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TWO FAMILIES OF STRONGLY WALK REGULAR GRAPHS FROM THREE-WEIGHT CODES OVER \mathbb{Z}_4

MINJIA SHI, WENJUN XU, XUAN WANG, YUE CHENG, HUAZHANG WU, AND PATRICK SOLÉ

ABSTRACT. A necessary condition for a \mathbb{Z}_4 -code to be a three-weight code for the Lee weight is given. Two special constructions of three-weight codes over \mathbb{Z}_4 are derived. The coset graphs of their duals are shown to be strongly 3-walk-regular, a generalization of strongly regular graphs.

1. INTRODUCTION

Strongly walk-regular graphs (SWRG) were introduced in [2] as a generalization of strongly regular graphs. Recently, a simple numerical condition bearing on the homogeneous weights of three-weight codes over rings was introduced to check if the coset graph of their dual codes is SWRG [4, 6]. In parallel, in [5, 8, 9, 10, 11, 15, 12, 13, 14], Shi *et al.* determined the algebraic structures of one-weight codes, two-weight and three-weight codes over different finite rings. Especially, Shi *et al.* considered the construction of one-Lee weight and two-Lee weight projective codes over \mathbb{Z}_4 in [9]. Later, the authors also determined the linearity of these codes completely in [12].

Inspired by the above works, we revisit \mathbb{Z}_4 codes. The alphabet \mathbb{Z}_4 has been an important and popular example of ring alphabet ever since the paper [3], where, already the coset graph of the Preparata code is used to construct a distance regular graph of diameter three.

We give two special constructions of projective three-weight codes over \mathbb{Z}_4 , with explicit weight distributions. Using the weight information, the spectrum of the coset graph of the dual codes are determined. Using a spectral condition of [2], these graphs are shown to be 3-SWRG.

This work is organized as follows. In Section 2, we recall some background and notations about linear codes over \mathbb{Z}_4 , and their coset graphs. In Section 3, some useful conditions for a linear code to have three-Lee weight over \mathbb{Z}_4 are given. The structures of three-Lee weight projective linear codes are discussed in Section 4. Moreover, we also give some examples to illustrate the results. In section 5, we discuss the linearity of the Gray images of these codes. Section 6 concludes the article.

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2. Definitions and Notation

2.1. \mathbb{Z}_4 codes. In this section, we first recall the codes over \mathbb{Z}_4 in [4]. Let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, and $\mathbb{Z}_4^n = \{(x_1, x_2, \dots, x_n) | x_i \in \mathbb{Z}_4, 1 \leq i \leq n\}$. A linear code C over \mathbb{Z}_4 of length n is a \mathbb{Z}_4 -submodule of \mathbb{Z}_4^n . The Lee weights of $0, 1, 2, 3 \in \mathbb{Z}_4$ are $0, 1, 2, 1$ respectively. For any $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{Z}_4^n$, we define $W_L(\mathbf{x}) = \sum_{i=1}^n W_L(x_i)$. Each element $x \in \mathbb{Z}_4$ has a 2-adic expansion $x = \alpha(x) + 2\beta(x)$, where $\alpha(x), \beta(x) \in \mathbb{F}_2$, the Gray map from \mathbb{Z}_4 to \mathbb{F}_2^2 is given by $\Phi(x) = (\beta(x), \alpha(x) + \beta(x))$. Define $\alpha(\mathbf{x}) = (\alpha(x_1), \dots, \alpha(x_n))$, where $\alpha(0) = \alpha(2) = 0$ and $\alpha(1) = \alpha(3) = 1$. This map can be extended to \mathbb{Z}_4^n naturally. Φ is a weight-preserving map from $(\mathbb{Z}_4^n, \text{Lee weight})$ to $(\mathbb{F}_2^{2n}, \text{Hamming weight})$.

Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ be two elements of \mathbb{Z}_4^n , the inner product of \mathbf{x} and \mathbf{y} in \mathbb{Z}_4^n is defined by $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n$, and the componentwise multiplication $*$ of \mathbf{x} and \mathbf{y} is $\mathbf{x} * \mathbf{y} = (x_1y_1, x_2y_2, \dots, x_ny_n)$, where the operation is performed in \mathbb{Z}_4 . The dual code of C is defined as $C^\perp = \{\mathbf{x} \in \mathbb{Z}_4^n | \mathbf{x} \cdot \mathbf{y} = 0, \forall \mathbf{y} \in C\}$. A Lee weight projective code C of length n over \mathbb{Z}_4 is a linear code such that the minimum Lee weight of its dual code is at least three.

It is well known that a nonzero linear code C over \mathbb{Z}_4 has a generator matrix which after a suitable permutation of the coordinates can be written in the form (see [3])

$$(1) \quad G_{k_1, k_2} = \begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2D \end{pmatrix}$$

where I_{k_1} and I_{k_2} denote the $k_1 \times k_1$ and $k_2 \times k_2$ identity matrices, respectively, A and D are \mathbb{Z}_2 -matrices, B is a \mathbb{Z}_4 -matrix, and $|C| = 4^{k_1}2^{k_2}$.

2.2. Graphs. An *eigenvalue* of a graph Γ (i.e., an eigenvalue of its adjacency matrix) is called a *restricted eigenvalue* if there is a corresponding eigenvector which is not a multiple of the all-one vector $\mathbf{1}$. Note that for an η -regular connected graph, the restricted eigenvalues are simply the eigenvalues different from η .

Definition 2.1. Let T be a finite abelian group and $S \subseteq T$ a subset satisfying $S = -S$ and $0_T \notin S$. The corresponding *Cayley graph* $C(T, S)$ has vertex set equal to T ; two vertices $g, h \in T$ are adjacent in $C(T, S)$ iff $g - h \in S$.

For a precise definition of a Cayley graph attached to a code in a canonical way we refer to [7]. Now, we recall the relation between the weight distribution of a linear code over \mathbb{Z}_4 and the eigenvalues of the syndrome graph of its dual code. This extension of Lemma 3.4 in [1] was derived in Theorem 3.4 in [7], thus its proof is omitted here.

Theorem 2.2. *Suppose that C is a regular, projective linear code over \mathbb{Z}_4 with Lee weights w_i and corresponding weight distribution $A_i = |\{x \in C; W_L(x) = w_i\}|$. Then the eigenvalues of $\Gamma(C^\perp)$ are $2n - 2w_i$ with multiplicity A_i .*

3. SOME PRELIMINARIES

We motivate the constructions of the next section by two necessary conditions for a \mathbb{Z}_4 -code to be a three-weight code.

Lemma 3.1. *Let C be a linear code over \mathbb{Z}_4 with type $4^{k_1}2^{k_2}$ of length n . There is no three-Lee weight projective linear code when $k_1 = 1$.*

Proof. Suppose there is a three-Lee weight projective linear code when $k_1 = 1$. Let $\mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_{k_2}$ be the rows of generator matrix G_{1,k_2} . Let $\mathbf{r}_0 = (r_{00}, r_{01}, \dots, r_{0,n-1})$. We claim that r_{0j} can only take 1 or 3 for $0 \leq j \leq n-1$. Otherwise, if $r_{0j} = 0$ or 2, there is $(0, \dots, 2, \dots, 0)$ in its dual code C^\perp , this is a contradiction since C is projective. Thus $\omega_1 = w_L(\mathbf{r}_0) = n$, $\omega_2 = w_L(2\mathbf{r}_0) = 2n$. We assume that $\omega_3 = 2n_2$, where n_2 is the number of 2 in the row \mathbf{r}_i for $1 \leq i \leq k_2$, thus $w_L(2\mathbf{r}_0 + \mathbf{r}_i) = 2n - 2n_2$ it is easy to check that $2n - 2n_2$ can't equal to ω_1, ω_2 and ω_3 . This completes the proof. \square

We can generalize Lemma 1 to the general case by the similar proof, thus we omit the proof here.

Lemma 3.2. *Let C be a linear code over \mathbb{Z}_4 with type $4^{k_1}2^{k_2}$ of length n , where $k_1 \geq 1$ and $k_2 \geq 1$. Let $\mathbf{r}_1, \dots, \mathbf{r}_{k_1}, \mathbf{r}_{k_1+1}, \dots, \mathbf{r}_{k_1+k_2}$ be the rows of the generator matrix G_{k_1,k_2} . If $\mathbf{r} = (r_{00}, r_{01}, \dots, r_{0,n-1})$ is a linear combination of the first k_1 rows and r_{0j} can only take 1 or 3 for $0 \leq j \leq n-1$, then there is no three-Lee weight projective linear code.*

Here are two fundamental lemmas.

Lemma 3.3. *Let C be an N -Lee weight linear code of length n with the nonzero Lee weights w_1, w_2, \dots, w_N over \mathbb{Z}_4 and G be its generator matrix. If C' is generated by*

$$G' = \begin{pmatrix} G & G & G & G \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{pmatrix},$$

where \mathbf{i} is the row vector (i, i, \dots, i) of length n , $i \in \mathbb{Z}_4$, then C' is an $(N+1)$ -Lee weight linear code of length $n' = 4n$ with the nonzero Lee weights $w'_1 = 4w_1, w'_2 = 4w_2, \dots, w'_N = 4w_N$ and $w'_{N+1} = 4n$. Moreover, $A'_{w_1} = A_{w_1}, A'_{w_2} = A_{w_2}, \dots, A'_{w_N} = A_{w_N}$, and $A'_{w_{N+1}} = 3|C|$, where A'_{w_i} is the number of the codewords of weight w'_i in C' , $1 \leq i \leq N+1$, and A_{w_j} is the number of the codewords of weight w_j in C , $1 \leq j \leq N$.

Proof. It is easily seen that $n' = 4n$, $w'_i = 4w_i$ and $A'_{w_i} = A_{w_i}$, $1 \leq i \leq N$. Without loss of generality, let $\mathbf{c}' = (\mathbf{c} \ \mathbf{c} \ \mathbf{c} \ \mathbf{c})$ and $\mathbf{r}_0 = (\mathbf{0} \ \mathbf{1} \ \mathbf{2} \ \mathbf{3})$, where $\mathbf{c} \in C$ and \mathbf{i} is the row vector (i, i, \dots, i) of length n , $i \in \mathbb{Z}_4$. Thus $W_L(\mathbf{r}_0) = W_L(2\mathbf{r}_0) = W_L(3\mathbf{r}_0) = 4n$. Denote the number of i in \mathbf{c} by n_i , $i \in \mathbb{Z}_4$.

(1) If the order of \mathbf{c} is 2, then $n = n_0 + n_2$, and

$$\begin{aligned} W_L(\mathbf{r}_0 + \mathbf{c}') &= W_L(\mathbf{c}) + W_L(\mathbf{c} + \mathbf{1}) + W_L(\mathbf{c} + \mathbf{2}) + W_L(\mathbf{c} + \mathbf{3}) \\ &= 2n_2 + n + 2(n - n_2) + n = 4n. \end{aligned}$$

(2) If the order of \mathbf{c} is 4, then $n = n_0 + n_1 + n_2 + n_3$, and

$$\begin{aligned} W_L(\mathbf{r}_0 + \mathbf{c}') &= W_L(\mathbf{c}) + W_L(\mathbf{c} + \mathbf{1}) + W_L(\mathbf{c} + \mathbf{2}) + W_L(\mathbf{c} + \mathbf{3}) \\ &= (n_1 + 2n_2 + n_3) + (n_0 + 2n_1 + n_2) \\ &\quad + (2n_0 + n_1 + n_3) + (n_0 + n_2 + 2n_3) \\ &= 4(n_1 + n_3 + n_2 + n_0) = 4n. \end{aligned}$$

Similarly, we can also prove $W_L(2\mathbf{r}_0 + \mathbf{c}') = W_L(3\mathbf{r}_0 + \mathbf{c}') = 4n = w'_{N+1}$. Therefore, $A'_{w_{N+1}} = 3(|C| - 1) + 3 = 3|C|$. \square

Remark. Obviously, if $w_i = n$ for some i , then C' is still an N -Lee weight code and $A'_{w_i} = A_{w_i} + 3|C|$.

Similar to the discussion of Lemma 3.3, we have the following lemma.

Lemma 3.4. *Let C be an N -Lee weight linear code of length n with the nonzero Lee weights w_1, w_2, \dots, w_N over \mathbb{Z}_4 and G be its generator matrix. If C' is generated by*

$$G' = \begin{pmatrix} G & G \\ \mathbf{0} & \mathbf{2} \end{pmatrix},$$

where \mathbf{i} is the row vector (i, i, \dots, i) of length n , $i \in 2\mathbb{Z}_4$, then C' is an $(N+1)$ -Lee weight linear code of length $n' = 2n$ with the nonzero Lee weights $w'_1 = 2w_1, w'_2 = 2w_2, \dots, w'_N = 2w_N$ and $w'_{N+1} = 2n$. Moreover, $A'_{w'_1} = A_{w_1}, A'_{w'_2} = A_{w_2}, \dots, A'_{w'_N} = A_{w_N}$, and $A'_{w'_{N+1}} = |C|$, where $A'_{w'_i}$ is the number of the codewords of weight w'_i in C' , $1 \leq i \leq N+1$, and A_{w_j} is the number of the codewords of weight w_j in C , $1 \leq j \leq N$.

Remark. Obviously, if $w_i = n$ for some i , then C' is still an N -Lee weight code and $A'_{w_i} = A_{w_i} + |C|$.

Let C be an N -Lee weight linear code of length n with type $4^s 2^t$ over \mathbb{Z}_4 and G be its generator matrix. Assume the nonzero Lee weights of C are w_1, w_2, \dots , and w_N . If $N = 2$, $w_1 \neq n$, and $w_2 \neq n$, or $N = 3$, $w_i = n$, for some $i \in \{1, 2, 3\}$, then we can construct a three-Lee weight code with type $4^{k_1} 2^{k_2}$, where $k_1 \geq s + 1$, and $k_2 \geq t + 1$.

4. Constructions

In this section, we will give the constructions of three-weight codes with type $4^{k_1} 2^{k_2}$ over \mathbb{Z}_4 .

Proposition 4.1. *Let C be a linear code over \mathbb{Z}_4 with type $4^2 2^0$ of length 3 with the following generator matrix:*

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}.$$

It is easy to check that C is a two-Lee weight linear code with the nonzero Lee weights $w_1 = 2$, and $w_2 = 4$. Moreover, $A_{w_1} = 6$, and $A_{w_2} = 9$.

We can generalize Proposition 4.1 to the general case by Lemma 3.3 and Lemma 3.4.

Theorem 4.2. *Let C_1 be the linear code over \mathbb{Z}_4 with type $4^2 2^{k_2}$ of length $n = 3 \cdot 2^{k_2}$ with generator matrix*

$$(2) \quad G_{k_2+2, n} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} \\ G_{0, k_2} & G_{0, k_2} & G_{0, k_2} \end{pmatrix}$$

where \mathbf{i} is the row vector $(\underbrace{i, i, \dots, i}_{2^{k_2}}, c_{3, j}, \dots, c_{k_2+2, j})^\top$, $i = 1$ or 2 and $(c_{3, j}, \dots, c_{k_2+2, j})^\top$, $c_{i, j} = 0$ or 2 , $3 \leq$

$i \leq k_2 + 2, 1 \leq j \leq 2^{k_2}$ runs over all distinct column vectors of G_{0, k_2} . Then C_1 is a three-Lee weight code with the nonzero weights $\omega_1 = 2^{k_2+1}, \omega_2 = 3 \cdot 2^{k_2}, \omega_3 = 2^{k_2+2}$, of respective frequencies $A_{\omega_1} = 6$, $A_{\omega_2} = 2^{k_2+4} - 16$, $A_{\omega_3} = 9$.

Proof. Let $\mathbf{r}_{01}, \mathbf{r}_{02}, \mathbf{r}_1, \dots, \mathbf{r}_{k_2}$ be the rows of generator matrix $G_{k_2+2, n}$, where

$$\mathbf{r}_{01} = (\underbrace{1, 1, \dots, 1}_{2^{k_2}}, \underbrace{1, 1, \dots, 1}_{2^{k_2}}, \underbrace{2, 2, \dots, 2}_{2^{k_2}}), \mathbf{r}_{02} = (\underbrace{2, 2, \dots, 2}_{2^{k_2}}, \underbrace{1, 1, \dots, 1}_{2^{k_2}}, \underbrace{1, 1, \dots, 1}_{2^{k_2}}).$$

It is easy to check that $W_L(p_{01}\mathbf{r}_{01} + p_{02}\mathbf{r}_{02}) = 2^{k_2+1}$ or 2^{k_2+2} unless $p_{01} = p_{02} = 0$ for any $p_{0i} \in \mathbb{Z}_4$, $1 \leq i \leq 2$. Moreover, we can also prove that $W_L(p_{01}\mathbf{r}_{01} + p_{02}\mathbf{r}_{02} + p_1\mathbf{r}_1 + \dots + p_{k_2}\mathbf{r}_{k_2}) = 3 \cdot 2^{k_2}$ unless $p_1 = \dots = p_{k_2} = 0$ for any $p_{0i} \in \mathbb{Z}_4$, $1 \leq i \leq 2$, $p_j \in \mathbb{F}_2$, $1 \leq j \leq k_2$. Hence C_1 contains 9 codewords with weight 2^{k_2+2} , 6 codewords of weight 2^{k_2+1} and $2^{k_2+4} - 16$ codewords with weight $3 \cdot 2^{k_2}$. \square

Example 1. If $k_1 = 2$ and $k_2 = 2$, then $n = 12$ and $\omega_1 = 8$, $\omega_2 = 12$, $\omega_3 = 16$, according to Theorem 4.2, there exists a three-Lee weight code with the generator matrix:

$$G_{2,2} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \end{pmatrix}.$$

Corollary 4.3. *The coset graph of C_1^\perp is a 3-SWRG.*

Proof. From [2] we know that a graph with three restricted eigenvalues is 3-SWRG iff they add up to zero. Translating in terms of weights of C_1 by Theorem 2.2 we see that this equivalent to check that $\omega_1 + \omega_2 + \omega_3 = 3n$. This is easy to check by the explicit formulas of Theorem 4.2. \square

In the above theorem, if we assume

$$(3) \quad G_{2,k_2} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} \\ G_{0,k_2} & G_{0,k_2} & G_{0,k_2} \end{pmatrix} = (G_1^2, G_2^2, G_3^2),$$

then we have the following result.

Theorem 4.4. *Let C_2 be the linear code over \mathbb{Z}_4 with type $4^{k_1}2^{k_2}$ of length $n = 3 \cdot 2^{2k_1+k_2-4}$ with a generator matrix G_{k_1, k_2} , defined inductively as follows. G_{2, k_2} is as above. Assume $G_{m, k_2} = (G_1^m, G_2^m, G_3^m)$, and if \mathbf{i} is the row vector (i, i, \dots, i) , $i \in \mathbb{Z}_4$, $m \geq 2$ define G_{m+1, k_2} as*

$$(4) \quad \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ G_1^m & G_1^m & G_1^m & G_1^m & G_2^m & G_2^m & G_2^m & G_2^m & G_3^m & G_3^m & G_3^m & G_3^m \end{pmatrix}.$$

Then C_2 is a three-Lee weight code with the nonzero weights $\omega_1 = 2^{2k_1+k_2-3}$, $\omega_2 = 3 \cdot 2^{2k_1+k_2-4}$, $\omega_3 = 2^{2k_1+k_2-2}$, of respective frequencies $A_{\omega_1} = 6$, $A_{\omega_2} = 2^{2k_1+k_2} - 16$, and $A_{\omega_3} = 9$, where $k_1 \geq 2$ and $k_2 \geq 1$.

Proof. If $k_1 = 2$, the result is proved by Theorem 2. Assume the result is valid for any $k_1 = m$. Namely, if $G_{m, k_2} = (G_1^m, G_2^m, G_3^m)$ is the generator matrix of a three-Lee weight code with the nonzero weights $\omega_1 = 2^{2m+k_2-3}$, $\omega_2 = 3 \cdot 2^{2m+k_2-4}$ and $\omega_3 = 2^{2m+k_2-2}$. Now we consider the case when $k_1 = m + 1$, the generator matrix of the linear code C is (4), it is easily seen that C is a three-Lee weight code with the nonzero weight $\bar{\omega}_1 = 2^{2m+k_2-1} = 4\omega_1$, $\bar{\omega}_2 = 3 \times 2^{2m+k_2-2} = 4\omega_2$, $\bar{\omega}_3 = 2^{2m+k_2} = 4\omega_3$. By induction hypothesis, C is a three-Lee weight code over \mathbb{Z}_4 . \square

Example 2. If $k_1 = 3, k_2 = 1$, then $n = 24, \omega_1 = 16, \omega_2 = 24, \omega_3 = 32$, according to Theorem 3, there is a three-Lee weight code with the generator matrix $G_{3,1}$:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \end{pmatrix}$$

where

$$G_{2,1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 \end{pmatrix}.$$

The analogue of Corollary 4.3 is the following result, the proof of which is omitted.

Corollary 4.5. *The coset graph of C_2^\perp is a 3-SWRG.*

Here we give another method to construct three-Lee weight codes.

Proposition 4.6. *Consider the code C generated by the following generator matrix:*

$$G_0 = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 1 & 1 & 2 \end{pmatrix}.$$

It is easy to check that C is a three-Lee weight linear code with the nonzero Lee weights $w_1 = 4, w_2 = 8$, and $w_3 = 6$. Moreover, $A_{w_1} = 18, A_{w_2} = 21$, and $A_{w_3} = 24$.

Theorem 4.7. *Let C_3 be a linear code over \mathbb{Z}_4 with type $4^{k_1}2^{k_2}$ of length n . If the columns of the generator matrix $G_{(k_1+k_2) \times n}$ are all distinct nonzero vectors*

$$(c_1, c_2, \dots, c_{k_1}, c_{k_1+1}, \dots, c_{k_1+k_2})^T,$$

where $(c_1, c_2, \dots, c_{k_1})^T$ is one column of G_0 defined above, $c_i \in \mathbb{Z}_4, 4 \leq i \leq k_1$ when $k_1 \geq 4$, and $c_j = 0$ or $2, k_1+1 \leq j \leq k_1+k_2$, then C_3 is a three-Lee weight code of length $n = 3 \times 2^{2k_1+k_2-5}$ with the nonzero Lee weights $w_1 = 2^{2k_1+k_2-4}, w_2 = 2^{2k_1+k_2-3}$ and $w_3 = 3 \times 2^{2k_1+k_2-5}$. Moreover, $A_{w_1} = 18, A_{w_2} = 21$, and $A_{w_3} = 4^{k_1}2^{k_2} - 40$, respectively.

Proof. Denote the generator matrix of C_3 by G_{k_1, k_2} .

(1) If $k_1 = 3, k_2 = 0$, then $G_{3,0} = G_0$, and the result is valid from the statement of Proposition 4.6. Assume the result is valid for any $k_2 = m$ and we denote the nonzero weights by w'_1, w'_2 and w'_3 . Since

$$G_{3, m+1} = \begin{pmatrix} G_{3, m} & G_{3, m} \\ \mathbf{0} & \mathbf{2} \end{pmatrix},$$

where \mathbf{i} is the row vector (i, i, \dots, i) of length $3 \times 2^{m+1}, i \in 2\mathbb{Z}_4$, according to Lemma 3.4 and the remark of Lemma 3.4, C_3 is a three-Lee weight code with the nonzero Lee weights $w_1 = 2w'_1, w_2 = 2w'_2$ and $w_3 = 2w'_3$. By induction hypothesis, the results are valid.

(2) It is easy to check that

$$G_{k_1+1, k_2} = \begin{pmatrix} G_{k_1, k_2} & G_{k_1, k_2} & G_{k_1, k_2} & G_{k_1, k_2} \\ \mathbf{0}' & \mathbf{1}' & \mathbf{2}' & \mathbf{3}' \end{pmatrix},$$

where \mathbf{i}' is the row vector (i, i, \dots, i) of length $3 \times 2^{2k_1+k_2-5}$, $i \in \mathbb{Z}_4$. Similar to the proof of (1), we can prove C_3 is a three-Lee weight code by Lemma 3.3 and the following remark of Lemma 3.3. \square

Example 3. If $k_1 = 3, k_2 = 1$, then $w_1 = 8, w_2 = 16$, and $w_3 = 12$, then, according to Theorem 4.7, there is a three-Lee weight code with the generator matrix:

$$(5) \quad G_1 = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Like Corollary 2, we have the following graphical consequence, whose proof is omitted.

Corollary 4.8. *The coset graph of C_3^\perp is 3-SWRG.*

Here we prove the codes we get are projective.

Theorem 4.9. *The codes C_2 and C_3 obtained in Theorem 3 and Theorem 4 are projective, respectively.*

Proof. Take C_3 for example. Let C_3^\perp be the dual code of C_3 , G be the generator matrix of C_3 , and $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ be the columns of G , where n is the length of C_3 . It's easy to check that every column of G contains 1 and any two columns of G are not multiple of each other by ± 1 . If $\mathbf{c} \in C_3^\perp$ and $\mathbf{c} \neq \mathbf{0}$, then we only need to prove $W_L(\mathbf{c}) \geq 3$. Obviously, $W_L(\mathbf{c})$ cannot be 1 because all columns are nonzero. Assume $W_L(\mathbf{c}) = 2$. Since every column of G contains at least a 1, \mathbf{c} cannot be written as $\mathbf{c} = (0, \dots, 0, 2, 0, \dots, 0)$. Then we have 2 cases:

Case 1 Without loss of generality, let $\mathbf{c} = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$, where $c_i = c_j = 1, i < j$. Then $\mathbf{x}_i + \mathbf{x}_j = \mathbf{0}$. Thus, $\mathbf{x}_j = 3\mathbf{x}_i$, \mathbf{x}_i and \mathbf{x}_j are in proportion by the unit 3, contradiction.

Case 2 Without loss of generality, let $\mathbf{c} = (0, \dots, 0, 1, 0, \dots, 0, 3, 0, \dots, 0)$, where $c_i = 1, c_j = 3, i < j$. Then $\mathbf{x}_i + \mathbf{x}_j = \mathbf{0}$. Thus, $\mathbf{x}_j = \mathbf{x}_i$, contradiction.

Therefore, C_3 is projective. Similarly, we can prove C_2 is projective. \square

5. Linearity of Gray Images

For our purpose, we first require the following classic lemma of [3, Th.4].

Lemma 5.1. *If C is a linear code over \mathbb{Z}_4 , with generators $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$, then $\Phi(C)$ is linear if and only if $2\alpha(\mathbf{x}_i) * \alpha(\mathbf{x}_j)$ in C for all i, j , satisfying $1 \leq i \leq j \leq m$.*

We can now state and prove the following results.

Theorem 5.2. *The Gray image $\phi(C_2)$ of C_2 is nonlinear.*

Proof. It is enough to prove the result is valid for $k_1 = 2$. Let \mathbf{x}_1 and \mathbf{x}_2 be two row vectors of $G_{k_2+2, n}$. Clearly, if either \mathbf{x}_1 or \mathbf{x}_2 is from the last k_2 rows, it is easy to check that $2\alpha(\mathbf{x}_1) * \alpha(\mathbf{x}_2) = 0 \in C_2$. When

$$\mathbf{x}_1 = (\underbrace{1, 1, \dots, 1}_{2^{k_2}}, \underbrace{1, 1, \dots, 1}_{2^{k_2}}, \underbrace{2, 2, \dots, 2}_{2^{k_2}}), \mathbf{x}_2 = (\underbrace{2, 2, \dots, 2}_{2^{k_2}}, \underbrace{1, 1, \dots, 1}_{2^{k_2}}, \underbrace{1, 1, \dots, 1}_{2^{k_2}}).$$

We have

$$2\alpha(\mathbf{x}_1) * \alpha(\mathbf{x}_2) = \underbrace{(0, 0, \dots, 0)}_{2^{k_2}} \underbrace{(2, 2, \dots, 2)}_{2^{k_2}} \underbrace{(0, 0, \dots, 0)}_{2^{k_2}}.$$

The weight of $2\alpha(\mathbf{x}_1) * \alpha(\mathbf{x}_2)$ is ω_1 . According to Theorem 4.2, $A_{\omega_1} = 6$. However, we can easily check that $2\alpha(\mathbf{x}_1) * \alpha(\mathbf{x}_2)$ is not in C_2 , because it is not the linear combination of the first two rows. Therefore C_2 is nonlinear. \square

Example 4. Consider the code defined in the Example 1. Let \mathbf{x}_1 and \mathbf{x}_2 denote the first and second rows of $G_{2,2}$, respectively. Then we can get

$$2\alpha(\mathbf{x}_1) * \alpha(\mathbf{x}_2) = 2\alpha(111111112222) * \alpha(222211111111) = (000022220000).$$

However, we can easily check that $2\alpha(\mathbf{x}_1) * \alpha(\mathbf{x}_2)$ is not in C_2 , because it is not the linear combination of the first two rows. Therefore C_2 is nonlinear.

Similar to Theorem 6, we have the following theorem.

Theorem 5.3. *The Gray image $\phi(C_3)$ of C_3 is nonlinear.*

Proof. It is easy to check that the first three rows of G_{k_1, k_2} can be written as:

$$\underbrace{G_0 \ G_0 \ \cdots \ G_0}_{4^{k_1-3}2^{k_2}},$$

which is permutation-equivalent to the matrix

$$(6) \quad G'_0 = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{2} \end{pmatrix},$$

where \mathbf{i} is the row vector (i, i, \dots, i) of length $4^{k_1-3}2^{k_2}$, $i \in \mathbb{Z}_4$. Let \mathbf{r}_1 and \mathbf{r}_3 be the first and third rows of G'_0 , respectively, then $W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_3)) = 2 \times 4^{k_1-3}2^{k_2} = 2^{2k_1+k_2-5}$. By Theorem 4.7, C_3 is a three-Lee weight code with the nonzero Lee weights $w_1 = 2^{2k_1+k_2-4}$, $w_2 = 2^{2k_1+k_2-3}$ and $w_3 = 3 \times 2^{2k_1+k_2-5}$. Thus, $2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_3) \notin C_3$. By Lemma 5.1, $\phi(C_3)$ is nonlinear. \square

Example 5. Consider the code defined in Example 3. Let \mathbf{r}_1 and \mathbf{r}_3 denote the first and third rows of G_1 in (5), respectively. Then the Lee weight of

$$2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_3) = 2\alpha(123000123000) * \alpha(112112112112) = (200000200000)$$

is 4, which can not be contained in C_3 since C_3 is a three-Lee weight code with $w_1 = 8$, $w_2 = 16$, and $w_3 = 12$.

6. Conclusion and open problems

In this paper we have constructed two infinite families of three-weight projective codes over \mathbb{Z}_4 . From there strongly walk-regular graphs have been built. There are two main directions of enquiry from that point on. One is to look at other families of three-weight \mathbb{Z}_4 -codes. The other is to extend this research to other rings. In particular, chain rings are the natural candidates to replace \mathbb{Z}_4 as an alphabet for our three-weight codes.

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