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Salem numbers less than the plastic constant

J.-M. Sac-Épée*

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Dedicated to the memory of Professor Georges Rhin

Abstract

A list of Salem numbers less than 1.3 is available on M. Mossinghoff's website ([2]). This list is certified complete up to degree 44 in [9], and it includes only one Salem number of degree 46. The objective of the present work is to advance the understanding of Salem numbers by extending the list [2] through the provision of a list of Salem numbers less than the plastic constant, denoted by η , which is approximately equal to 1.324718. The algorithmic approach used is based on Integer Linear Programming.

Keywords— Salem numbers, plastic constant, reciprocal polynomials

MSC: 11K16

1 Introduction

In the following lines, we recall some basic elements related to Salem numbers.

1.1 Definitions and examples

Definition 1.1. Let $\alpha \in \mathbb{C}$.

- We say that α is an *algebraic number* if α is a root of a non-zero polynomial in $\mathbb{Z}[x]$, the ring of polynomials in one indeterminate x with coefficients in \mathbb{Z} .
- We say that α is an *algebraic integer* if α is a root of a *monic* polynomial in $\mathbb{Z}[x]$.
- The *degree* of α is defined as the degree of its minimal polynomial.

Example 1.2. The algebraic integer i has degree 2, because its minimal polynomial is $x^2 + 1$.

Definition 1.3. A *Salem number* is an algebraic integer τ such that:

- τ is a real number strictly greater than 1,
- All of its Galois conjugates (i.e., the other roots of its minimal polynomial) have an absolute value less than or equal to 1,
- At least one of its Galois conjugates has an absolute value equal to 1.

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Definition 1.4. The minimal polynomial of a Salem number is called a *Salem polynomial*.

Example 1.5. The Salem number $\tau_0 = 1.176280\dots$ has the minimal polynomial $P_0(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$, known as the *Lehmer polynomial*. The real number τ_0 is the smallest known Salem number to date (D. Lehmer, 1933).

1.2 Some properties

Proposition 1.6.

- The minimal polynomial of a Salem number τ is reciprocal and has an even degree greater than or equal to 4.
- The real number $\frac{1}{\tau}$ is one of the Galois conjugates of τ .
- All the roots of the minimal polynomial of τ , except for τ and $\frac{1}{\tau}$, have an absolute value equal to 1.

A very rich and comprehensive reference on Salem numbers is [4].

1.3 Small Salem numbers

Salem numbers less than 1.3 are referred to as *small* in the literature. In 1977, D. Boyd provided a list of 39 such numbers in [5]. The following year, D. Boyd added 4 more small Salem numbers in [6], before 4 additional small Salem numbers were discovered and published by M. Mossinghoff in 1993 ([7]). Notably, among the 4 values provided by M. Mossinghoff, one has a degree of 46, which is the highest known degree for a small Salem number.

The compilation of these sources thus yields a list of 47 small Salem numbers, which is available on M. Mossinghoff's website ([2]). This list provides, for each Salem number, its degree and the coefficients of its minimal polynomial. Naturally, only the relevant part of the coefficients is given, as these polynomials are reciprocal.

This list was first certified complete up to degree 40 in [8], and later up to degree 44 in [9]. Only the degree 46 polynomial mentioned earlier does not belong to the certified complete list.

In this work, we propose to extend the list [2] by providing Salem numbers that are smaller than the plastic constant $1.324718\dots$. The choice of the plastic constant as the threshold may seem somewhat arbitrary, but in fact the aim is to go slightly beyond 1.3 in order to advance our understanding of Salem numbers by compiling an extended list. As a reminder, the plastic constant is the unique real root of the polynomial $x^3 - x - 1$. This constant is also the smallest Pisot number, where a Pisot number is a real algebraic integer strictly greater than 1 whose Galois conjugates all have absolute values strictly less than 1. The plastic constant plays an important role in number theory. For example, C. Smyth shows in [1] that if the Mahler measure of a polynomial in $\mathbb{Z}[x]$ is less than this constant, then the polynomial must be reciprocal.

To compile our extended list, we will use a variation of a method based on integer linear optimization, which has allowed us to obtain interesting results in various contexts. Notably, we followed this approach to provide, in [10], a list of degree 16 irreducible monic polynomials with integer coefficients, all real and strictly positive roots and minimal trace (specifically, with trace 29). Similarly, we provided examples of Salem numbers of degree 34 and trace -3 in [11] and [12].

Here, it is not the trace of the Salem number that interests us, but its size. Our approach, based on integer linear optimization, is well-suited for addressing this problem. In the following lines, we only outline the method, as it is described in detail in the aforementioned articles.

2 Algorithmic approach

2.1 Reduction to a half-degree problem

The minimal polynomial P of a Salem number τ , as mentioned above, is a reciprocal polynomial of even degree (say $2d$). It can be written as

$$P(x) = x^{2d} + c_1x^{2d-1} + c_2x^{2d-2} + \dots + c_{d-1}x^{d+1} + c_dx^d + c_{d-1}x^{d-1} + \dots + c_2x^2 + c_1x + 1,$$

and can be rewritten as

$$P(x) = x^d Q\left(x + \frac{1}{x}\right),$$

where Q is a degree d monic polynomial of $\mathbb{Z}[x]$, whose roots are all real numbers and lie within the interval $(-2, 2)$, except for one root that is strictly greater than 2. The polynomial Q is obtained explicitly as follows: the first coefficients of P are the integers $1, c_1, c_2, \dots, c_d$ (the other coefficients of P are determined by symmetry), and then

$$Q(x) = C_{d+1}(x) + c_1C_d(x) + c_2C_{d-1}(x) + \dots + c_dC_1(x),$$

where the C_i are the Chebyshev polynomials associated with the interval $(-2, 2)$. As a reminder, we have

$$\begin{aligned} C_1(x) &= 1 \\ C_2(x) &= x \\ C_3(x) &= x^2 - 2 \\ C_4(x) &= x^3 - 3x \\ C_5(x) &= x^4 - 4x^2 + 2 \\ C_6(x) &= x^5 - 5x^3 + 5x \\ &\vdots \end{aligned}$$

For example, the Lehmer polynomial $P_0(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ can be written as

$$P_0(x) = C_6(x) + c_1C_5(x) + c_2C_4(x) + c_3C_3(x) + c_4C_2(x) + c_5C_1(x) = x^5Q_0\left(x + \frac{1}{x}\right),$$

where $Q_0(x) = x^5 + x^4 - 5x^3 - 5x^2 + 4x + 3$. It is easy to verify that Q_0 has 4 roots in $(-2, 2)$ and a single root strictly greater than 2.

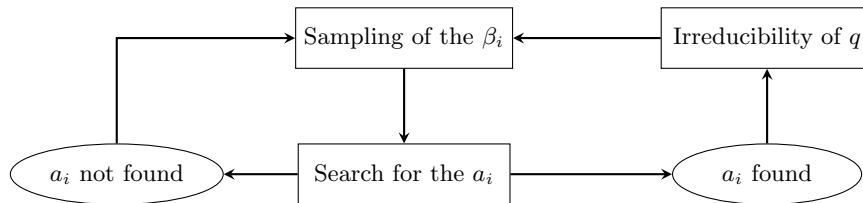
Since P can be easily obtained from Q and vice versa, the search for Salem numbers of degree $2d$, which is equivalent to the search for Salem polynomials of degree $2d$, can be reduced to the search for monic polynomials of degree d , with integer coefficients, and all real roots, where $d - 1$ roots lie in $(-2, 2)$ and the last one is strictly greater than 2. In this work, we have an additional constraint: we want the polynomials P (of degree $2d$) to be Salem polynomials, and furthermore, we want their unique real root greater than 1 to also be strictly less than η . When transitioning from the search for P to the search for Q , this constraint translates to the fact that the unique root of Q greater than 2 must also be strictly less than $\eta + \frac{1}{\eta}$.

2.2 Reformulation as an Optimization Problem

As mentioned earlier, we have an efficient algorithmic approach based on integer linear optimization for the search for polynomials of type Q , which we outline below.

Consider an irreducible polynomial in $\mathbb{Z}[x]$, $q(x) = x^d + \sum_{i=0}^{d-1} a_i x^i$, with all real roots, where $d-1$ roots lie in $(-2, 2)$ and a single root is strictly greater than 2 and less than $\eta + \frac{1}{\eta}$. For such a polynomial q , any $(d+1)$ -tuple $(-2, \beta_1, \dots, \beta_{d-2}, 2, \eta + \frac{1}{\eta})$ such that $\beta_1, \dots, \beta_{d-2}$ and 2 separate its d roots, satisfies the condition that $q(-2), q(\beta_1), q(\beta_2), \dots, q(\beta_{d-2}), q(2), q(\eta + \frac{1}{\eta})$ alternate in sign, which provides $d+1$ linear constraints with respect to the integer variables a_i . These linear constraints form an integer linear optimization problem whose feasibility we must investigate. In other words, we must test the existence of integers a_i that satisfy all the linear constraints. The idea is to randomly and independently sample a large number of $(d-2)$ -tuples $(\beta_1, \dots, \beta_{d-2})$ within the interval $(-2, 2)$ and use the $(d+1)$ -tuples $(-2, \beta_1, \dots, \beta_{d-2}, 2, \eta + \frac{1}{\eta})$ as root separators for the potential relevant polynomials. Solving the integer linear optimization problem (which we do for each sampled $(\beta_1, \dots, \beta_{d-2})$) does not involve maximization or minimization, as there is no objective function to optimize. The algorithm is reduced to a feasibility study for each randomly drawn $(d-2)$ -tuple.

In practice, once a $(d-2)$ -tuple $(\beta_1, \dots, \beta_{d-2})$ is drawn, we solve the problem. If no d -tuple (a_0, \dots, a_{d-1}) is found that satisfies all the constraints, we immediately draw another sample. Conversely, if such a d -tuple (a_0, \dots, a_{d-1}) is found, this means we have found a polynomial q (of degree d) of the type Q , which allows us to reconstruct a Salem polynomial P of the desired type. If q is irreducible in $\mathbb{Z}[x]$, we add the polynomial P reconstructed from q to our list. If q is not irreducible, we keep the factor of q , say q_1 , that has a root between 2 and $\eta + \frac{1}{\eta}$, and we reconstruct a Salem polynomial from q_1 . The following flowchart summarizes the approach used:



3 Implementation and results

3.1 Implementation

The program was entirely written in the *Julia* language. Polynomial manipulations were performed using the *Polynomials* library, while irreducibility tests were carried out with the *Nemo* library.

The integer linear optimization problems for each sample were solved using the *Gurobi* library ([13]).

The results were obtained on a multiprocessor server featuring two Intel Xeon Gold 6138 CPUs, each with 20 cores (2 threads per core), totaling 80 logical processors (hyperthreading is enabled).

3.2 Results

The implementation of the algorithmic approach presented in this work did not lead to the discovery of new Salem numbers smaller than 1.3. In the following table, we therefore only present the Salem numbers between 1.3 and η resulting from our

calculations. To make the table easy to read, we follow the presentation chosen in [2], in which the different columns respectively show the degrees of the Salem numbers, the numbers themselves, and the relevant coefficients of the minimal polynomials (since these are reciprocal)

2d	Salem numbers	Coefficients of Minimal Polynomial
12	1.302268805094	1 -1 0 0 0 -1 1
32	1.302721444014	1 -1 0 -1 0 1 0 0 0 -1 1 -1 1 0 0 0 -1
32	1.303283348964	1 1 0 -1 -2 -2 -1 0 1 1 0 -1 -1 -1 0 1 1
30	1.303385419369	1 -1 0 0 -1 0 0 0 1 0 0 1 -1 0 0 -1
26	1.304697625411	1 0 -1 -1 0 0 0 0 0 0 0 1 0 -1
22	1.305131378642	1 -1 0 0 -1 0 0 1 0 0 0 -1
38	1.306473537533	1 -2 1 0 0 0 0 0 -1 1 0 -1 1 0 -1 0 1 0 -1 1
44	1.308071085577	1 0 1 -1 0 -2 -1 -2 -1 -1 0 0 1 0 1 0 1 0 1 0 1 0 1
16	1.308409006213	1 1 0 -1 -1 -1 -1 -1 -1
36	1.308966300288	1 0 1 -1 0 -2 -1 -2 -1 -1 0 0 1 0 1 0 1 0 1
22	1.310180863375	1 0 0 -1 -1 -1 -1 0 0 1 1 1
22	1.312566632631	1 -1 -1 1 0 0 0 -1 0 1 0 -1
16	1.312773239526	1 0 0 -1 0 -1 0 -1 0
12	1.315914431926	1 0 0 0 -1 -1 -1
40	1.316069252718	1 -1 0 0 -1 0 1 -1 1 0 -1 0 0 -1 1 0 0 1 0 -1 1
14	1.318197504432	1 -1 0 -1 1 0 0 -1
20	1.319869661883	1 1 0 -1 -1 -1 -1 -1 -1 -1 -1
14	1.321101848259	1 -1 0 0 -1 0 1 -1
22	1.322014239618	1 1 0 -1 -1 -1 -1 -1 -1 -1 -1 -1
22	1.322692457903	1 0 0 -1 0 -1 0 -1 0 -1 0 -1
18	1.323198173512	1 -1 -1 1 0 0 0 -1 0 1
18	1.323576201647	1 0 0 0 0 -1 -1 -1 -1 -2
26	1.323859346186	1 1 0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
24	1.324071761641	1 -1 0 0 0 -1 1 -1 0 0 0 -1 1
28	1.324231319862	1 1 0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
20	1.324351296267	1 -1 0 -1 0 1 0 0 0 -1 1
24	1.324441583513	1 0 0 0 -1 -1 -1 -1 -1 0 0 0 1
30	1.324509572546	1 0 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1
32	1.324560796779	1 1 0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
30	1.324599405908	1 -1 0 0 0 -1 1 -1 0 0 0 -1 1 -1 0 0
30	1.324628516131	1 0 -1 -1 0 1 0 -1 -1 0 1 0 -1 -1 0 1
34	1.324650470079	1 0 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1
34	1.324667030364	1 0 -1 0 0 -1 0 0 -1 0 0 -1 0 0 -1 0 0 -1
32	1.324679524104	1 0 0 -1 -1 -1 0 0 1 0 0 -1 -1 -1 0 0 1
30	1.324688951068	1 0 0 0 0 0 -1 -1 -1 -1 -2 -2 -2 -2 -2 -3
32	1.324696064745	1 -2 1 0 0 0 0 -1 1 0 0 -1 1 -1 1 0 -1
40	1.324701433212	1 1 0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
40	1.324705484871	1 0 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0
36	1.324708542862	1 0 0 0 -1 -1 -1 -1 -1 0 0 0 1 0 0 0 -1 -1 -1
36	1.324710850968	1 0 0 0 0 -1 -1 -1 -1 -2 -1 -1 -1 -1 0 0 0 0 1
40	1.324712593127	1 0 -1 -1 0 1 0 -1 -1 0 1 0 -1 -1 0 1 0 -1 -1 0 1
38	1.324713908140	1 -1 0 0 -1 0 1 -1 1 0 -1 0 0 -1 1 0 0 0 0 -1
46	1.324714900752	1 1 0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
46	1.324715650017	1 0 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1
46	1.324716215600	1 0 -1 0 0 -1 0 0 -1 0 0 -1 0 0 -1 0 0 -1 0 0 -1 0 0 -1
44	1.324716642533	1 -1 0 -1 1 0 0 -1 0 0 1 -1 0 -1 1 0 0 -1 0 0 1 -1 0
50	1.324716964809	1 1 0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
48	1.324717208083	1 -1 0 0 0 -1 1 -1 0 0 0 -1 1 -1 0 0 0 -1 1 -1 0 0 0 -1 1

2d	Salem numbers	Coefficients of Minimal Polynomial
52	1.324717391722	1 1 0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
48	1.324717530347	1 0 0 -1 -1 -1 0 0 1 0 0 -1 -1 -1 0 0 1 0 0 -1 -1 -1 0 0 1
44	1.324717634990	1 -1 0 0 -1 1 -1 0 0 0 1 -1 1 -1 0 1 -2 1 -1 0 1 -1 1
54	1.324717713983	1 0 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1 0 -1
56	1.324717773612	1 1 0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
54	1.324717818625	1 -1 0 0 0 -1 1 -1 0 0 0 -1 1 -1 0 0 0 -1 1 -1 0 0 0 -1 1 -1 0 0
58	1.324717852604	1 1 0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
54	1.324717878254	1 -1 0 -1 1 0 0 -1 0 0 1 -1 0 -1 1 0 0 -1 0 0 1 -1 0 -1 1 0 0 -1
58	1.324717897616	1 0 -1 0 0 -1 0 0 -1 0 0 -1 0 0 -1 0 0 -1 0 0 -1 0 0 -1 0 0 -1
50	1.324717912232	1 0 0 0 -1 -1 -1 -1 0 -1 0 0 -1 0 0 0 1 0 1 0 -1 0 -1 -1 0 -1
60	1.324717946211	1 0 0 0 -1 -1 -1 -1 -1 0 0 0 1 0 0 0 -1 -1 -1 -1 -1 0 0 0 1 0 0 0 -1 -1 -1

4 Conclusion

A new Salem number smaller than 1.3 would of course have been welcome. In any case, our line of research has proven fruitful, as dozens of Salem numbers lower than the plastic constant are now available. It should be noted, of course, that this algorithmic approach does not guarantee the exhaustiveness of the tables it provides. But since it is based on random sampling, the repeated discovery of values already present in our list suggests that any missing values are likely to be few. In particular, the repeated discovery of known Salem numbers below 1.3 using this algorithmic approach, without finding new ones, suggests that List [2] might be complete well beyond degree 44 certified in [9].

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