



**HAL**  
open science

# Substructuring based FEM-BEM coupling for Helmholtz problems

Antonin Boisneault, Marcella Bonazzoli, Xavier Claeys, Pierre Marchand

► **To cite this version:**

Antonin Boisneault, Marcella Bonazzoli, Xavier Claeys, Pierre Marchand. Substructuring based FEM-BEM coupling for Helmholtz problems. WAVES 2024 - 16th International Conference on Mathematical and Numerical Aspects of Wave Propagation, Jun 2024, Berlin, Germany. 10.17617/3.MBE4AA . hal-04701091

**HAL Id: hal-04701091**

**<https://hal.science/hal-04701091v1>**

Submitted on 18 Sep 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## Substructuring based FEM-BEM coupling for Helmholtz problems

Antonin Boisneault<sup>1,\*</sup>, Marcella Bonazzoli<sup>2</sup>, Xavier Claeys<sup>3</sup>, Pierre Marchand<sup>1</sup>

<sup>1</sup>POEMS, CNRS, Inria, ENSTA Paris, Institut Polytechnique de Paris, 91120 Palaiseau, France

<sup>2</sup>Inria, UMA, ENSTA Paris, Institut Polytechnique de Paris, 91120 Palaiseau, France

<sup>3</sup>Sorbonne Université, CNRS, Université de Paris, Inria, Laboratoire Jacques-Louis Lions, F-75005 Paris, France

\*Email: antonin.boisneault@inria.fr

### Abstract

This talk concerns the solution of the Helmholtz equation in a medium composed of a bounded heterogeneous domain and an unbounded homogeneous one. Such problems can be expressed using classical FEM-BEM coupling techniques. We solve these coupled formulations using iterative solvers based on substructuring Domain Decomposition Methods (DDM), and aim to develop a convergence theory, with fast and guaranteed convergence.

A recent article [1] proposed a substructuring Optimized Schwarz Method, with a non-local exchange operator, for Helmholtz problems on a bounded domain with classical conditions on its boundary (Dirichlet, Neumann, Robin). The variational formulation of the problem can be written as a bilinear application associated with the volume and another with the surface, for which, under certain sufficient assumptions, convergence of the DDM strategy is guaranteed.

In this presentation we show how some specific FEM-BEM coupling methods fit, or not, the previous framework, in which we consider Boundary Integral Equations (BIEs) instead of classical boundary conditions. In particular, we prove that the symmetric Costabel coupling satisfies the framework assumptions, implying that the convergence is guaranteed.

**Keywords:** Helmholtz equation, substructuring DDM, FEM-BEM coupling, guaranteed convergence, symmetric Costabel coupling

### 1 DDM substructuring framework

We consider the scattering of an acoustic wave by a penetrable heterogeneous bounded domain  $\Omega_1$ , with refractive index  $n \in L^\infty(\Omega_1)$ , assuming that  $\Omega_0 := \mathbb{R}^d \setminus \overline{\Omega_1}$  is homogeneous.

For each  $\Omega_j$ , consider interior  $\gamma_N^j$  (resp.  $\gamma_D^j$ ) and exterior  $\gamma_{N,c}^j$  (resp.  $\gamma_{D,c}^j$ ) Neumann (resp. Dirichlet) traces, the orientation given by the unit normal vector to  $\partial\Omega_j$  outgoing with respect

to  $\Omega_j$ . Denote trace operators  $\gamma^j := (\gamma_D^j, \gamma_N^j)$ ,  $\gamma_c^j := (\gamma_{D,c}^j, \gamma_{N,c}^j)$  and the interface  $\Gamma := \partial\Omega_1$ .

We seek to solve the transmission problem

$$\begin{aligned} \Delta u_1 + n(x)k^2 u_1 &= f, & \text{in } \Omega_1, \\ \Delta u_0 + k^2 u_0 &= 0, & \text{in } \Omega_0, \\ \gamma_D^0(u_0) - \gamma_D^1(u_1) &= g, & \text{on } \Gamma, \\ \gamma_N^0(u_0) + \gamma_N^1(u_1) &= h, & \text{on } \Gamma, \end{aligned} \quad (1)$$

+ Sommerfeld condition

with  $k^2 > 0$ ,  $f \in H^{-1}(\Omega_1)$  a source term. It is known that there exists a unique solution in  $H_{\text{loc}}(\Delta, \mathbb{R}^d)$ .

Problem (1) can be reformulated equivalently as an equation posed only on the interface  $\Gamma$ , following the Optimized Schwarz Method proposed by [2, 4], and involving a non-local exchange operator  $\Pi$  — as in [1] — and a scattering operator  $S$ :

$$(\text{Id} + \Pi S)\mathbf{q} = \mathbf{b}$$

for  $\mathbf{q} \in H^{-\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ .

Ensuring  $\Pi S$  is a contraction implies strong coercivity of  $\text{Id} + \Pi S$ , and thus convergence of iterative schemes (at continuous level), such as Richardson or GMRes. The framework of [1] identifies several sufficient assumptions, which will be described afterward, to ensure this contractivity. Our contribution consists in using BIEs from FEM-BEM coupling techniques, instead of classical Dirichlet, Neumann or Robin boundary conditions, and studying if those assumptions are satisfied.

On the one hand, (1) is supposed to rewrite as: find  $\mathbf{u}$  in  $H(\Omega_1 \times \Gamma) := H^1(\Omega_1) \times H^{-\frac{1}{2}}(\Gamma)$  such that

$$A_{\Omega_1 \times \Gamma}(\mathbf{u}) = \mathbf{rhs}, \quad (2)$$

where  $\mathbf{rhs}$  stands for some source term, and  $A_{\Omega_1 \times \Gamma}$  takes the form:

$$\begin{aligned} \langle A_{\Omega_1 \times \Gamma}(u, p), (v, q) \rangle &:= \langle A_{\Omega_1}(u|_{\Omega_1}), v|_{\Omega_1} \rangle \\ &+ \langle A_\Gamma(\gamma_D^1(u), p), (\gamma_D^1(v), q) \rangle \end{aligned} \quad (3)$$

where  $A_{\Omega_1}$  corresponds to a volume part which accounts for the first line of Equation (1).  $A_\Gamma : H^B(\Gamma) \rightarrow H^B(\Gamma)^*$  corresponds to a pure surface part accounting for the outer boundary contribution, with  $H^B(\Gamma) := H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ . On the other hand, our convergence theory requires:

- (a)  $\text{Im}(\langle A_\Gamma(\phi, p), (\overline{\phi}, \overline{p}) \rangle) \leq 0 \quad \forall(\phi, p)$ ,
- (b)  $\text{range}(A_{\Omega_1 \times \Gamma})$  is closed in  $H(\Omega_1 \times \Gamma)^*$ .

## 2 Application with FEM-BEM coupling

The expression of  $A_\Gamma$  relies on Boundary Integral Operators. We introduce  $G_k$  the Helmholtz  $k$ -outgoing Green kernel, and

$$\begin{aligned} \text{SL}_k(q)(\mathbf{x}) &:= \int_\Gamma G_k(\mathbf{x} - \mathbf{y})q(\mathbf{y}) \, d\sigma(\mathbf{y}), \\ \text{DL}_k(v)(\mathbf{x}) &:= \int_\Gamma (\gamma_N^0 G_k)(\mathbf{x} - \mathbf{y})v(\mathbf{y}) \, d\sigma(\mathbf{y}), \\ \mathcal{A} &:= \begin{pmatrix} \{\gamma_D^0\} \circ \text{DL}_k & \{\gamma_D^0\} \circ \text{SL}_k \\ \{\gamma_N^0\} \circ \text{DL}_k & \{\gamma_N^0\} \circ \text{SL}_k \end{pmatrix}, \end{aligned}$$

with  $\{\gamma_\bullet^0\} := (\gamma_\bullet^0 + \gamma_{\bullet,c}^0)/2$ ,  $\bullet \in \{D, N\}$ , the mean of interior and exterior traces.

We choose here to focus on two classical FEM-BEM coupling methods: the Johnson-Nédélec one [5], and the symmetric Costabel one [3]. Then, the assumptions (2) and (3) on  $A_{\Omega_1 \times \Gamma}$  rewriting are fulfilled. Indeed,  $\langle A_{\Omega_1}(u), v \rangle := \int_{\Omega_1} \nabla u(x) \cdot \nabla v(x) \, dx - \int_{\Omega_1} k^2 n(x)u(x)v(x) \, dx$ , and the volume equation writes

$$\langle A_{\Omega_1}(u), v \rangle + \int_\Gamma \gamma_N^0(u)\gamma_D^1(v) = \int_{\Omega_1} f v + \int_\Gamma h \gamma_D^1(v).$$

Johnson-Nédélec method is then obtained by adding to the the volume equation a BIE derived using the first line of operator  $\mathcal{A}$ .

A second BIE can also be derived using the second line of operator  $\mathcal{A}$ , allowing to express  $\gamma_N^0(u)$  as a sum of BIEs. Symmetric Costabel method is then derived by replacing  $\gamma_N^0(u)$  in the volume equation with this new expression, and by adding to the volume equation the BIE derived using the first line of operator  $\mathcal{A}$  (the same as in Johnson-Nédélec method).

Using Dirichlet transmission condition for both methods, problem (1) is thus reformulated with  $A_{\Omega_1 \times \Gamma}$  satisfying (3) — see Theorems 1 and 2 for expressions of  $A_\Gamma$ . Defining the skew-symmetric pairing

$$[(u, p), (v, q)] := \langle q, u \rangle - \langle p, v \rangle,$$

which puts in duality  $H^B(\Gamma)$  with itself, we state our new contributions.

**Theorem 1** *The operator  $A_\Gamma$  for Johnson-Nédélec coupling writes:  $\forall(\phi, p) \in H^B(\Gamma), (v, q) \in H^B(\Gamma)$*

$$\begin{aligned} \langle A_\Gamma(\phi, p), (v, q) \rangle &:= \left[ \left( \frac{\text{Id} - \mathcal{A}}{2} \right) (\phi, p), (0, q) \right] \\ &\quad + [(\phi, p), \theta(v, 0)] \end{aligned}$$

with  $\theta(v, q) = (-v, q)$ . We show that

$$\exists(\Phi, P) \in H^B(\Gamma) \mid \text{Im}(\langle A_\Gamma(\Phi, P), (\overline{\Phi}, \overline{P}) \rangle) > 0.$$

In the case of Johnson-Nédélec coupling, assumption (a) appears too restrictive. Yet, this does not discard convergence of DDM schemes.

**Theorem 2** *The operator  $A_\Gamma$  for Costabel coupling writes:  $\forall(\phi, p) \in H^B(\Gamma), (v, q) \in H^B(\Gamma)$*

$$\begin{aligned} \langle A_\Gamma(\phi, p), (v, q) \rangle &:= - \left[ \frac{\mathcal{A}}{2} (\phi, p), (v, q) \right] \\ &\quad + \frac{1}{2} [(\phi, p), \theta(v, q)] \end{aligned}$$

from which it can be shown

$$\forall(\phi, p) \in H^B(\Gamma), \quad \text{Im}(\langle A_\Gamma(\phi, p), (\overline{\phi}, \overline{p}) \rangle) \leq 0.$$

In the case of Costabel coupling, a generalized Gårding inequality can be also derived, implying that  $A_{\Omega_1 \times \Gamma}$  is Fredholm with index 0. Thus, except for spurious resonances (Dirichlet eigenvalues of Laplacian problem in  $\Omega_1$ ), assumption (b) is fulfilled, and the convergence of the substructuring method can be proved.

## References

- [1] Claeys, Non-local optimized Schwarz method for the Helmholtz equation with physical boundaries, *SIAM J. Math. Anal.* **55** (2023), 7490–7512
- [2] Collino, Joly, Lecouvez, Exponentially convergent non overlapping domain decomposition methods for the Helmholtz equation, *ESAIM: Math. Model. and Numer. Anal.* **54** (2020), 775–810
- [3] M. Costabel, Symmetric Methods for the Coupling of Finite Elements and Boundary Elements, *Springer Berl. Heidelb.* (1987), 411–420.
- [4] Després, Domain decomposition method and the Helmholtz problem, *SIAM* (1991), 44–52.
- [5] Johnson and Nédélec, On the coupling of boundary integral and finite element methods, *Math. Comput.* **35** (1980), 1063–1079.