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Shengjun Fan, Ying Hu, Shanjian Tang. Multi-dimensional backward stochastic differential equations of non-Markovian interactively quadratic generators. 2024. hal-04701649

**HAL Id: hal-04701649**

**<https://hal.science/hal-04701649v1>**

Preprint submitted on 18 Sep 2024

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# Multi-dimensional backward stochastic differential equations of non-Markovian interactively quadratic generators\*

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## Abstract

This paper is devoted to the general solvability of multi-dimensional backward stochastic differential equations (BSDEs) with interactively quadratic generators in the non-Markovian setting. Some general structural conditions on the generator  $g$  of BSDEs are demonstrated to guarantee existence and uniqueness of the local and global solutions, which admit that  $g$  has a general growth in state variable  $y$ , and the  $i$ th component  $g^i$  of  $g$  depends quadratically not only on the  $i$ th row  $z^i$  of the state variable  $z$ , but also the  $j$ th row  $z^j$  of  $z$  for  $j \neq i$ . For these multi-dimensional BSDEs of non-Markovian interactively quadratic generators, we first establish an existence and uniqueness result on the local bounded solution and then several existence and uniqueness results on the global bounded and unbounded solutions. They unify and strengthen some existing works in the non-Markovian setting, and also incorporate some interesting examples, one of which partially answers an open problem posed in Jackson [25]. A comprehensive investigation on the bounded solution of one-dimensional quadratic BSDEs with unbounded stochastic parameters is carried on for deriving our main results.

*Keywords:* Multi-dimensional BSDE, Interactively quadratic generator, BMO martingale, Bounded solution, Unbounded solution, Existence and uniqueness.

*2010 MSC:* 60H10

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## 1. Introduction

Let  $T \in (0, +\infty)$ ,  $(B_t)_{t \in [0, T]}$  be a  $d$ -dimensional Brownian motion defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and  $(\mathcal{F}_t)_{t \in [0, T]}$  be the augmented filtration generated by  $B$ . Consider

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\*This work is supported by National Natural Science Foundation of China (Nos. 12171471, 12031009 and 11631004), by Key Laboratory of Mathematics for Nonlinear Sciences (Fudan University), Ministry of Education, Handan Road 220, Shanghai 200433, China; by Lebesgue Center of Mathematics “Investissements d’avenir” program-ANR-11-LABX-0020-01, by CAESARS-ANR-15-CE05-0024 and by MFG-ANR-16-CE40-0015-01.

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the following backward stochastic differential equation (BSDE in short):

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \quad (1.1)$$

where the terminal value  $\xi$  is a  $d$ -dimensional  $\mathcal{F}_T$ -measurable random vector, the generator function  $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n$  is  $(\mathcal{F}_t)$ -progressively measurable for each pair  $(y, z)$ , and the solution  $(Y_t, Z_t)_{t \in [0, T]}$  is a pair of  $(\mathcal{F}_t)$ -progressively measurable processes taking values in  $\mathbb{R}^n \times \mathbb{R}^{n \times d}$  which almost surely satisfies BSDE (1.1). Bismut [3] initially introduced this kind of equations in the linear version, Bismut [4] studied a specifically structural matrix-valued nonlinear case, and Pardoux and Peng [36] first established the existence and uniqueness of a multidimensional ( $n > 1$ ) and nonlinear BSDE, where the generator  $g$  is uniformly Lipschitz continuous with respect to the state variables  $(y, z)$ . Since then, there has been an increasing interest for BSDEs and the theoretical results have also been applied to a wide range of various fields such as mathematical finance, stochastic control, partial differential equations (PDEs), etc. Interested readers are referred to El Karoui et al. [13], Kobylanski [30], El Karoui and Hamadene [12], Hu et al. [23], Frei and Dos Reis [21], Kramkov and Pulido [32], Escauriaza et al. [14], Tian [39], Weston [40] among others.

For one-dimensional ( $n = 1$ ) quadratic BSDEs with bounded terminal values, where the generator  $g$  has a quadratic growth in the state variable  $z$ , the first existence and uniqueness result is due to Kobylanski [30], see also Tevzadze [38], Briand and Elie [6], Fan [15] and Luo and Fan [34] for more details. Existence and uniqueness of the unbounded solution for one-dimensional quadratic BSDEs with unbounded terminal values was further investigated in Briand and Hu [7, 8], Delbaen et al. [10], Barrieu and El Karoui [1], Delbaen et al. [11], Fan et al. [16] and Fan et al. [17]. However, for the case of multidimensional quadratic BSDEs, the situation is more complicated and there are no general existence theory. In particular, a counterexample of multidimensional quadratic BSDE with a simple generator and a bounded terminal value which fails to have a global bounded solution on  $[0, T]$  was constructed in Frei and Dos Reis [21]. At the same time, many applications have been found in various domains such as financial price-impact models, financial market equilibrium problems for several interacting agents, nonzero-sum risk-sensitive stochastic differential games, stochastic equilibria problems in incomplete financial markets and so on. This has attracted many efforts on this topic in recent years. Under a smallness assumption on the terminal value, Tevzadze [38] established the first existence and uniqueness result for multidimensional quadratic BSDEs with bounded terminal values, which inspired many scholars to investigate the bounded or unbounded solution under different types of “smallness” assumptions on the terminal value, the terminal time and the generator, see for example Frei [20], Jamneshan et al. [27], Kramkov and Pulido [32, 31], Harter

and Richou [22] and Kardaras et al. [28]. Furthermore, in the Markovian setting, Cheridito and Nam [9] proved the global solvability of a special multi-dimensional quadratic BSDE related to a forward BSDE, and Xing and Žitković [42] addressed a large class of system of quadratic BSDEs under the Bensoussan-Frehse (BF) condition initially put forward in Bensoussan and Frehse [2] and the a priori boundedness (AB) condition (see (AB) in Section 2 for details) on the generator, see Weston and Zitkovic [41] and Escauriaza et al. [14] for more information on this topic. Recently, Jackson and Zitkovic [26] and Jackson [25] further extended the result of [42] to the non-Markovian multi-dimensional quadratic BSDEs, while an additional Malliavin-regular condition on the generator is required except for the (BF) condition and the (AB) condition.

We have to mention the following several important works on the bounded solution in the non-Markovian setting and without the smallness assumptions and the Malliavin-regular condition on the terminal time, the terminal value and the generator. That is, Cheridito and Nam [9] considered multidimensional BSDEs with projectable quadratic generators and sub-quadratic generators, Hu and Tang [24] addressed the global solvability of a class of multi-dimensional BSDEs with diagonally quadratic generators, namely, the  $i$ th component  $g^i$  of the generator  $g$  admits only a quadratic growth on the  $i$ th row  $z^i$  of the matrix  $z$  and is bounded on  $z^j$  for each  $j \neq i$ , which answers an open problem arising from a nonzero-sum risk-sensitive stochastic differential game posed in El Karoui and Hamadene [12], Fan et al. [17] made a remarkable improvement on the result of [24] by imposing a strictly quadratic condition of  $g^i$  in  $z^i$  such that  $g^i$  can have a sub-quadratic growth on  $z^j$  for any  $j \neq i$ , and Luo [35] investigated a kind of multidimensional BSDEs with triangularly quadratic generators, in which  $g^i$  may depend quadratically on  $z^j$  for any  $1 \leq j \leq i$  in a special form. Very recently, Fan et al. [19] generalized the result of [17] to the case of multi-dimensional quadratic backward stochastic volterra integral equations, and Weston [40] applied the result of [17] to prove the global existence of a Radner equilibrium in a limited participation economy models.

The present paper focuses on the local and global solvability for a large class of multidimensional BSDEs with interacting quadratic generators and bounded and unbounded terminal values in the non-Markovian setting, in which  $g^i$  depends quadratically not only on  $z^i$ , but also  $z^j$  for each  $j \neq i$ . Our results unify and strengthen some corresponding known results in the non-Markovian setting such as [24], [35] and [17], and show that under some circumstance, existence and uniqueness of the solution can be guaranteed when some part of the generator  $g$  is small enough, which can be compared with some known corresponding results mentioned in the last second paragraph. More specifically, we first establish an existence and uniqueness result on the local bounded solution of the multidimensional quadratic BSDEs with bounded terminal

values (see Theorem 2.3 in Section 2) under general assumptions, see assumptions (B1) and (B2) in Section 2 for details. The assumption (B1) means that  $g^i$  may satisfy any one of the following three different cases with respect to  $z^i$ : a strictly quadratic growth, a general quadratic growth and a linear growth. The  $g^i$  can also have a general growth in  $y$ , a quadratic growth in  $z^j$  for  $j \neq i$  but with a small enough constant  $\theta$  when needed, and an interacting quadratic growth like the inner product of  $z^i$  and  $z^j$  for  $j < i$ . We would like to mention that generally speaking, the parameters of processes defined in assumptions (B1) and (B2) (namely,  $\alpha, \bar{\alpha}, \tilde{\alpha}$  and  $v$ ) are all unbounded and stochastic, and satisfy different integrability conditions, which is an endogenous requirement for desired conclusions, and on the other hand, provides convenience for the study on the unbounded solution of multidimensional quadratic BSDEs. Then, under, respectively, an a priori boundedness assumption and two stronger assumptions than (B1) (see (AB), (C1a) and (C1b) in Section 2), based on Theorem 2.3 we further establish three existence and uniqueness results on the global bounded solution of multi-dimensional quadratic BSDEs, see Theorems 2.6, 2.10 and 2.14 in Section 2 for details. These results totally cover the corresponding results posed in Hu and Tang [24], Luo [35] and Fan et al. [17], and some ideas to prove them are borrowed from there. The proofs involve the contraction mapping argument, the uniform a priori estimate on the solution and the utilization of the induction technique based on some delicate a priori estimates on the solution, where the problem on the bounded solution of one-dimensional quadratic BSDEs with unbounded stochastic parameters is comprehensively investigated by virtue of the BMO martingale (bounded oscillation martingale) tool, Girsanov's transform and some useful inequalities, and some new results are explored, see Propositions A.1 to A.3 in the Appendix for details. As a natural application of Theorems 2.10 and 2.14, by means of the invertible linear transformation method we address solvability of three kinds of special structured systems of BSDEs, see Theorems 2.20, 2.22 and 2.23 in Section 2 for more details, which can be respectively compared with Theorems 6.9 and 6.19 of Jackson [25] and Theorem 3.1 of Xing and Žitković [42]. By the way, by Proposition 2.24 and Corollary 2.25 in Section 2 we answer partially the open problem 6.25 of Jackson [25] on solvability of a special system of BSDEs. Finally, according to Theorems 2.6, 2.10 and 2.14, we establish three existence and uniqueness results for the unbounded solution of multi-dimensional quadratic BSDEs with the unbounded BMO terminal values, where the generator  $g$  needs to be bounded in  $y$ , see assumptions (D1) and (D2) as well as Theorems 2.26, 2.27 and 2.31 in Section 2 for details. The method of proof is to transfer the BSDE with an unbounded terminal value into one with a bounded terminal value and a generator satisfying the assumptions of Theorems 2.6, 2.10 and 2.14. These results also improve some existing results in the non-Markovian setting.

The rest of the paper is organized as follows. In the next section, we introduce some notations used throughout this paper, state the main results, and present some interesting examples and remarks to illustrate the novelty of our results. The proofs of some of the results are also provided in this section when they are not very long. In Section 3 we prove the solvability of local bounded solution (Theorem 2.3), and in Section 4 we prove solvability of global bounded solution (Theorems 2.10 and 2.14). The solvability of global unbounded solution (Theorems 2.26 and 2.27) is finally proved in Section 5. In Appendix, we present and prove some auxiliary results for the bounded solution of one-dimensional quadratic BSDEs with unbounded stochastic parameters, including existence, uniqueness and several important a priori estimates.

## 2. Notations and statement of main results

### 2.1. Notations

First, let us fix a terminal time  $T \in (0, \infty)$  and two positive integers  $n$  and  $d$ . Let  $a \wedge b := \min\{a, b\}$ ,  $a \vee b := \max\{a, b\}$ ,  $a^+ := a \vee 0$ ,  $a^- := -(a \wedge 0)$  and

$$\sum_{j=1}^0 b^j = \sum_{j=n+1}^n b^j = 0$$

for any real  $b^j$ . And, denote by  $|\cdot|$  the Euclidean norm,  $z^\top$  the transpose of vector (or matrix)  $z$ ,  $\mathbf{1}_A$  the indicator of set  $A$ , and  $\text{sgn}(x) := \mathbf{1}_{x>0} - \mathbf{1}_{x\leq 0}$ .

In the whole paper, all processes are  $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable, and all equalities and inequalities between random variables and processes are understood in the senses of  $\mathbb{P} - a.s.$  and  $d\mathbb{P} \times dt - a.e.$ , respectively. We need the following spaces of random variables and processes:

- $L^\infty(\mathbb{R}^n)$ : all  $\mathbb{R}^n$ -valued and  $\mathcal{F}_T$ -measurable random variables  $\xi$  satisfying

$$\|\xi\|_\infty := \text{ess sup}_{\omega \in \Omega} |\xi(\omega)| < +\infty.$$

- $\mathcal{S}^p(\mathbb{R}^n)$  for  $p \geq 1$ : all  $\mathbb{R}^n$ -valued continuous adapted processes  $(Y_t)_{t \in [0, T]}$  such that

$$\|Y\|_{\mathcal{S}^p} := \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t|^p \right]^{\frac{1}{p}} < +\infty.$$

- $\mathcal{S}^\infty(\mathbb{R}^n)$ : all  $Y \in \bigcap_{p \geq 1} \mathcal{S}^p(\mathbb{R}^n)$  such that

$$\|Y\|_{\mathcal{S}^\infty} := \text{ess sup}_{(\omega, t) \in \Omega \times [0, T]} |Y_t(\omega)| = \left\| \sup_{t \in [0, T]} |Y_t| \right\|_\infty < +\infty.$$

- $\mathcal{M}^\infty$ : all real-valued non-negative progressively measurable process  $(Y_t)_{t \in [0, T]}$  satisfying

$$\|Y\|_{\mathcal{M}^\infty} := \sup_{\tau \in \mathcal{T}_{[0, T]}} \left\| \mathbb{E}_\tau \left[ \int_\tau^T Y_s ds \right] \right\|_\infty < +\infty,$$

here and hereafter  $\mathcal{T}_{[a, b]}$  denotes the set of all  $(\mathcal{F}_t)$ -stopping times  $\tau$  valued in  $[a, b] \subset [0, T]$ , and  $\mathbb{E}_\tau$  stands for the conditional expectation with respect to  $\mathcal{F}_\tau$ .

- $\mathcal{L}^\infty$ : all  $Y \in \mathcal{M}^\infty$  satisfying

$$\|Y\|_{\mathcal{L}^\infty} := \left\| \int_0^T Y_s ds \right\|_\infty < +\infty.$$

- $\mathcal{H}^p(\mathbb{R}^{n \times d})$  for  $p \geq 1$ : all  $\mathbb{R}^{n \times d}$ -valued progressively measurable processes  $(Z_t)_{t \in [0, T]}$  such that

$$\|Z\|_{\mathcal{H}^p} := \mathbb{E} \left[ \left( \int_0^T |Z_s|^2 ds \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} < +\infty.$$

- $\text{BMO}(\mathbb{R}^{n \times d})$ : all  $Z \in \mathcal{H}^2(\mathbb{R}^{n \times d})$  such that

$$\|Z\|_{\text{BMO}} := \sup_{\tau \in \mathcal{T}_{[0, T]}} \left\| \mathbb{E}_\tau \left[ \int_\tau^T |Z_s|^2 ds \right] \right\|_\infty^{\frac{1}{2}} < +\infty.$$

We write  $L^\infty(\mathbb{R}) := L^\infty(\mathbb{R}^1)$ ,  $\mathcal{S}^\infty(\mathbb{R}) := \mathcal{S}^\infty(\mathbb{R}^1)$ ,  $\mathcal{S}^p(\mathbb{R}) := \mathcal{S}^p(\mathbb{R}^1)$  for  $p \geq 1$ ,  $\mathcal{H}^p(\mathbb{R}) := \mathcal{H}^p(\mathbb{R}^{1 \times 1})$  for  $p \geq 1$ , and  $\text{BMO}(\mathbb{R}) := \text{BMO}(\mathbb{R}^{1 \times 1})$ . By  $\mathcal{H}_{[a, b]}$ , we denote the restriction to the subinterval  $[a, b]$  of the space  $\mathcal{H}$  of processes on  $[0, T]$ , i.e. the space of all processes which are restrictions to the interval  $[a, b]$  of processes in  $\mathcal{H}$ . It is noted that the process  $\left( \int_0^t Z_s dB_s \right)_{t \in [0, T]}$  is an  $n$ -dimensional BMO martingale for each  $Z \in \text{BMO}(\mathbb{R}^{n \times d})$ . We refer to Kazamaki [29] for more details on the BMO theory. For the reader's convenience, here we would like to recall the well-known John-Nirenberg inequality (see for example Lemma A.1 in Hu and Tang [24]): Let the process  $Z$  belong to the space of  $\text{BMO}(\mathbb{R})$ . If  $\|Z\|_{\text{BMO}}^2 \leq 1/2$ , then for any  $\tau \in \mathcal{T}_{[0, T]}$ ,

$$\mathbb{E}_\tau \left[ \exp \left( \int_\tau^T |Z_s|^2 ds \right) \right] \leq \frac{1}{1 - \|Z\|_{\text{BMO}}^2} \leq 2.$$

Throughout the paper, we also always fix four real-valued non-negative progressively measurable processes

$$(v_t)_{t \in [0, T]} \in \text{BMO}(\mathbb{R}), \quad (\tilde{\alpha}_t)_{t \in [0, T]} \in \mathcal{L}^\infty, \quad (\bar{\alpha}_t)_{t \in [0, T]} \in \mathcal{M}^\infty$$

and  $(\alpha_t)_{t \in [0, T]}$  as well as a deterministic nondecreasing continuous function  $\phi(\cdot) : [0, +\infty) \rightarrow [0, +\infty)$  and several real constants  $\beta, \lambda, \bar{\lambda}, \theta, c, \bar{c} \geq 0$ ,  $0 < \bar{\gamma} \leq \gamma$  and  $\delta \in [0, 1)$ .

Finally, for  $i = 1, \dots, n$ , denote by  $z^i$ ,  $y^i$  and  $g^i$  respectively the  $i$ th row of matrix  $z \in \mathbb{R}^{n \times d}$  and the  $i$ th component of the vector  $y \in \mathbb{R}^n$  and the generator  $g \in \mathbb{R}^n$ . In addition, for some

real  $r > 0$  and some time sub-interval  $[a, b] \subset [0, T]$ , we let  $\mathcal{E}_{[a,b]}^\infty(r)$  represent the set of all real-valued non-negative progressively measurable process  $(\hat{\alpha}_t)_{t \in [0, T]}$  satisfies

$$\|\hat{\alpha}\|_{\mathcal{E}_{[a,b]}^\infty(r)} := \frac{1}{r} \ln \left( \sup_{\tau \in \mathcal{T}_{[a,b]}} \left\| \mathbb{E}_\tau \left[ \exp \left( r \int_\tau^b \hat{\alpha}_s ds \right) \right] \right\|_\infty \right) < +\infty.$$

We also denote  $\mathcal{E}_{[0, T]}^\infty(r)$  by  $\mathcal{E}^\infty(r)$  simply.

**Remark 2.1.** Let  $[a, b] \subset [0, T]$ . By virtue of the inequality  $\exp(x) > x$  for  $x \geq 0$ , the John-Nirenberg inequality and Hölder's inequality, it is not difficult to verify the following assertions.

(i) The assertion  $\hat{\alpha} \in \text{BMO}(\mathbb{R})$  holds if and only if  $\hat{\alpha}^2 \in \mathcal{M}^\infty$ , and  $\|\hat{\alpha}^2\|_{\mathcal{M}^\infty} = \|\hat{\alpha}\|_{\text{BMO}}^2$ .

If  $\hat{\alpha} \in \mathcal{L}_{[a,b]}^\infty$ , then  $\hat{\alpha} \in \mathcal{E}_{[a,b]}^\infty(r)$  for each  $r > 0$ , and  $\|\hat{\alpha}\|_{\mathcal{E}_{[a,b]}^\infty(r)} \leq \|\hat{\alpha}\|_{\mathcal{L}_{[a,b]}^\infty}$ .

If  $\hat{\alpha} \in \mathcal{E}_{[a,b]}^\infty(r)$  for some  $r > 0$ , then  $\hat{\alpha} \in \mathcal{M}_{[a,b]}^\infty$ , and  $\|\hat{\alpha}\|_{\mathcal{M}_{[a,b]}^\infty} \leq \frac{1}{r} \exp \left\{ r \|\hat{\alpha}\|_{\mathcal{E}_{[a,b]}^\infty(r)} \right\}$ .

In addition, for any  $0 < r < \bar{r}$ , we have  $\mathcal{E}^\infty(r) \subset \mathcal{E}_{[a,b]}^\infty(r)$  and  $\mathcal{E}_{[a,b]}^\infty(\bar{r}) \subset \mathcal{E}_{[a,b]}^\infty(r)$ .

(ii) If  $\hat{\alpha}^1 \in \mathcal{E}_{[a,b]}^\infty(r)$  for some  $r > 0$  and  $\hat{\alpha}^2 \in \mathcal{L}_{[a,b]}^\infty$ , then  $\hat{\alpha} := \hat{\alpha}^1 + \hat{\alpha}^2 \in \mathcal{E}_{[a,b]}^\infty(r)$ . Moreover,

$$\|\hat{\alpha}\|_{\mathcal{E}_{[a,b]}^\infty(r)} \leq \|\hat{\alpha}^1\|_{\mathcal{E}_{[a,b]}^\infty(r)} + \|\hat{\alpha}^2\|_{\mathcal{L}_{[a,b]}^\infty}.$$

(iii) If  $\hat{\alpha}^1 \in \mathcal{E}_{[a,b]}^\infty(pr)$  and  $\hat{\alpha}^2 \in \mathcal{E}_{[a,b]}^\infty(qr)$  for some  $r > 0$  and  $p, q > 1$  satisfying  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $\hat{\alpha}^3 \in \mathcal{L}_{[a,b]}^\infty$ , then  $\hat{\alpha} := \hat{\alpha}^1 + \hat{\alpha}^2 + \hat{\alpha}^3 \in \mathcal{E}_{[a,b]}^\infty(r)$ . Moreover, we have

$$\|\hat{\alpha}\|_{\mathcal{E}_{[a,b]}^\infty(r)} \leq \|\hat{\alpha}^1\|_{\mathcal{E}_{[a,b]}^\infty(pr)} + \|\hat{\alpha}^2\|_{\mathcal{E}_{[a,b]}^\infty(qr)} + \|\hat{\alpha}^3\|_{\mathcal{L}_{[a,b]}^\infty}.$$

Let the  $\mathbb{R}^n$ -valued function  $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n$  be  $(\mathcal{F}_t)$ -progressively measurable for each  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , and consider the following multidimensional BSDE:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \quad (2.1)$$

or, equivalently,

$$Y_t^i = \xi^i + \int_t^T g^i(s, Y_s, Z_s) ds - \int_t^T Z_s^i dB_s, \quad t \in [0, T], \quad i = 1, \dots, n, \quad (2.2)$$

where  $\xi$  is an  $\mathcal{F}_T$ -measurable  $\mathbb{R}^n$ -valued random vector, and the solution  $(Y, Z)$  is a pair of  $(\mathcal{F}_t)$ -progressively measurable processes with values in  $\mathbb{R}^n \times \mathbb{R}^{n \times d}$ .

## 2.2. Local bounded solution of multi-dimensional quadratic BSDEs

We first introduce the following two assumptions on the quadratic growth generator  $g$ .

(B1) For each fixed  $i = 1, \dots, n$ , either of the following three conditions holds:



(i)  $f := g^i$  or  $f := -g^i$  satisfies that  $d\mathbb{P} \times dt - a.e.$ , for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,

$$\begin{aligned} \frac{\bar{\gamma}}{2}|z^i|^2 - \bar{\alpha}_t(\omega) - \phi(|y|) - \sum_{j=i+1}^n (\bar{\lambda}|z^j|^{1+\delta} + \theta|z^j|^2) - \bar{c} \sum_{j=1}^{i-1} |z^j|^2 &\leq f(\omega, t, y, z) \\ &\leq \alpha_t(\omega) + \phi(|y|) + \sum_{j \neq i} (\lambda|z^j|^{1+\delta} + \theta|z^j|^2) + \frac{\gamma}{2}|z^i|^2; \end{aligned}$$

(ii) Almost everywhere in  $\Omega \times [0, T]$ , for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , we have

$$|g^i(\omega, t, y, z)| \leq \tilde{\alpha}_t(\omega) + \phi(|y|) + |z^i| \left( v_t(\omega) + \phi(|y|) + c \sum_{j=1}^{i-1} |z^j| \right) + \frac{\gamma}{2}|z^i|^2;$$

(iii) Almost everywhere in  $\Omega \times [0, T]$ , for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , we have

$$|g^i(\omega, t, y, z)| \leq \bar{\alpha}_t(\omega) + \phi(|y|) + \bar{\lambda}|z| + \theta \sum_{j \neq i} |z^j|^2.$$

(B2) For  $i = 1, \dots, n$ ,  $g^i$  satisfies that  $d\mathbb{P} \times dt - a.e.$ , for any  $(y, \bar{y}, z, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{n \times d}$ ,

$$\begin{aligned} &|g^i(\omega, t, y, z) - g^i(\omega, t, \bar{y}, \bar{z})| \\ &\leq \phi(|y| \vee |\bar{y}|) \left\{ \left( v_t(\omega) + |z|^{1+\delta} + |\bar{z}|^{1+\delta} \right) |y - \bar{y}| + (v_t(\omega) + |z| + |\bar{z}|) \sum_{j=1}^i |z^j - \bar{z}^j| \right. \\ &\quad \left. + \left[ \left( v_t(\omega) + |z|^\delta + |\bar{z}|^\delta \right) + \theta (v_t(\omega) + |z| + |\bar{z}|) \right] \sum_{j=i+1}^n |z^j - \bar{z}^j| \right\}. \end{aligned}$$

**Remark 2.2.** Concerning assumptions (B1) and (B2), we make the following remarks.

(i) It is easy to see that  $g^i$  satisfying assumptions (B1)(i), (B1)(ii) and (B1)(iii) together with assumption (B2) admits, respectively, a strictly quadratic growth, a general quadratic growth and a linear growth in  $z^i$ . The  $g^i$  can also have a general growth in  $y$ , a quadratic growth in  $z^j$  for  $j \neq i$  but with a small enough constant  $\theta$  when needed, and an interacting quadratic growth like the inner product of  $z^i$  and  $z^j$  for  $j < i$ .

(ii) Assumptions (B1)(i) and (B2) are more general than those of Luo [35, Theorem 2.2], where for each  $i = 1, \dots, n$ ,  $g^i$  has a strictly quadratic growth in  $z^i$  and a stronger growth and continuity in the unknown variables  $y$  and  $z$ .

(iii) The following assumptions (B1') and (B2') are used in Theorem 2.1 of Fan et al. [17]:

(B1') For each  $i = 1, \dots, n$ ,  $g^i$  satisfies that  $d\mathbb{P} \times dt - a.e.$ , for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,

$$|g^i(\omega, t, y, z)| \leq \tilde{\alpha}_t(\omega) + \phi(|y|) + \frac{\gamma}{2}|z^i|^2 + \lambda \sum_{j \neq i} |z^j|^{1+\delta};$$

(B2') For  $i = 1, \dots, n$ ,  $g^i$  satisfies that  $d\mathbb{P} \times dt - a.e.$ , for any  $(y, \bar{y}, z, \bar{z}) \in \mathbb{R}^{2n} \times \mathbb{R}^{2(n \times d)}$ ,

$$\begin{aligned} & |g^i(\omega, t, y, z) - g^i(\omega, t, \bar{y}, \bar{z})| \\ & \leq \phi(|y| \vee |\bar{y}|) \left[ (1 + |z| + |\bar{z}|) (|y - \bar{y}| + |z^i - \bar{z}^i|) + \left(1 + |z|^\delta + |\bar{z}|^\delta\right) \sum_{j \neq i} |z^j - \bar{z}^j| \right]. \end{aligned}$$

It is easy to check that assumption (B2) is strictly weaker than assumption (B2'), while assumptions (B1) and (B1') do not cover each other.

(iv) In assumption (B1), the set of integers  $\{1, \dots, n\}$  is divided into two disjoint parts  $I_1$  and  $I_2$ , both of which can be  $\emptyset$ , such that  $I_1 + I_2 = \{1, \dots, n\}$ ,  $g^i$  satisfies either of (B1)(i), (B1)(ii) and (B1)(iii) for  $i \in I_1$ , and  $-g^i$  satisfies (B1)(i) for  $i \in I_2$ . Now, we define, for each  $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,

$$\bar{g}^i(\omega, t, y, z) := \begin{cases} g^i(\omega, t, \bar{y}, \bar{z}), & i \in I_1; \\ -g^i(\omega, t, \bar{y}, \bar{z}), & i \in I_2 \end{cases} \quad (2.3)$$

with

$$\bar{y}^i := \begin{cases} y^i, & i \in I_1; \\ -y^i, & i \in I_2 \end{cases} \quad \text{and} \quad \bar{z}^i := \begin{cases} z^i, & i \in I_1; \\ -z^i, & i \in I_2. \end{cases}$$

It is not hard to verify that  $\bar{g}^i$  satisfies either of (B1)(i), (B1)(ii) and (B1)(iii) for all  $i = 1, \dots, n$ . Therefore, in assumption (B1) we can without loss of generality assume that  $g^i$  satisfies either of (B1)(i), (B1)(ii) and (B1)(iii) for all  $i = 1, \dots, n$ . Furthermore, it is clear that  $\bar{g}$  defined by (2.3) satisfies assumption (B2) as soon as  $g$  satisfies it.

The following Theorem 2.3 establishes a general existence and uniqueness result on the local bounded solution of multi-dimensional BSDEs with interacting quadratic generators.

**Theorem 2.3.** *Let  $\xi \in L^\infty(\mathbb{R}^n)$ ,  $\alpha \in \mathcal{E}^\infty(p\gamma)$  for some  $p > 1$  and the generator  $g$  satisfy assumptions (B1) and (B2). Then, there exist two constants  $\varepsilon_0 > 0$  and  $\theta_0 > 0$  depending only on  $(\|\xi\|_\infty, \|\alpha\|_{\mathcal{E}^\infty(p\gamma)}, \|\bar{\alpha}\|_{\mathcal{M}^\infty}, \|\tilde{\alpha}\|_{\mathcal{L}^\infty}, \|v\|_{\text{BMO}}, n, \gamma, \bar{\gamma}, \lambda, \bar{\lambda}, c, \delta, T, p)$  and  $\phi(\cdot)$  together with a bounded subset  $\mathcal{B}_{\varepsilon_0}$  of the product space  $\mathcal{S}_{[T-\varepsilon_0, T]}^\infty(\mathbb{R}^n) \times \text{BMO}_{[T-\varepsilon_0, T]}(\mathbb{R}^{n \times d})$  such that when  $\theta \in [0, \theta_0]$ , BSDE (2.1) has a unique local solution  $(Y, Z)$  on the time interval  $[T - \varepsilon_0, T]$  with  $(Y, Z) \in \mathcal{B}_{\varepsilon_0}$ . Moreover, the above conclusion holds still for  $p = 1$  when  $\lambda = 0$  and  $\theta_0 = 0$ .*

**Remark 2.4.** *It follows from (ii) and (iii) of Remark 2.2 that Theorem 2.3 strengthens Luo [35, Theorem 2.2], and that Theorem 2.3 and Fan et al. [17, Theorem 2.1] do not cover each other. In addition, it follows from Fan et al. [17, Remark 2.2] that Fan et al. [17, Theorem 2.1] extends Theorem 2.2 of Hu and Tang [24]. However, it can be easily checked that Theorem 2.3 and Hu and Tang [24, Theorem 2.2] also do not cover each other.*

### 2.3. Global bounded solution of multi-dimensional quadratic BSDEs

In this subsection, we will present three existence and uniqueness results on the global bounded solution of multi-dimensional BSDE (2.1). For the first one, let us introduce the following a priori boundedness assumption (AB) on the generator  $g$ , which was used in Jackson [25] and Jackson and Zitkovic [26]. Interested readers are referred to Xing and Žitković [42] and Escarriaza et al. [14] for the other versions of this assumption.

(AB) There exists a finite collection  $\{a_k\} = (a_1, \dots, a_K)$  of vectors in  $\mathbb{R}^n$  such that

- (i) the collection of  $(a_1, \dots, a_K)$  positively span  $\mathbb{R}^n$ , i.e., for any  $a \in \mathbb{R}^n$  there exist nonnegative constants  $\lambda_1, \dots, \lambda_K$  such that  $\lambda_1 a_1 + \dots + \lambda_K a_K = a$ ;
- (ii) Almost everywhere in  $\Omega \times [0, T]$ , for any  $k = 1, \dots, K$  and any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , we have  $a_k^\top g(\omega, t, y, z) \leq \tilde{\alpha}_t(\omega) + \gamma |a_k^\top z|^2$ .

The following proposition can be proved by an identical way as in Lemma 6.6 of Jackson [25] and Proposition 3.8 of Jackson and Zitkovic [26]. We omit its proof here.

**Proposition 2.5.** *Let  $\xi \in L^\infty(\mathbb{R}^n)$  and the generator  $g$  satisfy assumption (AB). Assume that for some  $h \in (0, T]$ , BSDE (2.1) has a solution  $(Y, Z) \in \mathcal{S}_{[T-h, T]}^\infty(\mathbb{R}^n) \times \text{BMO}_{[T-h, T]}(\mathbb{R}^{n \times d})$  on the time interval  $[T - h, T]$ . Then, there exists a positive constant  $\tilde{K} > 0$  depending only on  $(\|\xi\|_\infty, \|\tilde{\alpha}\|_{\mathcal{L}^\infty}, \{a_k\}, \gamma)$  and being independent of  $h$  such that*

$$\|Y\|_{\mathcal{S}_{[T-h, T]}^\infty} + \|Z\|_{\text{BMO}_{[T-h, T]}}^2 \leq \tilde{K}.$$

With Theorem 2.3 and Proposition 2.5 in hands, we can closely follow the proof of Theorem 4.1 in Cheridito and Nam [9] to prove the following Theorem 2.6, which is our first result on the global bounded solution of multi-dimensional BSDE (2.1). All details are omitted here.

**Theorem 2.6.** *Let  $\xi \in L^\infty(\mathbb{R}^n)$ ,  $\alpha \in \mathcal{E}^\infty(p\gamma)$  for some  $p > 1$  and the generator  $g$  satisfy assumptions (B1), (B2) and (AB). Then, there exists a positive constant  $\theta_0 > 0$  depending only on  $(\|\xi\|_\infty, \|\alpha\|_{\mathcal{E}^\infty(p\gamma)}, \|\bar{\alpha}\|_{\mathcal{M}^\infty}, \|\tilde{\alpha}\|_{\mathcal{L}^\infty}, \|v\|_{\text{BMO}}, n, \gamma, \bar{\gamma}, \lambda, \bar{\lambda}, c, \delta, T, p)$  and  $\phi(\cdot)$  such that when  $\theta \in [0, \theta_0]$ , BSDE (2.1) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  on  $[0, T]$ .*

By Theorem 2.6 and (i) of Remark 2.1, the following result is immediate.

**Corollary 2.7.** *Let  $\alpha, \bar{\alpha}, \tilde{\alpha} \in \mathcal{L}^\infty$  and the generator  $g$  satisfy assumptions (B1), (B2) and (AB) with  $\theta = 0$ . Then for each  $\xi \in L^\infty(\mathbb{R}^n)$ , BSDE (2.1) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  on the time interval  $[0, T]$ .*

**Remark 2.8.** *We have the following remarks.*

- (i) *Compared with Jackson and Zitkovic [26, Theorem 3.5], the Malliavin regular condition on the generator  $g$  is not required in Theorem 2.6 due to a distinct strategy of the proof. From this point of view, Theorem 2.6 strengthens Jackson and Zitkovic [26, Theorem 3.5].*
- (ii) *By [17, Theorem 2.1], Proposition 2.5 and an identical argument as that in the proof of Cheridito and Nam [9, Theorem 4.1], it can be checked that the conclusion of Corollary 2.7 still holds when (B1) and (B2) are replaced with (B1') and (B2') in (iii) of Remark 2.2.*
- (iii) *In our opinion, the (AB) condition of the generator  $g$  is not easily verified in the multi-dimension case. In particular, it seems that the generator  $g$  satisfying this condition is forced to be bounded in the state variable  $y$ , which is a strong restriction. Consequently, in the sequel we will search for some better conditions on the generator  $g$  to guarantee existence and uniqueness of the global solution of multi-dimensional BSDE (2.1).*

In order to obtain the second result on the global bounded solution of multi-dimensional BSDE (2.1), the assumption (B1) needs to be strengthened to the following assumption (C1a). Before that, we introduce the following notations. For each  $i = 1, \dots, n$  and any  $M \in \mathbb{R}^n$  and  $x \in \mathbb{R}$ , denote by  $M(x; i)$  the vector in  $\mathbb{R}^n$  whose  $i$ th component is  $x$  and whose  $j$ th component is  $M^j$  for  $j \neq i$ . And, For each  $i = 1, \dots, n$  and any  $H \in \mathbb{R}^{n \times d}$  and  $w \in \mathbb{R}^{1 \times d}$ , denote by  $H(w; i)$  the matrix in  $\mathbb{R}^{n \times d}$  whose  $i$ th row is  $w$  and whose  $j$ th row is  $H^j$  for  $j \neq i$ .

(C1a) For each fixed  $i = 1, \dots, n$ , either of the following three conditions holds:

- (i) The random field  $f := g^i$  or  $f(\omega, t, y, z) := -g^i(\omega, t, y(-y^i; i), z(-z^i; i))$  satisfies that  $d\mathbb{P} \times dt - a.e.$ , for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,

$$\begin{aligned} & \frac{\bar{\gamma}}{2}|z^i|^2 - \bar{\alpha}_t(\omega) - \beta|y| - \sum_{j=i+1}^n (\bar{\lambda}|z^j|^{1+\delta} + \theta|z^j|^2) - \bar{c} \sum_{j=1}^{i-1} |z^j|^2 \leq f(\omega, t, y, z) \\ & \leq \alpha_t(\omega) + [\beta|y| \mathbf{1}_{y^i > 0} + \phi(|y|) \mathbf{1}_{y^i < 0}] + \sum_{j \neq i} (\lambda|z^j|^{1+\delta} \mathbf{1}_{y^i < 0} + \theta|z^j|^2) + \frac{\gamma}{2}|z^i|^2; \end{aligned}$$

- (ii) Almost everywhere in  $\Omega \times [0, T]$ , for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , we have

$$\begin{aligned} - [\beta|y^i| \mathbf{1}_{y^i < 0} + \phi(|y|) \mathbf{1}_{y^i > 0}] - l_i(\omega, t, z) & \leq g^i(\omega, t, y, z) \\ & \leq [\beta|y^i| \mathbf{1}_{y^i > 0} + \phi(|y|) \mathbf{1}_{y^i < 0}] + l_i(\omega, t, z) \end{aligned}$$

with

$$l_i(\omega, t, z) := \tilde{\alpha}_t(\omega) + |z^i| \left( v_t(\omega) + c \sum_{j=1}^{i-1} |z^j| \right) + \frac{\gamma}{2}|z^i|^2;$$

(iii) Almost everywhere in  $\Omega \times [0, T]$ , for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , we have

$$\begin{aligned} -[\beta|y^i| \mathbf{1}_{y^i < 0} + \phi(|y|) \mathbf{1}_{y^i > 0}] - \bar{l}_i(\omega, t, z) &\leq g^i(\omega, t, y, z) \\ &\leq [\beta|y^i| \mathbf{1}_{y^i > 0} + \phi(|y|) \mathbf{1}_{y^i < 0}] + \bar{l}_i(\omega, t, z) \end{aligned}$$

with

$$\bar{l}_i(\omega, t, z) := \bar{\alpha}_t(\omega) + \bar{\lambda}|z| + \theta \sum_{j \neq i} |z^j|^2.$$

**Remark 2.9.** *It is clear that the generator  $g$  satisfying assumptions (C1a) and (B2) can have a general growth in the state variable  $y$  and a general quadratic growth in the state variable  $z$ . And, in assumption (C1a),  $g$  has a diagonally quadratic growth in  $z$  when  $\theta = 0$  and  $c = \bar{c} = 0$ , and a triangularly quadratic growth in  $z$  when  $\theta = 0$  and  $c > 0$  or  $\bar{c} > 0$ . In addition, by a similar argument to (iv) of Remark 2.2, in assumption (C1a) we can without loss of generality assume that  $g^i$  satisfies either of conditions (C1a)(i), (C1a)(ii) and (C1a)(iii) for all  $i = 1, \dots, n$ .*

**Theorem 2.10.** *Let  $\xi \in L^\infty(\mathbb{R}^n)$ ,  $\alpha \in \mathcal{E}^\infty(p\gamma \exp(\beta T))$  for some  $p > 1$  and the generator  $g$  satisfy assumptions (C1a) and (B2). Then, there exists a  $\theta_0 > 0$  depending only on  $(\|\xi\|_\infty, \|\alpha\|_{\mathcal{E}^\infty(p\gamma \exp(\beta T))}, \|\bar{\alpha}\|_{\mathcal{M}^\infty}, \|\tilde{\alpha}\|_{\mathcal{L}^\infty}, \|v\|_{\text{BMO}}, n, \beta, \gamma, \bar{\gamma}, \lambda, \bar{\lambda}, c, \delta, T, p)$  such that when  $\theta \in [0, \theta_0]$ , BSDE (2.1) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  on the time interval  $[0, T]$ . Moreover, the above conclusion holds still for  $p = 1$  when  $\lambda = 0$  and  $\theta_0 = 0$ .*

We would like to mention that Theorem 2.10 can be compared with Theorem 2.3 and Remark 2.5 of Jamneshan et al. [27] to observe the role of  $\theta$  in assumptions (C1a) and (B2) for existence and uniqueness of the bounded solution of multi-dimensional interacting quadratic BSDEs.

By virtue of Theorem 2.10 and (i) of Remark 2.1, the following corollary is immediate.

**Corollary 2.11.** *Let  $\alpha, \bar{\alpha}, \tilde{\alpha} \in \mathcal{L}^\infty$  and the generator  $g$  satisfy assumptions (C1a) and (B2) with  $\theta = 0$ . Then, for each  $\xi \in L^\infty(\mathbb{R}^n)$ , BSDE (2.1) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  on the time interval  $[0, T]$ .*

**Remark 2.12.** *It is easy to verify that if  $g$  satisfies those assumptions in Luo [35, Theorem 2.3], then assumption (C1a)(i) with  $\lambda = \theta = 0$  holds for each  $g^i$  with  $i = 1, \dots, n$ , and assumption (B2) with  $\theta = 0$  holds. Consequently, Corollary 2.11 strengthens Luo [35, Theorem 2.3]. In particular, in order to obtain the global solvability of BSDE (2.1), in Luo [35, Theorem 2.3] the interacting term  $(z^i)^\top z^j$  can not appear in  $g^i$  for  $1 \leq j < i \leq n$ . While the situation in Corollary 2.11 is different because  $g^i$  with this type of term can satisfy (C1a)(ii) and (C1a)(i), see the next subsection for more details. Finally, we especially mention that multidimensional*

BSDEs with this type of interacting term appear in many applications, such as price impact models (see Kramkov and Pulido [32]), incomplete stochastic equilibria (see Xing and Žitković [42], Kardaras et al. [28], Escarriaza et al. [14], Weston [40]), and risk-sensitive nonzero-sum stochastic games (see Xing and Žitković [42], Jackson and Zitkovic [26], Jackson [25]).

Finally, let us further demonstrate the third existence and uniqueness result on the global bounded solution. In stating it, the following assumption (C1b) on the generator will be used, which is strictly stronger than (B1) with  $\theta = 0$ . It should be noted that assumptions (C1b) and (C1a) with  $\theta = 0$  do not cover each other.

(C1b) The set of integers  $\{1, \dots, n\}$  is divided into three pairwise disjoint parts  $J_1, J_2$  and  $J_3$ , any of which can be  $\emptyset$ , such that  $J_1 + J_2 + J_3 = \{1, \dots, n\}$ . For each fixed  $i \in \{1, \dots, n\}$ , either of the following three conditions holds:

- (i) If  $i \in J_1$ , then the random field  $f := g^i$  or  $f(\omega, t, y, z) := -g^i(\omega, t, y(-y^i; i), z(-z^i; i))$  satisfies that  $d\mathbb{P} \times dt - a.e.$ , for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,

$$\begin{aligned} \frac{\bar{\gamma}}{2}|z^i|^2 - \bar{\alpha}_t(\omega) - \beta|y| - \bar{\lambda} \sum_{j \in J_1} |z^j|^{1+\delta} &\leq f(\omega, t, y, z) \\ &\leq \alpha_t(\omega) + [\beta|y| \mathbf{1}_{y^i > 0} + \phi(|y|) \mathbf{1}_{y^i < 0}] + \lambda \sum_{j \in J_1} |z^j|^{1+\delta} + \frac{\gamma}{2}|z^i|^2; \end{aligned}$$

- (ii) If  $i \in J_2$ , then  $d\mathbb{P} \times dt - a.e.$ , for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , we have

$$\begin{aligned} -[\beta|y| \mathbf{1}_{y^i < 0} + \phi(|y|) \mathbf{1}_{y^i > 0}] - \tilde{l}_i(\omega, t, z) &\leq g^i(\omega, t, y, z) \\ &\leq [\beta|y| \mathbf{1}_{y^i > 0} + \phi(|y|) \mathbf{1}_{y^i < 0}] + \tilde{l}_i(\omega, t, z) \end{aligned}$$

with

$$\tilde{l}_i(\omega, t, y, z) := \tilde{\alpha}_t(\omega) + |z^i| \left( v_t(\omega) + \phi(|y|) + c \sum_{j=1}^{i-1} |z^j| \right) + \frac{\gamma}{2}|z^i|^2;$$

- (iii) If  $i \in J_3$ , then  $d\mathbb{P} \times dt - a.e.$ , for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , we have

$$\begin{aligned} -[\beta|y| \mathbf{1}_{y^i < 0} + \phi(|y|) \mathbf{1}_{y^i > 0}] - \hat{l}_i(\omega, t, z) &\leq g^i(\omega, t, y, z) \\ &\leq [\beta|y| \mathbf{1}_{y^i > 0} + \phi(|y|) \mathbf{1}_{y^i < 0}] + \hat{l}_i(\omega, t, z) \end{aligned}$$

with

$$\hat{l}_i(\omega, t, z) := \bar{\alpha}_t(\omega) + \bar{\lambda} \sum_{j \in J_3} |z^j|.$$

**Remark 2.13.** By a similar analysis to (iv) of Remark 2.2, in assumption (C1b) we can without loss of generality assume that  $g^i$  satisfies either of conditions (C1b)(i), (C1b)(ii) and (C1b)(iii) for all  $i = 1, \dots, n$ . In addition, in assumption (C1b)(i) it creates no essential difference to replace the term  $\sum_{j \in J_1} |z^j|^{1+\delta}$  with  $\sum_{j \in J_1, j \neq i} |z^j|^{1+\delta}$ .

**Theorem 2.14.** *Let  $\alpha, \bar{\alpha}, \tilde{\alpha} \in \mathcal{L}^\infty$  and the generator  $g$  satisfy assumptions (C1b) and (B2) with  $\theta = 0$ . Then, for each  $\xi \in L^\infty(\mathbb{R}^n)$ , BSDE (2.1) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  on the time interval  $[0, T]$ .*

**Remark 2.15.** *We have the following remarks.*

(i) *It is easy to check that assumptions (C1b)(ii)-(iii) generalize assumption (H3.2) used in Fan et al. [19]. Consequently, Theorem 2.14 strengthens Fan et al. [19, Theorem 3.6], and then Fan et al. [17, Theorems 2.4] and Hu and Tang [24, Theorem 2.3].*

(ii) *The following two assumptions (B3') and (B4') together with (B1') and (B2') in (ii) of Remark 2.2 are used in Theorem 2.5 of Fan et al. [17]:*

(B3') *For each  $i = 1, \dots, n$ ,  $g^i$  satisfies that  $d\mathbb{P} \times dt - a.e.$ , for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,*

$$g^i(\omega, t, y, z) \operatorname{sgn}(y^i) \leq \alpha_t(\omega) + \beta|y| + \lambda \sum_{j \neq i} |z^j|^{1+\delta} + \frac{\gamma}{2}|z^i|^2;$$

(B4') *For  $i = 1, \dots, n$ ,  $f := g^i$  or  $f := -g^i$  satisfies that  $d\mathbb{P} \times dt - a.e.$ ,*

$$f(\omega, t, y, z) \geq \frac{\bar{\gamma}}{2}|z^i|^2 - \bar{\alpha}_t(\omega) - \beta|y| - \bar{\lambda} \sum_{j \neq i} |z^j|^{1+\delta}, \quad \forall (y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}.$$

*It is clear that assumptions (B1')-(B4') are equivalent to (C1b) with  $J_1 = \{1, \dots, n\}$  and  $J_2 = J_3 = \emptyset$ . Consequently, Theorem 2.14 strengthens Fan et al. [17, Theorem 2.5].*

(iii) *Let the assumptions of Theorem 2.14 be satisfied except that  $\theta = 0$ . It can also be proved that the conclusion of Theorem 2.14 holds still when  $\theta$  is smaller than a constant  $\theta_0 > 0$  depending only on  $\|\xi\|_\infty$  and those parameters in assumption (C1b).*

(iv) *Let  $l \in (1, 2]$ ,  $(a_t)_{t \in [0, T]}, (b_t)_{t \in [0, T]} \in \mathcal{S}^\infty(\mathbb{R}^n)$  and the generator be defined as follows:*

$$g(\omega, t, y, z) := a_t(\omega)|y| + b_t(\omega) \sin(|z|^l), \quad \forall (\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}.$$

*According to (iii) of this remark, we know that for each  $\xi \in L^\infty(\mathbb{R}^n)$ , there exists a constant  $\theta_0 > 0$  such that when  $\|b\|_{\mathcal{S}^\infty} \mathbf{1}_{l=2} \leq \theta_0$ , BSDE (2.1) with this generator  $g$  admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  on the time interval  $[0, T]$ . It seems that this conclusion can not be obtained by any existing results, to the best of our knowledge.*

#### 2.4. Examples of application

First of all, Jamneshan et al. [27] addressed the following two-dimensional BSDE:

$$\begin{cases} Y_t^1 = \xi^1 + \int_t^T (\theta_1 |Z_s^1|^2 + \vartheta_1 |Z_s^2|^2) ds - \int_t^T Z_s^1 dB_s, \\ Y_t^2 = \xi^2 + \int_t^T (\vartheta_2 |Z_s^1|^2 + \theta_2 |Z_s^2|^2) ds - \int_t^T Z_s^2 dB_s, \end{cases} \quad t \in [0, T], \quad (2.4)$$

where  $\xi^i \in L^\infty(\mathbb{R})$  and  $\theta_i, \vartheta_i \in \mathbb{R}$ ,  $i = 1, 2$ . According to Theorem 2.10, we see that if  $\theta_1\theta_2 \neq 0$ , then for each  $\xi \in L^\infty(\mathbb{R}^2)$ , there exists a constant  $\theta_0 > 0$  depending only on  $(\theta_1, \theta_2, \|\xi\|_\infty)$  such that for each pair  $(\vartheta_1, \vartheta_2)$  satisfying  $|\vartheta_1|, |\vartheta_2| \leq \theta_0$ , the system of BSDEs (2.4) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^2) \times \text{BMO}(\mathbb{R}^{2 \times d})$  on the time interval  $[0, T]$ . This conclusion can be compared with Theorem 2.3 and Remark 2.5 posed in Jamneshan et al. [27].

On the other hand, in the case of  $\theta_1 = \vartheta_1 = 0$ ,  $\vartheta_2 = 1$  and  $\theta_2 = 1/2$ , Frei and Dos Reis [21] showed that for some  $\xi \in L^\infty(\mathbb{R}^2)$ , the system of BSDEs (2.4) fails to have a global bounded solution on the time interval  $[0, T]$ , see Theorem 2.1 of Frei and Dos Reis [21] for more details. However, for the case of  $\vartheta_1 = 0$  and  $\vartheta_2\theta_2 < 0$  (for example,  $\vartheta_2 = 1$  and  $\theta_2 = -1/2$ ), by Corollary 2.11 we know that for each  $\xi \in L^\infty(\mathbb{R}^2)$ , the system of BSDEs (2.4) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^2) \times \text{BMO}(\mathbb{R}^{2 \times d})$  on the time interval  $[0, T]$ .

Furthermore, we consider the following variant of the system of BSDEs (2.4):

$$\begin{cases} Y_t^1 = \xi^1 + \int_t^T (\theta_1 |Z_s^1|^2 + \vartheta_1 |Z_s^2|^2) ds - \int_t^T Z_s^1 dB_s, \\ Y_t^2 = \xi^2 + \int_t^T (\vartheta_2 |Z_s^1|^2 + \theta_2 |Z_s^2|^2 + l Z_s^1 (Z_s^2)^\top) ds - \int_t^T Z_s^2 dB_s, \quad t \in [0, T], \end{cases} \quad (2.5)$$

where  $l \in \mathbb{R}$ ,  $\xi^i \in L^\infty(\mathbb{R})$  and  $\theta_i, \vartheta_i \in \mathbb{R}$ ,  $i = 1, 2$  such that  $\theta_2\vartheta_2 < 0$ . Without loss of generality, we assume that  $\vartheta_2 < 0$  and  $\theta_2 > 0$ . Observe that

$$-\frac{l^2}{2\theta_2} |Z_s^1|^2 - \frac{\theta_2}{2} |Z_s^2|^2 \leq l Z_s^1 (Z_s^2)^\top \leq -\frac{\vartheta_2}{2} |Z_s^1|^2 - \frac{l^2}{2\vartheta_2} |Z_s^2|^2$$

and then

$$\frac{\theta_2}{2} |Z_s^2|^2 - \left(\frac{l^2}{2\theta_2} - \vartheta_2\right) |Z_s^1|^2 \leq \vartheta_2 |Z_s^1|^2 + \theta_2 |Z_s^2|^2 + l Z_s^1 (Z_s^2)^\top \leq \left(\theta_2 - \frac{l^2}{2\vartheta_2}\right) |Z_s^2|^2. \quad (2.6)$$

We get that the generator  $f := g^1$  satisfies (C1a)(i) for  $\theta_1 \neq 0$  and (C1a)(iii) for  $\theta_1 = 0$ , and  $f := g^2$  satisfies (C1a)(i). It follows from Corollary 2.11 that for each  $\xi \in L^\infty(\mathbb{R}^2)$ , the system of BSDEs (2.5) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^2) \times \text{BMO}(\mathbb{R}^{2 \times d})$  on the time interval  $[0, T]$ . In addition, it is clear that when  $\vartheta_2 = 0$ , the above conclusion holds still.

We proceed by considering the following three-dimensional BSDEs, which is another variant of the system of BSDEs (2.4):

$$\begin{cases} t \in [0, T]; \quad Y_t^1 = \xi^1 + \int_t^T (\vartheta_1 |Z_s^1|^2 + \theta_1 |Z_s^3|^2) ds - \int_t^T Z_s^1 dB_s, \\ Y_t^2 = \xi^2 + \int_t^T (\vartheta_2 |Z_s^1|^2 + \theta_2 |Z_s^2|^2 + l_{21} Z_s^2 (Z_s^1)^\top + k_2 |Z_s^3|^2) ds - \int_t^T Z_s^2 dB_s, \\ Y_t^3 = \xi^3 + \int_t^T (\vartheta_3 |Z_s^1|^2 + \theta_3 |Z_s^2|^2 + \kappa_3 |Z_s^3|^2) ds - \int_t^T Z_s^3 dB_s \\ \quad + \int_t^T (l_{31} Z_s^3 (Z_s^1)^\top + l_{32} Z_s^3 (Z_s^2)^\top + l_{33} Z_s^1 (Z_s^2)^\top + k_3 |Z_s^2|^2) ds, \end{cases} \quad (2.7)$$



where  $\xi^i \in L^\infty(\mathbb{R})$ ,  $\theta_i, \vartheta_i \in \mathbb{R}$  for  $i = 1, 2, 3$ ,  $\kappa_3, l_{21}, l_{31}, l_{32}, l_{33}, k_2, k_3 \in \mathbb{R}$  with

$$\theta_2\vartheta_2 < 0, \quad \kappa_3\theta_3 < 0, \quad \kappa_3\vartheta_3 < 0 \quad \text{and} \quad l_{33}^2 < 4\theta_3\vartheta_3.$$

Without loss of generality, we assume that  $\theta_2 > 0$ ,  $\vartheta_2 < 0$ ,  $\kappa_3 > 0$ ,  $\theta_3 < 0$  and  $\vartheta_3 < 0$ . Observe that there exists a unique real  $\varepsilon \in (0, 1]$  satisfying

$$l_{33}^2 = 4(1 - \varepsilon)^2\theta_3\vartheta_3$$

and

$$\vartheta_3(1 - \varepsilon)|Z_s^1|^2 + \theta_3(1 - \varepsilon)|Z_s^2|^2 + l_{33}Z_s^1(Z_s^2)^\top = -(1 - \varepsilon) \left| \sqrt{|\vartheta_3|}Z_s^1 - \text{sgn}(l_{33})\sqrt{|\theta_3|}Z_s^2 \right|^2.$$

Combining a similar argument to (2.6), we can directly verify that

$$\Delta_s \leq \left( \kappa_3 - \frac{l_{31}^2}{4\varepsilon\vartheta_3} - \frac{l_{32}^2}{4\varepsilon\theta_3} \right) |Z_s^3|^2$$

and

$$\Delta_s \geq \frac{\kappa_3}{2}|Z_s^3|^2 - \left( \frac{1}{2} + \frac{l_{31}^2}{\kappa_3} - \vartheta_3 \right) |Z_s^1|^2 - \left( \frac{l_{33}^2}{2} + \frac{l_{32}^2}{\kappa_3} - \theta_3 \right) |Z_s^2|^2,$$

where

$$\Delta_s := \vartheta_3|Z_s^1|^2 + \theta_3|Z_s^2|^2 + \kappa_3|Z_s^3|^2 + l_{31}Z_s^3(Z_s^1)^\top + l_{32}Z_s^3(Z_s^2)^\top + l_{33}Z_s^1(Z_s^2)^\top.$$

Consequently, the generator  $g$  of the system of BSDEs (2.7) satisfies assumption (C1a). It follows from Corollary 2.11 that for each  $\xi \in L^\infty(\mathbb{R}^2)$ , the system of BSDEs (2.7) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^2) \times \text{BMO}(\mathbb{R}^{2 \times d})$  on the time interval  $[0, T]$ . In addition, it can also be proved that when  $\vartheta_2 = k_2 = 0$ ,  $\kappa_3\theta_3 < 0$ ,  $\kappa_3\vartheta_3 < 0$ ,  $l_{31} = l_{32} = 0$  and  $l_{33}^2 = 4\theta_3\vartheta_3$  holds or  $\vartheta_2 = k_2 = 0$  and  $\theta_3 = \vartheta_3 = l_{33} = k_3 = 0$  holds, the above conclusion holds still.

Finally, let us present several specific examples of multi-dimensional solvable BSDEs with interacting quadratic generators, to which one of Theorem 2.10, Corollary 2.11 and Theorem 2.14 can apply, but any existing results can not, to the best of our knowledge.

**Example 2.16.** *We have the following assertions.*

- (i) *Assume that the generator  $g := (g^1, \dots, g^n)^\top$  is defined as follows: for each  $i = 1, \dots, n$ , and  $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,*

$$g^i(\omega, t, y, z) := \tilde{\alpha}_t(\omega) + (-1)^i \left( \sum_{j=1}^{i-1} a_{ij}|z^j|^2 - a_{ii}|z^i|^2 + \sum_{j=i+1}^n a_{ij}|z^j|^{1+\delta} \right) + h^i(y, z),$$

*where  $A := (a_{ij})_{n \times n}$  is any nonnegative matrix with  $a_{i,i} > 0$  for each  $i = 1, \dots, n$ , and for each  $i = 1, \dots, n$ ,  $h^i(y, z) : \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}$  is any Lipschitz continuous function.*

It is not hard to verify that this generator  $g$  satisfies assumption (B2) with  $\theta = 0$ , and for each  $i = 1, \dots, n$ ,  $f := g^i$  or  $f(\omega, t, y, z) := -g^i(\omega, t, y(-y^i; i), z(-z^i; i))$  satisfies assumption (C1a)(i) with  $\theta = 0$ . Then, from Corollary 2.11 we can conclude that for each  $\xi \in L^\infty(\mathbb{R}^n)$ , BSDE (2.1) with this generator  $g$  admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  on the time interval  $[0, T]$ . We would like to mention that in order to satisfy assumption (C1a)(i), it is generally required that the signs of  $|z^i|^2$  and  $|z^j|^2$  for  $j < i$  should be opposite.

(ii) Let  $n = d$  and the generator  $g(y, z) := zy$  for each  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}$ . It is easy to check that this generator  $g$  satisfies assumption (B2) with  $\theta = 0$ , and for each  $i = 1, \dots, n$ ,  $f := g^i(y, z) = z^i y$  satisfies assumption (C1b)(ii). It then follows from Theorem 2.14 that for each  $\xi \in L^\infty(\mathbb{R}^n)$ , BSDE (2.1) with this generator  $g$  admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  on the time interval  $[0, T]$ .

(iii) Assume that the generator  $g := (g^1, \dots, g^n)^\top$  is defined as follows: for each  $i = 1, \dots, n$ , and  $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,

$$g^i(\omega, t, y, z) := \tilde{\alpha}_t(\omega) + |z^i| \left( v_t(\omega) + e^{|y|} \right) + e^{-y^i} |z^i|^{\frac{3}{2}} + z^i \sum_{j=1}^i c_{ij} (z^j)^\top,$$

where  $C = (c_{ij})_{n \times n}$  is any real matrix. It is easy to verify that this generator  $g$  satisfies assumption (B2) with  $\theta = 0$ , and for each  $i = 1, \dots, n$ ,  $f := g^i$  satisfies assumption (C1b)(ii). Then, according to Theorem 2.14, we can conclude that for each  $\xi \in L^\infty(\mathbb{R}^n)$ , BSDE (2.1) with this generator  $g$  admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  on the time interval  $[0, T]$ .

(iv) Let  $n = 5$  and  $d = 2$ . Assume that for each  $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , the generator  $g$  is defined as follows:

$$g(\omega, t, y, z) := \begin{pmatrix} g^1 \\ g^2 \\ g^3 \\ g^4 \\ g^5 \end{pmatrix} (\omega, t, y, z) := \begin{pmatrix} e^{-y^1} - |y| + |z^1|^2 - |z^2|^{\frac{4}{3}} + \sin |z^3| \\ |y| \cos |y| - |z^2|^2 + |z^1|^{\frac{5}{4}} - \cos |z^4| \\ |y| + z^3(2z^1 - 3z^2)^\top + z^3 A (z^3)^\top - \arcsin |z^5| \\ 2|y| \sin |y| + |z^4| - |z^5| + \arccos |z^1| \\ y^1 + 3y^3 - y^4 + y^5 - |z^4| + 2|z^5| - \arctan |z^2| \end{pmatrix}$$

with the matrix

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

It is easy to check that this generator  $g$  satisfies (B2) with  $\theta = 0$ , and that  $f := g^i$  satisfies (C1b)(i) for  $i = 1, 2$ , (C1b)(ii) for  $i = 3$ , and (C1b)(iii) for  $i = 4, 5$ . Then, according to Theorem 2.14, we can conclude that for each  $\xi \in L^\infty(\mathbb{R}^n)$ , BSDE (2.1) with this generator  $g$  admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  on  $[0, T]$ .

## 2.5. Connections to existing results

With an invertible linear transformation as in Xing and Žitković [42] and Weston [40], we immediately have the following result after an application of Itô formula.

**Proposition 2.17.** *Let  $A \in \mathbb{R}^{n \times n}$  be a real invertible matrix, and  $A^{-1}$  its inverse. Then,  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  is a solution of BSDE (2.1) if and only if  $(\bar{Y}, \bar{Z}) := (AY, AZ) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  is a solution of the following BSDE*

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{g}(s, \bar{Y}_s, \bar{Z}_s) ds - \int_t^T \bar{Z}_s dB_s, \quad t \in [0, T], \quad (2.8)$$

where  $\bar{\xi} := A\xi$  and

$$\bar{g}(\omega, t, \bar{y}, \bar{z}) := Ag(\omega, t, A^{-1}\bar{y}, A^{-1}\bar{z}), \quad \forall (\omega, t, \bar{y}, \bar{z}) \in \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}.$$

Since the condition (AB) is invariant under an invertible linear transformation of  $\mathbb{R}^n$ , as shown in Remark 2.12 of Xing and Žitković [42], we immediately have from Corollary 2.7 and Proposition 2.17 the following assertion.

**Theorem 2.18.** *Let  $\alpha, \bar{\alpha}, \tilde{\alpha} \in \mathcal{L}^\infty$  and the generator  $g$  satisfy (AB). If there is a real invertible matrix  $A \in \mathbb{R}^{n \times n}$  such that the generator  $\bar{g}$  defined in Proposition 2.17 satisfies (B1) and (B2) with  $\theta = 0$ , then for each  $\xi \in L^\infty(\mathbb{R}^n)$ , the system of BSDEs (2.1) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$ .*

From Proposition 2.17, we easily see the following slight extension of Theorem 2.1 of Frei and Dos Reis [21].

**Theorem 2.19.** *Let  $\theta_1 = \vartheta_1 = 0$  and  $\theta_2 \vartheta_2 > 0$ . Then, there is a terminal value  $\xi \in L^\infty(\mathbb{R}^2)$  such that the system of BSDEs (2.4) has no global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^2) \times \text{BMO}(\mathbb{R}^{2 \times d})$ .*

*Proof.* We use a contradiction argument. Suppose that for any  $\xi \in L^\infty(\mathbb{R}^2)$ , the system of BSDEs (2.4) admits a global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^2) \times \text{BMO}(\mathbb{R}^{2 \times d})$ . Define

$$(\bar{Y}, \bar{Z}) := \begin{pmatrix} \sqrt{2\theta_2 \vartheta_2} & 0 \\ 0 & 2\vartheta_2 \end{pmatrix} (Y, Z).$$

Then, we have from Proposition 2.17 that  $(\bar{Y}, \bar{Z}) \in \mathcal{S}^\infty(\mathbb{R}^2) \times \text{BMO}(\mathbb{R}^{2 \times d})$  solves the following system of BSDEs

$$\begin{cases} \bar{Y}_t^1 = \sqrt{2\theta_2\vartheta_2}\xi^1 - \int_t^T \bar{Z}_s^1 dB_s, \\ \bar{Y}_t^2 = 2\vartheta_2\xi^2 + \int_t^T \left( |\bar{Z}_s^1|^2 + \frac{1}{2}|\bar{Z}_s^2|^2 \right) ds - \int_t^T \bar{Z}_s^2 dB_s, \quad t \in [0, T], \end{cases}$$

which is a contradiction to Theorem 2.1 of Frei and Dos Reis [21] as  $\xi$  is arbitrary. The proof is then complete.  $\square$

The following theorem is a nonlinear extension of Theorem 6.9 of Jackson [25] with a different proof.

**Theorem 2.20.** *Define the generator  $g$  as follows:  $\forall(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,*

$$g(\omega, t, y, z) := f(\omega, t, y, z) + zh(b^\top z),$$

where  $b := (b_1, \dots, b_n)^\top \in \mathbb{R}^n$  with  $b_1 \neq 0$ , the vector function  $h : \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}^d$  is Lipschitz continuous, and  $f : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n$  has the following stronger continuity than assumption (B2):  $d\mathbb{P} \times dt - a.e.$ , for each  $(y, \bar{y}, z, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{n \times d}$ ,

$$|f(\omega, t, y, z) - f(\omega, t, \bar{y}, \bar{z})| \leq \phi(|y| \vee |\bar{y}|) \left( v_t(\omega) + |z|^\delta + |\bar{z}|^\delta \right) (|y - \bar{y}| + |z - \bar{z}|), \quad (2.9)$$

and the following growth:  $d\mathbb{P} \times dt - a.e.$ ,

$$|b^\top f(\omega, t, y, z)| \leq \tilde{\alpha}_t(\omega) + \beta|y| + \frac{\gamma}{2}|b^\top z|^2, \quad \forall(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d} \quad (2.10)$$

and

$$|f^i(\omega, t, y, z)| \leq \tilde{\alpha}_t(\omega) + \beta|y| + \frac{\gamma}{2}|z^i|^2, \quad i = 2, \dots, n. \quad (2.11)$$

Then, for each  $\xi \in L^\infty(\mathbb{R}^n)$ , BSDE (2.1) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$ .

*Proof.* The following matrix

$$A := \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \quad (2.12)$$

is invertible. For each  $(\bar{y}, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , let  $(y, z) := A^{-1}(\bar{y}, \bar{z})$ , and then  $(\bar{y}, \bar{z}) = A(y, z)$ . Clearly,  $\bar{y}^1 = b^\top y$ ,  $\bar{z}^1 = b^\top z$ ,  $\bar{y}^i = y^i$  and  $\bar{z}^i = z^i$  for each  $i = 2, \dots, n$ . By (2.9)-(2.13) together

with the Lipschitz continuity of  $h$ , the following generator

$$\bar{g}(\omega, t, \bar{y}, \bar{z}) := Ag(\omega, t, A^{-1}\bar{y}, A^{-1}\bar{z}) = Af(\omega, t, A^{-1}\bar{y}, A^{-1}\bar{z}) + \begin{pmatrix} \bar{z}^1 h(\bar{z}^1) \\ \bar{z}^2 h(\bar{z}^1) \\ \vdots \\ \bar{z}^n h(\bar{z}^1) \end{pmatrix} \quad (2.13)$$

satisfies Assumptions (B2) with  $\theta = 0$ , and (C1b)(ii). From Theorem 2.14 we see that for each  $\xi \in L^\infty$ , the system of BSDEs (2.8) admits a unique global solution  $(\bar{Y}, \bar{Z}) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$ . Finally, by Proposition 2.17 we know that  $(A^{-1}\bar{Y}, A^{-1}\bar{Z}) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  is just the desired unique solution of (2.1). The proof is complete.  $\square$

**Remark 2.21.** *If the generator  $f$  is Lipschitz continuous in the last two variables  $(y, z)$  and bounded in the last variable  $z$ , then (2.9)-(2.11) hold naturally. On the other hand, if  $b_1 = 0$  but  $b_{i_0} \neq 0$  for some  $i_0 > 1$ , Theorem 2.20 is still true when (2.11) is satisfied for all integer  $i \neq i_0$  instead of for all  $i \geq 2$ , just using an obvious invertible transformation in the proof.*

Similar to the proof of Theorem 2.20, we have the following Theorem 2.22.

**Theorem 2.22.** *Consider the following generator  $g: \forall(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,*

$$g(\omega, t, y, z) := f(\omega, t, y, z) + zh(b^\top z) + a\bar{h}(b^\top z),$$

where  $a := (a_1, \dots, a_n)^\top \in \mathbb{R}^n$  and  $b := (b_1, \dots, b_n)^\top \in \mathbb{R}^n$  with  $a_1 \neq 0$  and  $b^\top a \neq 0$ , the vector function  $h: \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}^d$  is Lipschitz continuous, the function  $\bar{h}: \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$  satisfies that  $\bar{h}(0) = 0$  and

$$|\bar{h}(w_1) - \bar{h}(w_2)| \leq L(1 + |w_1| + |w_2|)|w_1 - w_2|, \quad \forall w_1, w_2 \in \mathbb{R}^{1 \times d}, \quad (2.14)$$

and  $f: \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n$  has the continuity (2.9) and the following growth:  $d\mathbb{P} \times dt - a.e.$ ,

$$|f(\omega, t, y, z)| \leq \tilde{\alpha}_t(\omega) + \beta|y|, \quad \forall(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}. \quad (2.15)$$

Then, for each  $\xi \in L^\infty(\mathbb{R}^n)$ , BSDE (2.1) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$ .

It is quite related to Theorem 6.19 of Jackson [25], which requires the a priori boundedness condition (AB) and the Malliavin regular condition on the generator  $g$ . However, both do not cover each other.

*Proof.* The following matrix

$$A := \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ -a_2 & a_1 & & & \\ -a_3 & & a_1 & & \\ \vdots & & & \ddots & \\ -a_n & & & & a_1 \end{pmatrix} \quad (2.16)$$

has the determinant  $\det(A) = a_1^{n-2} b^\top a \neq 0$ , and is thus invertible. For each  $(\bar{y}, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , let  $(y, z) := A^{-1}(\bar{y}, \bar{z})$ , and then  $(\bar{y}, \bar{z}) = A(y, z)$ . Clearly,  $\bar{y}^1 = b^\top y$ ,  $\bar{z}^1 = b^\top z$ ,  $\bar{y}^i = -a_i y^1 + a_1 y^i$  and  $\bar{z}^i = -a_i z^1 + a_1 z^i$  for each  $i = 2, \dots, n$ . From (2.9) and (2.14)-(2.17) together with the Lipschitz continuity of  $h$ , we see that the following generator

$$\bar{g}(\omega, t, \bar{y}, \bar{z}) := Af(\omega, t, A^{-1}\bar{y}, A^{-1}\bar{z}) + \begin{pmatrix} \bar{z}^1 h(\bar{z}^1) + b^\top a \bar{h}(\bar{z}^1) \\ \bar{z}^2 h(\bar{z}^1) \\ \vdots \\ \bar{z}^n h(\bar{z}^1) \end{pmatrix} \quad (2.17)$$

satisfies Assumptions (B2) with  $\theta = 0$ , and (C1b)(ii). Proceeding identically as in Theorem 2.20, we have the desired assertion.  $\square$

Furthermore, the following Theorem 2.23 and Theorem 3.1 of Xing and Žitković [42] do not cover each other.

**Theorem 2.23.** *Define the generator  $g := (g^1, \dots, g^n)^\top$  as follows: for each  $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,*

$$\begin{cases} g^1(\omega, t, y, z) := f^1(\omega, t, y, z) + z^1 h(b^\top z) - \bar{h}_1(b^\top z) - \frac{1}{b_1} \sum_{j=2}^n a_j b_j |z^j|^2; \\ g^i(\omega, t, y, z) := f^i(\omega, t, y, z) + z^i h(b^\top z) - \bar{h}_i(b^\top z) + a_i |z^i|^2, \quad i = 2, \dots, n, \end{cases} \quad (2.18)$$

where  $a := (0, a_2, \dots, a_n)^\top \in \mathbb{R}^n$  and  $b := (b_1, \dots, b_n)^\top \in \mathbb{R}^n$  with  $b_1 \neq 0$ , the vector function  $h : \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}^d$  is Lipschitz continuous with Lipschitz constant  $L > 0$  and  $|h(0)| \leq L$ , the vector function  $\bar{h} = (\bar{h}_1, \dots, \bar{h}_n)^\top : \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}^n$  satisfies that  $\bar{h}(0) = 0$  and

$$|\bar{h}(w_1) - \bar{h}(w_2)| \leq L(1 + |w_1| + |w_2|)|w_1 - w_2|, \quad \forall w_1, w_2 \in \mathbb{R}^{1 \times d}, \quad (2.19)$$

and  $f = (f^1, \dots, f^n)^\top : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n$  satisfies that  $d\mathbb{P} \times dt - a.e.$ ,  $|f(\omega, t, 0, 0)| \leq \tilde{\alpha}_t(\omega)$  and for each  $(y_1, y_2, z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{n \times d}$ ,

$$|f(\omega, t, y_1, z_1) - f(\omega, t, y_2, z_2)| \leq \beta|y_1 - y_2| + \gamma|z_1 - z_2|. \quad (2.20)$$

Assume further that

$$\text{either } \inf_{w \in \mathbb{R}^{1 \times d}} \left[ wh(w) - b^\top \bar{h}(w) - \frac{\bar{\gamma}}{2} |w|^2 \right] \geq -c \quad \text{or} \quad \sup_{w \in \mathbb{R}^{1 \times d}} \left[ wh(w) - b^\top \bar{h}(w) + \frac{\bar{\gamma}}{2} |w|^2 \right] \leq c, \quad (2.21)$$

and for each  $i = 2, \dots, n$ ,

$$\text{either } a_i > 0 \text{ and } \inf_{w \in \mathbb{R}^{1 \times d}} \left[ \bar{h}_i(w) - \frac{\bar{\gamma}}{2} |w|^2 \right] \geq -c \quad \text{or} \quad a_i < 0 \text{ and } \sup_{w \in \mathbb{R}^{1 \times d}} \left[ \bar{h}_i(w) + \frac{\bar{\gamma}}{2} |w|^2 \right] \leq c. \quad (2.22)$$

Then, for each  $\xi \in L^\infty(\mathbb{R}^n)$ , BSDE (2.1) with the generator  $g$  defined in (2.18) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$ .

*Proof.* We only give the proof when the first conditions of (2.21) and (2.22) are satisfied. The other cases are proved identically.

Let the invertible matrix  $A$  be defined in (2.12). For each  $(\bar{y}, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , let  $(y, z) := A^{-1}(\bar{y}, \bar{z})$ , and then  $(\bar{y}, \bar{z}) = A(y, z)$ . i.e.,  $\bar{y}^1 = b^\top y$ ,  $\bar{z}^1 = b^\top z$ ,  $\bar{y}^i = y^i$  and  $\bar{z}^i = z^i$  for each  $i = 2, \dots, n$ . From (2.19), (2.20), (2.23) and the Lipschitz continuity of  $h$ , the following generator

$$\bar{g}(\omega, t, \bar{y}, \bar{z}) := Af(\omega, t, A^{-1}\bar{y}, A^{-1}\bar{z}) + \begin{pmatrix} \bar{z}^1 h(\bar{z}^1) - b^\top \bar{h}(\bar{z}^1) \\ \bar{z}^2 h(\bar{z}^1) - \bar{h}_i(\bar{z}^1) + a_i |\bar{z}^2|^2 \\ \vdots \\ \bar{z}^n h(\bar{z}^1) - \bar{h}_n(\bar{z}^1) + a_n |\bar{z}^n|^2 \end{pmatrix} \quad (2.23)$$

satisfies Assumption (B2) with  $\theta = 0$ . Furthermore, we have from (2.20) that

$$|Af(\omega, t, A^{-1}\bar{y}, A^{-1}\bar{z})| \leq |A|\alpha_t(\omega) + \beta|\bar{y}| + \gamma|\bar{z}|. \quad (2.24)$$

From (2.19), (2.21), (2.23) and (2.24) together with the Lipschitz continuity of  $h$ , we see that  $\bar{g}^1$  satisfies assumption (C1a)(i) with  $\theta = 0$  since

$$\bar{g}^1(\omega, t, \bar{y}, \bar{z}) \geq \frac{\bar{\gamma}}{2} |\bar{z}^1|^2 - c - |A|\alpha_t(\omega) - \beta|\bar{y}| - \gamma|\bar{z}|$$

and

$$\bar{g}^1(\omega, t, \bar{y}, \bar{z}) \leq |A|\alpha_t(\omega) + \beta|\bar{y}| + \gamma|\bar{z}| + L|\bar{z}^1|(1 + |\bar{z}^1|) + L|b|(1 + |\bar{z}^1|)|\bar{z}^1|.$$

From (2.19), (2.22), (2.23) and (2.24) together with the Lipschitz continuity of  $h$  and Young's inequality, we see that for each  $i = 2, \dots, n$ ,  $\bar{g}^i$  satisfies (C1a)(i) with  $\theta = 0$  since

$$\begin{aligned} \bar{g}^i(\omega, t, \bar{y}, \bar{z}) &\geq a_i |\bar{z}^i|^2 - L|\bar{z}^i|(1 + |\bar{z}^1|) - L(1 + |\bar{z}^1|)|\bar{z}^1| - |A|\alpha_t(\omega) - \beta|\bar{y}| - \gamma|\bar{z}| \\ &\geq \frac{a_i}{2} |\bar{z}^i|^2 - \left( \frac{L^2}{2a_i} + L \right) |\bar{z}^1|^2 - L|\bar{z}^i| - L|\bar{z}^1| - |A|\alpha_t(\omega) - \beta|\bar{y}| - \gamma|\bar{z}| \end{aligned}$$

and

$$\begin{aligned}\bar{g}^i(\omega, t, \bar{y}, \bar{z}) &\leq a_i |\bar{z}^i|^2 + L |\bar{z}^i| (1 + |\bar{z}^1|) - \frac{\bar{\gamma}}{2} |\bar{z}^1|^2 + c + |A| \alpha_t(\omega) + \beta |\bar{y}| + \gamma |\bar{z}| \\ &\leq \left( a_i + \frac{L^2}{2\bar{\gamma}} \right) |\bar{z}^i|^2 + L |\bar{z}^i| + c + |A| \alpha_t(\omega) + \beta |\bar{y}| + \gamma |\bar{z}|.\end{aligned}$$

Consequently, we have from Corollary 2.11 that for each  $\xi \in L^\infty$ , the system of BSDEs (2.8) with the generator  $\bar{g}$  defined in (2.23) admits a unique global solution  $(\bar{Y}, \bar{Z}) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$ . Finally, by Proposition 2.17 we know that  $(A^{-1}\bar{Y}, A^{-1}\bar{Z}) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  is just the desired unique solution of the system of BSDEs (2.1). The proof is complete.  $\square$

The following proposition and its corollary partially answer the problem 6.25 of Jackson [25].

**Proposition 2.24.** *Assume that  $n = 2$  and  $d = 1$ . Consider the following generator  $g$*

$$g(z) := \begin{pmatrix} g^1(z) \\ g^2(z) \end{pmatrix} = \begin{pmatrix} z^\top A_1 z + z^\top k_1 + l_1 \\ z^\top A_2 z + z^\top k_2 + l_2 \end{pmatrix}, \quad z \in \mathbb{R}^2,$$

where  $A_i \in \mathbb{R}^{2 \times 2}$ ,  $k_i \in \mathbb{R}^2$  and  $l_i \in \mathbb{R}$  for  $i = 1, 2$ . Assume further that there exist three constants  $a, b, \iota \in \mathbb{R}$  with  $a \neq 0$  such that

$$aA_1 + bA_2 = \iota \begin{pmatrix} a \\ b \end{pmatrix} (a, b) = \begin{pmatrix} a^2\iota & ab\iota \\ ab\iota & b^2\iota \end{pmatrix} \quad (2.25)$$

and either  $\alpha_{11} = 0$  and  $\alpha_{22} \neq 0$  or  $\alpha_{11}\alpha_{22} < 0$  with

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix} := \frac{1}{a^2} \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix} A_2 \begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$$

Then for each  $\xi \in L^\infty(\mathbb{R}^2)$ , BSDE (2.1) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^2) \times \text{BMO}(\mathbb{R}^{2 \times 1})$ .

*Proof.* For simplicity, we only prove the case of  $k_1 = k_2 = 0$  and  $l_1 = l_2 = 0$ . Let

$$A := \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

For each  $\bar{z} \in \mathbb{R}^2$ , let

$$z := A^{-1}\bar{z} = \frac{1}{a} \begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix} \bar{z}.$$

Then, in view of (2.25), we have

$$\bar{g}^1(\bar{z}) := ag^1(z) + bg^2(z) = \frac{1}{a^2} \bar{z}^\top \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix} (aA_1 + bA_2) \begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix} \bar{z} = \iota (z^1)^2$$



and

$$\bar{g}^2(\bar{z}) = g^2(z) = z^\top A_2 z = \frac{1}{a^2} \bar{z}^\top \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix} A_2 \begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix} \bar{z}.$$

According to the conditions of Proposition 2.24 combined with an argument similar to (2.6), we know that  $\bar{g}$  satisfies assumptions (B2) and (C1b) with  $\theta = 0$ . Proceeding identically as in Theorem 2.20, we have the desired assertion.  $\square$

The following result is a direct consequence of Proposition 2.24

**Corollary 2.25.** *Assume that  $n = 2$  and  $d = 1$ . Consider the following generator  $g$*

$$g(z) := \begin{pmatrix} g^1(z) \\ g^2(z) \end{pmatrix} = \begin{pmatrix} z^1(z^1 + z^2) - \frac{\alpha}{2}(z^1)^2 \\ z^2(z^1 + z^2) - \frac{\beta}{2}(z^2)^2 \end{pmatrix}, \quad z \in \mathbb{R}^2,$$

where  $\alpha, \beta \in \mathbb{R}^*$  are two constants. If  $1/\alpha + 1/\beta = 1$ , then for each  $\xi \in L^\infty(\mathbb{R}^2)$ , BSDE (2.1) admits a unique global solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^2) \times \text{BMO}(\mathbb{R}^{2 \times 1})$ .

*Proof.* In Proposition 2.24, let  $k_1 = k_2 = 0$ ,  $l_1 = l_2 = 0$ ,

$$A_1 := \begin{pmatrix} 1 - \frac{\alpha}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad A_2 := \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 - \frac{\beta}{2} \end{pmatrix}.$$

There are two cases:  $\alpha = 2$  and  $\alpha \neq 2$ . In the case of  $\alpha = 2$ , we have  $A_1 = A_2$ . When  $a = 1$ ,  $b = -1$  and  $\iota = 0$ , we have (2.25),  $\alpha_{11} = 0$  and  $\alpha_{22} = 1$ , and thus all assumptions of Proposition 2.24 are satisfied. In the other case of  $\alpha \neq 2$ , we take  $a = 1 - \alpha/2 \neq 0$ ,  $b = 1 - \beta/2$  and  $\iota = 1$ . Since  $\alpha + \beta = \alpha\beta$ , we have

$$\frac{a+b}{2} = 1 - \frac{\alpha+\beta}{4} = 1 - \frac{\alpha+\beta}{2} + \frac{\alpha\beta}{4} = \left(1 - \frac{\alpha}{2}\right) \left(1 - \frac{\beta}{2}\right) = ab.$$

It can also be directly verified that all of (2.25),  $\alpha_{11} = 0$  and  $\alpha_{22} \neq 0$  hold, and then all assumptions of Proposition 2.24 are satisfied. Thus, the desired assertion follows immediately from Proposition 2.24.  $\square$

## 2.6. Global unbounded solution of multi-dimensional quadratic BSDEs

In this subsection, we will present three existence and uniqueness results on the global unbounded solution of multi-dimensional BSDE (2.1). For this, let us first introduce the following assumption (D2), which is strictly stronger than the previous assumption (B2). In particular, the generator  $g$  satisfying (B2) can have a general growth in the state variable  $y$ , while the generator  $g$  satisfying (D2) can only have a linear growth in  $y$ .

(D2) For  $i = 1, \dots, n$ ,  $g^i$  satisfies that  $d\mathbb{P} \times dt$ -a.e., for each  $(y, \bar{y}, z, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{n \times d}$ ,

$$|g^i(\omega, t, y, z) - g^i(\omega, t, \bar{y}, \bar{z})| \leq \gamma(v_t(\omega) + |z| + |\bar{z}|) \left( |y - \bar{y}| + \sum_{j=1}^i |z^j - \bar{z}^j| \right) \\ + \left[ \gamma \left( v_t(\omega) + |z|^\delta + |\bar{z}|^\delta \right) + \theta(v_t(\omega) + |z| + |\bar{z}|) \right] \sum_{j=i+1}^n |z^j - \bar{z}^j|.$$

The following is the first existence and uniqueness result of this subsection.

**Theorem 2.26.** *Let the generator  $g$  satisfy Assumptions (B1), (D2) and (AB) with  $\theta = 0$  and  $\phi(\cdot) \equiv 0$ , and let*

$$\xi = \bar{\xi} + \int_0^T H_s dB_s$$

with  $\bar{\xi} \in L^\infty(\mathbb{R}^n)$  and  $H \in \text{BMO}(\mathbb{R}^{n \times d})$ . When  $g^i$  satisfies (ii) in assumption (B1) and violates (B1) (i) and (iii), and  $-g^i$  violates (B1) (i), we further assume that  $H^i \equiv 0$ . If  $\alpha, |H|^2 \in \mathcal{L}^\infty$ , then BSDE (2.1) admits a unique global solution  $(Y, Z)$  on the time interval  $[0, T]$  such that

$$\left( Y - \int_0^\cdot H_s dB_s, Z \right) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d}).$$

The second existence and uniqueness result requires the following assumption (D1), which is strictly stronger than the previous assumption (C1a), for  $g$  admits a general growth in the state variable  $y$  in (C1a), instead of only a linear growth in  $y$  in (D1).

(D1) For each fixed  $i = 1, \dots, n$ , either of the following three conditions holds:

(i)  $f := g^i$  or  $f := -g^i$  satisfies that  $d\mathbb{P} \times dt$ -a.e., for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,

$$\frac{\bar{\gamma}}{2}|z^i|^2 - \bar{\alpha}_t(\omega) - \beta|y| - \sum_{j=i+1}^n (\bar{\lambda}|z^j|^{1+\delta} + \theta|z^j|^2) - \bar{c} \sum_{j=1}^{i-1} |z^j|^2 \leq f(\omega, t, y, z) \\ \leq \alpha_t(\omega) + \beta|y| + \sum_{j \neq i} (\lambda|z^j| + \theta|z^j|^2) + \frac{\gamma}{2}|z^i|^2;$$

(ii) Almost everywhere in  $\Omega \times [0, T]$ , for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , we have

$$|g^i(\omega, t, y, z)| \leq \bar{\alpha}_t(\omega) + \beta|y^i| + |z^i| \left( v_t(\omega) + c \sum_{j=1}^{i-1} |z^j| \right) + \frac{\gamma}{2}|z^i|^2;$$

(iii) Almost everywhere in  $\Omega \times [0, T]$ , for any  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , we have

$$|g^i(\omega, t, y, z)| \leq \bar{\alpha}_t(\omega) + \beta|y^i| + \bar{\lambda}|z| + \theta \sum_{j \neq i} |z^j|^2.$$

The following Theorem 2.27 and Jamneshan et al. [27, Theorem 2.6] do not cover each other, and so are Theorem 2.27 and Frei [20, Proposition 2.1].

**Theorem 2.27.** *Let the generator  $g$  satisfy assumptions (D1) and (D2) with  $\beta = 0$ , and let*

$$\xi = \bar{\xi} + \int_0^T H_s dB_s$$

*with  $\bar{\xi} \in L^\infty(\mathbb{R}^n)$  and  $H \in \text{BMO}(\mathbb{R}^{n \times d})$ . When  $g^i$  satisfies (ii) in assumption (D1) and violates (D1) (iii) and (i), and  $-g^i$  violates (D1) (i), we further assume that  $H^i \equiv 0$ . If  $\alpha \in \mathcal{E}^\infty(p\gamma)$  for some  $p > 1$ , and  $|H|^2 \in \mathcal{E}^\infty(2\bar{p}(q\gamma)^2)$  for some  $\bar{p} > 1$  with  $q = p/(p-1)$  such that  $1/p + 1/q = 1$ , then there exists a positive constant  $\theta_0 > 0$  depending only on  $(\|\bar{\xi}\|_\infty, \|\alpha\|_{\mathcal{E}^\infty(p\gamma)}, \|\bar{\alpha}\|_{\mathcal{M}^\infty}, \|\bar{\alpha}\|_{\mathcal{L}^\infty}, \|v\|_{\text{BMO}}, \| |H|^2 \|_{\mathcal{E}^\infty(2\bar{p}(q\gamma)^2)}, n, \gamma, \bar{\gamma}, \lambda, \bar{\lambda}, c, \delta, T, p)$  such that when  $\theta \in [0, \theta_0]$ , BSDE (2.1) admits a unique global solution  $(Y, Z)$  on  $[0, T]$  such that*

$$\left( Y - \int_0^\cdot H_s dB_s, Z \right) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d}).$$

**Remark 2.28.** *By the John-Nirenberg inequality, if  $\|H\|_{\text{BMO}} \leq 1/(\sqrt{2\bar{p}q\gamma})$ , then  $|H|^2 \in \mathcal{E}^\infty(2\bar{p}(q\gamma)^2)$ . And, the existence and uniqueness of the solution for the multi-dimensional BSDE appearing in Theorem 3.1 of Kramkov and Pulido [32] can be derived from Theorem 2.27.*

In view of (i) of Remark 2.1, the following corollary follows immediately from Theorem 2.27.

**Corollary 2.29.** *Let the generator  $g$  satisfy (D1) and (D2) with  $\beta = 0$  and  $\theta = 0$ , and let*

$$\xi = \bar{\xi} + \int_0^T H_s dB_s$$

*with  $\bar{\xi} \in L^\infty(\mathbb{R}^n)$  and  $H \in \text{BMO}(\mathbb{R}^{n \times d})$ . When  $g^i$  satisfies (ii) in assumption (D1) and violates (D1) (iii) and (i), and  $-g^i$  violates (D1) (i), we further assume that  $H^i \equiv 0$ . If  $\alpha, |H|^2 \in \mathcal{L}^\infty$ , then BSDE (2.1) admits a unique global solution  $(Y, Z)$  on the time interval  $[0, T]$  such that*

$$\left( Y - \int_0^\cdot H_s dB_s, Z \right) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d}).$$

**Remark 2.30.** *It can be verified that if the constant  $c$  appearing in (ii) of assumption (D1) vanishes, then the condition of  $H_i \equiv 0$  appearing in Theorem 2.27 and Corollary 2.29 can be weakened to the condition that both  $|H^i|$  and  $|H^i|v$  belongs to  $\mathcal{L}^\infty$ .*

The following Theorem 2.31 is the last existence and uniqueness result of this subsection. It can be identically proved as Theorem 2.27 via Theorem 2.14, and the proof is omitted here.

**Theorem 2.31.** *Let the generator  $g$  satisfy assumptions (B1) and (D2) with  $\beta = \theta = c = \bar{c} = 0$  and  $\phi(\cdot) \equiv 0$ , and*

$$\xi = \bar{\xi} + \int_0^T H_s dB_s$$

with  $\bar{\xi} \in L^\infty(\mathbb{R}^n)$  and  $H \in \text{BMO}(\mathbb{R}^{n \times d})$ . When  $g^i$  satisfies (ii) in assumption (B1) and violates (B1) (iii) and (i), and  $-g^i$  violates (B1) (i), we further assume that  $|H^i|^2, |H^i|v \in \mathcal{L}^\infty$ . If  $\alpha \in \mathcal{E}^\infty(p\gamma)$  for some  $p > 1$ , and  $|H|^2 \in \mathcal{E}^\infty(2\bar{p}(q\gamma)^2)$  for some  $\bar{p} > 1$  with  $q = p/(p-1)$  such that  $1/p + 1/q = 1$ , then BSDE (2.1) admits a unique global solution  $(Y, Z)$  such that

$$\left(Y - \int_0^\cdot H_s dB_s, Z\right) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d}).$$

### 3. Local unbounded solution: proof of Theorem 2.3

Let us begin with the following technical lemma which will be used later.

**Lemma 3.1.** *Let  $r > 0$ ,  $\bar{\delta} \in [0, 1)$ ,  $i \in \{1, \dots, n\}$ ,  $\bar{t} \in [0, T]$  and  $V, \bar{V} \in \text{BMO}_{[\bar{t}, T]}(\mathbb{R}^{n \times d})$  such that*

$$2n\theta \left( \sum_{j=1}^{i-1} \|\bar{V}^j\|_{\text{BMO}_{[\bar{t}, T]}}^2 + \|V\|_{\text{BMO}_{[\bar{t}, T]}}^2 \right) \leq 1.$$

(i) Define

$$\check{\alpha}_s := \hat{\alpha}_s + \sum_{j=1}^{i-1} \left( \lambda |\bar{V}_s^j|^{1+\bar{\delta}} + \theta |\bar{V}_s^j|^2 \right) + \sum_{j=i+1}^n \left( \lambda |V_s^j|^{1+\bar{\delta}} + \theta |V_s^j|^2 \right), \quad s \in [\bar{t}, T]. \quad (3.1)$$

If  $\hat{\alpha} \in \mathcal{E}_{[\bar{t}, T]}^\infty(pr)$  for some  $p > 1$ , then  $\check{\alpha} \in \mathcal{E}_{[\bar{t}, T]}^\infty(r)$ . And, for each  $t \in [\bar{t}, T]$ , we have

$$\begin{aligned} \|\check{\alpha}\|_{\mathcal{E}_{[\bar{t}, T]}^\infty(r)} &\leq \|\hat{\alpha}\|_{\mathcal{E}_{[\bar{t}, T]}^\infty(pr)} + \frac{\ln 2}{r} + C_{p,n,r,\lambda,\bar{\delta}} \sum_{j=1}^{i-1} \|\bar{V}^j\|_{\text{BMO}_{[\bar{t}, T]}}^{2\frac{1+\bar{\delta}}{1-\bar{\delta}}}(T-t) \\ &\quad + C_{p,n,r,\lambda,\bar{\delta}} \sum_{j=i+1}^n \|V\|_{\text{BMO}_{[\bar{t}, T]}}^{2\frac{1+\bar{\delta}}{1-\bar{\delta}}}(T-t), \end{aligned} \quad (3.2)$$

where

$$C_{p,n,r,\lambda,\bar{\delta}} := \lambda \frac{1-\bar{\delta}}{2} \left( \frac{pnr\lambda(1+\bar{\delta})}{p-1} \right)^{\frac{1+\bar{\delta}}{1-\bar{\delta}}}. \quad (3.3)$$

Moreover, it is clear that the previous conclusion holds also for  $p = 1$  when  $\lambda = 0$  and  $\theta = 0$  if we let  $C_{p,n,r,\lambda,\bar{\delta}} := 0$  when  $\lambda = 0$  and  $p = 1$ .

(ii) Define

$$\check{\alpha}_s := \hat{\alpha}_s + c \sum_{j=1}^{i-1} |\bar{V}_s^j|^2 + \sum_{j=i+1}^n \left( \lambda |V_s^j|^{1+\bar{\delta}} + \theta |V_s^j|^2 \right), \quad s \in [\bar{t}, T]. \quad (3.4)$$

If  $\hat{\alpha} \in \mathcal{M}_{[\bar{t}, T]}^\infty$ , then  $\check{\alpha} \in \mathcal{M}_{[\bar{t}, T]}^\infty$ . And, for each  $t \in [\bar{t}, T]$ ,

$$\|\check{\alpha}\|_{\mathcal{M}_{[\bar{t}, T]}^\infty} \leq 1 + \|\hat{\alpha}\|_{\mathcal{M}_{[\bar{t}, T]}^\infty} + c \sum_{j=1}^{i-1} \|\bar{V}^j\|_{\text{BMO}_{[\bar{t}, T]}}^2 + n\bar{\lambda} \|V\|_{\text{BMO}_{[\bar{t}, T]}}^{1+\bar{\delta}}(T-t)^{\frac{1-\bar{\delta}}{2}}. \quad (3.5)$$

(iii) Define

$$\check{\alpha}_s := \hat{\alpha}_s + \sum_{j=1}^{i-1} (\bar{\lambda} |\bar{V}_s^j| + \theta |\bar{V}_s^j|^2) + \sum_{j=i+1}^n (\bar{\lambda} |V_s^j| + \theta |V_s^j|^2), \quad s \in [\bar{t}, T]. \quad (3.6)$$

If  $\hat{\alpha} \in \mathcal{M}_{[\bar{t}, T]}^\infty$ , then  $\check{\alpha} \in \mathcal{M}_{[\bar{t}, T]}^\infty$ . And, for each  $t \in [\bar{t}, T]$ , we have

$$\|\check{\alpha}\|_{\mathcal{M}_{[\bar{t}, T]}^\infty} \leq 1 + \|\hat{\alpha}\|_{\mathcal{M}_{[\bar{t}, T]}^\infty} + \bar{\lambda} \left( \sum_{j=1}^{i-1} \|\bar{V}^j\|_{\text{BMO}_{[t, T]}} + \sum_{j=i+1}^n \|V^j\|_{\text{BMO}_{[t, T]}} \right) (T - t)^{\frac{1}{2}}. \quad (3.7)$$

*Proof.* (i) Since  $V, \bar{V} \in \text{BMO}(\mathbb{R}^{n \times d})$ , it follows from Young's inequality that for  $p > 1$ ,  $\lambda \geq 0$ ,  $0 \leq \bar{t} \leq t \leq s \leq T$  and  $j = 1, \dots, n$ , we have

$$\begin{aligned} \lambda |V_s^j|^{1+\bar{\delta}} &= \lambda \left( \left( \frac{pnr\lambda(1+\bar{\delta})}{p-1} \right)^{\frac{1+\bar{\delta}}{1-\bar{\delta}}} \|V^j\|_{\text{BMO}_{[t, T]}}^{2\frac{1+\bar{\delta}}{1-\bar{\delta}}} \right)^{\frac{1-\bar{\delta}}{2}} \\ &\quad \times \left( \frac{(p-1)|V_s^j|^2}{pnr\lambda(1+\bar{\delta})\|V^j\|_{\text{BMO}_{[t, T]}}^2} \right)^{\frac{1+\bar{\delta}}{2}} \\ &\leq \frac{p-1}{2pnr\|V^j\|_{\text{BMO}_{[t, T]}}^2} |V_s^j|^2 + C_{p,n,r,\lambda,\bar{\delta}} \|V^j\|_{\text{BMO}_{[t, T]}}^{2\frac{1+\bar{\delta}}{1-\bar{\delta}}} \end{aligned} \quad (3.8)$$

and

$$\lambda |\bar{V}_s^j|^{1+\bar{\delta}} \leq \frac{p-1}{2pnr\|\bar{V}^j\|_{\text{BMO}_{[t, T]}}^2} |\bar{V}_s^j|^2 + C_{p,n,r,\lambda,\bar{\delta}} \|\bar{V}^j\|_{\text{BMO}_{[t, T]}}^{2\frac{1+\bar{\delta}}{1-\bar{\delta}}}, \quad (3.9)$$

where the constant  $C_{p,n,r,\lambda,\bar{\delta}}$  is defined in (3.3). For  $p > 1$ , let  $q := \frac{p}{p-1} > 1$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . By (3.8) and (3.9) we know that for each  $t \in [\bar{t}, T]$ ,  $\check{\alpha}_s \leq \hat{\alpha}_s + \check{\alpha}_s^{1,t} + \check{\alpha}_s^{2,t}$ ,  $s \in [t, T]$ , where

$$\check{\alpha}_s^{1,t} := \sum_{j=1}^{i-1} \frac{1}{2qnr\|\bar{V}^j\|_{\text{BMO}_{[t, T]}}^2} |\bar{V}_s^j|^2 + \sum_{j=i+1}^n \frac{1}{2qnr\|V^j\|_{\text{BMO}_{[t, T]}}^2} |V_s^j|^2 + \theta \left( \sum_{j=1}^{i-1} |\bar{V}_s^j|^2 + |V_s|^2 \right)$$

and

$$\check{\alpha}_s^{2,t} := C_{p,n,r,\lambda,\bar{\delta}} \left( \sum_{j=1}^{i-1} \|\bar{V}^j\|_{\text{BMO}_{[t, T]}}^{2\frac{1+\bar{\delta}}{1-\bar{\delta}}} + \sum_{j=i+1}^n \|V^j\|_{\text{BMO}_{[t, T]}}^{2\frac{1+\bar{\delta}}{1-\bar{\delta}}} \right).$$

It follows from Hölder's inequality and the John-Nirenberg inequality that for each  $t \in [\bar{t}, T]$  and  $\tau \in \mathcal{T}_{[t, T]}$ ,

$$\begin{aligned} \mathbb{E}_\tau \left[ \exp \left( qr \int_\tau^T \check{\alpha}_s^{1,t} ds \right) \right] &\leq \prod_{j=1}^{i-1} \left( \mathbb{E}_\tau \left[ \exp \left( \frac{1}{2\|\bar{V}^j\|_{\text{BMO}_{[t, T]}}^2} \int_\tau^T |\bar{V}_s^j|^2 ds \right) \right] \right)^{\frac{1}{n}} \\ &\quad \times \left( \mathbb{E}_\tau \left[ \exp \left( n\theta \int_\tau^T \left( \sum_{j=1}^{i-1} |\bar{V}_s^j|^2 + |V_s|^2 \right) ds \right) \right] \right)^{\frac{1}{n}} \\ &\quad \times \prod_{j=i+1}^n \left( \mathbb{E}_\tau \left[ \exp \left( \frac{1}{2\|V^j\|_{\text{BMO}_{[t, T]}}^2} \int_\tau^T |V_s^j|^2 ds \right) \right] \right)^{\frac{1}{n}} \\ &\leq 2, \end{aligned}$$

which means that  $\check{\alpha}^{1,t} \in \mathcal{E}_{[t,T]}^\infty(qr)$  and  $\|\check{\alpha}^{1,t}\|_{\mathcal{E}_{[t,T]}^\infty(qr)} \leq \frac{\ln 2}{qr}$ . On the other hand, it is clear that

$$\|\check{\alpha}^{2,t}\|_{\mathcal{L}_{[t,T]}^\infty} = C_{p,n,r,\lambda,\delta} \left( \sum_{j=1}^{i-1} \|\bar{V}^j\|_{\text{BMO}_{[t,T]}}^{2\frac{1+\delta}{1-\delta}} + \sum_{j=i+1}^n \|V^j\|_{\text{BMO}_{[t,T]}}^{2\frac{1+\delta}{1-\delta}} \right) (T-t), \quad t \in [\bar{t}, T].$$

It then follows from (iii) of Remark 2.1 that  $\check{\alpha} \in \mathcal{E}^\infty(r)$  and for each  $t \in [\bar{t}, T]$ ,

$$\begin{aligned} \|\check{\alpha}\|_{\mathcal{E}_{[t,T]}^\infty(r)} &\leq \|\hat{\alpha}\|_{\mathcal{E}_{[t,T]}^\infty(pr)} + \frac{\ln 2}{qr} \\ &\quad + C_{p,n,r,\lambda,\delta} \left( \sum_{j=1}^{i-1} \|\bar{V}^j\|_{\text{BMO}_{[t,T]}}^{2\frac{1+\delta}{1-\delta}} + \sum_{j=i+1}^n \|V^j\|_{\text{BMO}_{[t,T]}}^{2\frac{1+\delta}{1-\delta}} \right) (T-t), \end{aligned}$$

from which the desired conclusion (3.2) follows immediately.

(ii) It follows from Hölder's inequality that for each  $t \in [\bar{t}, T]$ ,  $j = 1, \dots, n$  and  $\tau \in \mathcal{T}_{[t,T]}$ ,

$$\mathbb{E}_\tau \left[ \int_\tau^T |V_s^j|^{1+\delta} ds \right] \leq \left( \mathbb{E}_\tau \left[ \int_\tau^T |V_s^j|^2 ds \right] \right)^{\frac{1+\delta}{2}} (T-t)^{\frac{1-\delta}{2}} \leq \|V^j\|_{\text{BMO}_{[t,T]}}^{1+\delta} (T-t)^{\frac{1-\delta}{2}}.$$

Then, the desired conclusion (3.5) follows from (3.4) and the previous inequality.

(iii) It follows from Hölder's inequality that for each  $t \in [\bar{t}, T]$ ,  $j = 1, \dots, n$  and  $\tau \in \mathcal{T}_{[t,T]}$ ,

$$\mathbb{E}_\tau \left[ \int_\tau^T |V_s^j| ds \right] \leq \|V^j\|_{\text{BMO}_{[t,T]}} (T-t)^{\frac{1}{2}} \quad \text{and} \quad \mathbb{E}_\tau \left[ \int_\tau^T |\bar{V}_s^j| ds \right] \leq \|\bar{V}^j\|_{\text{BMO}_{[t,T]}} (T-t)^{\frac{1}{2}}.$$

Then, the desired conclusion (3.7) follows from (3.6) and the previous inequality.  $\square$

Now, we can give the proof of Theorem 2.3. Assume first that  $\xi \in L^\infty(\mathbb{R}^n)$ ,  $\alpha \in \mathcal{E}^\infty(p\gamma)$  for some real  $p > 1$  and the generator  $g$  satisfies assumptions (B1) and (B2). Define

$$\begin{aligned} C_1 &:= \frac{2+\bar{\gamma}}{\bar{\gamma}} \left\{ 7 + \frac{\ln 2}{\bar{\gamma}} + 3\|\xi\|_\infty + \|\alpha\|_{\mathcal{E}^\infty(p\gamma)} + 2\|\bar{\alpha}\|_{\mathcal{M}^\infty} + C_{p,n,\gamma,\lambda,\delta}T + 2\bar{c} \right\} \\ &\quad + \frac{4(\gamma+1)}{\gamma^2} \exp\{2\gamma(\|\xi\|_\infty + \|\bar{\alpha}\|_{\mathcal{L}^\infty} + 1)\} (5 + \|\bar{\alpha}\|_{\mathcal{L}^\infty} + 2n\|v\|_{\text{BMO}}^2 + 2nc^2) \\ &\quad + 2c_0 \exp(2\bar{\lambda}^2T) (73 + \|\xi\|_\infty^2 + 4\|\bar{\alpha}\|_{\mathcal{M}^\infty}^2 + n\bar{\lambda}^2T) \end{aligned}$$

and

$$\begin{aligned} C_2 &:= \frac{2+\bar{\gamma}}{\bar{\gamma}} (2C_{p,n,\gamma,\lambda,\delta}T + 4\bar{c}) + \frac{8nc^2(\gamma+1)}{\gamma^2} \exp\{4\gamma(\|\xi\|_\infty + \|\bar{\alpha}\|_{\mathcal{L}^\infty} + 1)\} \\ &\quad + 2nc_0\bar{\lambda}^2T \exp(2\bar{\lambda}^2T), \end{aligned}$$

where the uniform constant  $c_0 > 0$  is defined in (i) of Proposition A.3 in Appendix, and

$$C_{p,n,\gamma,\lambda,\delta} := \lambda \frac{1-\delta}{2} \left( \frac{pn\gamma\lambda(1+\delta)}{p-1} \right)^{\frac{1+\delta}{1-\delta}} + 1. \quad (3.10)$$

Furthermore, let  $C_1^0 := 0$  and for  $i = 1, \dots, n$ , recursively define

$$C_1^i := C_1^{i-1} + C_1 + C_2 [C_1^{i-1}]^{\frac{1+\delta}{1-\delta}}. \quad (3.11)$$

Finally, define  $K := C_1^n$  and let the constant  $\varepsilon$  satisfy

$$0 < \varepsilon \leq \min \left\{ T, \frac{1}{[1 + \phi(K)]^2}, \frac{1}{nC_{p,n,\gamma,\lambda,\delta} K^{\frac{1+\delta}{1-\delta}}}, \left( \frac{1}{1 + 2n\bar{\lambda}K^{\frac{1+\delta}{2}}} \right)^{\frac{2}{1-\delta}}, \frac{1}{1 + n^2\bar{\lambda}^2 K} \right\}. \quad (3.12)$$

It is clear that all these constants defined above (except  $\varepsilon$ ) depend only on  $\|\xi\|_\infty$ ,  $\|\alpha\|_{\mathcal{E}^\infty(p\gamma)}$ ,  $\|\bar{\alpha}\|_{\mathcal{M}^\infty}$ ,  $\|\tilde{\alpha}\|_{\mathcal{L}^\infty}$ ,  $\|v\|_{\text{BMO}}$ ,  $n, \gamma, \bar{\gamma}, \lambda, \bar{\lambda}, c, \delta, T$  and  $p$ , and that  $\varepsilon$  also depends on  $\phi(\cdot)$ .

In the sequel, for each  $H \in \mathbb{R}^{n \times d}$ ,  $i \in \{1, \dots, n\}$  and  $z^{(1)} \dots, z^{(i)} \in \mathbb{R}^{1 \times d}$ , define by

$$[z^{(1)}, \dots, z^{(i)}, H^{i+1}, \dots, H^n]$$

the matrix in  $\mathbb{R}^{n \times d}$  whose  $j$ th row is  $z^{(j)}$  for  $j = 1, \dots, i$  and  $H^j$  for  $j = i+1, \dots, n$  with  $i \neq n$ .

Given a pair of processes  $(U, V) \in \mathcal{S}_{[T-\varepsilon, T]}^\infty(\mathbb{R}^n) \times \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}^{n \times d})$  and a real  $\theta$  satisfying

$$\|U\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty} + \|V\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \leq K \quad \text{and} \quad 0 \leq \theta \leq \frac{1}{4nK}. \quad (3.13)$$

We will first prove that for each  $i = 1, \dots, n$ , the following one-dimensional BSDE

$$Y_t^i = \xi^i + \int_t^T f^i(s, Z_s^i) ds - \int_t^T Z_s^i dB_s, \quad t \in [T - \varepsilon, T] \quad (3.14)$$

admits, successively, a unique solution  $(Y^i, Z^i)$  in the space  $\mathcal{S}_{[T-\varepsilon, T]}^\infty(\mathbb{R}) \times \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}^{1 \times d})$ , where for each  $(\omega, t, z) \in \Omega \times [T - \varepsilon, T] \times \mathbb{R}^{1 \times d}$ ,

$$\begin{cases} f^1(\omega, t, z) := g^1(\omega, t, U_t(\omega), [z, V_t^2(\omega), \dots, V_t^n(\omega)]), \\ f^i(\omega, t, z) := g^i(\omega, t, U_t(\omega), [Z_t^1(\omega), \dots, Z_t^{i-1}(\omega), z, V_t^{i+1}(\omega), \dots, V_t^n(\omega)]), \quad i = 2, \dots, n. \end{cases}$$

Moreover, it holds that for each  $i = 1, \dots, n$ ,

$$\|Y^i\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty} + \|Z^i\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \leq C_1 + C_2 \sum_{j=1}^{i-1} \|Z^j\|_{\text{BMO}_{[T-\varepsilon, T]}}^{2\frac{1+\delta}{1-\delta}}, \quad (3.15)$$

$$\sum_{j=1}^i \left( \|Y^j\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty} + \|Z^j\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \right) \leq C_1^i \leq K \quad (3.16)$$

and

$$2n\theta \left( \sum_{j=1}^i \|Z^j\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 + \|V\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \right) \leq 1. \quad (3.17)$$

In particular, letting  $i = n$  in (3.16) yields that

$$\|Y\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty} + \|Z\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \leq K. \quad (3.18)$$

For this, let us use an induction argument to prove the following proposition.

**Proposition 3.2.** *We have the following two assertions.*

(a) For  $i = 1$ , BSDE (3.14) admits a unique solution  $(Y^1, Z^1)$  in the space  $\mathcal{S}_{[T-\varepsilon, T]}^\infty(\mathbb{R}) \times \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}^{1 \times d})$ , and inequalities (3.15)-(3.17) hold.

(b) For  $i = 2, \dots, n$ , assume that for each  $l = 1, \dots, i-1$ , the following BSDE

$$Y_t^l = \xi^l + \int_t^T f^l(s, Z_s^l) ds - \int_t^T Z_s^l dB_s, \quad t \in [T-\varepsilon, T] \quad (3.19)$$

admits, successively, a unique solution  $(Y^l, Z^l)$  in the space  $\mathcal{S}_{[T-\varepsilon, T]}^\infty(\mathbb{R}) \times \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}^{1 \times d})$ , and the following inequalities hold:

$$\|Y^l\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty} + \|Z^l\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \leq C_1 + C_2 \sum_{j=1}^{l-1} \|Z^j\|_{\text{BMO}_{[T-\varepsilon, T]}}^{2\frac{1+\delta}{1-\delta}}, \quad (3.20)$$

$$\sum_{j=1}^l \left( \|Y^j\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty} + \|Z^j\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \right) \leq C_1^l \leq K \quad (3.21)$$

and

$$2n\theta \left( \sum_{j=1}^l \|Z^j\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 + \|V\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \right) \leq 1. \quad (3.22)$$

Then, BSDE (3.14) also admits a unique solution  $(Y^i, Z^i) \in \mathcal{S}_{[T-\varepsilon, T]}^\infty(\mathbb{R}) \times \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}^{1 \times d})$ , and the inequalities (3.15)-(3.17) also hold.

*Proof.* We first prove the assertion (b). Assume that for some  $i \in \{2, \dots, n\}$  and each  $l = 1, \dots, i-1$ , BSDE (3.19) admits a unique solution  $(Y^l, Z^l) \in \mathcal{S}_{[T-\varepsilon, T]}^\infty(\mathbb{R}) \times \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}^{1 \times d})$ , and the inequalities (3.20)-(3.22) hold. Since the generator  $g$  satisfies assumption (B2), it follows from the definition of  $f^i$  that  $d\mathbb{P} \times dt - a.e.$  on  $\Omega \times [T-\varepsilon, T]$ , for each  $(z, \bar{z}) \in \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d}$ ,

$$|f^i(t, z) - f^i(t, \bar{z})| \leq \phi(\|U\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty}) \left( v_t + 2 \sum_{j=1}^{i-1} |Z_t^j| + 2|V_t| + |z| + |\bar{z}| \right) |z - \bar{z}|,$$

which means that on  $\Omega \times [T-\varepsilon, T]$ ,  $f^i$  satisfies assumption (A5) in Appendix with  $\bar{\beta} = 0$ ,

$$k = \phi(\|U\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty}) \quad \text{and} \quad \bar{v} = v + 2 \sum_{j=1}^{i-1} |Z^j| + 2|V| \in \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}).$$

And, since  $g$  satisfies assumption (B1), in view of (ii) of Remark 2.2, we need only to consider the following three cases:

(1)  $g^i$  satisfies (i) of assumption (B1). For this case, it follows from the definition of  $f^i$  that  $d\mathbb{P} \times dt - a.e.$  on  $\Omega \times [T-\varepsilon, T]$ , for each  $z \in \mathbb{R}^{1 \times d}$ , we have

$$\frac{\bar{\gamma}}{2}|z|^2 - \check{\alpha}_t \leq f^i(t, z) \leq \check{\alpha}_t + \frac{\gamma}{2}|z|^2, \quad (3.23)$$

where

$$\check{\alpha}_t := \alpha_t + \phi(|U_t|) + \sum_{j=1}^{i-1} \left( \lambda |Z_t^j|^{1+\delta} + \theta |Z_t^j|^2 \right) + \sum_{j=i+1}^n \left( \lambda |V_t^j|^{1+\delta} + \theta |V_t^j|^2 \right)$$



and

$$\dot{\alpha}_t := \bar{\alpha}_t + \phi(|U_t|) + \bar{c} \sum_{j=1}^{i-1} |Z_t^j|^2 + \sum_{j=i+1}^n \left( \bar{\lambda} |V_t^j|^{1+\delta} + \theta |V_t^j|^2 \right).$$

Since  $\alpha \in \mathcal{E}_{[T-\varepsilon, T]}^\infty(p\gamma)$ ,  $\bar{\alpha} \in \mathcal{M}_{[T-\varepsilon, T]}^\infty$  and  $\phi(|U|) \in \mathcal{L}_{[T-\varepsilon, T]}^\infty$ , in view of (ii) of Remark 2.1, (3.13) and (3.22) with  $l = i - 1$ , using (i) and (ii) of Lemma 3.1 with  $r = \gamma$ ,  $\bar{\delta} = \delta$ ,  $\bar{t} = T - \varepsilon$ ,  $\bar{V} = Z$ ,  $\check{\alpha} = \bar{\alpha}$  and  $\hat{\alpha} = \alpha + \phi(|U|)$  (resp.  $\check{\alpha} = \dot{\alpha}$  and  $\hat{\alpha} = \bar{\alpha} + \phi(|U|)$ ) we can deduce that

$$\begin{aligned} \|\check{\alpha}\|_{\mathcal{E}_{[t, T]}^\infty(\gamma)} &\leq \|\alpha\|_{\mathcal{E}_{[t, T]}^\infty(p\gamma)} + \phi(K)(T-t) + \frac{\ln 2}{\gamma} + nC_{p, n, \gamma, \lambda, \delta} K^{\frac{1+\delta}{1-\delta}}(T-t) \\ &\quad + C_{p, n, \gamma, \lambda, \delta} T \sum_{j=1}^{i-1} \|Z^j\|_{\text{BMO}_{[t, T]}}^{2\frac{1+\delta}{1-\delta}} < +\infty, \quad t \in [T-\varepsilon, T] \end{aligned} \quad (3.24)$$

and

$$\begin{aligned} \|\dot{\alpha}\|_{\mathcal{M}_{[t, T]}^\infty} &\leq 1 + \|\bar{\alpha}\|_{\mathcal{M}_{[t, T]}^\infty} + \phi(K)(T-t) + n\bar{\lambda} K^{\frac{1+\delta}{2}}(T-t)^{\frac{1-\delta}{2}} \\ &\quad + c \sum_{j=1}^{i-1} \|Z^j\|_{\text{BMO}_{[t, T]}}^2 < +\infty, \quad t \in [T-\varepsilon, T], \end{aligned} \quad (3.25)$$

where the constant  $C_{p, n, \gamma, \lambda, \delta}$  is defined in (3.10). Combining (3.23), (3.24) and (3.25) yields that on  $\Omega \times [T-\varepsilon, T]$ , the generator  $f^i$  satisfies assumption (A1) in Appendix with  $\bar{\beta} = 0$  and  $\varphi(\cdot) \equiv 0$ . Since  $f^i$  also satisfies assumption (A5), it follows from Proposition A.1 that BSDE (3.14) admits a unique solution  $(Y^i, Z^i) \in \mathcal{S}_{[T-\varepsilon, T]}^\infty(\mathbb{R}) \times \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}^{1 \times d})$ . Moreover, we have

$$\begin{aligned} \|Y^i\|_{\mathcal{S}_{[t, T]}^\infty} + \|Z^i\|_{\text{BMO}_{[t, T]}}^2 &\leq \frac{2 + \bar{\gamma}}{\bar{\gamma}} \left\{ 2 + \frac{\ln 2}{\gamma} + 3\|\xi^i\|_\infty + \|\alpha\|_{\mathcal{E}_{[t, T]}^\infty(p\gamma)} + 2\|\bar{\alpha}\|_{\mathcal{M}_{[t, T]}^\infty} \right. \\ &\quad \left. + 3\phi(K)(T-t) + (C_{p, n, \gamma, \lambda, \delta} T + 2\bar{c}) \sum_{j=1}^{i-1} \left( \|Z^j\|_{\text{BMO}_{[t, T]}}^{2\frac{1+\delta}{1-\delta}} + \|Z^j\|_{\text{BMO}_{[t, T]}}^2 \right) \right. \\ &\quad \left. + nC_{p, n, \gamma, \lambda, \delta} K^{\frac{1+\delta}{1-\delta}}(T-t) + 2n\bar{\lambda} K^{\frac{1+\delta}{2}}(T-t)^{\frac{1-\delta}{2}} \right\}, \quad t \in [T-\varepsilon, T]. \end{aligned} \quad (3.26)$$

Then, it follows from (3.26) and (3.12) together with the definitions of  $C_1$  and  $C_2$  that the desired inequality (3.15) holds. Moreover, in view of the following inequality

$$a^{\frac{1+\delta}{1-\delta}} + b^{\frac{1+\delta}{1-\delta}} \leq (a+b)^{\frac{1+\delta}{1-\delta}}, \quad \forall a, b \geq 0,$$

combining (3.15) and (3.21) with  $l = i - 1$  as well as (3.11) we can derive that inequality (3.16) also holds. Finally, the desired inequality (3.17) follows from (3.13) and (3.16). Consequently, the assertion (b) is proved in this case.

(2)  $g^i$  satisfies (ii) of assumption (B1). For this case, it follows from the definition of  $f^i$  that  $d\mathbb{P} \times dt - a.e.$  on  $\Omega \times [T-\varepsilon, T]$ , for each  $z \in \mathbb{R}^{1 \times d}$ , we have

$$|f^i(t, z)| \leq \check{\alpha}_t + \bar{u}_t |z| + \frac{\gamma}{2} |z|^2,$$

where

$$\ddot{\alpha} := \tilde{\alpha} + \phi(|U|) \in \mathcal{L}_{[T-\varepsilon, T]}^\infty \quad \text{and} \quad \bar{u} := c \sum_{j=1}^{i-1} |Z^j| + \phi(|U|) + v \in \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}).$$

This means that on  $\Omega \times [T - \varepsilon, T]$ , the generator  $f^i$  satisfies assumption (A3) in Appendix with  $\bar{\beta} = 0$  and  $\varphi(\cdot) \equiv 0$ . Since  $f^i$  also satisfies assumption (A5), it follows from Proposition A.2 in Appendix that BSDE (3.14) has a unique solution  $(Y^i, Z^i) \in \mathcal{S}_{[T-\varepsilon, T]}^\infty(\mathbb{R}) \times \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}^{1 \times d})$ . Moreover, in view of (3.13) and (3.12), we have for each  $t \in [T - \varepsilon, T]$ ,

$$\begin{aligned} \|Y^i\|_{\mathcal{S}_{[t, T]}^\infty} + \|Z^i\|_{\text{BMO}_{[t, T]}}^2 &\leq \frac{4(\gamma + 1)}{\gamma^2} \exp \left\{ 4\gamma \left( \|\xi^i\|_\infty + \|\tilde{\alpha}\|_{\mathcal{L}_{[t, T]}^\infty} + 1 \right) \right\} \\ &\times \left( 5 + \|\tilde{\alpha}\|_{\mathcal{L}_{[t, T]}^\infty} + 2n\|v\|_{\text{BMO}_{[t, T]}}^2 + 2nc^2 \sum_{j=1}^{i-1} \|Z^j\|_{\text{BMO}_{[t, T]}}^2 \right). \end{aligned} \quad (3.27)$$

Then, it follows from (3.27) together with the definitions of constants  $C_1$  and  $C_2$  that the desired inequality (3.15) holds. And, in the same way as in (1) we can deduce that (3.16) and (3.17) also hold. Thus, the assertion (b) is proved in this case.

(3)  $g^i$  satisfies (iii) of assumption (B1). For this case, it follows from the definition of  $f^i$  that  $d\mathbb{P} \times dt - a.e.$  on  $\Omega \times [T - \varepsilon, T]$ , for each  $z \in \mathbb{R}^{1 \times d}$ , we have

$$|f^i(t, z)| \leq \check{\alpha}_t + \bar{\lambda}|z|, \quad (3.28)$$

where

$$\check{\alpha}_t := \bar{\alpha}_t + \phi(|U_t|) + \sum_{j=1}^{i-1} \left( \bar{\lambda}|Z_t^j| + \theta|Z_t^j|^2 \right) + \sum_{j=i+1}^n \left( \bar{\lambda}|V_t^j| + \theta|V_t^j|^2 \right).$$

Since  $\bar{\alpha} \in \mathcal{M}_{[T-\varepsilon, T]}^\infty$  and  $\phi(|U|) \in \mathcal{L}_{[T-\varepsilon, T]}^\infty$ , in view of (3.13) and (3.22) with  $l = i - 1$ , using (iii) of Lemma 3.1 with  $\bar{t} = T - \varepsilon$ ,  $\bar{V} = Z$  and  $\hat{\alpha} = \bar{\alpha} + \phi(|U|)$  we can deduce that, in view of (3.12),

$$\begin{aligned} \|\check{\alpha}\|_{\mathcal{M}_{[t, T]}^\infty} &\leq 1 + \|\bar{\alpha}\|_{\mathcal{M}_{[t, T]}^\infty} + \phi(K)(T - t) + n\bar{\lambda}K^{\frac{1}{2}}(T - t)^{\frac{1}{2}} + \bar{\lambda}\sqrt{T} \sum_{j=1}^{i-1} \|Z^j\|_{\text{BMO}_{[t, T]}} \\ &\leq 3 + \|\bar{\alpha}\|_{\mathcal{M}_{[t, T]}^\infty} + \bar{\lambda}\sqrt{T} \sum_{j=1}^{i-1} \|Z^j\|_{\text{BMO}_{[t, T]}} < +\infty, \quad t \in [T - \varepsilon, T]. \end{aligned} \quad (3.29)$$

Combining (3.28) and (3.29) yields that on  $\Omega \times [T - \varepsilon, T]$ , the generator  $f^i$  satisfies assumption (A4) in Appendix with  $\dot{\alpha} = \check{\alpha}$ ,  $\bar{\beta} = 0$  and  $\varphi(\cdot) \equiv 0$ . Since  $f^i$  also satisfies assumption (A5), it follows from Proposition A.3 in Appendix that BSDE (3.14) admits a unique solution  $(Y^i, Z^i)$  in the space  $\mathcal{S}_{[T-\varepsilon, T]}^\infty(\mathbb{R}) \times \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}^{1 \times d})$ . Moreover, we have for each  $t \in [T - \varepsilon, T]$ ,

$$\begin{aligned} \|Y^i\|_{\mathcal{S}_{[t, T]}^\infty} + \|Z^i\|_{\text{BMO}_{[t, T]}}^2 &\leq 2c_0 \exp(2\bar{\lambda}^2 T) \left( 73 + \|\xi^i\|_\infty^2 + 4\|\bar{\alpha}\|_{\mathcal{M}_{[t, T]}^\infty}^2 \right) \\ &+ 2nc_0 \bar{\lambda}^2 T \exp(2\bar{\lambda}^2 T) \sum_{j=1}^{i-1} \|Z^j\|_{\text{BMO}_{[t, T]}}^2, \end{aligned} \quad (3.30)$$

where the uniform constant  $c_0 > 0$  is defined in (i) of Proposition A.3. Then, it follows from (3.30) together with the definitions of  $C_1$  and  $C_2$  that the desired inequality (3.15) holds. And, in the same way as in (1) we can deduce that (3.16) and (3.17) hold. Thus, the assertion (b) is also proved in this case.

Next, we prove the assertion (a). Indeed, in view of (3.13), by applying the above argument to  $i = 1$  we can deduce that for  $i = 1$ , BSDE (3.14) admits a unique solution  $(Y^1, Z^1)$  in the space  $\mathcal{S}_{[T-\varepsilon, T]}^\infty(\mathbb{R}) \times \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}^{1 \times d})$  and (3.15) holds. In addition, in the case of  $i = 1$ , (3.16) is just (3.15), and (3.17) is trivially satisfied by (3.13). Thus, the assertion (a) is also true, and the proof of Proposition 3.2 is then complete.  $\square$

Now, for each real  $\varepsilon > 0$  satisfying (3.12), define the following complete metric space

$$\mathcal{B}_\varepsilon := \left\{ (U, V) \in \mathcal{S}_{[T-\varepsilon, T]}^\infty(\mathbb{R}^n) \times \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}^{n \times d}) : \|U\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty} + \|V\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \leq K \right\},$$

which is a closed convex subset in the Banach space  $\mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  with the norm

$$\|(U, V)\|_{\mathcal{B}_\varepsilon} := \sqrt{\|U\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty}^2 + \|V\|_{\text{BMO}_{[T-\varepsilon, T]}}^2}, \quad \forall (U, V) \in \mathcal{B}_\varepsilon.$$

Based on Proposition 3.2, we know that those assertions from (3.14) to (3.18) are all true. Thus, in the case of  $0 \leq \theta \leq 1/4nK$  we can define the following map from  $\mathcal{B}_\varepsilon$  to itself:

$$\Gamma : (U, V) \in \mathcal{B}_\varepsilon \mapsto \Gamma(U, V) := (Y, Z) \in \mathcal{B}_\varepsilon,$$

where for each  $i = 1, \dots, n$ ,  $(Y^i, Z^i)$  (the  $i$ th component of  $Y$  and the  $i$ th row of  $Z$ ) is the unique solution of BSDE (3.14) in the space  $\mathcal{S}_{[T-\varepsilon, T]}^\infty(\mathbb{R}) \times \text{BMO}_{[T-\varepsilon, T]}(\mathbb{R}^{1 \times d})$ .

It remains to show that there exists a real  $\varepsilon_0 > 0$  satisfying (3.12) and a real  $\theta_0 \in (0, 1/4nK]$  (both depending only on  $\|\xi\|_\infty$ ,  $\|\alpha\|_{\mathcal{E}^\infty(p\gamma)}$ ,  $\|\bar{\alpha}\|_{\mathcal{M}^\infty}$ ,  $\|\tilde{\alpha}\|_{\mathcal{L}^\infty}$ ,  $\|v\|_{\text{BMO}}$ ,  $n, \gamma, \bar{\gamma}, \lambda, \bar{\lambda}, c, \delta, T, p$  and  $\phi(\cdot)$ ) such that in the case of  $\theta \in [0, \theta_0]$ ,  $\Gamma$  is a contraction in  $\mathcal{B}_{\varepsilon_0}$ .

In the sequel, let  $0 \leq \theta \leq 1/4nK$ . For any fixed  $\varepsilon$  satisfying (3.12) as well as  $(U, V) \in \mathcal{B}_\varepsilon$  and  $(\tilde{U}, \tilde{V}) \in \mathcal{B}_\varepsilon$ , we set

$$(Y, Z) := \Gamma(U, V), \quad (\tilde{Y}, \tilde{Z}) := \Gamma(\tilde{U}, \tilde{V}).$$

That is, for  $i = 1, \dots, n$  and  $t \in [T - \varepsilon, T]$ , we have

$$Y_t^i = \xi^i + \int_t^T g^i(s, U_s, V_s(Z_s, Z_s^i; i)) ds - \int_t^T Z_s^i dB_s$$

and

$$\tilde{Y}_t^i = \xi^i + \int_t^T g^i(s, \tilde{U}_s, \tilde{V}_s(\tilde{Z}_s, \tilde{Z}_s^i; i)) ds - \int_t^T \tilde{Z}_s^i dB_s.$$

Here and hereafter, for each  $i = 1, \dots, n$ ,  $w \in \mathbb{R}^{1 \times d}$  and  $H, \bar{H} \in \mathbb{R}^{n \times d}$ , we denote by  $H(\bar{H}, w; i)$  the matrix in  $\mathbb{R}^{n \times d}$  whose  $i$ th row is  $w$ , and whose  $j$ th row is  $\bar{H}^j$  for  $j = 1, \dots, i-1$  with  $i \neq 1$  and  $H^j$  for  $j = i+1, \dots, n$  with  $i \neq n$ . Then, for each  $i = 1, \dots, n$  and  $\tau \in \mathcal{T}_{[T-\varepsilon, T]}$ , we have

$$\begin{aligned} Y_\tau^i - \tilde{Y}_\tau^i + \int_\tau^T (Z_s^i - \tilde{Z}_s^i) dB_s \\ - \int_\tau^T \underbrace{\left( g^i(s, U_s, V_s(Z_s, Z_s^i; i)) - g^i(s, U_s, V_s(Z_s, \tilde{Z}_s^i; i)) \right)}_{:= \Delta_s^{1,i}} ds \\ = \int_\tau^T \underbrace{\left( g^i(s, U_s, V_s(Z_s, \tilde{Z}_s^i; i)) - g^i(s, \tilde{U}_s, \tilde{V}_s(\tilde{Z}_s, \tilde{Z}_s^i; i)) \right)}_{:= \Delta_s^{2,i}} ds. \end{aligned} \quad (3.31)$$

It follows from assumption (B2) that for each  $s \in [T - \varepsilon, T]$  and each  $i = 1, \dots, n$ , we have

$$|\Delta_s^{1,i}| \leq \phi(|U_s|) \left( v_s + 2|V_s| + 2|Z_s| + |\tilde{Z}_s| \right) |Z_s^i - \tilde{Z}_s^i| \quad (3.32)$$

and

$$|\Delta_s^{2,i}| \leq \phi(|U_s| \vee |\tilde{U}_s|) \left[ \tilde{v}_s |U_s - \tilde{U}_s| + \hat{v}_s \sum_{j=1}^{i-1} |Z_s^j - \tilde{Z}_s^j| + \sqrt{n}(\tilde{v}_s + \theta \hat{v}_s) |V_s - \tilde{V}_s| \right] \quad (3.33)$$

with

$$\tilde{v}_s := v_s + 3|V_s|^{1+\delta} + 2|\tilde{V}_s|^{1+\delta} + 3|Z_s|^{1+\delta} + 5|\tilde{Z}_s|^{1+\delta}, \quad (3.34)$$

$$\hat{v}_s := v_s + |V_s| + |\tilde{V}_s| + |Z_s| + 2|\tilde{Z}_s| \quad (3.35)$$

and

$$\check{v}_s := v_s + |V_s|^\delta + |\tilde{V}_s|^\delta + |Z_s|^\delta + 2|\tilde{Z}_s|^\delta. \quad (3.36)$$

For  $i = 1, \dots, n$ , define  $G_s(i) \equiv 0$ ,  $s \in [0, T - \varepsilon]$  and

$$G_s(i) := \frac{(Z_s^i - \tilde{Z}_s^i)^\top}{|Z_s^i - \tilde{Z}_s^i|^2} \mathbf{1}_{|Z_s^i - \tilde{Z}_s^i| \neq 0} \Delta_s^{1,i}, \quad s \in [T - \varepsilon, T].$$

By (3.32) we know that for each  $i = 1, \dots, n$  and  $s \in [T - \varepsilon, T]$ ,

$$\Delta_s^{1,i} = (Z_s^i - \tilde{Z}_s^i) G_s(i) \quad \text{and} \quad |G_s(i)| \leq \phi(|U_s|) \left( v_s + 2|V_s| + 2|Z_s| + |\tilde{Z}_s| \right). \quad (3.37)$$

Then for each  $i = 1, \dots, n$ ,  $\tilde{B}_t(i) := B_t - \int_0^t G_s(i) ds$  is a Brownian motion under the probability measure  $\mathbb{P}^i$  defined by

$$\frac{d\mathbb{P}^i}{d\mathbb{P}} := \exp \left\{ \int_0^T G_s(i) dB_s - \frac{1}{2} \int_0^T |G_s(i)|^2 ds \right\},$$

and from the definition of  $\mathcal{B}_\varepsilon$ , there exists a constant  $\bar{K} > 0$  such that

$$\|G(i)\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \leq \bar{K} := 4[\phi(K)]^2 (\|v\|_{\text{BMO}}^2 + 9K).$$

Furthermore, it follows from Theorem 3.3 in Kazamaki [29] that there exist constants  $0 < L_1 \leq 1$  and  $L_2 \geq 1$  depending only on  $\bar{K}$  such that for any  $M \in \text{BMO}(\mathbb{R}^{n \times d})$  or  $M \in \text{BMO}(\mathbb{R}^{1 \times d})$ , we have, for each  $i = 1, \dots, n$ ,

$$L_1 \|M\|_{\text{BMO}_{[T-\varepsilon, T]}} \leq \|M\|_{\text{BMO}_{[T-\varepsilon, T]}(\mathbb{P}^i)} \leq L_2 \|M\|_{\text{BMO}_{[T-\varepsilon, T]}}, \quad (3.38)$$

where

$$\|M\|_{\text{BMO}_{[T-\varepsilon, T]}(\mathbb{P}^i)} := \sup_{\tau \in \mathcal{T}_{[T-\varepsilon, T]}} \left\| \mathbb{E}_\tau^i \left[ \int_\tau^T |Z_s|^2 ds \right] \right\|_\infty^{\frac{1}{2}}$$

and  $\mathbb{E}_\tau^i$  denotes the conditional expectation with respect to  $\mathcal{F}_\tau$  under the measure  $\mathbb{P}^i$ .

It follows from (3.31) and (3.37) that

$$Y_\tau^i - \tilde{Y}_\tau^i + \int_\tau^T (Z_s^i - \tilde{Z}_s^i) d\tilde{B}_s(i) = \int_\tau^T \Delta_s^{2,i} ds, \quad i = 1, \dots, n, \quad \tau \in \mathcal{T}_{[T-\varepsilon, T]}.$$

Taking square and then the conditional mathematical expectation under  $\mathbb{P}^i$  on both sides of the last equation, in view of (3.33) and the definition of  $\mathcal{B}_\varepsilon$  together with Hölder's inequality we can deduce that for each  $i = 1, \dots, n$  and  $\tau \in \mathcal{T}_{[T-\varepsilon, T]}$ ,

$$\begin{aligned} |Y_\tau^i - \tilde{Y}_\tau^i|^2 + \mathbb{E}_\tau^i \left[ \int_\tau^T |Z_s^i - \tilde{Z}_s^i|^2 ds \right] &\leq 3[\phi(K)]^2 \mathbb{E}_\tau^i \left[ \left( \int_\tau^T \tilde{v}_s ds \right)^2 \right] \|U - \tilde{U}\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty}^2 \\ &+ 3[\phi(K)]^2 \left\{ \mathbb{E}_\tau^i \left[ \left( \int_\tau^T |\hat{v}_s|^2 ds \right)^2 \right] \right\}^{\frac{1}{2}} \sum_{j=1}^{i-1} \left\{ \mathbb{E}_\tau^i \left[ \left( \int_\tau^T |Z_s^j - \tilde{Z}_s^j|^2 ds \right)^2 \right] \right\}^{\frac{1}{2}} \\ &+ 3n[\phi(K)]^2 \left\{ \mathbb{E}_\tau^i \left[ \left( \int_\tau^T |\check{v}_s + \theta \hat{v}_s|^2 ds \right)^2 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E}_\tau^i \left[ \left( \int_\tau^T |V_s - \tilde{V}_s|^2 ds \right)^2 \right] \right\}^{\frac{1}{2}}. \end{aligned} \quad (3.39)$$

It follows from the energy inequality for BMO martingales (see for example Section 2.1 in Kazamaki [29]) together with (3.38) that for each  $i = 1, \dots, n$  and  $\tau \in \mathcal{T}_{[T-\varepsilon, T]}$ ,

$$\begin{aligned} \sum_{j=1}^{i-1} \left\{ \mathbb{E}_\tau^i \left[ \left( \int_\tau^T |Z_s^j - \tilde{Z}_s^j|^2 ds \right)^2 \right] \right\}^{\frac{1}{2}} &\leq 5 \sum_{j=1}^{i-1} \|Z^j - \tilde{Z}^j\|_{\text{BMO}_{[T-\varepsilon, T]}(\mathbb{P}^i)}^2 \\ &\leq 5L_2^2 \sum_{j=1}^{i-1} \|Z^j - \tilde{Z}^j\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \end{aligned}$$

and

$$\left\{ \mathbb{E}_\tau^i \left[ \left( \int_\tau^T |V_s - \tilde{V}_s|^2 ds \right)^2 \right] \right\}^{\frac{1}{2}} \leq 5 \|V - \tilde{V}\|_{\text{BMO}_{[T-\varepsilon, T]}(\mathbb{P}^i)}^2 \leq 5L_2^2 \|V - \tilde{V}\|_{\text{BMO}_{[T-\varepsilon, T]}}^2.$$

Furthermore, in view of (3.34)-(3.36), using Hölder's inequality and the energy inequality for BMO martingales together with (3.38) and the definition of  $\mathcal{B}_\varepsilon$  we can derive, see the argument in pages 1078-1079 of Hu and Tang [24] for details, that for each  $i = 1, \dots, n$  and  $\tau \in \mathcal{T}_{[T-\varepsilon, T]}$ ,

$$\mathbb{E}_\tau^i \left[ \left( \int_\tau^T \tilde{v}_s ds \right)^2 \right] \leq 5\varepsilon L_2^2 \|v\|_{\text{BMO}}^2 + 235\varepsilon^{1-\delta} (1 + 5L_2^4 K^2),$$

$$\left\{ \mathbb{E}_\tau^i \left[ \left( \int_\tau^T |\hat{v}_s|^2 ds \right)^2 \right] \right\}^{\frac{1}{2}} \leq 25L_2^2 (\|v\|_{\text{BMO}}^2 + 7K),$$

and

$$\begin{aligned} & \left\{ \mathbb{E}_\tau^i \left[ \left( \int_\tau^T |\check{v}_s + \theta \hat{v}_s|^2 ds \right)^2 \right] \right\}^{\frac{1}{2}} \\ & \leq 50\theta^2 L_2^2 (\|v\|_{\text{BMO}}^2 + 7K) + 5\sqrt{\varepsilon} L_2^2 \|v\|_{\text{BMO}}^2 + 5\varepsilon^{1-\delta} (4 + 35L_2^2 K). \end{aligned} \quad (3.40)$$

Combining those inequalities from (3.38) to (3.40) yields that for each  $i = 1, \dots, n$ ,

$$\begin{aligned} & \|Y^i - \tilde{Y}^i\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty}^2 + L_1^2 \|Z^i - \tilde{Z}^i\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \\ & \leq C_3 \varepsilon^{\frac{1-\delta}{2}} \left( \|U - \tilde{U}\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty}^2 + \|V - \tilde{V}\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \right) + 2n\theta^2 C_4 \|V - \tilde{V}\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \\ & \quad + C_4 \sum_{j=1}^{i-1} \|Z^j - \tilde{Z}^j\|_{\text{BMO}_{[T-\varepsilon, T]}}^2, \end{aligned}$$

and then,

$$\begin{aligned} & \|Y^i - \tilde{Y}^i\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty}^2 + \|Z^i - \tilde{Z}^i\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \\ & \leq \frac{C_3 \varepsilon^{\frac{1-\delta}{2}} + 2nC_4\theta^2}{L_1^2} \left( \|U - \tilde{U}\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty}^2 + \|V - \tilde{V}\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \right) \\ & \quad + \frac{C_4}{L_1^2} \sum_{j=1}^{i-1} \|Z^j - \tilde{Z}^j\|_{\text{BMO}_{[T-\varepsilon, T]}}^2, \end{aligned} \quad (3.41)$$

where

$$\begin{aligned} C_3 := & 15L_2^2 [\phi(K)]^2 \left\{ 5T^{\frac{1+\delta}{2}} L_2^2 \|v\|_{\text{BMO}}^2 + 235T^{\frac{1-\delta}{2}} (1 + 5L_2^4 K^2) \right. \\ & \left. + 5nT^{\frac{\delta}{2}} L_2^2 \|v\|_{\text{BMO}}^2 + 5nT^{\frac{1-\delta}{2}} (4 + 35L_2^2 K) \right\} \end{aligned}$$

and

$$C_4 := 375L_2^4 [\phi(K)]^2 \{ \|v\|_{\text{BMO}}^2 + 7K \}.$$

Next, in view of (3.41), by induction for  $i$  we can deduce that for each  $i = 1, \dots, n$ ,

$$\begin{aligned} & \sum_{j=1}^{i-1} \|Y^j - \tilde{Y}^j\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty}^2 + \sum_{j=1}^{i-1} \|Z^j - \tilde{Z}^j\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \\ & \leq \frac{C_3 \varepsilon^{\frac{1-\delta}{2}} + 2nC_4\theta^2}{L_1^2} C_5^i \left( \|U - \tilde{U}\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty}^2 + \|V - \tilde{V}\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \right), \end{aligned}$$

where

$$C_5^i := i \left[ 1 + \frac{C_4}{L_1^2} + \dots + \left( \frac{C_4}{L_1^2} \right)^{i-2} \right] + \left( \frac{C_4}{L_1^2} \right)^{i-1}.$$

In particular, letting  $i = n$  in the last equation yields that

$$\left\| (Y - \tilde{Y}, Z - \tilde{Z}) \right\|_{\mathcal{B}_\varepsilon}^2 \leq \frac{C_3 \varepsilon^{\frac{1-\delta}{2}} + 2nC_4\theta^2}{L_1^2} C_5^n \left\| (U - \tilde{U}, V - \tilde{V}) \right\|_{\mathcal{B}_\varepsilon}^2.$$

Consequently, there exist two very small positive numbers  $\varepsilon_0$  and  $\theta_0$  such that for each  $\theta \in [0, \theta_0]$ , the solution map  $\Gamma$  is a contraction on the previously given set  $\mathcal{B}_{\varepsilon_0}$ .

Finally, we remark that in view of (i) of Lemma 3.1, all above arguments remain valid for  $p = 1$  when  $\lambda = \theta = 0$ . The proof of Theorem 2.3 is then completed.

#### 4. Global bounded solution: proof of Theorems 2.10 and 2.14

To prove the existence of global bounded solution, we need to establish some uniform estimates of the solution.

##### 4.1. Proof of Theorem 2.10

In order to prove Theorem 2.10, we need to the following proposition.

**Proposition 4.1.** *Let  $\xi \in L^\infty(\mathbb{R}^n)$ ,  $\alpha \in \mathcal{E}^\infty(p\gamma \exp(\beta T))$  for some  $p > 1$  and the generator  $g$  satisfy assumption (C1a). Assume that for some  $h \in (0, T]$ , BSDE (2.1) has a solution  $(Y, Z) \in \mathcal{S}_{[T-h, T]}^\infty(\mathbb{R}^n) \times \text{BMO}_{[T-h, T]}(\mathbb{R}^{n \times d})$  on the time interval  $[T - h, T]$ . Then, there exists a  $\tilde{K} > 0$  depending only on  $(\|\xi\|_\infty, \|\alpha\|_{\mathcal{E}^\infty(p\gamma \exp(\beta T))}, \|\bar{\alpha}\|_{\mathcal{M}^\infty}, \|\tilde{\alpha}\|_{\mathcal{L}^\infty}, \|v\|_{\text{BMO}}, n, \beta, \gamma, \bar{\gamma}, \lambda, \bar{\lambda}, c, \delta, T, p)$  and being independent of  $h$  such that for  $\theta \in [0, 1/(4n\tilde{K})]$ , we have*

$$\|Y\|_{\mathcal{S}_{[T-h, T]}^\infty} + \|Z\|_{\text{BMO}_{[T-h, T]}}^2 \leq \tilde{K}.$$

Moreover, the above conclusion holds still for  $p = 1$  when  $\lambda = 0$  and  $\theta_0 = 0$ .

*Proof.* We only prove the case of  $p > 1$ . The other case can be proved in the same way.

First of all, define an increasing non-negative real-valued function  $\Phi : [0, +\infty) \rightarrow [0, +\infty)$  as follows:

$$\begin{aligned} \Phi(x) := & \frac{2(1 + \beta T) + \bar{\gamma}}{\bar{\gamma}} \exp(2\beta T) \left[ 2 + 2n\bar{\lambda}T^{\frac{1-\delta}{2}} + \frac{\ln 2}{\gamma} + \|\alpha\|_{\mathcal{E}^\infty(p\gamma \exp(\beta T))} + 2\|\bar{\alpha}\|_{\mathcal{M}^\infty} + 3x \right] \\ & + \frac{4(\gamma + 1)}{\gamma^2} \exp\{2\gamma \exp(\beta T) (\|\xi\|_\infty + \|\tilde{\alpha}\|_{\mathcal{L}^\infty})\} \left[ 2 + \beta T \exp(\beta T) (\|\xi\|_\infty + \|\tilde{\alpha}\|_{\mathcal{L}^\infty}) \right. \\ & \left. + \|\tilde{\alpha}\|_{\mathcal{L}^\infty} + n\|v\|_{\text{BMO}_{[t, T]}}^2 \right] + 2c_0 \exp(2\beta T + 2\bar{\lambda}^2 T) (7 + 6\|\bar{\alpha}\|_{\mathcal{M}^\infty}^2 + x^2), \end{aligned}$$

and define the following four constants:

$$\begin{aligned} \bar{C}_1 &:= \Phi(\|\xi\|_\infty), & \bar{C}_2 &:= \frac{6\beta(1 + \beta T) + 3\beta\bar{\gamma}}{\bar{\gamma}} \exp(2\beta T), \\ \bar{C}_3 &:= \frac{2(1 + \beta T) + \bar{\gamma}}{\bar{\gamma}} \exp(2\beta T) (C_{p, n, \beta, \gamma, \lambda, T} + 2n\bar{\lambda}) + 12nc_0\bar{\lambda}^2 T \exp(2\bar{\lambda}^2 T) \end{aligned}$$

and

$$\bar{C}_4 := \frac{4\bar{c}(1 + \beta T) + 2c\bar{\gamma}}{\bar{\gamma}} \exp(2\beta T) + \frac{4nc^2(\gamma + 1)}{\gamma^2} \exp\{2\gamma \exp(\beta T) (\|\xi\|_\infty + \|\tilde{\alpha}\|_{\mathcal{L}^\infty})\},$$

where the uniform constant  $c_0 > 0$  is defined in (i) of Proposition A.3 in Appendix, and

$$C_{p,n,\beta,\gamma,\lambda,T} := \frac{n\gamma(p\lambda)^2}{2(p-1)} \exp(\beta T). \quad (4.1)$$

Note that  $(Y, Z) \in \mathcal{S}_{[T-h,T]}^\infty \times \text{BMO}_{[T-h,T]}$  is a solution of BSDE (2.2) on  $[T-h, T]$ . Then for each  $i = 1, \dots, n$ ,  $(Y, Z)$  also solves the following BSDE:

$$Y_t^i = \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i) ds - \int_t^T Z_s^i dB_s, \quad t \in [T-h, T], \quad (4.2)$$

where for each  $(\omega, t, y, z) \in \Omega \times [T-h, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d}$ ,

$$f^i(t, y, z) := g^i(t, Y_t(y; i), Z_t(z; i)). \quad (4.3)$$

In the sequel, let the constant  $\theta$  always satisfy

$$4n\theta \|Z\|_{\text{BMO}_{[T-h,T]}}^2 \leq 1. \quad (4.4)$$

We will first prove that for each  $i = 1, \dots, n$  and each  $t \in [T-h, T]$ , if  $T-t \leq 1$ , then

$$\begin{aligned} & \|Y^i\|_{\mathcal{S}_{[T-t,T]}^\infty} + \|Z^i\|_{\text{BMO}_{[T-t,T]}}^2 \\ & \leq \bar{C}_1 + \bar{C}_2 \|Y\|_{\mathcal{S}_{[T-t,T]}^\infty} (T-t) + \bar{C}_3 \|Z\|_{\text{BMO}_{[T-t,T]}}^2 (T-t)^{\frac{1-\delta}{2}} + \bar{C}_4 \sum_{j=1}^{i-1} \|Z^j\|_{\text{BMO}_{[T-t,T]}}^2. \end{aligned} \quad (4.5)$$

Indeed, for any fixed  $i = 1, \dots, n$ , since  $g$  satisfies assumption (C1a), in view of (ii) of Remark 2.9, we need to consider the following three cases.

(1)  $g^i$  satisfies (i) of assumption (C1a). In this case, it follows from (4.3) that  $d\mathbb{P} \times dt - a.e.$  on  $\Omega \times [T-h, T]$ , for each  $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ , we have

$$f^i(t, y, z) \mathbf{1}_{y>0} \leq \check{\alpha}_t + \beta|y| + \frac{\gamma}{2}|z|^2 \quad \text{and} \quad f^i(t, y, z) \geq \frac{\bar{\gamma}}{2}|z|^2 - \check{\alpha}_t - \beta|y|, \quad (4.6)$$

where

$$\check{\alpha}_t := \alpha_t + \beta|Y_t| + \sum_{j \neq i} \left( \lambda |Z_t^j| + \theta |Z_t^j|^2 \right)$$

and

$$\hat{\alpha}_t := \bar{\alpha}_t + \beta|Y_t| + \bar{c} \sum_{j=1}^{i-1} |Z_t^j|^2 + \sum_{j=i+1}^n \left( \bar{\lambda} |Z_t^j|^{1+\delta} + \theta |Z_t^j|^2 \right).$$

Since  $\alpha \in \mathcal{E}_{[T-h,T]}^\infty(p\gamma \exp(\beta T))$ ,  $\bar{\alpha} \in \mathcal{M}_{[T-h,T]}^\infty$  and  $|Y| \in \mathcal{L}_{[T-h,T]}^\infty$ , in view of (ii) of Remark 2.1 as well as (4.4), applying (i) of Lemma 3.1 with

$$r = p\gamma \exp(\beta T), \quad \bar{\delta} = 0, \quad \bar{t} = T-h, \quad V = \bar{V} = Z, \quad \check{\alpha} = \check{\alpha} \quad \text{and} \quad \hat{\alpha} = \alpha + \beta|Y|$$



and (ii) of Lemma 3.1 with

$$\bar{\delta} = \delta, \bar{t} = T - h, V = \bar{V} = Z, \check{\alpha} = \dot{\alpha} \text{ and } \hat{\alpha} = \bar{\alpha} + \beta|Y|$$

we can deduce that

$$\begin{aligned} \|\check{\alpha}\|_{\mathcal{E}_{[t,T]}^\infty(\gamma \exp(\beta T))} &\leq \|\alpha\|_{\mathcal{E}_{[t,T]}^\infty(p\gamma \exp(\beta T))} + \beta\|Y\|_{\mathcal{S}_{[t,T]}^\infty}(T-t) + \frac{\ln 2}{\gamma \exp(\beta T)} \\ &+ C_{p,n,\beta,\gamma,\lambda,T}\|Z\|_{\text{BMO}_{[t,T]}^2}^2(T-t) < +\infty, \quad t \in [T-h, T] \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} \|\dot{\alpha}\|_{\mathcal{M}_{[t,T]}^\infty} &\leq 1 + \|\bar{\alpha}\|_{\mathcal{M}_{[t,T]}^\infty} + \beta\|Y\|_{\mathcal{S}_{[t,T]}^\infty}(T-t) + n\bar{\lambda}\|Z\|_{\text{BMO}_{[t,T]}^{1+\delta}}^{1+\delta}(T-t)^{\frac{1-\delta}{2}} \\ &+ \bar{c} \sum_{j=1}^{i-1} \|Z^j\|_{\text{BMO}_{[t,T]}^2}^2 < +\infty, \quad t \in [T-h, T], \end{aligned} \quad (4.8)$$

where the constant  $C_{p,n,\beta,\gamma,\lambda,T}$  is defined in (4.1). Combining (4.6), (4.7) and (4.8) yields that on  $\Omega \times [T-h, T]$ , the generator  $f^i$  satisfies the second inequality for the case of  $y > 0$  and the first inequality in assumption (A1) with  $\bar{\beta} = \beta$  and  $\varphi(\cdot) \equiv 0$ . It then follows from (i) of Proposition A.1 that for each  $t \in [T-h, T]$  such that  $T-t \leq 1$ , we have

$$\begin{aligned} \|Y^i\|_{\mathcal{S}_{[t,T]}^\infty} + \|Z^i\|_{\text{BMO}_{[t,T]}^2}^2 &\leq \frac{2(1+\beta T) + \bar{\gamma}}{\bar{\gamma}} \exp(2\beta T) \left\{ 2 + 2n\bar{\lambda}T^{\frac{1-\delta}{2}} + \frac{\ln 2}{\gamma} \right. \\ &+ 3\|\xi^i\|_\infty + \|\alpha\|_{\mathcal{E}^\infty(p\gamma \exp(\beta T))} + 2\|\bar{\alpha}\|_{\mathcal{M}^\infty} + 3\beta\|Y\|_{\mathcal{S}_{[t,T]}^\infty}(T-t) \\ &\left. + (C_{p,n,\beta,\gamma,\lambda,T} + 2n\bar{\lambda})\|Z\|_{\text{BMO}_{[t,T]}^2}^2(T-t)^{\frac{1-\delta}{2}} + 2\bar{c} \sum_{j=1}^{i-1} \|Z^j\|_{\text{BMO}_{[t,T]}^2}^2 \right\}. \end{aligned} \quad (4.9)$$

Then, it follows from (4.9) together with the definitions of constants  $\bar{C}_1, \bar{C}_2, \bar{C}_3$  and  $\bar{C}_4$  that the desired inequality (4.5) holds in this case.

(2)  $g^i$  satisfies (ii) of assumption (C1a). For this case, it follows from (4.3) that  $d\mathbb{P} \times dt - a.e.$  on  $\Omega \times [T-h, T]$ , for each  $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ , we have

$$f^i(t, y, z) \text{sgn}(y) \leq \tilde{\alpha}_t + \beta|y| + \bar{u}_t|z| + \frac{\gamma}{2}|z|^2,$$

where

$$\bar{u} := c \sum_{j=1}^{i-1} |Z^j| + v \in \text{BMO}_{[T-h, T]}(\mathbb{R}).$$

This means that on  $\Omega \times [T-h, T]$ , the generator  $f^i$  satisfies the first inequality for the case of  $y < 0$  and the second inequality for the case of  $y > 0$  in (A3) with  $\check{\alpha} = \tilde{\alpha}$  and  $\bar{\beta} = \beta$ . It then follows from (i) of Proposition A.2 that for each  $t \in [T-h, T]$ ,

$$\begin{aligned} \|Y^i\|_{\mathcal{S}_{[t,T]}^\infty} + \|Z^i\|_{\text{BMO}_{[t,T]}}^2 &\leq \frac{4(\gamma+1)}{\gamma^2} \exp\{2\gamma \exp(\beta T) (\|\xi^i\|_\infty + \|\tilde{\alpha}\|_{\mathcal{L}^\infty})\} (2 + \|\tilde{\alpha}\|_{\mathcal{L}^\infty} \\ &\quad + \beta T \exp(\beta T) (\|\xi^i\|_\infty + \|\tilde{\alpha}\|_{\mathcal{L}^\infty}) + n\|v\|_{\text{BMO}_{[t,T]}}^2 + nc^2 \sum_{j=1}^{i-1} \|Z^j\|_{\text{BMO}_{[t,T]}}^2). \end{aligned} \quad (4.10)$$

Then, it follows from (4.10) together with the definitions of constants  $\bar{C}_1, \bar{C}_2, \bar{C}_3$  and  $\bar{C}_4$  that the desired inequality (4.5) holds in this case.

(3)  $g^i$  satisfies (iii) of assumption (C1a). For this case, it follows from (4.3) that  $d\mathbb{P} \times dt - a.e.$  on  $\Omega \times [T-h, T]$ , for each  $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ , we have

$$f^i(t, y, z) \operatorname{sgn}(y) \leq \check{\alpha}_t + \beta|y| + \bar{\lambda}|z|, \quad (4.11)$$

where

$$\check{\alpha}_t := \bar{\alpha}_t + \sum_{j \neq i} \left( \bar{\lambda}|Z_t^j| + \theta|Z_t^j|^2 \right).$$

In view of (4.4), using (iii) of Lemma 3.1 with  $\bar{\delta} = \delta, \bar{t} = T-h, V = \bar{V} = Z$  and  $\hat{\alpha} = \bar{\alpha}$  we can deduce that

$$\|\check{\alpha}\|_{\mathcal{M}_{[t,T]}^\infty} \leq 1 + \|\bar{\alpha}\|_{\mathcal{M}_{[t,T]}^\infty} + \bar{\lambda}\sqrt{n}\|Z\|_{\text{BMO}_{[t,T]}}(T-t)^{\frac{1}{2}} < +\infty, \quad t \in [T-h, T]. \quad (4.12)$$

Combining (4.11) and (4.12) yields that on  $\Omega \times [T-h, T]$ , the generator  $f^i$  satisfies the first inequality for the case of  $y < 0$  and the second inequality for the case of  $y > 0$  in assumption (A4) with  $\check{\alpha} = \check{\alpha}$  and  $\check{\beta} = \beta$ . It then follows from (i) of Proposition A.3 that for each  $t \in [T-h, T]$ ,

$$\begin{aligned} \|Y^i\|_{\mathcal{S}_{[t,T]}^\infty} + \|Z^i\|_{\text{BMO}_{[t,T]}}^2 &\leq 2c_0 \exp(2\beta T + 2\bar{\lambda}^2 T) (7 + \|\xi^i\|_\infty^2 + 6\|\bar{\alpha}\|_{\mathcal{M}^\infty}^2) \\ &\quad + 12nc_0\bar{\lambda}^2 T \exp(2\bar{\lambda}^2 T) \|Z\|_{\text{BMO}_{[t,T]}}^2 (T-t). \end{aligned} \quad (4.13)$$

Then, it follows from (4.13) together with the definitions of constants  $\bar{C}_1, \bar{C}_2, \bar{C}_3$  and  $\bar{C}_4$  that the desired inequality (4.5) holds in this case.

Furthermore, in view of (4.5), by induction for  $i$  it is not difficult to derive that for each  $i = 2, \dots, n$  and each  $t \in [T-h, T]$ , if  $T-t \leq 1$ , then

$$\begin{aligned} &\sum_{j=1}^i \left( \|Y^j\|_{\mathcal{S}_{[t,T]}^\infty} + \|Z^j\|_{\text{BMO}_{[t,T]}}^2 \right) \\ &\leq \left( \bar{C}_1 + \bar{C}_2 \|Y\|_{\mathcal{S}_{[T-t,T]}^\infty} (T-t) + \bar{C}_3 \|Z\|_{\text{BMO}_{[T-t,T]}}^2 (T-t)^{\frac{1-\delta}{2}} \right) \bar{C}_5^i, \end{aligned} \quad (4.14)$$

where

$$\bar{C}_5^i := i \left( 1 + \bar{C}_4 + \dots + (\bar{C}_4)^{i-2} \right) + (\bar{C}_4)^{i-1}. \quad (4.15)$$

In particular, letting  $i = n$  in (4.14) yields that for each  $t \in [T-h, T]$ , if  $T-t \leq 1$ , then

$$\begin{aligned} \|Y\|_{\mathcal{S}_{[t,T]}^\infty} + \|Z\|_{\text{BMO}_{[t,T]}}^2 &\leq \bar{C}_1 \bar{C}_5^n + \bar{C}_2 \bar{C}_5^n \|Y\|_{\mathcal{S}_{[T-t,T]}^\infty} (T-t) \\ &\quad + \bar{C}_3 \bar{C}_5^n \|Z\|_{\text{BMO}_{[T-t,T]}}^2 (T-t)^{\frac{1-\delta}{2}}. \end{aligned}$$

Then, we have

$$\|Y\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty} + \|Z\|_{\text{BMO}_{[T-\varepsilon, T]}}^2 \leq 2\bar{C}_1\bar{C}_5^n, \quad (4.16)$$

where

$$\varepsilon := \min \left\{ h, 1, \frac{1}{2\bar{C}_2\bar{C}_5^n}, \left( \frac{1}{2\bar{C}_3\bar{C}_5^n} \right)^{\frac{2}{1-\delta}} \right\} > 0.$$

Finally, for  $m \geq 1$ , we define successively the following constants:

$$\bar{C}_1^1 := 2\bar{C}_1\bar{C}_5^n \quad \text{and} \quad \bar{C}_1^{m+1} := \bar{C}_1^m + 2\Phi(\bar{C}_1^m)\bar{C}_5^n, \quad (4.17)$$

where  $\bar{C}_5^n$  is defined in (4.15). And, we let  $m_0$  be the unique positive integer satisfying  $T - h \in [T - m_0\varepsilon, T - (m_0 - 1)\varepsilon)$ , or equivalently,

$$\frac{h}{\varepsilon} \leq m_0 < \frac{h}{\varepsilon} + 1. \quad (4.18)$$

If  $m_0 = 1$ , it then follows from (4.16) and (4.17) that

$$\|Y\|_{\mathcal{S}_{[T-h, T]}^\infty} + \|Z\|_{\text{BMO}_{[T-h, T]}}^2 \leq \bar{C}_1^1 = \bar{C}_1^{m_0}.$$

If  $m_0 = 2$ , it then follows from (4.16) and (4.17) that

$$\|Y_{T-\varepsilon}\|_\infty \leq \|Y\|_{\mathcal{S}_{[T-\varepsilon, T]}^\infty} \leq \bar{C}_1^1. \quad (4.19)$$

Now, consider the following system of BSDEs

$$Y_t = Y_{T-\varepsilon} + \int_t^{T-\varepsilon} g(s, Y_s, Z_s) ds - \int_t^{T-\varepsilon} Z_s dB_s, \quad t \in [T-h, T-\varepsilon].$$

In view of (4.19) and the definition of the function  $\Phi(\cdot)$ , by virtue of Propositions A.1 to A.3 we can use a similar argument as that obtaining (4.5) to get that for each  $i = 1, \dots, n$ ,

$$\begin{aligned} & \|Y^i\|_{\mathcal{S}_{[T-h, T-\varepsilon]}^\infty} + \|Z^i\|_{\text{BMO}_{[T-h, T-\varepsilon]}}^2 \\ & \leq \Phi(\bar{C}_1^1) + \bar{C}_2\|Y\|_{\mathcal{S}_{[T-h, T-\varepsilon]}^\infty} \varepsilon + \bar{C}_3\|Z\|_{\text{BMO}_{[T-h, T-\varepsilon]}}^2 \varepsilon^{\frac{1-\delta}{2}} + \bar{C}_4 \sum_{j=1}^{i-1} \|Z^j\|_{\text{BMO}_{[T-h, T-\varepsilon]}}^2. \end{aligned}$$

And, in view of the definition of  $\varepsilon$ , by induction and a similar argument as that from (4.14) to (4.16) we can further get that

$$\|Y\|_{\mathcal{S}_{[T-h, T-\varepsilon]}^\infty} + \|Z\|_{\text{BMO}_{[T-h, T-\varepsilon]}}^2 \leq 2\Phi(\bar{C}_1^1)\bar{C}_5^n.$$

Combining the last inequality and (4.16) yields that, in view of (4.17),

$$\|Y\|_{\mathcal{S}_{[T-h, T]}^\infty} + \|Z\|_{\text{BMO}_{[T-h, T]}}^2 \leq \bar{C}_1^1 + 2\Phi(\bar{C}_1^1)\bar{C}_5^n = \bar{C}_1^2 = \bar{C}_1^{m_0},$$

Proceeding the above computation gives that if  $m_0$  satisfies (4.18), then

$$\|Y\|_{\mathcal{S}_{[T-h, T]}^\infty} + \|Z\|_{\text{BMO}_{[T-h, T]}}^2 \leq \bar{C}_1^{m_0} = \bar{C}_1^{\lceil \frac{h}{\varepsilon} \rceil} =: \tilde{K}, \quad (4.20)$$

where  $\bar{C}_1^{m_0}$  is defined in (4.17), and  $\lceil x \rceil$  stands for the minimum of integers which is equal to or bigger than  $x \in \mathbb{R}$ . Then, the desired conclusion follows from (4.4) and (4.20). The proof is complete.  $\square$

*Proof of Theorem 2.10.* With Theorem 2.3 and Proposition 4.1 in hands, we can closely follow the proof of Theorem 4.1 in Cheridito and Nam [9] to prove our Theorem 2.10. All the details are omitted here.  $\square$

#### 4.2. Proof of Theorem 2.14

To prove Theorem 2.14, we need to the following proposition.

**Proposition 4.2.** *Let  $\xi \in L^\infty(\mathbb{R}^n)$ ,  $\alpha, \bar{\alpha}, \tilde{\alpha} \in \mathcal{L}^\infty$  and the generator  $g$  satisfy assumption (C1b). Assume that for some  $h \in (0, T]$ , BSDE (2.1) has a solution  $(Y, Z) \in \mathcal{S}_{[T-h, T]}^\infty(\mathbb{R}^n) \times \text{BMO}_{[T-h, T]}(\mathbb{R}^{n \times d})$  on the time interval  $[T-h, T]$ . Then, there exists a  $\tilde{K} > 0$  depending only on  $(\|\xi\|_\infty, \|\alpha\|_{\mathcal{L}^\infty}, \|\bar{\alpha}\|_{\mathcal{L}^\infty}, \|\tilde{\alpha}\|_{\mathcal{L}^\infty}, \|v\|_{\text{BMO}}, n, \beta, \gamma, \bar{\gamma}, \lambda, \bar{\lambda}, \delta, T)$  and being independent of  $h$  such that*

$$\|Y\|_{\mathcal{S}_{[T-h, T]}^\infty} \leq \tilde{K}.$$

*Proof.* Let  $\xi \in L^\infty(\mathbb{R}^n)$  and  $\alpha, \bar{\alpha}, \tilde{\alpha} \in \mathcal{L}^\infty$  such that

$$\|\xi\|_\infty \leq C_1 \quad \text{and} \quad \left\| \int_0^T (\alpha_t + \bar{\alpha}_t + \tilde{\alpha}_t) dt \right\|_\infty \leq C_2 \quad (4.21)$$

for two positive constants  $C_1$  and  $C_2$ . Denote respectively by  $n_1$ ,  $n_2$  and  $n_3$  the number of elements in  $J_1$ ,  $J_2$  and  $J_3$  such that  $n_1 + n_2 + n_3 = n$ . Since the generator  $g$  satisfies assumption (C1b), in view of (i) of Remark 2.13, we need to consider the following three cases.

(1) For  $i \in J_1$ ,  $g^i$  satisfies (i) of assumption (C1b). In this case, for each  $t \in [T-h, T]$ , we have, with  $\tilde{\lambda} := \lambda + \bar{\lambda}$  and  $\hat{\alpha}_t(\omega) := \alpha_t(\omega) + \bar{\alpha}_t(\omega)$ ,

$$g^i(\omega, t, Y_t(\omega), Z_t(\omega)) \operatorname{sgn}(Y_t^i(\omega)) \leq \hat{\alpha}_t(\omega) + \beta|Y_t(\omega)| + \tilde{\lambda} \sum_{j \in J_1} |Z_t^j(\omega)|^{1+\delta} + \frac{\gamma}{2} |Z_t^i(\omega)|^2$$

and

$$g^i(\omega, t, Y_t(\omega), Z_t(\omega)) \geq \frac{\bar{\gamma}}{2} |Z_t^i(\omega)|^2 - \hat{\alpha}_t(\omega) - \beta|Y_t(\omega)| - \tilde{\lambda} \sum_{j \in J_1} |Z_t^j(\omega)|^{1+\delta}.$$

Note that  $(Y^i, Z^i) \in \mathcal{S}_{[T-h, T]}^\infty(\mathbb{R}) \times \text{BMO}_{[T-h, T]}(\mathbb{R}^{1 \times d})$  is a solution of BSDE (2.2) on  $[T-h, T]$ . According to Lemma A.2 in Fan et al. [17] and in view of the last two inequalities together with (4.21), we can get that for each  $i \in J_1$  and  $t \in [T-h, T]$ ,

$$\begin{aligned} \exp(\gamma|Y_t^i|) &\leq \exp\left(\gamma(C_1 + C_2) + \beta\gamma \int_t^T \|Y\|_{\mathcal{S}_{[s, T]}^\infty} ds\right) \\ &\quad \times \mathbb{E}_t \left[ \exp\left(\gamma\tilde{\lambda} \sum_{j \in J_1} \int_t^T |Z_s^j|^{1+\delta} ds\right) \right] \end{aligned} \quad (4.22)$$

and

$$\begin{aligned}
& \mathbb{E}_t \left[ \exp \left( \frac{\bar{\gamma}}{2} \varepsilon_0 \int_t^T |Z_s^i|^2 ds \right) \right] \\
& \leq \mathbb{E}_t \left[ \exp \left( 6\varepsilon_0 \|Y^i\|_{\mathcal{S}_{[t,T]}^\infty} + 3\varepsilon_0 \left\| \int_0^T \hat{\alpha}_t ds \right\|_\infty \right. \right. \\
& \quad \left. \left. + 3\varepsilon_0 \beta \int_t^T |Y_s| ds + 3\varepsilon_0 \tilde{\lambda} \sum_{j \in J_1} \int_t^T |Z_s^j|^{1+\delta} ds \right) \right] \\
& \leq \exp \left( 6\varepsilon_0 \sum_{j \in J_1} \|Y^j\|_{\mathcal{S}_{[t,T]}^\infty} + 3\varepsilon_0 C_2 + 3\varepsilon_0 \beta \int_t^T \|Y\|_{\mathcal{S}_{[s,T]}^\infty} ds \right) \\
& \quad \times \mathbb{E}_t \left[ \exp \left( 3\varepsilon_0 \tilde{\lambda} \sum_{j \in J_1} \int_t^T |Z_s^j|^{1+\delta} ds \right) \right],
\end{aligned} \tag{4.23}$$

where

$$\varepsilon_0 := \frac{\bar{\gamma}}{9} \wedge \frac{\gamma}{24} > 0.$$

Furthermore, in view of (4.23), by Hölder's inequality we obtain that for  $t \in [T-h, T]$ ,

$$\begin{aligned}
& \mathbb{E}_t \left[ \exp \left( \frac{\bar{\gamma} \varepsilon_0}{2n_1} \sum_{j \in J_1} \int_t^T |Z_s^j|^2 ds \right) \right] \\
& \leq \exp \left( 6\varepsilon_0 \sum_{j \in J_1} \|Y^j\|_{\mathcal{S}_{[t,T]}^\infty} + 3\varepsilon_0 C_2 + 3\varepsilon_0 \beta \int_t^T \|Y\|_{\mathcal{S}_{[s,T]}^\infty} ds \right) \\
& \quad \times \mathbb{E}_t \left[ \exp \left( 3\varepsilon_0 \tilde{\lambda} \sum_{j \in J_1} \int_t^T |Z_s^j|^{1+\delta} ds \right) \right].
\end{aligned} \tag{4.24}$$

By Young's inequality, observe that for each pair of  $a, b > 0$ ,

$$ab^{1+\delta} = \left( \left( \frac{1+\delta}{2} \right)^{\frac{1+\delta}{1-\delta}} a^{\frac{2}{1-\delta}} \right)^{\frac{1-\delta}{2}} \left( \frac{2}{1+\delta} b^2 \right)^{\frac{1+\delta}{2}} \leq b^2 + \frac{1-\delta}{2} \left( \frac{1+\delta}{2} \right)^{\frac{1+\delta}{1-\delta}} a^{\frac{2}{1-\delta}}. \tag{4.25}$$

Letting  $a = 12n_1 \tilde{\lambda} / \bar{\gamma}$  and  $b = |Z_s^j|$  in (4.25), we have for each  $j \in J_1$ ,

$$3\varepsilon_0 \tilde{\lambda} |Z_s^j|^{1+\delta} = \frac{\bar{\gamma} \varepsilon_0}{4n_1} \left( \frac{12n_1 \tilde{\lambda}}{\bar{\gamma}} |Z_s^j|^{1+\delta} \right) \leq \frac{\bar{\gamma} \varepsilon_0}{4n_1} |Z_s^j|^2 + C_3, \quad s \in [0, T], \tag{4.26}$$

with

$$C_3 := \frac{\bar{\gamma} \varepsilon_0 (1-\delta)}{8n_1} \left( \frac{1+\delta}{2} \right)^{\frac{1+\delta}{1-\delta}} \left( \frac{12n_1 \tilde{\lambda}}{\bar{\gamma}} \right)^{\frac{2}{1-\delta}}.$$

Coming back to (4.24), by (4.26) and Hölder's inequality we derive that for  $t \in [T-h, T]$ ,

$$\begin{aligned}
& \mathbb{E}_t \left[ \exp \left( \frac{\bar{\gamma} \varepsilon_0}{2n_1} \sum_{j \in J_1} \int_t^T |Z_s^j|^2 ds \right) \right] \\
& \leq \exp \left( 12\varepsilon_0 \sum_{j \in J_1} \|Y^j\|_{\mathcal{S}_{[t,T]}^\infty} + 6\varepsilon_0 C_2 + 2C_3 T + 6\varepsilon_0 \beta \int_t^T \|Y\|_{\mathcal{S}_{[s,T]}^\infty} ds \right).
\end{aligned} \tag{4.27}$$

On the other hand, it follows from (4.22) and Jensen's inequality that

$$\begin{aligned}
\exp \left( \gamma \sum_{j \in J_1} |Y_t^j| \right) & \leq \exp \left( n_1 \gamma (C_1 + C_2) + n_1 \gamma \beta \int_t^T \|Y\|_{\mathcal{S}_{[s,T]}^\infty} ds \right) \\
& \quad \times \mathbb{E}_t \left[ \exp \left( n_1 \gamma \tilde{\lambda} \sum_{j \in J_1} \int_t^T |Z_s^j|^{1+\delta} ds \right) \right], \quad t \in [T-h, T].
\end{aligned} \tag{4.28}$$

By letting  $a = 2n_1^2\gamma\tilde{\lambda}/\bar{\gamma}\varepsilon_0$  and  $b = |Z_s|$  in (4.25), we have for each  $j \in J_1$ ,

$$n_1\gamma\tilde{\lambda}|Z_s^j|^{1+\delta} = \frac{\bar{\gamma}\varepsilon_0}{2n_1} \left( \frac{2n_1^2\gamma\tilde{\lambda}}{\bar{\gamma}\varepsilon_0} |Z_s|^{1+\delta} \right) \leq \frac{\bar{\gamma}\varepsilon_0}{2n_1} |Z_s^j|^2 + C_4, \quad s \in [0, T], \quad (4.29)$$

where

$$C_4 := \frac{\bar{\gamma}\varepsilon_0(1-\delta)}{4n_1} \left( \frac{1+\delta}{2} \right)^{\frac{1+\delta}{1-\delta}} \left( \frac{2n_1^2\tilde{\lambda}\gamma}{\bar{\gamma}\varepsilon_0} \right)^{\frac{2}{1-\delta}}.$$

Combining (4.27)-(4.29) yields that for each  $t \in [T-h, T]$ ,

$$\begin{aligned} \sum_{j \in J_1} |Y_t^j| &\leq n_1(C_1 + C_2) + \frac{C_4 T}{\gamma} + \frac{6\varepsilon_0 C_2 + 2C_3 T}{\gamma} + \frac{12\varepsilon_0}{\gamma} \sum_{j \in J_1} \|Y^j\|_{\mathcal{S}_{[t, T]}^\infty} \\ &\quad + \beta \left( n_1 + \frac{6\varepsilon_0}{\gamma} \right) \int_t^T \|Y\|_{\mathcal{S}_{[s, T]}^\infty} ds. \end{aligned}$$

And, it follows from the definition of  $\varepsilon_0$  that

$$\sum_{j \in J_1} \|Y^j\|_{\mathcal{S}_{[t, T]}^\infty} \leq C_5 + 2\beta \left( n_1 + \frac{1}{4} \right) \int_t^T \|Y\|_{\mathcal{S}_{[s, T]}^\infty} ds, \quad t \in [T-h, T], \quad (4.30)$$

where

$$C_5 := 2n_1(C_1 + C_2) + \frac{2C_4 T}{\gamma} + \frac{12\varepsilon_0 C_2 + 4C_3 T}{\gamma}.$$

By taking square in both sides of (4.30) and using Hölder's inequality, we can conclude that for each  $t \in [T-h, T]$ ,

$$\sum_{j \in J_1} \|Y^j\|_{\mathcal{S}_{[t, T]}^\infty}^2 \leq \left( \sum_{j \in J_1} \|Y^j\|_{\mathcal{S}_{[t, T]}^\infty} \right)^2 \leq 2C_5^2 + 8\beta^2 T (n_1 + 1)^2 \int_t^T \|Y\|_{\mathcal{S}_{[s, T]}^\infty}^2 ds. \quad (4.31)$$

(2) For  $i \in J_2$ ,  $g^i$  satisfies (ii) of assumption (C1b). In this case, for each  $t \in [T-h, T]$ , we have

$$g^i(\omega, t, Y_t(\omega), Z_t(\omega)) \operatorname{sgn}(Y_t^i(\omega)) \leq \tilde{\alpha}_t(\omega) + \beta|Y_t(\omega)| + \bar{v}_t(\omega)|Z_t^i(\omega)| + \frac{\gamma}{2}|Z_t^i(\omega)|^2$$

with

$$\bar{v}_t(\omega) := v_t(\omega) + \phi(|Y_t(\omega)|) + c \sum_{j=1}^{i-1} |Z_t^j(\omega)| \in \operatorname{BMO}_{[T-h, T]}(\mathbb{R}).$$

Note that  $(Y^i, Z^i) \in \mathcal{S}_{[T-h, T]}^\infty(\mathbb{R}) \times \operatorname{BMO}_{[T-h, T]}(\mathbb{R}^{1 \times d})$  is a solution of BSDE (2.2) on  $[T-h, T]$ .

In view of the last inequality, by using Itô-Tanaka's formula and Girsanov's transform, a similar argument to that from (A.11) to (A.12) yields that for each  $j \in J_2$  and  $t \in [T-h, h]$ ,

$$|Y_t^i| \leq \|\xi^i\|_\infty + \|\tilde{\alpha}\|_{\mathcal{L}^\infty} + \beta \int_t^T \|Y\|_{\mathcal{S}_{[s, T]}^\infty} ds,$$

and then, in view of (5.2),

$$\sum_{j \in J_2} \|Y^j\|_{\mathcal{S}_{[t, T]}^\infty} \leq n_2(C_1 + C_2) + n_2\beta \int_t^T \|Y\|_{\mathcal{S}_{[s, T]}^\infty} ds.$$

By taking square in both sides of the last inequality and using Hölder's inequality, we can conclude that for each  $t \in [T - h, T]$ ,

$$\sum_{j \in J_2} \|Y^j\|_{\mathcal{S}_{[t,T]}^\infty}^2 \leq \left( \sum_{j \in J_2} \|Y^j\|_{\mathcal{S}_{[t,T]}^\infty} \right)^2 \leq 2n_2^2(C_1 + C_2)^2 + 2Tn_2^2\beta^2 \int_t^T \|Y\|_{\mathcal{S}_{[s,T]}^\infty}^2 ds. \quad (4.32)$$

(3) For  $i \in J_3$ ,  $g^i$  satisfies (iii) of assumption (C1b). In this case, for each  $t \in [T - h, T]$ , we have

$$g^i(\omega, t, Y_t(\omega), Z_t(\omega)) \operatorname{sgn}(Y_t^i(\omega)) \leq \bar{\alpha}_t(\omega) + \beta|Y_t(\omega)| + \bar{\lambda} \sum_{j \in J_3} |Z_t^j(\omega)|,$$

and then, in view of inequality  $2ab \leq 2\varepsilon a^2 + \frac{1}{2\varepsilon}b^2$  for each  $a, b \geq 0$  and  $\varepsilon > 0$ ,

$$\begin{aligned} & 2 \sum_{j \in J_3} Y_t^j(\omega) g^j(\omega, t, Y_t(\omega), Z_t(\omega)) \\ & \leq 2 \sum_{j \in J_3} |Y_t^j(\omega)| \left( \bar{\alpha}_t(\omega) + \beta|Y_t(\omega)| + \bar{\lambda} \sum_{j \in J_3} |Z_t^j(\omega)| \right) \\ & \leq \sum_{j \in J_3} \left[ 2\bar{\alpha}_t(\omega) |Y_t^j(\omega)| + 2\beta|Y_t(\omega)|^2 + \sum_{j \in J_3} \left( 2n_3\bar{\lambda}^2 |Y_t^j(\omega)|^2 + \frac{1}{2n_3} |Z_t^j(\omega)|^2 \right) \right] \\ & \leq 2\bar{\alpha}_t(\omega) \sum_{j \in J_3} |Y_t^j(\omega)| + 2n_3(\beta + n_3\bar{\lambda}^2) |Y_t(\omega)|^2 + \frac{1}{2} \sum_{j \in J_3} |Z_t^j(\omega)|^2. \end{aligned}$$

Note that for each  $i \in J_3$ ,  $(Y^i, Z^i) \in \mathcal{S}_{[T-h, T]}^\infty(\mathbb{R}) \times \text{BMO}_{[T-h, T]}(\mathbb{R}^{1 \times d})$  is a adapted solution of BSDE (2.2) on  $[T - h, T]$ . In view of the last inequality together with (4.21), applying Itô's formula to  $\sum_{j \in J_3} |Y_s^j|^2$  yields that for each  $t \in [T - h, h]$ ,

$$\begin{aligned} & \sum_{j \in J_3} |Y_t^j|^2 + \frac{1}{2} \mathbb{E}_t \left[ \int_t^T \sum_{j \in J_3} |Z_s^j(\omega)|^2 ds \right] \\ & \leq \mathbb{E}_t \left[ \sum_{j \in J_3} |\xi^j|^2 \right] + \mathbb{E}_t \left[ \int_t^T \left( 2\bar{\alpha}_s \sum_{j \in J_3} |Y_s^j| + 2n_3(\beta + n_3\bar{\lambda}^2) |Y_s|^2 \right) ds \right] \\ & \leq C_1 + 2C_2 \sum_{j \in J_3} \|Y^j\|_{\mathcal{S}_{[t,T]}^\infty} + 2n_3(\beta + n_3\bar{\lambda}^2) \int_t^T \|Y\|_{\mathcal{S}_{[s,T]}^\infty}^2 ds, \end{aligned}$$

and then

$$\sum_{j \in J_3} \|Y^j\|_{\mathcal{S}_{[t,T]}^\infty}^2 \leq 2C_1 + 4C_2^2 + 4n_3(\beta + n_3\bar{\lambda}^2) \int_t^T \|Y\|_{\mathcal{S}_{[s,T]}^\infty}^2 ds. \quad (4.33)$$

Finally, adding (4.31), (4.32) and (4.33) up together yields that

$$\|Y\|_{\mathcal{S}_{[t,T]}^\infty}^2 \leq C_6 + C_7 \int_t^T \|Y\|_{\mathcal{S}_{[s,T]}^\infty}^2 ds, \quad t \in [T - h, T]$$

with  $C_6 := 2C_5^2 + 2n_2^2(C_1 + C_2)^2 + 2C_1 + 4C_2^2$  and

$$C_7 := 8\beta^2 T (n_1 + 1)^2 + 2Tn_2^2\beta^2 + 4n_3(\beta + n_3\bar{\lambda}^2).$$

It then follows from Gronwall's inequality that

$$\|Y\|_{\mathcal{S}_{[t,T]}^\infty}^2 \leq C_6 \exp(C_7(T - t)) \leq C_6 \exp(C_7 T) =: \tilde{K}, \quad t \in [T - h, T],$$

which yields the desired conclusion. The proof is complete.  $\square$

*Proof of Theorem 2.14.* With Theorem 2.3 and Proposition 4.2 in hands, we can closely follow the proof of Theorem 4.1 in Cheridito and Nam [9] to prove our Theorem 2.10. All the details are omitted here.  $\square$

## 5. Global unbounded solution: proof of Theorems 2.26 and 2.27

### 5.1. Proof of Theorem 2.26

*Proof of Theorem 2.26.* Define

$$\bar{g}(t, y, z) := g\left(t, y + \int_0^t H_s dB_s, z + H_t\right), \quad (\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}. \quad (5.1)$$

Consider the following multi-dimensional BSDE

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{g}(s, \bar{Y}_s, \bar{Z}_s) ds - \int_t^T \bar{Z}_s dB_s, \quad t \in [0, T], \quad (5.2)$$

or, equivalently,

$$\bar{Y}_t^i = \bar{\xi}^i + \int_t^T \bar{g}^i(s, \bar{Y}_s, \bar{Z}_s) ds - \int_t^T \bar{Z}_s^i dB_s, \quad t \in [0, T], \quad i = 1, \dots, n. \quad (5.3)$$

It is not difficult to check that BSDE (5.2) (or BSDE (5.3)) admits a unique global solution  $(\bar{Y}, \bar{Z}) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$  if and only if BSDE (2.1) (or BSDE (2.2)) admits a unique global solution  $(Y, Z) := (\bar{Y} + \int_0^\cdot H_s dB_s, \bar{Z} + H)$  on the time interval  $[0, T]$  such that

$$\left(Y - \int_0^\cdot H_s dB_s, Z\right) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d}).$$

Consequently, in view of Corollary 2.7, for completing the proof of Theorem 2.26 it suffices to prove that the generator  $\bar{g}$  defined in (5.1) also satisfies assumptions (B1), (D2) and (AB) with  $\theta = 0$ . In view of assumptions of the generator  $g$  and parameters  $(\alpha, \bar{\alpha}, \tilde{\alpha}, H)$ , it is straightforward to verify the above assertion. The proof is then complete.  $\square$

### 5.2. Proof of Theorem 2.27

*Proof of Theorem 2.27.* By an identical argument as in the proof of Theorem 2.26, we can conclude that in view of Theorem 2.10, for completing the proof of Theorem 2.27 it suffices to prove that the generator  $\bar{g}$  defined in (5.1) also satisfies assumptions (D1) and (D2) with  $\beta = 0$  and some other appropriate parameters when the constant  $\theta$  is smaller than a given constant  $\bar{\theta}_0$  depending only on  $p, \bar{p}$  and  $\gamma$ . Clearly,  $\bar{g}$  satisfies (D2) since  $g$  satisfies it.



In the sequel, we will prove that the generator  $\bar{g}$  also satisfies assumption (D1) with  $\beta = 0$  and some other appropriate parameters under the given conditions. Assume that  $\alpha \in \mathcal{E}^\infty(p\gamma)$  for some real  $p > 1$ , and  $|H|^2 \in \mathcal{E}^\infty(2\bar{p}(q\gamma)^2)$  for some  $\bar{p} > 1$  with  $q = p/(p-1)$  such that  $1/p + 1/q = 1$ . Define the following constants:

$$\varepsilon := \frac{p-1}{p+1}, \quad \hat{p} := \frac{(p+3)\bar{p} + p - 1}{2(p+2\bar{p}-1)} > 1, \quad \tilde{p} := \frac{p+1}{2\hat{p}} > 1 \quad \text{and} \quad \tilde{q} := \frac{p+1}{p-2\hat{p}+1} > 1.$$

It is clear that  $1/\tilde{p} + 1/\tilde{q} = 1$ . Note that  $g$  satisfies assumption (D1) with  $\beta = 0$ . By a similar argument to (iv) of Remark 2.2, in assumption (D1) we can without loss of generality assume that  $f = g^i$  satisfies either of conditions (D1)(i), (D1)(ii) and (D1)(iii) with  $\beta = 0$  for all  $i = 1, \dots, n$ . Thus, for  $i = 1, \dots, n$ , we need to consider the following three cases.

(1)  $g^i$  satisfies (D1)(i) with  $\beta = 0$ . For this case, it follows from the definition of  $\bar{g}$  that  $d\mathbb{P} \times dt - a.e.$ , for each  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , we have

$$\bar{g}^i(\omega, t, y, z) \leq \alpha_t(\omega) + \sum_{j \neq i} \left( \lambda |z^j + H_t^j(\omega)| + \theta |z^j + H_t^j(\omega)|^2 \right) + \frac{\gamma}{2} |z^i + H_t^i(\omega)|^2$$

and

$$\begin{aligned} \bar{g}^i(\omega, t, y, z) &\geq \frac{\bar{\gamma}}{2} |z^i + H_t^i(\omega)|^2 - \bar{\alpha}_t(\omega) - \sum_{j=i+1}^n \left( \bar{\lambda} |z^j + H_t^j(\omega)|^{1+\delta} + \theta |z^j + H_t^j(\omega)|^2 \right) \\ &\quad - \bar{c} \sum_{j=1}^{i-1} |z^j + H_t^j(\omega)|^2. \end{aligned}$$

Note that for each  $a, b \geq 0$ , it holds that

$$\begin{aligned} (a+b)^2 &\leq 2a^2 + 2b^2, \quad (a+b)^{1+\delta} \leq 2a^{1+\delta} + 2b^{1+\delta}, \\ (a+b)^2 &\leq (1+\varepsilon)a^2 + \left(1 + \frac{1}{\varepsilon}\right)b^2 \quad \text{and} \quad (a+b)^2 \geq \frac{1}{2}a^2 - b^2. \end{aligned}$$

We know that  $d\mathbb{P} \times dt - a.e.$ , for each  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,

$$\begin{aligned} \frac{\bar{\gamma}}{4} |z^i|^2 - \hat{\alpha}_t(\omega) - \sum_{j=i+1}^n \left( 2\bar{\lambda} |z^j|^{1+\delta} + 2\theta |z^j|^2 \right) - 2\bar{c} \sum_{j=1}^{i-1} |z^j|^2 &\leq \bar{g}^i(\omega, t, y, z) \\ &\leq \check{\alpha}_t(\omega) + \sum_{j \neq i} \left( \lambda |z^j| + 2\theta |z^j|^2 \right) + \frac{\gamma(1+\varepsilon)}{2} |z^i|^2 \end{aligned} \tag{5.4}$$

with

$$\check{\alpha}_t(\omega) := \alpha_t(\omega) + \sum_{j \neq i} \left( \lambda |H_t^j(\omega)| + 2\theta |H_t^j(\omega)|^2 \right) + \frac{\gamma(1+\varepsilon)}{2\varepsilon} |H_t^i(\omega)|^2$$

and

$$\hat{\alpha}_t(\omega) := \bar{\alpha}_t(\omega) + 2\bar{c} \sum_{j=1}^{i-1} |H_t^j(\omega)|^2 + \frac{\bar{\gamma}}{2} |H_t^i(\omega)|^2 + \sum_{j=i+1}^n \left( 2\bar{\lambda} |H_t^j(\omega)|^{1+\delta} + 2\theta |H_t^j(\omega)|^2 \right).$$

By virtue of  $\bar{\alpha} \in \mathcal{M}^\infty$  and  $H \in \text{BMO}(\mathbb{R}^{n \times d})$  together with the fact that

$$\hat{\alpha} \leq 2n\bar{\lambda} + \bar{\alpha} + \left(2\bar{c} + \frac{\bar{\gamma}}{2} + 2n\bar{\lambda} + 2\theta\right) |H|^2,$$

we can deduce that

$$\hat{\alpha} \in \mathcal{M}^\infty. \quad (5.5)$$

On the other hand, from the definitions of constants  $q, \hat{p}, \tilde{p}, \tilde{q}, \varepsilon$  and  $\check{\alpha}$  it is not difficult to verify that  $p\gamma = \tilde{p}\hat{p}\gamma(1 + \varepsilon)$  and that there exists two very small positive constants  $\bar{\varepsilon}$  and  $\bar{\theta}_0$  such that

$$\check{\alpha} \leq \alpha + \frac{n\lambda^2}{4\bar{\varepsilon}} + (\bar{\varepsilon} + 2\theta + q\gamma)|H|^2 \quad \text{and} \quad (\bar{\varepsilon} + 2\bar{\theta}_0 + q\gamma)\tilde{q}\hat{p}\gamma(1 + \varepsilon) = 2\bar{p}(q\gamma)^2.$$

Then, it follows from the integrability condition of  $\alpha$  and  $|H|^2$  that  $\alpha \in \mathcal{E}^\infty(\tilde{p}\hat{p}\gamma(1 + \varepsilon))$  and  $(\bar{\varepsilon} + 2\theta + q\gamma)|H|^2 \in \mathcal{E}^\infty(\tilde{q}\hat{p}\gamma(1 + \varepsilon))$  for each  $\theta \in [0, \bar{\theta}_0]$ . Note that  $1/\tilde{p} + 1/\tilde{q} = 1$ . By (iii) of Remark 2.1 we know that for each  $\theta \in [0, \bar{\theta}_0]$ ,

$$\check{\alpha} \in \mathcal{E}^\infty(\hat{p}\gamma(1 + \varepsilon)). \quad (5.6)$$

Combining (5.4), (5.5) and (5.6) yields that the generator  $\bar{g}^i$  satisfies (i) of assumption (D1) with parameters  $(\check{\alpha}, \hat{\alpha}, 0, \gamma(1 + \varepsilon), \bar{\gamma}/2, 2c, \lambda, 2\bar{\lambda}, 2\theta)$  instead of  $(\alpha, \bar{\alpha}, \beta, \gamma, \bar{\gamma}, c, \lambda, \bar{\lambda}, \theta)$ .

(2)  $g^i$  satisfies (D1)(ii) with  $\beta = 0$ , but does not satisfy (i) or (iii) in assumption (D1). Note that we have  $H^i \equiv 0$  in this case. It follows from the definition of  $\bar{g}$  that  $d\mathbb{P} \times dt - a.e.$ , for each  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , we have

$$|\bar{g}^i(\omega, t, y, z)| \leq \check{\alpha}_t(\omega) + |z^i| \left( \bar{v}_t(\omega) + c \sum_{j=1}^{i-1} |z^j| \right) + \frac{\gamma}{2} |z^i|^2$$

with

$$\bar{v}_t(\omega) := v_t(\omega) + c \sum_{j=1}^{i-1} |H_t^j(\omega)| \leq v_t(\omega) + \sqrt{nc} |H_t(\omega)|.$$

And, it follows from  $v \in \text{BMO}(\mathbb{R})$  and  $H \in \text{BMO}(\mathbb{R}^{n \times d})$  that  $\bar{v} \in \text{BMO}(\mathbb{R})$ , which means that  $\bar{g}^i$  satisfies (ii) of assumption (D1) with parameters  $(\check{\alpha}, 0, \bar{v}, c, \gamma)$  instead of  $(\check{\alpha}, \beta, v, c, \gamma)$ .

(3)  $g^i$  satisfies (D1)(iv) with  $\beta = 0$ . In this case, it follows from the definition of  $\bar{g}$  that  $d\mathbb{P} \times dt - a.e.$ , for each  $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ , we have

$$|g^i(\omega, t, y, z)| \leq \check{\alpha}_t(\omega) + \bar{\lambda}|z| + 2\theta \sum_{j \neq i} |z^j|^2$$

with

$$\check{\alpha}_t(\omega) := \bar{\alpha}_t(\omega) + \bar{\lambda}|H_t(\omega)| + 2\theta \sum_{j \neq i} |H_t^j(\omega)|^2 \leq \bar{\alpha}_t(\omega) + \bar{\lambda}|H_t(\omega)| + 2\theta|H_t(\omega)|^2.$$

And, it follows from  $\bar{\alpha} \in \mathcal{M}^\infty$  and  $H \in \text{BMO}(\mathbb{R}^{n \times d})$  that  $\check{\alpha} \in \mathcal{M}^\infty$ , which means that  $\bar{g}^i$  satisfies (iv) of assumption (D1) with parameters  $(\check{\alpha}, 0, \bar{\lambda}, 2\theta)$  instead of  $(\bar{\alpha}, \beta, \bar{\lambda}, \theta)$ .

All in all, we have proved the desired conclusion. Theorem 2.27 is then proved.  $\square$

## Appendix A. Several auxiliary results on the bounded solution of one-dimensional BSDEs with unbounded coefficients

We collect here some general results concerning the uniform estimate of (bounded) solution to scalar-valued BSDEs. We give some brief proofs for completeness. Let the real-valued function  $f(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$  be  $(\mathcal{F}_t)$ -progressively measurable for each  $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ , and consider the following one-dimensional BSDE:

$$Y_t = \eta + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \quad (\text{A.1})$$

where  $\eta \in L^\infty(\mathbb{R})$ , and the solution  $(Y, Z)$  is a pair of  $(\mathcal{F}_t)$ -progressively measurable processes with values in  $\mathbb{R} \times \mathbb{R}^{1 \times d}$  such that  $Y \in \mathcal{S}^\infty(\mathbb{R})$ .

Assume that  $\bar{\beta}, \bar{\delta} \geq 0$  are two given constants,  $\varphi(\cdot) : [0, +\infty) \rightarrow [0, +\infty)$  is a nondecreasing continuous function, and  $\check{\alpha} \in \mathcal{E}^\infty(\gamma \exp(\bar{\beta}T))$ ,  $\dot{\alpha} \in \mathcal{M}^\infty$ ,  $\ddot{\alpha} \in \mathcal{L}^\infty$ ,  $\bar{u} \in \text{BMO}(\mathbb{R})$ ,  $\bar{v} \in \text{BMO}(\mathbb{R})$  are five real-valued non-negative progressively measurable processes.

We introduce the following assumptions on the generator  $f$ :

(A1) Almost everywhere in  $\Omega \times [0, T]$ , for any  $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ , we have

$$\frac{\bar{\gamma}}{2}|z|^2 - \dot{\alpha}_t(\omega) - \bar{\beta}|y| \leq f(\omega, t, y, z) \leq \check{\alpha}_t(\omega) + [\bar{\beta}|y| \mathbf{1}_{y>0} + \varphi(|y|) \mathbf{1}_{y<0}] + \frac{\bar{\gamma}}{2}|z|^2.$$

(A2) Almost everywhere in  $\Omega \times [0, T]$ , for any  $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ , we have

$$\frac{\bar{\gamma}}{2}|z|^2 - \dot{\alpha}_t(\omega) - \bar{\beta}|y| \leq -f(\omega, t, -y, -z) \leq \check{\alpha}_t(\omega) + [\bar{\beta}|y| \mathbf{1}_{y>0} + \varphi(|y|) \mathbf{1}_{y<0}] + \frac{\bar{\gamma}}{2}|z|^2.$$

(A3) Almost everywhere in  $\Omega \times [0, T]$ , for any  $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ , we have

$$\begin{aligned} -[\bar{\beta}|y| \mathbf{1}_{y<0} + \varphi(|y|) \mathbf{1}_{y>0}] - h(\omega, t, z) &\leq f(\omega, t, y, z) \\ &\leq [\bar{\beta}|y| \mathbf{1}_{y>0} + \varphi(|y|) \mathbf{1}_{y<0}] + h(\omega, t, z) \end{aligned}$$

with

$$h(\omega, t, z) := \ddot{\alpha}_t(\omega) + \bar{u}_t(\omega)|z| + \frac{\bar{\gamma}}{2}|z|^2.$$

(A4) Almost everywhere in  $\Omega \times [0, T]$ , for any  $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ , we have

$$\begin{aligned} -\dot{\alpha}_t(\omega) - [\bar{\beta}|y| \mathbf{1}_{y<0} + \varphi(|y|) \mathbf{1}_{y>0}] - \bar{\lambda}|z| &\leq f(\omega, t, y, z) \\ &\leq \dot{\alpha}_t(\omega) + [\bar{\beta}|y| \mathbf{1}_{y>0} + \varphi(|y|) \mathbf{1}_{y<0}] + \bar{\lambda}|z|. \end{aligned}$$

(A5) Almost everywhere in  $\Omega \times [0, T]$ , for any  $(y, \bar{y}, z, \bar{z}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d}$ , we have for some  $k > 0$ ,

$$|f(\omega, t, y, z) - f(\omega, t, \bar{y}, \bar{z})| \leq \bar{\beta}|y - \bar{y}| + k(\bar{v}_t(\omega) + |z| + |\bar{z}|)|z - \bar{z}|.$$

**Proposition A.1.** Assume that the generator  $f$  satisfies assumption (A1) (resp. (A2)).

(i) For any solution  $(Y, Z)$  of BSDE (A.1) such that  $Y \in \mathcal{S}^\infty(\mathbb{R})$ , we have  $Z \in \text{BMO}(\mathbb{R}^{1 \times d})$  and for each  $t \in [0, T]$ ,

$$\begin{aligned} & \|Y\|_{\mathcal{S}_{[t,T]}^\infty} + \|Z\|_{\text{BMO}_{[t,T]}^2} \\ & \leq \frac{2(1 + \bar{\beta}T) + \bar{\gamma}}{\bar{\gamma}} \exp(2\bar{\beta}T) \left( 3\|\eta\|_\infty + \|\dot{\alpha}\|_{\mathcal{E}_{[t,T]}^\infty(\gamma \exp(\bar{\beta}T))} + 2\|\dot{\alpha}\|_{\mathcal{M}_{[t,T]}^\infty} \right). \end{aligned} \quad (\text{A.2})$$

And, if the generator  $f$  only satisfies the second inequality for the case of  $y > 0$  and the first inequality in (A1) (resp. (A2)), the above conclusion (A.2) still holds.

(ii) BSDE (A.1) admits a minimal (resp. maximal) solution  $(Y, Z)$  such that  $Y \in \mathcal{S}^\infty(\mathbb{R})$  in the sense that for any solution  $(\bar{Y}, \bar{Z})$  of BSDE (A.1) such that  $\bar{Y} \in \mathcal{S}^\infty(\mathbb{R})$ , we have for each  $t \in [0, T]$ ,  $\mathbb{P} - a.s.$ ,  $Y_t \leq \bar{Y}_t$  (resp.  $Y_t \geq \bar{Y}_t$ ). Moreover,  $Z \in \text{BMO}(\mathbb{R}^{1 \times d})$ .

(iii) If the generator  $f$  further satisfies assumption (A5), then BSDE (A.1) admits a unique solution  $(Y, Z)$  such that  $Y \in \mathcal{S}^\infty(\mathbb{R})$ . Moreover,  $Z \in \text{BMO}(\mathbb{R}^{1 \times d})$ .

*Proof.* We only give the proof when the generator  $f$  satisfies assumption (A1). The other case can be proved in the same way.

(i) Let  $(Y, Z)$  be a solution of BSDE (A.1) such that  $Y \in \mathcal{S}^\infty(\mathbb{R})$ . For each integer  $m \geq 1$  and each stopping time  $\tau \in \mathcal{T}_{[0,T]}$ , define the following stopping time

$$\sigma_m^\tau := T \wedge \inf \left\{ s \in [\tau, T] : \int_\tau^s |Z_s|^2 ds \geq m \right\}$$

with convention  $\inf \emptyset = \infty$ . It follows from the first inequality in assumption (A1) that for each  $m \geq 1$  and each  $t \in [0, T]$ ,

$$\frac{\bar{\gamma}}{2} \mathbb{E}_\tau \left[ \int_\tau^{\sigma_m^\tau} |Z_s|^2 ds \right] \leq |Y_\tau| + \mathbb{E}_\tau \left[ |Y_{\sigma_m^\tau}| + \int_\tau^{\sigma_m^\tau} (\dot{\alpha}_s + \bar{\beta}|Y_s|) ds \right], \quad \tau \in \mathcal{T}_{[t,T]}.$$

Sending  $m \rightarrow +\infty$  in previous inequality and using Fatou's lemma yields that for each  $t \in [0, T]$ ,

$$\frac{\bar{\gamma}}{2} \mathbb{E}_\tau \left[ \int_\tau^T |Z_s|^2 ds \right] \leq \|\eta\|_\infty + \|\dot{\alpha}\|_{\mathcal{M}_{[t,T]}^\infty} + (1 + \bar{\beta}T) \|Y\|_{\mathcal{S}_{[t,T]}^\infty}, \quad \tau \in \mathcal{T}_{[t,T]},$$

which means that  $Z \in \text{BMO}(\mathbb{R}^{1 \times d})$ , and for each  $t \in [0, T]$ ,

$$\frac{\bar{\gamma}}{2} \|Z\|_{\text{BMO}_{[t,T]}^2} \leq \|\eta\|_\infty + \|\dot{\alpha}\|_{\mathcal{M}_{[t,T]}^\infty} + (1 + \bar{\beta}T) \|Y\|_{\mathcal{S}_{[t,T]}^\infty}. \quad (\text{A.3})$$

Furthermore, define the function

$$u(t, x) := \exp \left( \gamma \exp(\bar{\beta}t)x + \gamma \int_0^t \exp(\bar{\beta}s) \dot{\alpha}_s ds \right), \quad (t, x) \in [0, T] \times \mathbb{R}.$$

In view of the second inequality for the case of  $y > 0$  in assumption (A1), by applying Itô-Tanaka's formula to  $u(s, Y_s^+)$  we can deduce that for each  $m \geq 1$  and  $t \in [0, T]$ ,

$$\begin{aligned} \exp(\gamma Y_t^+) &\leq \mathbb{E}_t \left[ \exp \left( \gamma \exp(\bar{\beta}T) \eta^+ + \gamma \exp(\bar{\beta}T) \int_t^T \check{\alpha}_s ds \right) \right], \\ &\leq \exp(\gamma \exp(\bar{\beta}T) \|\eta\|_\infty) \exp \left( \gamma \exp(\bar{\beta}T) \|\check{\alpha}\|_{\mathcal{E}_{[t,T]}^\infty(\gamma \exp(\bar{\beta}T))} \right) \end{aligned}$$

and then,

$$Y_t^+ \leq \exp(\bar{\beta}T) \left( \|\eta\|_\infty + \|\check{\alpha}\|_{\mathcal{E}_{[t,T]}^\infty(\gamma \exp(\bar{\beta}T))} \right). \quad (\text{A.4})$$

On the other hand, from the first inequality in assumption (A1) we can also get that for each  $m \geq 1$  and  $t \in [0, T]$ ,

$$Y_t^- = (-Y_t)^+ \leq \mathbb{E}_t \left[ \eta^- + \int_t^T (\dot{\alpha}_s + \bar{\beta}|Y_s|) ds \right],$$

which together with (A.4) yields that for each  $t \in [0, T]$ ,

$$|Y_t| \leq \exp(\bar{\beta}T) \left( \|\eta\|_\infty + \|\check{\alpha}\|_{\mathcal{E}_{[t,T]}^\infty(\gamma \exp(\bar{\beta}T))} \right) + \|\eta\|_\infty + \|\dot{\alpha}\|_{\mathcal{M}_{[t,T]}^\infty} + \bar{\beta} \mathbb{E}_t \left[ \int_t^T |Y_s| ds \right].$$

And, it follows from Gronwall's inequality that

$$\|Y\|_{\mathcal{S}_{[t,T]}^\infty} \leq \exp(2\bar{\beta}T) \left( 2\|\eta\|_\infty + \|\check{\alpha}\|_{\mathcal{E}_{[t,T]}^\infty(\gamma \exp(\bar{\beta}T))} + \|\dot{\alpha}\|_{\mathcal{M}_{[t,T]}^\infty} \right), \quad t \in [0, T]. \quad (\text{A.5})$$

Finally, the desired conclusion (A.2) follows from (A.5) and (A.3) immediately.

(ii) In view of assumption (A1), it is easy to verify that for each integer  $m \geq 1$ , the following function

$$\begin{aligned} f^m(\omega, t, y, z) &:= \inf \left\{ f(\omega, t, \bar{y}, \bar{z}) + (m + \bar{\beta})|y - \bar{y}| + m|z - \bar{z}| : (\bar{y}, \bar{z}) \in \mathbb{R} \times \mathbb{R}^{1 \times d} \right\}, \\ &(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d} \end{aligned}$$

is well defined and an  $(\mathcal{F}_t)$ -progressively measurable process for each  $(y, z)$ . It is also not difficult to prove that  $f^m$  is uniformly Lipschitz continuous in the state variables  $(y, z)$  and also satisfies assumption (A1) with the same parameters for each  $m \geq 1$ , and that the sequence  $\{f^m\}_{m=1}^\infty$  converges increasingly uniformly on compact sets to the generator  $f$  as  $m$  tends to  $+\infty$ . Then,  $d\mathbb{P} \times dt - a.e.$ , for each  $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$  and  $m \geq 1$ , we have

$$|f^m(\omega, t, y, z)| \leq \check{\alpha}_t(\omega) + \dot{\alpha}_t(\omega) + \varphi(|y|) + \bar{\beta}|y| + \frac{\gamma}{2}|z|^2, \quad (\text{A.6})$$

and then

$$|f^m(\omega, t, 0, 0)| \leq \check{\alpha}_t(\omega) + \dot{\alpha}_t(\omega) + \varphi(0), \quad (\text{A.7})$$

which means that

$$\mathbb{E} \left[ \left( \int_0^T |f^m(s, 0, 0)| ds \right)^2 \right] < +\infty.$$

Consequently, by the classical results (see for example Theorems 3 and 2 in Fan et al. [18]) we know that for each  $m \geq 1$ , the following BSDE

$$Y_t^m = \eta + \int_t^T f^m(s, Y_s^m, Z_s^m) ds - \int_t^T Z_s^m dB_s, \quad t \in [0, T]$$

admits a unique solution  $(Y^m, Z^m) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^{1 \times d})$ , and  $Y^m$  converges increasing pointwisely to a process  $Y$ . Moreover, by the classical a priori estimate on the  $L^2$  solution (see for example Proposition 3.2 in Briand et al. [5]) we know that there exists a uniform constant  $c_0 > 0$  such that for each  $m \geq 1$  and  $t \in [0, T]$ ,

$$\mathbb{E}_t \left[ \sup_{s \in [t, T]} |Y_s^m|^2 \right] \leq c_0 \exp(2(m + \bar{\beta})T + 2m^2T) \mathbb{E}_t \left[ |\eta|^2 + \left( \int_t^T |f^m(s, 0, 0)| ds \right)^2 \right],$$

which together with (A.7) and the facts that  $\eta \in L^\infty(\mathbb{R})$ ,  $\check{\alpha} \in \mathcal{E}^\infty(\gamma)$  and  $\dot{\alpha} \in \mathcal{M}^\infty$  yields that  $Y^m \in \mathcal{S}^\infty(\mathbb{R})$  for each  $m \geq 1$ , and then  $Z^m \in \text{BMO}(\mathbb{R}^{1 \times d})$  by (i).

In the sequel, it follows from (i) that there exists a uniform constant  $A > 0$  which is independent of  $m$  such that  $d\mathbb{P} \times dt - a.e.$ , we have  $\sup_{m \geq 1} |Y_t^m(\omega)| \leq A$  and, in view of (A.6),

$$\forall (y, z) \in [-A, A] \times \mathbb{R}^{1 \times d}, \quad |f^m(\omega, t, y, z)| \leq \check{\alpha}_t(\omega) + \dot{\alpha}_t(\omega) + \varphi(A) + \bar{\beta}A + \frac{\gamma}{2}|z|^2.$$

Thus, we can apply the monotonic stability result Proposition 3.1 in Luo and Fan [34] to obtain the existence of a process  $Z \in \mathcal{H}^2(\mathbb{R}^{1 \times d})$  such that  $Y \in \mathcal{S}^\infty(\mathbb{R})$  and  $(Y, Z)$  is a solution of BSDE (A.1). And, it follows from (i) that  $Z \in \text{BMO}(\mathbb{R}^{1 \times d})$ .

It remains to show that  $(Y, Z)$  is the minimal solution. For this, let  $(\bar{Y}, \bar{Z})$  be any solution of BSDE (A.1) such that  $\bar{Y} \in \mathcal{S}^\infty(\mathbb{R})$ . By (i) again we know that  $\bar{Z} \in \text{BMO}(\mathbb{R}^{1 \times d})$ . This means that  $(\bar{Y}, \bar{Z}) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^{1 \times d})$ . Then, since  $f^m$  is uniformly Lipschitz continuous in  $(y, z)$  and  $f^m \leq f$  for each  $m \geq 1$ , it follows from the classical comparison theorem on the  $L^2$ -solution that for each  $m \geq 1$  and  $t \in [0, T]$ ,  $\mathbb{P} - a.s.$ ,  $Y_t^m \leq \bar{Y}_t$ , and letting  $m \rightarrow \infty$  yields that  $Y \leq \bar{Y}$ , which is the desired conclusion.

(iii) Let the generator  $f$  further satisfy assumption (A5), and  $(Y, Z)$  and  $(\bar{Y}, \bar{Z})$  be the solution of BSDE (A.1) such that  $Y \in \mathcal{S}^\infty(\mathbb{R})$  and  $\bar{Y} \in \mathcal{S}^\infty(\mathbb{R})$ . First of all, from (i) we know that  $Z \in \text{BMO}(\mathbb{R}^{1 \times d})$  and  $\bar{Z} \in \text{BMO}(\mathbb{R}^{1 \times d})$ . Furthermore, define  $\hat{Y} := Y - \bar{Y}$  and  $\hat{Z} := Z - \bar{Z}$ . By virtue of Itô-Tanaka's formula and (A5) we can deduce that for each  $t \in [0, T]$ ,

$$\begin{aligned} |\hat{Y}_t| &\leq \int_t^T \left( \bar{\beta} |\hat{Y}_s| + k(\bar{v}_s + |Z_s| + |\bar{Z}_s|) |\hat{Z}_s| \right) ds - \int_t^T \text{sgn}(\hat{Y}_s) \hat{Z}_s dB_s \\ &\leq \bar{\beta} \int_t^T |\hat{Y}_s| ds - \int_t^T \text{sgn}(\hat{Y}_s) \hat{Z}_s \left[ dB_s - k \text{sgn}(\hat{Y}_s) (\bar{v}_s + |Z_s| + |\bar{Z}_s|) \frac{\hat{Z}_s^\top}{|\hat{Z}_s|} \mathbf{1}_{|\hat{Z}_s| \neq 0} ds \right]. \end{aligned} \quad (\text{A.8})$$

Since all of processes  $\bar{v}, Z, \bar{Z}$  belong to  $\text{BMO}(\mathbb{R}^{1 \times d})$ , it is easy to verify that the process

$$M_t := k \int_0^t \text{sgn}(\hat{Y}_s)(\bar{v}_s + |Z_s| + |\bar{Z}_s|) \frac{\hat{Z}_s}{|\hat{Z}_s|} \mathbf{1}_{|\hat{Z}_s| \neq 0} dB_s, \quad t \in [0, T]$$

is a BMO martingale. Define

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} := \exp \left\{ M_T - \frac{1}{2} \langle M \rangle_T \right\}$$

and

$$\tilde{B}_t := B_t - k \int_0^t \text{sgn}(\hat{Y}_s)(\bar{v}_s + |Z_s| + |\bar{Z}_s|) \frac{\hat{Z}_s^\top}{|\hat{Z}_s|} \mathbf{1}_{|\hat{Z}_s| \neq 0} ds, \quad t \in [0, T].$$

Then,  $\tilde{\mathbb{P}}$  is a new probability, and  $\tilde{B}$  is a Brownian motion with respect to  $\tilde{\mathbb{P}}$ . Then, taking the obvious conditional mathematical expectation with respect to  $\tilde{\mathbb{P}}$  in (A.8) and utilizing Gronwall's inequality yields that  $|\hat{Y}_t| = 0$  for each  $t \in [0, T]$ , which is the desired conclusion.  $\square$

**Proposition A.2.** *Assume that the generator  $f$  satisfies assumption (A3).*

- (i) *For any solution  $(Y, Z)$  of BSDE (A.1) such that  $Y \in \mathcal{S}^\infty(\mathbb{R})$ , we have  $Z \in \text{BMO}(\mathbb{R}^{1 \times d})$  and for each  $0 \leq t \leq r \leq T$ , it holds that*

$$\begin{aligned} \|Y\|_{\mathcal{S}_{[t, T]}^\infty} + \|Z\|_{\text{BMO}_{[t, r]}}^2 &\leq \frac{4(\gamma + 1)}{\gamma^2} \exp \left\{ 4\gamma \exp(\bar{\beta}T) \left( \|\eta\|_\infty + \|\ddot{\alpha}\|_{\mathcal{L}_{[t, T]}^\infty} \right) \right\} \\ &\times \left( 2 + \bar{\beta}T \exp(\bar{\beta}T) \left( \|\eta\|_\infty + \|\ddot{\alpha}\|_{\mathcal{L}_{[t, T]}^\infty} \right) + \|\ddot{\alpha}\|_{\mathcal{L}_{[t, T]}^\infty} + \|\bar{u}\|_{\text{BMO}_{[t, T]}}^2 \right). \end{aligned} \quad (\text{A.9})$$

*And, if the generator  $f$  only satisfies the first inequality for the case of  $y < 0$  and the second inequality for the case of  $y > 0$  in (A3), the above conclusion (A.9) still holds.*

- (ii) *BSDE (A.1) admits a solution  $(Y, Z)$  such that  $Y \in \mathcal{S}^\infty(\mathbb{R})$ . Moreover,  $Z \in \text{BMO}(\mathbb{R}^{1 \times d})$ .*

- (iii) *If the generator  $f$  further satisfies assumption (A5), then BSDE (A.1) admits a unique solution  $(Y, Z)$  such that  $Y \in \mathcal{S}^\infty(\mathbb{R})$ . Moreover,  $Z \in \text{BMO}(\mathbb{R}^{1 \times d})$ .*

*Proof.* (i) Let  $(Y, Z)$  be any solution of BSDE (A.1) such that  $Y \in \mathcal{S}^\infty(\mathbb{R})$ . For each stopping time  $\tau \in \mathcal{T}_{[0, T]}$  and each integer  $m \geq 1$ , define the following stopping time

$$\sigma_m^\tau := T \wedge \inf \left\{ s \in [\tau, T] : \int_\tau^s |Z_s|^2 ds \geq m \right\}.$$

Using Itô-Tanaka's formula to compute  $\exp(2\gamma|Y_t|)$  and utilizing the first inequality for the case of  $y < 0$  and the second inequality for the case of  $y > 0$  in assumption (A3), we see that for each  $m \geq 1$ ,  $t \in [0, T]$  and  $\tau \in \mathcal{T}_{[t, T]}$ ,

$$\begin{aligned} \exp(2\gamma|Y_\tau|) + 2\gamma^2 \mathbb{E}_\tau \left[ \int_\tau^{\sigma_m^\tau} \exp(2\gamma|Y_s|) |Z_s|^2 ds \right] \\ \leq \mathbb{E}_\tau [\exp(2\gamma|\eta|)] + 2\gamma \mathbb{E}_\tau \left[ \int_\tau^{\sigma_m^\tau} \exp(2\gamma|Y_s|) \left( \ddot{\alpha}_s + \bar{\beta}|Y_s| + \bar{u}_t|Z_s| + \frac{\gamma}{2}|Z_s|^2 \right) ds \right]. \end{aligned}$$

Therefore, in view of the basic inequality that  $2ab \leq 2a^2 + b^2/2$  for each  $a, b \geq 0$ ,

$$\begin{aligned} \gamma^2 \mathbb{E}_\tau \left[ \int_\tau^{\sigma_m^T} |Z_s|^2 ds \right] &\leq \exp(2\gamma \|\eta\|_\infty) + 2\gamma \exp\left(2\gamma \|Y\|_{\mathcal{S}_{[t,T]}^\infty}\right) \left( \|\ddot{\alpha}\|_{\mathcal{L}_{[t,T]}^\infty} + \bar{\beta} T \|Y\|_{\mathcal{S}_{[t,T]}^\infty} \right) \\ &\quad + 2 \exp\left(4\gamma \|Y\|_{\mathcal{S}_{[t,T]}^\infty}\right) \|\bar{u}\|_{\text{BMO}_{[t,T]}}^2 + \frac{\gamma^2}{2} \mathbb{E}_\tau \left[ \int_\tau^{\sigma_m^T} |Z_s|^2 ds \right]. \end{aligned}$$

Sending  $m \rightarrow +\infty$  and using Fatou's lemma yields that  $Z \in \text{BMO}(\mathbb{R}^{1 \times d})$ , and for each  $t \in [0, T]$ , we have

$$\begin{aligned} \|Z\|_{\text{BMO}_{[t,T]}}^2 &\leq \frac{2}{\gamma^2} \exp(2\gamma \|\eta\|_\infty) + \frac{4}{\gamma} \exp\left(2\gamma \|Y\|_{\mathcal{S}_{[t,T]}^\infty}\right) \left( \|\ddot{\alpha}\|_{\mathcal{L}_{[t,T]}^\infty} + \bar{\beta} T \|Y\|_{\mathcal{S}_{[t,T]}^\infty} \right) \\ &\quad + \frac{4}{\gamma^2} \exp\left(4\gamma \|Y\|_{\mathcal{S}_{[t,T]}^\infty}\right) \|\bar{u}\|_{\text{BMO}_{[t,T]}}^2. \end{aligned} \quad (\text{A.10})$$

Furthermore, using Itô-Tanaka's formula we also have, for each  $t \in [0, T]$ ,

$$\begin{aligned} |Y_t| &\leq |\eta| + \int_t^T \left( \ddot{\alpha}_s + \bar{\beta} |Y_s| + \bar{u}_t |Z_s| + \frac{\gamma}{2} |Z_s|^2 \right) ds - \int_t^T \text{sgn}(Y_s) Z_s dB_s \\ &\leq \|\eta\|_\infty + \|\ddot{\alpha}\|_{\mathcal{L}_{[t,T]}^\infty} + \bar{\beta} \int_t^T |Y_s| ds \\ &\quad - \int_t^T \text{sgn}(Y_s) Z_s \left[ dB_s - \text{sgn}(Y_s) \left( \bar{u}_s \frac{1}{|Z_s|} \mathbf{1}_{|Z_s| \neq 0} + \frac{\gamma}{2} \right) Z_s^\top ds \right]. \end{aligned} \quad (\text{A.11})$$

Since both of processes  $\bar{u}$  and  $Z$  belong to  $\text{BMO}(\mathbb{R}^{1 \times d})$ , it is easy to verify that the process

$$M_t := \int_0^t \text{sgn}(Y_s) \left( \bar{u}_s \frac{1}{|Z_s|} \mathbf{1}_{|Z_s| \neq 0} + \frac{\gamma}{2} \right) Z_s dB_s, \quad t \in [0, T]$$

is a BMO martingale. Define

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} := \exp \left\{ M_T - \frac{1}{2} \langle M \rangle_T \right\}$$

and

$$\tilde{B}_t := B_t - \int_0^t \text{sgn}(Y_s) \left( \bar{u}_s \frac{1}{|Z_s|} \mathbf{1}_{|Z_s| \neq 0} + \frac{\gamma}{2} \right) Z_s^\top ds, \quad t \in [0, T].$$

Then,  $\tilde{\mathbb{P}}$  is a new probability, and  $\tilde{B}$  is a Brownian motion with respect to  $\tilde{\mathbb{P}}$ . Then, taking the obvious conditional mathematical expectation with respect to  $\tilde{\mathbb{P}}$  in (A.11) and utilizing Gronwall's inequality yields that for each  $t \in [0, T]$ ,

$$\|Y\|_{\mathcal{S}_{[t,T]}^\infty} \leq \exp(\bar{\beta} T) \left( \|\eta\|_\infty + \|\ddot{\alpha}\|_{\mathcal{L}_{[t,T]}^\infty} \right). \quad (\text{A.12})$$

Then, the desired conclusion (A.9) follows from (A.10) and (A.12) immediately.

(ii) It is easy to check that for each pair of integers  $m, l \geq 1$ , the following function

$$\begin{aligned} f^{m,l}(\omega, t, y, z) &:= \inf \left\{ f^+(\omega, t, \bar{y}, \bar{z}) + m|y - \bar{y}| + m|z - \bar{z}| : (\bar{y}, \bar{z}) \in \mathbb{R} \times \mathbb{R}^{1 \times d} \right\} \\ &\quad - \inf \left\{ f^-(\omega, t, \bar{y}, \bar{z}) + l|y - \bar{y}| + l|z - \bar{z}| : (\bar{y}, \bar{z}) \in \mathbb{R} \times \mathbb{R}^{1 \times d} \right\}, \\ &\quad (\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d} \end{aligned}$$



is well defined and an  $(\mathcal{F}_t)$ -progressively measurable process for each  $(y, z)$ . In view of assumption (A3), it is also not hard to verify that  $f^{m,l}$  is uniformly Lipschitz continuous in the state variables  $(y, z)$  and also satisfies assumption (A3) with the same parameters for  $m, l \geq 1$ , and that for each  $l \geq 1$ , the sequence  $\{f^{m,l}\}_{m=1}^\infty$  converges increasingly and uniformly on compact sets to a function  $f^{\infty,l}$  as  $m \rightarrow +\infty$ , and  $\{f^{\infty,l}\}_{l=1}^\infty$  converges decreasingly and uniformly on compact sets to the generator  $f$  as  $l \rightarrow +\infty$ . Then,  $d\mathbb{P} \times dt - a.e.$ , for each  $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$  and  $m, l \geq 1$ , we have

$$|f^{m,l}(\omega, t, y, z)| \leq \ddot{\alpha}_t(\omega) + \frac{1}{2}\bar{u}_t^2(\omega) + \varphi(|y|) + \frac{\gamma+1}{2}|z|^2, \quad (\text{A.13})$$

and then

$$|f^{m,l}(\omega, t, 0, 0)| \leq \ddot{\alpha}_t(\omega) + \frac{1}{2}\bar{u}_t^2(\omega) + \varphi(0), \quad (\text{A.14})$$

which means that

$$\mathbb{E} \left[ \left( \int_0^T |f^{m,l}(s, 0, 0)| ds \right)^2 \right] < +\infty.$$

Consequently, by the classical results we know that for each  $m, l \geq 1$ , the following BSDE

$$Y_t^{m,l} = \eta + \int_t^T f^{m,l}(s, Y_s^{m,l}, Z_s^{m,l}) ds - \int_t^T Z_s^{m,l} dB_s, \quad t \in [0, T]$$

admits a unique solution  $(Y^{m,l}, Z^{m,l}) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^{1 \times d})$ ,  $Y^{m,l}$  converges increasing pointwisely to a process  $Y^{\infty,l}$  as  $m \rightarrow +\infty$ , and  $Y^{\infty,l}$  converges decreasing pointwisely to a process  $Y$  as  $l \rightarrow +\infty$ . Moreover, by the classical a priori estimate on the  $L^2$  solution (see Proposition 3.2 in Briand et al. [5]) we know that there exists a uniform constant  $c_0 > 0$  such that for each  $m, l \geq 1$  and  $t \in [0, T]$ ,

$$\mathbb{E}_t \left[ \sup_{s \in [t, T]} |Y_s^{m,l}|^2 \right] \leq c_0 \exp(2mT + 2m^2T) \mathbb{E}_t \left[ |\eta|^2 + \left( \int_t^T |f^{m,l}(s, 0, 0)| ds \right)^2 \right],$$

which together with (A.14), the energy inequality for BMO martingales (see for example Section 2.1 in Kazamaki [29]) and the facts that  $\eta \in L^\infty(\mathbb{R})$  and  $\ddot{\alpha} \in \mathcal{L}^\infty$  yields that  $Y^{m,l} \in \mathcal{S}^\infty(\mathbb{R})$  for each  $m, l \geq 1$ , and then  $Z^{m,l} \in \text{BMO}(\mathbb{R}^{1 \times d})$  by (i).

Finally, it follows from (i) that there exists a uniform constant  $A > 0$  which is independent of  $m$  and  $l$  such that  $d\mathbb{P} \times dt - a.e.$ , we have

$$\sup_{m, l \geq 1} |Y_t^{m,l}(\omega)| \leq A$$

and, in view of (A.13),

$$\forall (y, z) \in [-A, A] \times \mathbb{R}^{1 \times d}, \quad |f^{m,l}(\omega, t, y, z)| \leq \ddot{\alpha}_t(\omega) + \frac{1}{2}\bar{u}_t^2(\omega) + \varphi(A) + \frac{\gamma+1}{2}|z|^2.$$

Thus, we can apply twice the monotonic stability result Proposition 3.1 in [34] to obtain the existence of a process  $Z \in \mathcal{H}^2(\mathbb{R}^{1 \times d})$  such that  $Y \in \mathcal{S}^\infty(\mathbb{R})$  and  $(Y, Z)$  is a desired solution of BSDE (A.1). And, it follows from (i) that  $Z \in \text{BMO}(\mathbb{R}^{1 \times d})$ .

(iii) In view of (i), (iii) is proved in the same way as (iii) of Proposition A.1.  $\square$

In the following proposition, the generator  $g$  has a linear growth in the state variable  $z$ .

**Proposition A.3.** *Assume that the generator  $f$  satisfies assumption (A4).*

(i) *For any solution  $(Y, Z)$  of BSDE (A.1) such that  $Y \in \mathcal{S}^2(\mathbb{R})$ , we have  $Y \in \mathcal{S}^\infty(\mathbb{R})$  and  $Z \in \text{BMO}(\mathbb{R}^{1 \times d})$  and for each  $t \in [0, T]$ , there exists a uniform constant  $c_0 > 1$  such that*

$$\|Y\|_{\mathcal{S}_{[t,T]}^\infty} + \|Z\|_{\text{BMO}_{[t,T]}^2} \leq 2c_0 \exp(2\bar{\beta}T + 2\bar{\lambda}^2T) \left(1 + \|\eta\|_\infty^2 + 2\|\dot{\alpha}\|_{\mathcal{M}_{[t,T]}^\infty}^2\right). \quad (\text{A.15})$$

*And, if the generator  $f$  only satisfies the first inequality for the case of  $y < 0$  and the second inequality for the case of  $y > 0$  in (A4), the above conclusion (A.15) still holds.*

(ii) *BSDE (A.1) admits a minimal (resp. maximal) solution  $(Y, Z)$  such that  $Y \in \mathcal{S}^2(\mathbb{R})$  in the sense that for any solution  $(\bar{Y}, \bar{Z})$  of BSDE (A.1) such that  $\bar{Y} \in \mathcal{S}^2(\mathbb{R})$ , we have for each  $t \in [0, T]$ ,  $\mathbb{P}$ -a.s.,  $Y_t \leq \bar{Y}_t$  (resp.  $Y_t \geq \bar{Y}_t$ ). Moreover,  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}) \times \text{BMO}(\mathbb{R}^{1 \times d})$ .*

(iii) *If the generator  $f$  further satisfies assumption (A5), then BSDE (A.1) admits a unique solution  $(Y, Z)$  such that  $Y \in \mathcal{S}^2(\mathbb{R})$ . Moreover,  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}) \times \text{BMO}(\mathbb{R}^{1 \times d})$ .*

*Proof.* (i) Let  $(Y, Z)$  be any solution of BSDE (A.1) such that  $Y \in \mathcal{S}^2(\mathbb{R})$ . In view of the first inequality for the case of  $y < 0$  and the second inequality for the case of  $y > 0$  in assumption (A4) and by virtue of the classical a priori estimate on the  $L^2$  solution (see Proposition 3.2 in Briand et al. [5]) we can deduce the existence of a uniform constant  $c_0 > 1$  such that for each  $t \in [0, T]$  and each  $\tau \in \mathcal{T}_{[t,T]}$ ,

$$\mathbb{E}_\tau \left[ \sup_{s \in [\tau, T]} |Y_s^m|^2 + \int_\tau^T |Z_s|^2 ds \right] \leq c_0 \exp(2\bar{\beta}T + 2\bar{\lambda}^2T) \mathbb{E}_\tau \left[ |\eta|^2 + \left( \int_\tau^T \dot{\alpha}_s ds \right)^2 \right],$$

and then, by the energy inequality for BMO martingales and Hölder's inequality,

$$\|Y\|_{\mathcal{S}_{[t,T]}^\infty}^2 + \|Z\|_{\text{BMO}_{[t,T]}^2}^2 \leq c_0 \exp(2\bar{\beta}T + 2\bar{\lambda}^2T) \left( \|\eta\|_\infty^2 + 2\|\dot{\alpha}\|_{\mathcal{M}_{[t,T]}^\infty}^2 \right).$$

Then, the desired inequality (A.15) follows from the previous inequality immediately, and then  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}) \times \text{BMO}(\mathbb{R}^{1 \times d})$ .

(ii) Define  $M := 2c_0 \exp(2\bar{\beta}T + 2\bar{\lambda}^2T) (1 + \|\eta\|_\infty^2 + 2\|\dot{\alpha}\|_{\mathcal{M}^\infty}^2)$ ,  $\rho^M(x) := \frac{Mx}{M \vee |x|}$ ,  $x \in \mathbb{R}$  and

$$f^M(\omega, t, y, z) := f(\omega, t, \rho^M(y), z), \quad (\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d}.$$

It is easy to verify that the generator  $f^M$  also satisfies assumption (A4), and that  $d\mathbb{P} \times dt - a.e.$ , for each  $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$ ,  $|f^M(\omega, t, y, z)| \leq \dot{\alpha}_t(\omega) + \varphi(M) + \bar{\lambda}|z|$ . Then, by Lepeltier and San Martin [33] we know that the following BSDE

$$Y_t = \eta + \int_t^T f^M(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T] \quad (\text{A.16})$$

admits a maximal solution  $(\bar{Y}, \bar{Z})$  and a minimal solution  $(\underline{Y}, \underline{Z})$  in the space  $\mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^{1 \times d})$ .

We now show that  $(\bar{Y}, \bar{Z})$  and  $(\underline{Y}, \underline{Z})$  are also the desired maximal and minimal solution of BSDE (A.1). Indeed, since  $f^M$  satisfies assumption (A4), it follows from (i) and the definition of  $f^M$  that both of them belong to the space  $\mathcal{S}^\infty(\mathbb{R}) \times \text{BMO}(\mathbb{R}^{1 \times d})$  and are also solutions of BSDE (A.1). Furthermore, let  $(Y, Z)$  be any solution of BSDE (A.1) such that  $Y \in \mathcal{S}^2(\mathbb{R})$ . Then, it follows from (i) and the definition of  $f^M$  again that  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}) \times \text{BMO}(\mathbb{R}^{1 \times d})$  and it is also a solution of BSDE (A.16) in the space  $\mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^{1 \times d})$ . Consequently, for each  $t \in [0, T]$ , we have  $\underline{Y}_t \leq Y_t \leq \bar{Y}_t$ ,  $\mathbb{P} - a.s.$ , which is the desired conclusion.

(iii) In view of (i), (iii) can be proved in the same way as (iii) of Proposition A.1.  $\square$

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