+s) \text{ for } K > 1 \text{ and } a \in \mathbb{R}, \\ \star \text{ the iterated logarithms } \log_m(K+s),^2 \\ \star\ \exp(|\log s|^\nu) \text{ with } \nu \in (0, 1/2), \\ \star \text{ the strongly oscillating functions} \\ \qquad \big[\log(3+s)\big]^{\sin[\log\log(3+s)]} \quad \text{and} \quad \exp\big[|\log s|^\nu \cos(|\log s|^\gamma)\big], \quad \nu, \gamma > 0,\ \nu + \gamma < 1/2, \\ \star\ 1 + a\sin\big(\log^\nu(2+s)\big) \text{ with } \nu \in (0, 1/2) \text{ and } |a| < 1. \end{cases}$$

Our results thus cover a large class of non scale invariant nonlinearities. However, it so far remains an open problem what is the largest possible class of f for which the conclusions of Theorem 2.1 hold. In particular the condition $\alpha > \frac{1}{2}$ in (2.3) *is required for the existence of the key energy functional used in the proof (see Lemma 3.1).*

- *For radial decreasing solutions of problem* (1.1)*, for a large class of regularly varying nonlinearities, the final blow-up profile and refined blow-up behavior are obtained in the companion paper [4]. Theorem 2.1 of the present paper is used as an important tool in the proofs in [4].*

Our next main result, for the case $\Omega = \mathbb{R}^n$, shows the compactness of the blowup set for initial data decaying at infinity. Note that this assumption is essentially optimal, in view of examples in [11] (see also [17, Remark 24.6(ii)]) of solutions blowing up at space infinity for nondecaying bounded initial data. To this end we recall the notation $C_0(\mathbb{R}^n) = \{\phi \in C(\mathbb{R}^n);\ \lim_{|x|\to\infty} \phi(x) = 0\}$.

Theorem 2.2. *Let $1 < p < p_S$, assume* (2.2)-(2.3)*, $f(0) = 0$, and let $u_0 \in C_0(\mathbb{R}^n)$ with $u_0 \geq 0$ be such that $T < \infty$. Then the blow-up set of u is compact. More precisely, there exists $R > 0$ such that*

$$\sup_{|x|>R,\, t\in(0,T)} u < \infty.$$

As a consequence of Theorem 2.1 we also get the following "no-needle" property for blow-up solutions.

Proposition 2.1. *Under the assumptions of Theorem 2.1, if $a \in \Omega$ is a blowup point, then*

$$\lim_{(x,t)\to(a,T)} u(x,t) = \infty. \tag{2.6}$$

[2] where $\log_m = \log \circ \cdots \circ \log$ (m times), $m \in \mathbb{N}^*$ and $K > [\exp \circ \cdots \circ \exp](0)$.

Remark 2.2. *The analogues of Theorems 2.1 and 2.2 are valid for sign changing solutions (assuming for instance that f is an odd function on $\mathbb{R}$), provided the solution is known to be of type I; see Theorems 3.1 and 4.1 below.*

Throughout this paper, we shall use the notation

$$F(X) := \int_X^\infty \frac{ds}{f(s)}. \tag{2.7}$$

Note that, under assumptions (2.2)-(2.3) with $p > 1$, there exists a large $A > 0$ such that $F : [A, \infty) \to (0, F(A)]$ is well defined and decreasing. Moreover, ψ is defined on some interval $(T - \eta, T)$ and we have

$$\psi(t) = F^{-1}(T - t), \quad t \in (T - \eta, T). \tag{2.8}$$

The rest of the paper is organizezd as follows. In Subsection 2.2, we present the main ideas of our proofs. Sections 3 and 4 are respectively devoted to the proof of Theorems 2.1 and 2.2, and of their extension to sign-changing solutions. Finally in Appendix, based on [20], we provide the necessary type I blowup estimate for nonnegative solutions.

2.2 Ideas of proofs

To prove Theorem 2.1 (and its extension to possibly sign-changing solutions given in Theorem 3.1 below), we shall adapt the methods of [8, 9, 10] to equations with genuinely non-scale invariant nonlinearities by taking advantage of the slow variation property of L in an appropriate manner. Since we want to compare u with the solution of ODE ψ near the blowup point a, we first rescale the equation by *similarity variables and ODE renormalization* by setting $y = (x - a)/\sqrt{T - t}$, $s = -\log(T - t)$ and defining the rescaled function w_a by $u(x, t) = \psi(t)w_a(y, s)$. Such a rescaling, which extends the seminal idea from [9] in the case $f(u) = u^p$, was applied in [6] in the particular case of a logarithmic nonlinearity in order to construct special blow-up solutions with prescribed profile. The function w_a is then a global solution of

$$\partial_s w_a - \Delta w_a + \frac{1}{2} y \cdot \nabla w_a = e^{-s}\psi^{p-1}\Big(|w|^{p-1}wL(|w|\psi) - L(\psi)w\Big) \tag{2.9}$$

(where we omit the variables without confusion) in $\mathcal{W}_a := \{(y, s) : s_0 < s < \infty,\ y \in D(s)\}$, with $D(s) := e^{s/2}(\Omega - a)$ and $s_0 > 0$ large. Moreover, w_a and $|\nabla w_a|$ are bounded under our assumptions. We thus want to show that w_a is attracted by the set of equilibria of (2.9), which turns out to be the same as in the pure power case, namely $\{0, -1, +1\}$. As a significant source of difficulty, some new, nonautonomous factors arise from the slowly varying part of the nonlinearity. To handle them, we rewrite (2.9) as the pure power plus a perturbative term, namely

$$\partial_s w_a - \Delta w_a + \frac{1}{2} y \cdot \nabla w_a = |w|^{p-1}w + H(y, s).$$

By suitably exploiting the slow variation hypothesis, we can show that H converges to 0 in L^∞-norm as $s \to \infty$ and is globally square integrable in space-time with respect to the Gaussian measure. This enables us to construct a weighted energy $G_a(s)$, which is a modified version of that in [9] and can be used as a Liapunov functional to show the desired convergence to a constant steady-state.

The nondegeneracy of blow-up (i.e. ruling out the case $w_a \to 0$) is then obtained by a similar argument as in [10], based on weighted energy, but using our modified energy functional, and a removable singularity property, namely a local lower bound on the blow-up rate. Under our assumptions on f, the latter (valid for all $p > 1$) takes the following form: if $|u(x, t)| \le \varepsilon\psi(t)$ in some neighbourhood of (a, T) with $\varepsilon > 0$ sufficiently small, then a is not a blow-up point. This is proved by a comparison argument extending that from [17, Proposition 25.1] for the pure power case.

As for Theorem 2.2, it is obtained by combining the above arguments with the fact that, under the assumption $u_0 \in C_0(\mathbb{R}^n)$, the weighted energy $G_a(s)$ can be shown to decay as $|a| \to \infty$.

3 Extension and proof of Theorem 2.1

As mentioned before, we have the following extension of Theorem 2.1 for possibly sign-changing solutions under a type I blowup assumption.

Theorem 3.1. *Let $1 < p \le p_S$. Assume that $f \in C^1(\mathbb{R})$ is an odd function, with $f > 0$ and C^2 for large $s > 0$, and that f satisfies (2.3). Let $u_0 \in L^\infty(\Omega)$ satisfy $T := T_{\max}(u_0) < \infty$ and (2.5). If $a \in \Omega$ is a blow-up point of u, then*

$$\lim_{t\to T} \frac{u(a + y\sqrt{T-t}, t)}{\psi(t)} = \pm 1, \tag{3.1}$$

uniformly on compact sets $|y| \le C$.

Remark 3.1. *The oddness assumption is made here only for simplicity. Similar conclusions could be obtained under more general hypotheses.*

The proof of Theorem 3.1 is divided into several subsections for clarity. In subsection 3.1, we prove a local lower bound on the blow-up rate. In subsection 3.2, we give an upper estimate of $|\nabla u|$ which is needed for the existence of the key weighted energy functional and its properties. The latter are derived in subsection 3.3. With the help of these tools, we then conclude the proof of Theorem 3.1 by showing the convergence and the nondegeneracy of blow-up in subsections 3.4 and 3.5 respectively.

3.1 Local lower bound on the blow-up rate

Proposition 3.1. *Let $f \in C^1(\mathbb{R})$ be odd. Assume that there exist $q, z_0 > 1$ such that*

$$f(z) > 0 \text{ and } g(z) := z^{-q} f(z) \text{ is increasing for } z > z_0. \tag{3.2}$$

Let $T, \delta > 0$, $a \in \mathbb{R}^n$ and set $\mathcal{Q} := B(a, \delta) \times (T - \delta^2, T)$. There exist $\varepsilon_0, \delta_0 > 0$ depending only on n, f such that if $\delta \in (0, \delta_0]$ and u is a classical solution of

$$u_t - \Delta u = f(u), \quad (x,t) \in \mathcal{Q}, \tag{3.3}$$

and satisfies

$$|u(x,t)| \le \varepsilon_0 \psi(t), \quad (x,t) \in \mathcal{Q}, \tag{3.4}$$

then u is uniformly bounded in a neighborhood of (a, T). More precisely,

$$\sup_{|x-a|<\delta/4} |u(x,t)| \le C, \quad \text{for all} \quad t \in \Big(T - \frac{\delta^2}{2}, T\Big),$$

where $C > 0$ depends only on n, f, δ and $\|u(T - \frac{\delta^2}{2})\|_{L^\infty(B_a(\delta))}$.

Remark 3.1. *(i) We note that under assumption (3.2), the ODE $y' = f(y)$ indeed has a positive solution $\psi(t)$ on some interval $(T - \delta_0^2, T)$ which blows up at $t = T$.*

(ii) The result in Proposition 3.1 is of independent interest since it is valid for any $p > 1$ and more general nonlinearities than Theorem 3.1. Note that no boundary conditions are assumed and that this is a purely local result.

(iii) Proposition 3.1 is the analogue of [10, Theorem 2.1] for the pure power case. See also [17, Theorem 25.3], which provides a different proof, based on comparison, a quadratic change of unknown and a cut-off. It turns out that, by suitable modifications of the arguments in [17], we can handle rather general nonlinearities without scale invariance.

Proof of Proposition 3.1. By assumption (3.2), we may pick $\delta_0, C_0 > 0$ depending only on f, such that

$$\psi(t) \ge z_0 > 1 \quad \text{and} \quad f(\psi(t)) \ge C_0 \psi^q(t) \quad \text{for all } t \in (T - \delta_0^2, T). \tag{3.5}$$

Let $\delta \in (0, \delta_0]$. By a space-time translation, we may assume $a = 0$ and $T = \delta^2$ and then $\mathcal{Q} = B_\delta \times (0, \delta^2)$. Set

$$\mu = \min\{1/2, (q-1)/4\}. \tag{3.6}$$

For given $R > 0$, we may find $\phi \in C^2(\mathbb{R}^n)$ such that $0 \le \phi \le 1$,

$$\begin{cases} \phi(x) = 0 \text{ for } |x| \ge R/\sqrt{2} \\ \phi(x) = 1 \text{ for } |x| \le R/2 \end{cases} \tag{3.7}$$

and

$$|\nabla\phi|^2 + |\Delta\phi^2| \le C(R,n)\phi^{2(1-\mu)}. \tag{3.8}$$

We choose $R = \delta$ and set

$$v = u^2\phi^2.$$

For $(x,t) \in \mathcal{Q}$, we have

$$v_t - \Delta v = 2uu_t\phi^2 - 2\phi^2(u\Delta u + |\nabla u|^2) - 8u\phi\nabla u\cdot\nabla\phi - u^2\Delta\phi^2.$$

Since $4|u\phi\nabla u\cdot\nabla\phi| \le \phi^2|\nabla u|^2 + 4u^2|\nabla\phi|^2$, therefore

$$v_t - \Delta v \le 2\phi^2|uf(u)| + u^2(8|\nabla\phi|^2 + |\Delta\phi^2|). \tag{3.9}$$

Let $\varepsilon \in (0,1]$ to be chosen below. Setting $M_0 = \sup_{|z|\le z_0} |zf(z)|$, using (3.2), (3.4) and writing $|uf(u)| = |u|^{q-1}g(|u|)u^2$, we have

$$|uf(u)| \le \begin{cases} \varepsilon^{q-1}\psi^{q-1}g(\psi)u^2 & \text{if } |u| \ge z_0 \\ M_0 & \text{otherwise.} \end{cases}$$

Note that $q - 1 \ge \frac{2\mu}{1-\mu} > 2\mu$ owing to (3.6). Using (3.4) again and (3.8), it follows that

$$\begin{aligned} v_t - \Delta v &\le 2\big(\varepsilon^{q-1}\psi^{q-1}g(\psi)u^2 + M_0\big)\phi^2 + |u|^{2\mu}v^{1-\mu}\phi^{-2(1-\mu)}(8|\nabla\phi|^2 + |\Delta\phi^2|) \\ &\le \varepsilon^{2\mu}\Big(2\frac{f(\psi)}{\psi}v + C(n,R)\psi^{2\mu}v^{1-\mu}\Big) + 2M_0. \end{aligned}$$

Next using Young's inequality and (3.5), we may estimate the second term by

$$C(n,R)\psi^{2\mu}v^{1-\mu} \le \psi^{\frac{2\mu}{1-\mu}}v + C^{\frac{1}{\mu}}(n,R) \le \psi^{q-1}v + C^{\frac{1}{\mu}}(n,R) \le C_0^{-1}\frac{f(\psi)}{\psi}v + C^{\frac{1}{\mu}}(n,R).$$

Consequently, we obtain, for some $C_1 = C_1(f) > 0$ and $C_2 = C_2(f,R,n) > 0$,

$$v_t - \Delta v \le C_1\varepsilon^{2\mu}\frac{f(\psi)}{\psi}v + C_2, \quad (x,t) \in \mathcal{Q}. \tag{3.10}$$

Let $\overline{v} = K\psi^\theta$ for $t \in (0,T)$, with $K,\theta > 0$ to be chosen. Using (3.5), we see that

$$\overline{v}_t = K\theta\psi^{\theta-1}f(\psi) \ge \frac{\theta}{2}\Big(\frac{f(\psi)}{\psi}\overline{v} + KC_0\Big).$$

Choosing $\theta = \min(2C_1\varepsilon^{2\mu}, 1)$ and $K = \max\{2C_2(C_0\theta)^{-1}, \|u(\cdot,T/2)\|_\infty\}$, it follows that $\overline{v}$ is a supersolution to (3.10) and that $\overline{v}(T/2) \ge \|v(\cdot,T/2)\|_\infty$. Since $v = 0$ on $\partial B_\delta \times (0,T)$, we deduce from the comparison principle that $v \le \overline{v}$ in $B_\delta \times [T/2,T)$, hence

$$u^2 \le K\psi^\theta \quad \text{in } B_{\delta/2}\times[T/2,T). \tag{3.11}$$

At points (x,t) where $|u| \ge z_1 := \max(z_0,K)$, inequality (3.11) implies $\psi^\theta \ge K$ hence $|u| \le \psi^\theta \le \psi$ so that, by Young's inequality, (3.2) and (3.5),

$$2|uf(u)| + C(n,R)u^2 \le 3|u|^{q+1}g(|u|) + C(n,R) \le 3\psi^{(q+1)\theta}g(\psi) + C(n,R).$$

It follows that

$$2|uf(u)| + C(n,R)u^2 \le 3\psi^{(q+1)\theta}g(\psi) + K_1 \quad \text{in } B_{\delta/2}\times[T/2,T), \tag{3.12}$$

with $K_1 = K_1\big(n,R,f,\theta,\|u(\cdot,T/2)\|_\infty\big) > 0$.

Now consider $v = u^2\phi^2$ with $R = \delta/2$ instead of $R = \delta$ in (3.7). Taking $\varepsilon = \varepsilon(f) > 0$ sufficiently small so that $\theta = 2C_1\varepsilon^{2\mu} < (q-1)/2(q+1)$, inequalities (3.9) and (3.12) imply

$$v_t - \Delta v \le 3\psi^{(q-1)/2}g(\psi) + \tilde{K}_1 \quad \text{in } B_{\delta/2}\times[T/2,T),$$

with $\tilde{K}_1 = \tilde{K}_1\big(n, R, f, \|u(\cdot, T/2)\|_\infty\big) > 0$. Let $\tilde{v} = K_2 + \tilde{K}_1 t - \frac{6}{q-1}\psi^{(1-q)/2}$ with $K_2 > 0$ to be chosen. By a simple computation, we have

$$\tilde{v}_t = 3\psi^{-(q+1)/2} f(\psi) + \tilde{K}_1 = 3\psi^{(q-1)/2} g(\psi) + \tilde{K}_1.$$

Choosing

$$K_2 = \|u(\cdot, T/2)\|_\infty + \frac{6}{q-1}\psi^{(1-q)/2}(T/2)$$

and noting that $\psi' \geq 0$ on $(T/2, T)$, it follows that $v \leq \tilde{v}$ on $\partial_p\Big(B_{\delta/2} \times [T/2, T)\Big)$. Using the comparison principle once again, we obtain

$$v \leq \tilde{v} \leq K_3 := K_2 + \tilde{K}_1 T, \quad \text{in } B_{\delta/2} \times [T/2, T).$$

By the definition of v and (3.7), we conclude that

$$\sup_{|x|<\delta/4} u^2(t) \leq K_3, \quad \frac{T}{2} < t < T.$$

□

3.2 Upper estimate of $|\nabla u|$

In this subsection we show that the upper blow-up estimate (2.5) implies a similar estimate for the gradient. The latter is needed for the existence and properties of our energy functional in section 3.3 below.

Proposition 3.2. *Let $f \in C^1(\mathbb{R})$ be odd and assume that there exist $m > q > 1$ and $z_0 > 1$ such that*

$$f(z) > 0,\ z^{-q} f(z) \text{ is increasing and } z^{-m} f(z) \text{ is decreasing for } z > z_0. \tag{3.13}$$

Let $u_0 \in L^\infty(\Omega)$ and assume that $T := T_{\max}(u_0) < \infty$ and that u satisfies (2.5) for some $M, \delta > 0$. Then, there exists $t_0 = t_0(f, T) \in (T - \delta, T)$ such that

$$\|\nabla u(t)\|_\infty \leq M_1 (T-t)^{-1/2} \psi(t), \quad t_0 < t < T, \tag{3.14}$$

for some $M_1 = M_1(\Omega, M, f) > 0$.

Remark 3.2. *Under assumption (2.3) of Theorem 2.1, property (3.13) is satisfied for any q, m such that $1 \leq q < p < m$, owing to*

$$\big(z^{p-q} L(z)\big)' = (p-q) z^{p-q-1} L(z) + z^{p-q} L'(z) = z^{p-q-1} L(z)\Big(p - q + \frac{z L'(z)}{L(z)}\Big).$$

The proof of Proposition 3.2 is based on modifications of arguments in [9, 17], suitably adapted to cover larger classes of nonlinearities.

Proof of Proposition 3.2. Let $t_1 \in (\max(0, T-1), T)$ be such that $\psi(t) \geq z_0$ in (t_1, T). We first claim that

$$\|f(u(t))\|_\infty \leq C(M, f) \frac{\psi(t)}{T-t}, \quad t_1 < t < T. \tag{3.15}$$

To this end, we firstly show that

$$\sup_{X \geq k^{-1} z_0} \frac{f(X)}{X} F(kX) \leq \frac{k^{1-m}}{q-1} \quad \text{for each } k \in (0, 1]. \tag{3.16}$$

By the increasing property in (3.13), we note that, for all $X \geq z_0$,

$$\frac{f(X)}{X} F(X) = \frac{f(X)}{X} \int_X^\infty \frac{ds}{f(s)} = \frac{f(X)}{X} \int_X^\infty \frac{s^q ds}{s^q f(s)} \leq \frac{f(X)}{X} \frac{X^q}{f(X)} \int_X^\infty \frac{ds}{s^q} = \frac{1}{q-1}. \tag{3.17}$$

On the other hand, the decreasing property in (3.13) yields $f(kz) \geq k^m f(z)$ for all $z \geq k^{-1}z_0$ hence, by the change of variable $s = kz$,

$$F(kX) = \int_{kX}^{\infty} \frac{ds}{f(s)} = k \int_X^{\infty} \frac{dz}{f(kz)} \leq k^{1-m} \int_X^{\infty} \frac{dz}{f(z)} = k^{1-m} F(X), \quad X \geq k^{-1}z_0,$$

and this combined with (3.17) implies (3.16). Next, at points where $|u(x,t)| \geq \max(M,1)z_0$, applying (3.16) with $k = \min(1, M^{-1})$ and using (2.5), the decreasing monotonicity of F and (2.8), we have

$$|f(u(x,t))| = \Big| \frac{f(u(x,t))}{u(x,t)} u(x,t) \Big| \leq \frac{C(M,f)\psi(t)}{F(k|u(x,t)|)} \leq \frac{C(M,f)\psi(t)}{F(\psi(t))} = \frac{C(M,f)\psi(t)}{T-t}.$$

Since $\psi(t) \geq 1 \geq T - t$, this implies

$$\|f(u(t))\|_\infty \leq C(M,f)\Big(1 + \frac{\psi(t)}{T-t}\Big) \leq 2C(M,f)\frac{\psi(t)}{T-t}, \quad t_1 < t < T,$$

hence (3.15).

Let $t_1 < s < t < T$. Denote by $(e^{-\tau A})_{\tau \geq 0}$ the heat semigroup on Ω (with Dirichlet conditions for $\Omega \neq \mathbb{R}^n$). By the variation of constants formula, the gradient estimate for $e^{-\tau A}$ (see, e.g, [17, Proposition 48.7*]) and (2.5), we have

$$\begin{aligned}
\|\nabla u(t)\|_\infty &\leq \|\nabla e^{-(t-s)A} u(s)\|_\infty + \int_s^t \|\nabla e^{-(t-\tau)A} f(u(\tau))\|_\infty d\tau \\
&\leq C(\Omega)(t-s)^{-1/2}\|u(s)\|_\infty + C(\Omega)\int_s^t (t-\tau)^{-1/2}\big\|f(u(\tau))\big\|_\infty d\tau \\
&\leq C(\Omega,M)(t-s)^{-1/2}\psi(s) + C(\Omega,M,f)\int_s^t (t-\tau)^{-1/2}(T-\tau)^{-1}\psi(\tau)d\tau, \\
&\leq C(\Omega,M,f)\Big((t-s)^{-1/2}\psi(s) + \psi(t)(T-t)^{-1}\int_s^t (t-\tau)^{-1/2}d\tau\Big), \\
&\leq C(\Omega,M,f)\Big((t-s)^{-1/2} + 2(T-t)^{-1}(t-s)^{1/2}\Big)\psi(t).
\end{aligned}$$

Assume $t > t_0 := (t_1 + T)/2$ and choose $s := 2t - T$. Since $t_1 < s < t < T$ and $t - s = T - t$, it follows that

$$\|\nabla u(t)\|_\infty \leq 3C(\Omega,M,f)(T-t)^{-1/2}\psi(t).$$

□

3.3 Weighted energy functional and its properties

In subsections 3.3-3.5, we shall work under the assumptions of Theorem 3.1. As mentioned in Subsection 2.2, our proof is essentially based on transformation by similarity variables and ODE renormalization and on weighted energy functional, after appropriate modifications of ideas in [8, 9, 10] (see also [17, section 23.4]). With this transformation we obtain a rescaled equation that we rewrite (cf (3.28)) as the pure power case plus an additional term H coming from L. Here new difficulties arise in the case of non scale invariant nonlinearities ($L \not\cong 1$). Under assumption (2.3), we shall show that $\|H(s,\cdot)\|_\infty \to 0$ as $s \to \infty$ and is globally square integrable in space-time with respect to the Gaussian measure, which will guarantee the existence of the energy and its properties (cf. Lemma 3.3). To this end, we divide this section in three parts as follows.

3.3.1 Preliminaries.

Let u be a solution of (1.1) with blow-up time $T \in (0,\infty)$. Let t_0 be given by Proposition 3.2, and take $s_0 > \max(2, -\log(T-t_0))$ such that $e^{\beta s} \geq s^{4\alpha} \geq 16$ for all $s \geq s_0$ and such that $\psi_1(s) := \psi(T - e^{-s})$ exists for all $s \geq s_0$ and satisfies

$$\psi_1(s) \geq e^{\beta s/2}, \quad s \geq s_0 \tag{3.18}$$

(inequality (3.18) follows from (2.7), (2.8) and the fact that, by Remark 3.2, $f(s) \leq s^{p+\varepsilon}$ as $s \to \infty$ for any $\varepsilon > 0$).

Let $a \in \Omega$. We set

$$y := \frac{x-a}{\sqrt{T-t}}, \quad s := -\log(T-t) \tag{3.19}$$

and define the rescaled function

$$w(y,s) = w_a(y,s) := \frac{u(a + ye^{-s/2}, T - e^{-s})}{\psi_1(s)}. \tag{3.20}$$

Note that it is equivalent to $u(x,t) = \psi(t)w(y,s)$. By a simple computation using $\psi_1' = e^{-s}f(\psi_1)$, we have

$$\nabla w = \frac{e^{-s/2}}{\psi_1}\nabla u, \qquad \Delta w = \frac{e^{-s}}{\psi_1}\Delta u,$$

and

$$w_s = -\frac{y}{2}\cdot\nabla w + e^{-s}\frac{u_t}{\psi_1} - e^{-s}\frac{f(\psi_1)}{\psi_1}w, \tag{3.21}$$

hence

$$w_s - \Delta w + \frac{1}{2}y\cdot\nabla w = \frac{e^{-s}}{\psi_1}\Big(u_t - \Delta u - f(\psi_1)w\Big).$$

Then w is a global (in time) solution of

$$w_s - \Delta w + \frac{1}{2}y\cdot\nabla w = h(s)\Big(|w|^{p-1}w\frac{L(\psi_1|w|)}{L(\psi_1)} - w\Big), \quad (y,s) \in \mathcal{W}_a, \tag{3.22}$$

where

$$\mathcal{W}_a := \{(y,s) : s_0 < s < \infty,\ y \in D(s)\}, \quad D(s) = D_a(s) := e^{s/2}(\Omega - a), \quad h(s) := e^{-s}\psi_1^{p-1}L(\psi_1). \tag{3.23}$$

Note that $\lim_\infty D(s) = \mathbb{R}^n$ and that $\mathcal{W}_a = \mathbb{R}^n \times (s_0, \infty)$ for $\Omega = \mathbb{R}^n$. In term of the variables y and s and rescaled function w, we observe that (2.5) and Proposition 3.2 imply the uniform estimates

$$|w(y,s)| \le M, \tag{3.24}$$

$$|\nabla w(y,s)| \le M_1, \tag{3.25}$$

where $M_1 = M_1(\Omega, M, p, T)$, and the desired result (3.1) can be restated as

$$\lim_{s\to\infty} w(y,s) = \pm 1, \text{ uniformly on compact sets } |y| \le C.$$

As in [9], we shall apply dynamical systems arguments to show that the global bounded solution w of (3.22) is attracted by the set of equilibria, and it will turn out that these are the same as in the pure power case, namely they are solutions of

$$\Delta w - \frac{1}{2}y\cdot\nabla w - \beta w + \beta|w|^{p-1}w = 0 \quad \text{in } \mathbb{R}^n. \tag{3.26}$$

As a main difference with the case $f(u) = |u|^{p-1}u$ in [9], equation (3.22) is not autonomous and the obtention of a Liapunov functional property requires additional effort.

3.3.2 Control of nonautonomous terms.

We rewrite equation (3.22) as

$$w_s - \Delta w + \frac{1}{2}y\cdot\nabla w = \beta|w|^{p-1}w - \beta w + H(s,y), \tag{3.27}$$

equivalently

$$\rho w_s - \nabla\cdot(\rho\nabla w) = \rho\beta(|w|^{p-1} - 1)w + \rho H(s,y), \tag{3.28}$$

where the nonautonomous term is given by

$$H(s,y) := \big(h(s) - \beta\big)\big(|w|^{p-1} - 1\big)w + h(s)|w|^{p-1}w\left(\frac{L(\psi_1|w|)}{L(\psi_1)} - 1\right) =: H_1(s,y) + H_2(s,y), \tag{3.29}$$

and the Gaussian weight ρ is defined by $\rho(y) := e^{-|y|^2/4}$. Under assumption (2.3) a key property for the existence of a Liapunov functional and the control of the nonautonomous terms is the following:

Lemma 3.1. *Under the assumptions of Theorem 3.1 with any $p > 1$, we have*

$$\|H(s,\cdot)\|_\infty \le C_0 s^{-\alpha}\log s, \quad s > s_0, \tag{3.30}$$

with $C_0 > 0$ independent of a.

Proof. It suffices to show that H_1 and H_2 satisfy (3.30). To prove it for H_1, we first claim that the function $h(s) = e^{-s}\psi_1^{p-1}L(\psi_1)$ satisfies

$$|h(s) - \beta| \le Cs^{-\alpha}, \quad s > s_0 \tag{3.31}$$

(here and in the rest of the proof, C denotes a generic positive constant independent of a, s, y). Indeed, integrating by parts, we have

$$F(X) = \int_X^\infty \frac{dz}{f(z)} = \int_X^\infty \frac{z^{-p}dz}{L(z)} = \beta\frac{X^{1-p}}{L(X)} - \beta\int_X^\infty z^{1-p}\frac{L'(z)}{L^2(z)}dz$$

hence, owing to (2.3),

$$\big|X^{p-1}L(X)F(X) - \beta\big| \le CX^{p-1}L(X)\int_X^\infty \frac{z^{-p}}{\log^\alpha z\, L(z)}dz \le C\frac{X^{p-1}L(X)F(X)}{\log^\alpha X}, \quad X \ge 2. \tag{3.32}$$

In particular $\lim\limits_{X\to\infty} X^{p-1}L(X)F(X) = \beta$ and, going back to (3.32), we get

$$\big|X^{p-1}L(X)F(X) - \beta\big| \le C\log^{-\alpha}X, \quad X \ge 2.$$

Since $F(\psi_1(s)) = F(\psi(T - e^{-s})) = e^{-s}$ in view of (2.8), claim (3.31) follows from (3.18). Properties (3.24) and (3.31) guarantee that H_1 satisfies (3.30).

Let us show that H_2 satisfies (3.30). To this end we define $E = \big\{(y,s) \in \mathcal{W}_a;\ |w(y,s)| \le s^{-\alpha}\big\}$. We have

$$|H_2(s,y)| \le C|w| \le Cs^{-\alpha}, \quad (y,s) \in E. \tag{3.33}$$

Next consider the case when $(y,s) \in \mathcal{W}_a \setminus E$. Then, recalling the definition of s_0 before (3.18), we have $|w|\psi_1 \ge s^{-\alpha}e^{\beta s/2} \ge e^{\beta s/4} \ge 2$. By assumption (2.3),

$$\Sigma(X) := \sup_{z\ge X}\Big|\frac{zL'(z)}{L(z)}\Big| \le C(\log X)^{-\alpha}, \quad X \ge 2.$$

Therefore, $\Sigma(e^{\beta s/4}) \le C(\log(e^{\beta s/4}))^{-\alpha} \le Cs^{-\alpha}$, as well as $M \ge |w| \ge s^{-\alpha}$. Since $\min(|w|\psi_1, \psi_1) \ge e^{\beta s/4}$, it follows that

$$\Big|\log\Big(\frac{L(|w|\psi_1)}{L(\psi_1)}\Big)\Big| = \Big|\int_{|w|\psi_1}^{\psi_1}\frac{L'(z)}{L(z)}dz\Big| \le \Sigma\big(e^{\beta s/4}\big)\Big|\big[\log z\big]_{|w|\psi_1}^{\psi_1}\Big| \le \Sigma\big(e^{\beta s/4}\big))|\log|w|| \le Cs^{-\alpha}\log s,$$

hence

$$|H_2(s,y)| \le C\Big|\frac{L(|w|\psi_1)}{L(\psi_1)} - 1\Big| \le Cs^{-\alpha}\log s.$$

Combining with (3.33), we deduce that

$$\|H_2(s,\cdot)\|_\infty \le Cs^{-\alpha} + Cs^{-\alpha}\log s \le Cs^{-\alpha}\log s, \quad s \ge s_0.$$

Consequently H_2, and hence H, satisfies (3.30). □

3.3.3 Weighted energy functional and its properties.

The Liapunov functional for equation (3.28) will be given by the perturbed weighted energy defined as follows:

$$G[w](s) := E[w](s) + C_1 s^{-\gamma}, \tag{3.34}$$

where $\gamma = \alpha - \frac{1}{2} > 0$,

$$E[w] := \int_{D(s)}\Big(\frac{1}{2}|\nabla w|^2 + \frac{\beta}{2}w^2 - \frac{\beta}{p+1}|w|^{p+1}\Big)\rho dy,$$

and the constant $C_1 > 0$ is independent of a. The term $C_1 s^{-\gamma}$ is designed to handle the effect of the perturbation H on the energy, making crucial use of its square integrability in space-time (cf. (3.48)), made possible by the assumption $\alpha > 1/2$ in (2.3). This energy enjoys the following properties

Lemma 3.2. *Under the assumptions of Theorem 3.1 with any $p > 1$, we have $G[w] \in C^1(s_0, \infty)$ and, for all $s > s_0$,*

$$\frac{d}{ds} G[w](s) \le -\frac{1}{2} \int_{D(s)} w_s^2 \rho \le 0, \tag{3.35}$$

$$G[w](s) \ge 0, \tag{3.36}$$

$$\int_{D(s)} w^2 \rho dy \le C(n,p) \big[G(w)(s)\big]^{2/(p+1)}. \tag{3.37}$$

Moreover, for each $s > s_0$,

$$a \mapsto G[w_a](s) \text{ is continuous.} \tag{3.38}$$

Proof. We first check the required regularity of w. For all $0 < \varepsilon < \tau < T$, denoting $Q_{\varepsilon,\tau} := \Omega \times (\varepsilon, \tau)$, it follows from standard parabolic L^q and Schauder regularity (see, e.g., [17, Section 48.1]) that

$$u_t, D^2 u \in BC(\bar{Q}_{\varepsilon,\tau}) \tag{3.39}$$

(where BC denotes the set of bounded continuous functions). Then, setting $g := f'(u) u_t \in L^\infty(Q_{\varepsilon,\tau})$, $v := u_t$ is the solution of the problem $v_t - \Delta v = g$ in $Q_{\varepsilon,\tau}$ with $v = 0$ on $\partial\Omega \times (\varepsilon, \tau)$ and $v(\varepsilon) = u_t(\varepsilon)$. By parabolic L^q regularity, we deduce that, for all $1 < q < \infty$ and $0 < \varepsilon < \tau < T$,

$$\sup_{x_0 \in \Omega} \|v_t\|_{L^q(Q_{\varepsilon,\tau,x_0})} + \|D^2 v\|_{L^q(Q_{\varepsilon,\tau,x_0})} < \infty,$$

where $Q_{\varepsilon,\tau,x_0} = (\Omega \cap B_1(x_0)) \times (\varepsilon, \tau)$ (the set Q_{ε,τ,x_0} can be replaced by $Q_{\varepsilon,\tau}$ and the supremum over x_0 omitted in case Ω is bounded). By interpolation inequalities it follows that

$$\nabla u_t \in BC(\bar{Q}_{\varepsilon,\tau}). \tag{3.40}$$

We deduce from (3.20), (3.21), (3.24), (3.25), (3.39) and (3.40) that, for all $s_1 \in (s_0, \infty)$,

$$D^2 w,\ (1+|y|)^{-1} w_s,\ (1+|y|)^{-1} \nabla w_s \in BC(\overline{\mathcal{W}}_{a,s_1}), \tag{3.41}$$

where $\mathcal{W}_{a,s_1} = \mathcal{W}_a \cap (\mathbb{R}^n \times (s_0, s_1))$. In view of the exponential decay of ρ, this guarantees the convergence and the differentiability of the various integrals and justifies the integrations by parts in the rest of the proof.

We next compute the variation of the first part $E[w](s)$ of the energy and derive a differential inequality for the weighted L^2 norm. This is essentially the same argument as in [8, 9], but we give details for completeness and convenience. For all $s > s_0$, we have for $q \ge 2$

$$\frac{1}{q} \frac{d}{ds} \int_{D(s)} |w|^q \rho dy = \int_{D(s)} \rho w_s |w|^{q-2} w dy + \frac{1}{2q} \int_{\partial D(s)} |w|^q \rho (y \cdot \nu) d\sigma, \tag{3.42}$$

where $d\sigma$ denotes the surface measure on $\partial D(s)$ and ν the exterior unit normal on $\partial D(s)$. Since $w = 0$ on $\partial D(s)$, the boundary term vanishes, hence

$$\frac{1}{q} \frac{d}{ds} \int_{D(s)} |w|^q \rho dy = \int_{D(s)} \rho w_s |w|^{q-2} w dy. \tag{3.43}$$

Next, to compute the variation of the term involving ∇w, we integrate by parts to get

$$\begin{aligned} \frac{d}{ds} \int_{D(s)} |\nabla w|^2 \rho &= 2 \int_{D(s)} \nabla w_s \cdot (\rho \nabla w) + \frac{1}{2} \int_{\partial D(s)} |\nabla w|^2 \rho (y \cdot \nu) d\sigma \\ &= -2 \int_{D(s)} w_s \nabla \cdot (\rho \nabla w) + 2 \int_{\partial D(s)} \rho w_s (\nabla w \cdot \nu) d\sigma + \frac{1}{2} \int_{\partial D(s)} .\rho |\nabla w|^2 (y \cdot \nu) d\sigma, \end{aligned}$$

Using that, on $\partial D(s)$, we have $w_s = -\frac{y}{2} \cdot \nabla w$ (owing to (3.21)) and $\nabla w = (\nabla w \cdot \nu)\nu$, hence $w_s (\nabla w \cdot \nu) = -\frac{1}{2} |\nabla w|^2 (y \cdot \nu)$, this yields

$$\frac{d}{ds} \int_{D(s)} |\nabla w|^2 \rho = -2 \int_{D(s)} w_s \nabla \cdot (\rho \nabla w) - \frac{1}{2} \int_{\partial D(s)} \rho |\nabla w|^2 (y \cdot \nu) d\sigma. \tag{3.44}$$

Also, we have $\partial D(s) = e^{s/2}\partial(\Omega - a)$ hence $\int_{\partial D(s)} d\sigma \le C(\Omega)e^{s/2}$ and, since $a \in \Omega$, we have $|y| \ge de^{s/2}$ on $\partial D(s)$ with $d = \mathrm{dist}(a, \partial\Omega) > 0$. Consequently, using (3.25), we get

$$\int_{\partial D(s)} \rho|\nabla w|^2 |y\cdot\nu| d\sigma \le CM_1^2 e^{s/2}\exp\big[-(de^{s/2})^2/4\big] \le C\exp\big(-ce^s\big)$$

(here and in the rest of the proof, C denotes a generic positive constant independent of a, s, y). Combining this with (3.28), (3.43), (3.44) we obtain

$$\begin{aligned}\frac{d}{ds}E[w](s) &= -\int_{D(s)} w_s\Big(\nabla\cdot(\rho\nabla w) - \beta\rho w + \beta\rho|w|^{p-1}w\Big) - \frac12\int_{\partial D(s)}\rho|\nabla w|^2(y\cdot\nu)d\sigma \\ &\le -\int_{D(s)} w_s^2\rho + \int_{D(s)} w_s H(s,y)\rho + C\exp\big(-ce^s\big). \end{aligned} \tag{3.45}$$

On the other hand, setting $\Psi(s) := \int_{\mathbb{R}^n} w^2\rho dy$ and using (3.42) with $q = 2$, (3.28) and integration by parts, we get

$$\begin{aligned}\frac12\Psi'(s) &= \int_{D(s)}\rho w_s w dy = \int_{D(s)}\Big(\nabla\cdot(\rho\nabla w) + \beta\rho|w|^{p-1}w - \beta\rho w + \rho H(s,y)\Big)w \\ &= \int_{D(s)}\big(-|\nabla w|^2 + \beta|w|^{p+1} - \beta w^2\big)\rho + \int_{D(s)} wH(s,y)\rho dy \\ &= -2E[w](s) + \frac{1}{p+1}\int_{D(s)}|w|^{p+1}\rho dy + \int_{D(s)} wH(s,y)\rho dy\end{aligned}$$

hence, by Jensen's inequality,

$$\Psi'(s) \ge -4E[w](s) - 2M\int_{D(s)}|H(s,y)|\rho dy + C(n,p)\Psi^{(p+1)/2}(s). \tag{3.46}$$

We shall now make use of the key decay property in Lemma 3.1 to handle the terms involving the perturbation H. By (3.45), we have

$$\frac{d}{ds}E[w](s) \le -\frac12\int_{D(s)} w_s^2\rho + \frac12\int_{D(s)} H^2(s,y)\rho + C\exp\big(-ce^s\big) \tag{3.47}$$

and (3.30) and $\alpha > \frac12$ guarantee that, for some $C_1 > 0$ (independent of a),

$$\frac12\int_{D(s)} H^2(s,y)\rho dy + C\exp\big(-ce^s\big) \le \gamma C_1 s^{-\alpha-\frac12} \quad\text{and}\quad 2M\int_{D(s)}|H(s,y)|\rho dy \le C_1 s^{-\gamma}, \tag{3.48}$$

for all $s \ge s_0$. It then follows from (3.34), (3.46) and (3.47) that

$$\frac{d}{ds}G[w](s) = \frac{d}{ds}E[w](s) - \gamma C_1 s^{-\alpha-\frac12} \le -\frac12\int_{D(s)} w_s^2\rho,$$

i.e. (3.35), and

$$\Psi'(s) \ge -4G[w](s) + C(n,p)\Psi^{(p+1)/2}(s).$$

The latter, combined with (3.35), implies

$$\Psi'(s) \ge -4G[w](s_1) + C(n,p)\Psi^{(p+1)/2}(s), \quad s \ge s_1 \ge s_0,$$

This guarantees (3.36) and (3.37), since otherwise Ψ has to blow up in finite time.

Finally, changing variables to write

$$\begin{aligned}E[w_a(s)] = e^{s/2}\int_\Omega\Bigg(&e^{-s/2}\Big|\frac{\nabla u(x,T-e^{-s})}{\psi_1(s)}\Big|^2 \\ &+\frac{\beta}{2}\Big|\frac{u(x,T-e^{-s})}{\psi_1(s)}\Big|^2 - \frac{\beta}{p+1}\Big|\frac{u(x,T-e^{-s})}{\psi_1(s)}\Big|^{p+1}\Bigg)\rho((x-a)e^{s/2})dx,\end{aligned}$$

property (3.38) follows from (3.24), (3.31) and dominated convergence. □

3.4 Convergence

Denote the set of bounded steady-states by

$$\mathcal{S} = \big\{ z \in C^2 \cap L^\infty(\mathbb{R}^n);\ z \text{ is a solution of (3.26)} \big\}.$$

We first recall the classification result from [9].

Proposition 3.3. *If $1 < p \le p_S$, then $\mathcal{S} = \{0, 1, -1\}$.*

The next lemma shows that w converges to $\mathcal{S}$ as $s \to \infty$.

Lemma 3.3. *Under the assumptions of Theorem 3.1, for any $a \in \Omega$, there exists $\ell \in \{0, 1, -1\}$ such that*

$$\lim_{s\to\infty} w_a(y, s) = \ell,$$

uniformly for $|y|$ bounded. Moreover, we have $\lim_{s\to\infty} G(w_a(s)) = 0$ if $\ell = 0$ and $\lim_{s\to\infty} G(w_a(s)) = \eta(n, p) > 0$ otherwise.

Proof. By (3.35)-(3.36), we have

$$l := \lim_{s\to\infty} G[w](s) \in [0, \infty). \tag{3.49}$$

Pick any sequence $s_j \to \infty$ and set $z_j := w(y, s + s_j)$. It follows from (3.24), (3.25), (3.27), (3.30) and parabolic estimates that the sequence $(z_j)_j$ is precompact in $C^{2,1}_{loc}\big(\mathbb{R}^n \times [0, 1]\big)$.

Consequently, there exists a subsequence (still denoted s_j) and a function z such that $w(\cdot, \cdot + s_j) \to z$ in $C^{2,1}_{loc}\big(\mathbb{R}^n \times [0, 1]\big)$. In view of (3.30) in Lemma 3.1, it follows that z is a classical solution of

$$w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \beta w + \beta |w|^{p-1} w = 0 \quad \text{in } \mathbb{R}^n \times (0, 1). \tag{3.50}$$

Moreover z and $|\nabla z|$ are bounded in $\mathbb{R}^n \times [0, 1]$. Let $R > 0$. Using (3.35), (3.49) and the fact that $B_R \subset D(s)$ for all sufficiently large s, we deduce that

$$\int_0^1 \int_{B_R} \big(\partial_s z_j\big)^2 \rho dy ds \le \int_{s_j}^{s_j+1} \int_{D(s)} \big(\partial_s w\big)^2 \rho dy ds \le 2G[w](s_j) - 2G[w](s_j + 1) \to 0, \quad j \to \infty.$$

By Fatou's lemma, it follows that $\partial_s z = 0$ in B_R and, since $R > 0$ is arbitrary, we deduce from (3.50) that $z \in \mathcal{S}$. Since $\mathcal{S}$ is discrete, the first assertion follows from an immediate connectedness argument. For the second assertion, we assume that $w(\cdot, s_j) \to 0$ or ± 1 uniformly for $|y|$ bounded. Using dominated convergence theorem and (3.24), (3.25), we have $G[w(s_j)] \to 0$ if $w(s_j) \to 0$ and when $w(s_j) \to \pm 1$ we have

$$G[w(s_j)] \underset{j\to\infty}{\longrightarrow} \left(\int_{\mathbb{R}^n} \rho dy \right) \left(\frac{\beta}{2} - \frac{\beta}{p+1} \right) = \frac{(4\pi)^{n/2}}{2(p+1)} =: \eta(n, p) > 0.$$

The assertion then follows from the monotonicity of $G[w](s)$. □

3.5 Nondegeneracy of blow-up and proof of Theorems 2.1 and 3.1

In this paragraph, we shall complete the proof of Theorems 2.1 and 3.1 by showing that $0 \notin \omega(w_a)$ if a is a blowup point of u,

Lemma 3.4. *Under the assumptions of Theorem 3.1, if $0 \in \omega(w_a)$, then a is not a blow-up point.*

Proof. Let $\theta = 1/(n+2)$ if $n \ge 2$ and $\theta = 1/2$ if $n = 1$. We shall use the interpolation inequality (cf. [17, p.250]):

$$|v(0)| \le C(n) \Big[\|v\|^\theta_{L^2(B_1)} \|\nabla v\|^{1-\theta}_{L^\infty(B_1)} + \|v\|_{L^2(B_1)} \Big], \quad v \in C^1(\bar{B}_1). \tag{3.51}$$

Set $d = \frac{1}{2}\text{dist}(a, \partial\Omega)$ and $\bar{s}_0 = \max(s_0, -2\log d)$. Let $b \in B_d(a)$. For all $s > \bar{s}_0$, since $\text{dist}(b, \partial\Omega) \ge d$, we have $B_1 \subset D_b(s)$ hence, by (3.37),

$$\|w_b(\cdot, s)\|^2_{L^2(B_1)} \le C(n) \int_{D_b(s)} w_b^2 \rho dy \le C(n, p) \big(G[w_b](s)\big)^{2/(p+1)}, \quad s \ge \bar{s}_0. \tag{3.52}$$

Combining (3.25), (3.35), (3.51) and (3.52), we obtain

$$|w_b(0,s)| \le C(n,p)\Big[M_1^{1-\theta}\big(G[w_b](s_1)\big)^{\theta/(p+1)} + \big(G[w_b](s_1)\big)^{1/(p+1)}\Big], \quad s \ge s_1 > \bar{s}_0.$$

Therefore, for $\varepsilon_0 = \varepsilon_0(n,f) > 0$ given by Proposition 3.1, there exists $\varepsilon_1 = \varepsilon_1(n,f,M_1) > 0$ such that

$$G[w_b](s_1) \le \varepsilon_1 \text{ for some } s_1 > \bar{s}_0 \quad \Longrightarrow \quad |w_b(0,s)| \le \varepsilon_0 \text{ for all } s \ge s_1. \tag{3.53}$$

Assume that $0 \in \omega(w_a)$. Then, by the second part of Lemma 3.3, there exists $s_1 > \bar{s}_0$ such that $G[w_a](s_1) < \varepsilon_1$ and, by the continuous dependence property (3.38), there exists $d_1 \in (0,d)$ such that $G[w_b(s_1)] < \varepsilon_1$ for all $b \in B_{d_1}(a)$. It follows from (3.53) that $|w_b(0,s)| \le \varepsilon_0$ for all $b \in B_{d_1}(a)$ and all $s \ge s_1$. Going back to original variables, we thus have $|u(b,t)| \le \varepsilon_0\psi(t)$ for all (b,t) sufficiently close to (a,T). By Proposition 3.1 we conclude that a is not a blow-up point. □

Proof of Theorem 3.1. Since, by (3.20),

$$\frac{u(a + y\sqrt{T-t}, t)}{\psi(t)} = w(y, -\log(T-t)),$$

the result is a direct consequence of Lemmas 3.3 and 3.4. □

Proof of Theorem 2.1. By Theorem 5.1, we know that u is of type I, i.e. (2.5) holds for some $M, \delta > 0$. Also, when $u_0 \ge 0$ and f is only defined for $s \ge 0$ and satisfies (2.2), it is easily checked that all the arguments in the proof of Theorem 3.1 remain valid (cf. Propositions 3.1-3.2 and Lemmas 3.1-3.4). Consequently, Theorem 2.1 follows. □

3.6 Proof of Proposition 2.1

We here prove the "no-needle" property (2.6). By a space translation, we may assume $a = 0$. Then, from Theorem 3.1, there exists $T_1 \in (0,T)$ such that

$$\frac{u(x,t)}{\psi(t)} \ge \frac{1}{2}, \quad T_1 \le t < T,\ |x| \le \sqrt{T-t}. \tag{3.54}$$

Fix any $t_0 \in [T_1, T)$ and let $\delta = \sqrt{T-t_0}$. Denoting by φ the first eigenfunction of the negative Dirichlet Laplacian in B_1 normalized by $\|\varphi\|_\infty = 1$ and $\lambda_1 > 0$ the corresponding eigenvalue, we set $\phi_\delta(x) = \varphi(x/\delta)$ and $\lambda_\delta = \frac{\lambda_1}{\delta^2}$. On the other hand, by assumption (2.2) there exists $C = C(f) > 0$ such that $f(s) \ge -Cs$ for all $s \ge 0$. Let v be solution of

$$\begin{cases} v_t - \Delta v = -Cv & \text{in } B_\delta \times (t_0, \infty), \\ v(x,t) = 0 & \text{on } \partial B_\delta \times (t_0, \infty), \\ v(x,t_0) = \dfrac{1}{2}\psi(t_0)\phi_\delta(x) & \text{in } B_\delta. \end{cases} \tag{3.55}$$

By (3.54) we have $u(\cdot,t_0) \ge \frac{1}{2}\psi(t_0) \ge \frac{1}{2}\psi(t_0)\phi_\delta(x)$ in B_δ. The comparison principle then guarantees that $u \ge v$ in $B_\delta \times [t_0, T)$. We note that the explicit solution of (3.55) is given by $v(x,t) = \frac{1}{2}\psi(t_0)\phi_\delta(x)\exp(-Ct - \lambda_\delta(t-t_0))$. Denoting $\eta = \min_{\bar{B}_{1/2}} \varphi > 0$, we obtain

$$u(x,t) \ge \frac{\eta}{2}\psi(t_0)\exp\Big(-CT - \frac{\lambda_1}{\delta^2}(t-t_0)\Big) \ge \frac{\eta}{2}\psi(t_0)\exp(-CT-\lambda_1), \quad |x| \le \frac{\delta}{2},\ t_0 \le t < T.$$

Since t_0 can be taken arbitrarily close to T and $\lim_{t\to T}\psi(t) = \infty$, property (2.6) follows.

4 Extension and proof of Theorem 2.2

Theorem 2.2 will be a consequence of the following result for possibly sign-changing solutions.

Theorem 4.1. *Let $1 < p \le p_S$. Assume that*

$$f \in C^1(\mathbb{R}) \text{ is an odd function, with } f > 0 \text{ and } C^2 \text{ for large } s > 0, \tag{4.1}$$

and that f satisfies (2.3). Let $u_0 \in C_0(\mathbb{R}^n)$ satisfy $T < \infty$ and (2.5). Then the blow-up set of u is compact. More precisely, there exists $R > 0$ such that

$$\sup_{|x|>R,\, t\in(0,T)} |u| < \infty. \tag{4.2}$$

Moreover the conclusion remains valid if $u_0 \ge 0$ and, instead of (4.1), f satisfies (2.2) and $f(0) = 0$.

Under the assumption $u_0 \in C_0(\mathbb{R}^n)$, we shall show that, at a suitable shifted time s_1, the weighted energy functional $G_a[w](s_1)$ (cf. (3.34)) becomes small enough for large $|a|$. Then we can conclude by using the nondegeneracy analysis in section 3.5. The time shift, which is made necessary by the second term $C_1 s^{-\gamma}$ of the energy, will be handled by means of the following lemma.

Lemma 4.1. *Let $0 < t_1 < T$. Under assumptions of Theorem 4.1, with $p > 1$, we have*

$$|\nabla u(x, t_1)| + |u(x, t_1)| \to 0, \quad \text{as } |x| \to \infty.$$

The proof of Lemma 4.1 relies on the following simple decay propagation property for the heat semigroup $S(t)$ in $\mathbb{R}^n$.

Lemma 4.2. *For all $v \in C_0(\mathbb{R}^n)$, we have*

$$\lim_{R\to\infty} \Big(\sup_{t>0,\ |x|>R} \big[S(t)|u_0|\big](x) \Big) = 0.$$

Proof of Lemma 4.2. Let $v \in C_0(\mathbb{R}^n)$ and $\varepsilon > 0$. Fix $R > 0$ such that $\sup_{|x|\ge R} |v(x)| \le \varepsilon$. We first claim that

$$\sup_{|x|\ge 2R} |(S(t)v)(x)| \le \varepsilon + C_1 \|v\|_\infty e^{-\frac{C_2 R^2}{t}}, \quad t > 0. \tag{4.3}$$

Indeed, we can write

$$\begin{aligned} \big|\big(S(t)v\big)(x)\big| &= (4\pi t)^{-n/2} \Big| \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} v(y)dy \Big| \\ &\le \varepsilon \int_{|y|>R} (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}} dy + \|v\|_\infty (4\pi t)^{-n/2} \int_{|y|\le R} e^{-\frac{|x-y|^2}{4t}} dy =: \varepsilon J_1 + \|v\|_\infty J_2. \end{aligned}$$

We have $J_1 \le 1$ and, for $|x| > 2R$, $|y| < R$ implies $|x - y| > R$, so that

$$J_2 \le (4\pi t)^{-n/2} \int_{|\xi|>R} e^{-\frac{|\xi|^2}{4t}} d\xi \le \int_{|z|>\frac{R}{\sqrt{t}}} e^{-\frac{|z|^2}{4}} dz \le C_1 e^{-C_2 R^2/t},$$

hence (4.3).

Next, observing that $|v| \le \varepsilon + \|v\|_\infty \chi_{B_R}$ and $v_0 := \|v\|_\infty \chi_{B_R} \in L^1$, it follows that, for $t_0 = t_0(\varepsilon, v) > 0$ large enough, we have

$$|(S(t)v)(x)| \le (S(t)|v|)(x) \le \varepsilon + (4\pi t)^{-n/2} \|v_0\|_1 \le 2\varepsilon, \quad x \in \mathbb{R}^n,\ t \ge t_0. \tag{4.4}$$

Now taking $R_0 = R_0(\varepsilon, v) > 2R$ large enough, we deduce from (4.3) that

$$|(S(t)v)(x)| \le 2\varepsilon, \quad 0 < t \le t_0,\ |x| \ge R_0, \tag{4.5}$$

and the conclusion follows by combining (4.4) and (4.5). □

Proof of Lemma 4.1. By the representation of $S(t)$ by the heat kernel $(4\pi t)^{-n/2} \exp(-|x|^2/(4t))$, we see that

$$\big|[\nabla_x S(t)v](x)\big| \le C(n) t^{-1/2} \big[S(2t)|v|\big](x), \quad v \in L^\infty(\Omega),\ t > 0,\ x \in \mathbb{R}^n. \tag{4.6}$$

Fix $t_1 < T$. Recalling $f(0) = 0$, there exists $K = K(t_1) > 0$ such that $-K|u| \le u_t - \Delta u = f(u) \le K|u|$ in $\mathbb{R}^n \times (0, t_1]$. By the comparison principle it follows that

$$|u| \le e^{Kt} S(t)|u_0| \le CS(t)|u_0| \quad \text{in } \mathbb{R}^n \times (0, t_1]. \tag{4.7}$$

(Here and in the rest of the proof, C denotes a generic positive constant, possibly depending on t_1.) Using (4.6), (4.7) and the variation of constant formula $u(t_1) = S(t_1)u_0 + \int_0^{t_1} S(t_1 - \tau) f(u(\tau))d\tau$, we deduce that, pointwise in $\mathbb{R}^n$,

$$\begin{aligned}|\nabla u(t_1)| &\le Ct_1^{-1/2} S(2t_1)|u_0| + C\int_0^{t_1} (t_1 - \tau)^{-1/2} S(2(t_1 - \tau))S(\tau)|u_0|d\tau \\ &= Ct_1^{-1/2} S(2t_1)|u_0| + C\int_0^{t_1} (t_1 - \tau)^{-1/2} S(2t_1 - \tau)|u_0|d\tau \le C\big(t_1^{-1/2} + t_1^{1/2}\big) \sup_{t\in[0,2t_1]} S(t)|u_0|\end{aligned}$$

hence, using again (4.7),

$$|u(t_1)| + |\nabla u(t_1)| \le C\big(1 + t_1^{-1/2} + t_1^{1/2}\big) \sup_{t\in[0,2t_1]} S(t)|u_0|.$$

This combined with Lemma 4.2 concludes the proof. □

Proof of Theorem 4.1. Recall the definition (3.34) of the weighted energy:

$$G[w_a(s)] = E[w_a](s) + C_1 s^{-\gamma}, \quad s > s_0,$$

for all $a \in \mathbb{R}^n$. Also, for $\varepsilon_0 = \varepsilon_0(n, f) > 0$ given by Proposition 3.1, by (3.53) there exists $\varepsilon_1 = \varepsilon_1(n, f, M_1) > 0$ such that

$$G[w_a](s_1) \le \varepsilon_1 \text{ for some } s_1 > s_0 \quad \Longrightarrow \quad |w_a(0, s)| \le \varepsilon_0 \text{ for all } s \ge s_1 \tag{4.8}$$

(noticing that $\bar{s}_0 = s_0$ for $\Omega = \mathbb{R}^n$). Choose $s_1 > s_0$ such that $C_1 s_1^{-\gamma} < \varepsilon_1/2$. Rewriting $E[w_a(s_1)]$ as

$$E[w_a(s_1)] = \int_{\mathbb{R}^n} \left(\frac{e^{-s_1}|\nabla u(a + e^{-\frac{s_1}{2}} y, t_1)|^2 + \beta|u(a + e^{-\frac{s_1}{2}} y, t_1)|^2}{2\psi^2(t_1)} - \frac{\beta|u(a + e^{-\frac{s_1}{2}} y, t_1)|^{p+1}}{\psi^{p+1}(t_1)} \right) \rho(y)dy$$

with $t_1 = T - e^{-s_1}$, and using Lemma 4.1 and dominated convergence, there exists $R > 0$ such that, for all a such that $|a| \ge R - 1$, we have $E[w_a](s_1) < \frac{\varepsilon_1}{2}$, hence $G[w_a](s_1) < \varepsilon_1$. By (4.8), we deduce that

$$|w_a(0, s)| \le \varepsilon_0, \quad \text{for all } s \ge s_1 \text{ and } |a| > R - 1,$$

i.e.

$$|u(a, t)| \le \varepsilon_0 \psi(t), \quad \text{for all } t \in (t_1, T) \text{ and } |a| > R - 1.$$

Applying Proposition 3.1 with $\delta = \min(\delta_0, 1, \sqrt{T - t_1})$. It follows that

$$\sup_{|x|>R,\ t\in(T-\delta^2/2,T)} |u| < \infty,$$

hence (4.2). It is easily checked that the final assertion (for $u_0 \ge 0$) follows from the same argument. □

Proof of Theorem 2.2. It follows directly from Theorems 4.1 and 5.1. □

5 Appendix: Type-I blowup

We have used the following result, which in particular guarantees the type I blowup estimate (2.5) under the assumptions of Theorem 2.1 (and the constant M is actually independent of the solution, although this fact is not used in our proofs).

Theorem 5.1. *Let Ω be a uniformly C^2 domain of $\mathbb{R}^n$, $f \in C([0,\infty))$ be positive for $s > 0$ large and assume that f has regular variation at ∞ with index $p \in (1, p_S)$. For each $\tau > 0$, there exist $M = M(\Omega, f) > 0$ and $t_0 = t_0(\tau, f) \in (0, \tau)$ such that, if $u \geq 0$ is a strong solution of*

$$\begin{cases} u_t - \Delta u = f(u), & x \in \Omega,\ t_0 < t < \tau, \\ u = 0, & x \in \partial\Omega,\ t_0 < t < \tau, \end{cases}$$

then

$$u(x,t) \leq MF^{-1}(\tau - t) \quad \text{in } \Omega \times [t_0, \tau). \tag{5.1}$$

Theorem 5.1 was essentially established in [20, Theorem 3.1]. We note that the assumptions on f, which are those from [20], are more general than those in Theorem 2.1 of the present paper (see after (2.1)). On the other hand, the estimate in [20] is given there in a different form (cf. (5.2) below). We therefore provide a proof of Theorem 5.1, where we derive (5.1) as a consequence of (5.2).

Proof. Under the assumptions of Theorem 5.1, we know from [20, Theorem 3.1 and Remark 1(i)] that

$$\frac{f(u(x,t))}{u(x,t)} \leq \frac{C_0}{\tau - t}, \quad \text{for all } (x,t) \in \Omega \times [\tau/2, \tau) \text{ such that } u(x,t) \geq 1, \tag{5.2}$$

with $C_0 = C_0(\Omega, f) > 0$. Also, there exist $\delta, A > 0$ such that $f \geq 1$ on $[A, \infty)$ and

$$F \text{ is a decreasing bijection from } [A, \infty) \text{ to } (0, \delta]. \tag{5.3}$$

We shall show that (5.2) implies (5.1).

To this end, we first claim that there exist $k > 1$ and $s_0 > kA$ such that

$$F\Big(\frac{s}{k}\Big) \geq \frac{C_0 s}{f(s)}, \quad \text{for all } s \geq s_0. \tag{5.4}$$

Indeed, by the representation theorem for slowly varying functions (see [18, Theorem 1.2]), there exist continuous functions τ, ξ such that

$$f(s) = \tau(s) f_0(s), \quad \text{where } f_0(s) := s^p \exp\Big[\int_1^s \frac{\xi(z)}{z} dz\Big], \quad \text{for all } s \geq A,$$

with $\lim_{s\to\infty} \tau(s) = \ell > 0$ and $\lim_{s\to\infty} \xi(s) = 0$. Moreover there exist $C_1, C_2 > 0$ such that $C_1 \leq \tau(s) \leq C_2$ for all $s \geq A$. Fixing $1 < m < p < q$, we see that there exists $s_1 > A$ such that $s^{-m} f_0(s)$ is increasing and $s^{-q} f_0$ is decreasing on $[s_1, \infty)$. For $z \geq s_1$, we have

$$F(z) \geq C_2^{-1} \int_z^\infty \frac{ds}{f_0(s)} = C_2^{-1} \int_z^\infty s^{-q} \frac{s^q}{f_0(s)} ds \geq C_2^{-1} \frac{z^q}{f_0(z)} \int_z^\infty s^{-q} ds = \frac{1}{C_2(q-1)} \frac{z}{f_0(z)} =: \frac{C_3 z}{f_0(z)}.$$

Let $k > 1$. In view of this control of $F(z)$, we have, for all $s > ks_1$,

$$F(s/k) \geq \frac{C_3 s}{k f_0(s/k)} = C_3 \Big(\frac{s}{k}\Big)^{1-m} \frac{\big(\frac{s}{k}\big)^m}{f_0(s/k)} \geq C_3 \Big(\frac{s}{k}\Big)^{1-m} \frac{s^m}{f_0(s)} = C_3 k^{m-1} \frac{s}{f_0(s)} \geq C_3 C_1 k^{m-1} \frac{s}{f(s)}.$$

Choosing $k > 1$ large enough, so that $C_3 C_1 k^{m-1} \geq C_0$, and then $s_0 = ks_1$, claim (5.4) follows.

Now set $t_0 := \max(\tau - \delta, \tau/2)$. For any $(x,t) \in \Omega \times [t_0, \tau)$ such that $u(x,t) \geq s_0$, it follows from (5.2), (5.4) that

$$F\Big(\frac{u(x,t)}{k}\Big) \geq C \frac{u(x,t)}{f(u(x,t))} \geq \tau - t,$$

and (5.3) then implies $u(x,t) \leq kF^{-1}(\tau - t)$. Consequently, we have $u(x,t) \leq s_0 + kF^{-1}(\tau - t)$ in $\Omega \times [t_0, \tau)$, which implies the desired result. □

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