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Collisionless kinetic theory for saltation over a rigid, bumpy bed

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We employ the methods of statistical mechanics to obtain closures for the balance equations of momentum and fluctuation kinetic energy that govern the ballistic motion of grains rebounding at a bed driven by turbulent or nonturbulent shearing fluids, in the absence of mid-trajectory collisions. We obtain semi-analytical solutions to steady and fully developed saltation over horizontal, rigid, bumpy beds for the vertical profiles of particle concentration and stresses, and fluid and particle velocity; these compare favorably with measurements in discrete-element numerical simulations in a wide range of conditions that apply to Earth and other planetary environments. The predictions of the particle horizontal mass flux and its scaling with the amount of particles in the system, the properties of the carrier fluid and the intensity of the shearing also agree with numerical simulations and wind tunnel experiments.

I. Introduction

Saltation, that is the motion of solid particles in a gravitational field through successive jumps and rebounds from a base driven by a shearing flow, has been established as the main mode of transport of wind-blown sand (Andreotti 2004; Bagnold 1941; Charru *et al.* 2013; Owen 1964; Valance *et al.* 2015) and is crucial in the dynamics of dunes (Sauermann *et al.* 2001). Although originally conceived as the transport of sand grains by the wind on Earth, it has been recognized as significant also for the transport of sand or gravel in water on Earth (Abbott and Francis 1977; Ancey *et al.* 2002; Fernandez Luque and Van Beek 1976; Niño and García 1998), basalt particles on Venus (Greeley *et al.* 1984; Iversen and White 1982) and on Mars (Iversen *et al.* 1976; Iversen and White 1982), and ice particles on Titan (Burr *et al.* 2015). While these studies considered the carrier fluid to be turbulent, saltation in viscous, non-turbulent fluids has also been investigated (Charru and Mouilleron-Arnould 2002; Ouriemi *et al.* 2009; Seizilles *et al.* 2014), given its practical relevance in limiting the transport capacity of oil in pipes (Dall'Acqua *et al.* 2017; Leporini *et al.* 2019).

There is a large body of mathematical models of saltation in the literature (see, e.g., the reviews of Kok *et al.* 2012; Pähz et al. 2020; Valance *et al.* 2015). As highlighted by Valance *et al.* (2015), the different models can be grouped into two main categories, Lagrangian or Eulerian approaches, based upon the point of view adopted in the description of the particle motion.

In many Lagrangian approaches (Anderson *et al.* 1988; Creyssels *et al.* 2009; Kok and Renno 2009; Werner 1990), an assigned distribution of particle ballistic trajectories is calculated by solving their differential momentum balances, governed by fluid drag and gravity, when immersed in a fluid with a prescribed velocity profile and for a given distribution of initial take-off velocity at the basal boundary. The calculation is repeated once the fluid velocity profile and the distribution of the take-off velocities are updated using a simple constitutive relation for the fluid shear stress (in the case of turbulent fluid, usually based on a mixing length approach) and a suitable set of boundary conditions that govern the impact of the particles at the base (Beladjine *et al.* 2007; Crassous *et al.* 2007; Oger *et al.* 2005), until a steady state is attained.

Simpler Lagrangian approaches in which the distribution of the particle trajectories is replaced by one (Jenkins and Valance 2014) or two (Andreotti 2004) modes have also been recently proposed and

1 successfully compared against experiments and numerical simulations. In particular, the one species models,
2 also called Periodic Trajectory (PT) models, have been applied to saltation in both turbulent (Berzi *et al.* 2016)
3 and viscous (Valance and Berzi 2022) shearing flows, for values of the ratio of the grain-to-fluid mass
4 densities encountered on Earth and other planetary bodies (Berzi *et al.* 2017). PT models are simple enough
5 to allow even for fully analytical solutions of steady and fully developed saltation, and permit the
6 determination, with sufficient accuracy, of global quantities, such as the particle mass flux, as a function of
7 the intensity of the shearing flows. On the other hand, many variables of interest, such as the distribution
8 with height of the particle concentration and the total depth of the saltation layer, are poorly predicted. It is
9 also not obvious how to extend this model to unsteady and/or boundary-valued (inhomogeneous) problems.

10 Another family of Lagrangian approaches is based on the framework of Discrete Element Method
11 (DEM, Cundall and Strack 1979), and solves the Newton's laws of motion for the individual grains that are
12 allowed to collide with other particles and with the base, while the surrounding fluid is replaced by discrete
13 forces (such as drag and buoyancy) acting on the particles themselves (Tsuji *et al.* 1993). The forces
14 transmitted by the fluid on the particles are then introduced, after a change of sign, in the fluid momentum
15 balance, ensuring an instantaneous two way coupling. These Discrete-Continuum (DC) numerical simulations
16 are a powerful tool, in that they greatly reduce the number of assumptions necessary to solve for the
17 transport process. For instance, the dynamics of the impact of the particles with the base must not be
18 modelled in advance, but is an output of the simulations. Likewise, the possibility of interparticle collisions
19 above the base is naturally accounted for. On the other hand, the number of particles that is feasible to
20 simulate is severely limited by computational power and nowhere near to the actual number of grains
21 involved in real-scale applications. DC simulations have been applied to saltation in turbulent (Durán *et al.*
22 2012; Pähz *et al.* 2015; Pähz and Durán 2020b; Ralaiarisoa *et al.* 2020) and viscous flows (Valance and Berzi
23 2022) and provide a large quantity of measurements, some of which are simply unattainable in physical
24 experiments. Hence, they serve as severe tests of more sophisticated approaches.

25 Eulerian approaches, in which both the fluid and the saltating particles are treated as two
26 superimposed continuum phases offer the most promising perspective for modelling large scale phenomena.
27 In the model of Sauermaun *et al.* (2001), the motion of the two phases is depth-averaged over the saltation
28 layer, and cannot therefore predict the distribution of the variables of interest with height. A key point of the
29 model is that the wind profile in the saltation layer is determined by assuming an exponential distribution of
30 the particle shear stress there. The same assumption is adopted in more recent Eulerian models (Lämmel *et al.*
31 2012; Pähz *et al.* 2012). Assuming an *a priori* distribution is crucially different from phrasing a constitutive
32 relation for the particle shear stress, and obtaining the distribution from the usual momentum balance.

33 Jenkins *et al.* (2010) were the first to propose a constitutive relation for the particle shear stress
34 based on averaging the equations governing the trajectory of the single particles and substituting averaging
35 of products with products of averaging. They also equated the particle pressure to the product of the particle
36 concentration and the granular temperature, the mean square of the particle velocity fluctuations. They
37 assumed, as in Creyssels *et al.* (2009), that the latter is uniformly distributed with the distance from the base,
38 which holds only if the vertical drag on the particles is negligible. From the uniform distribution of the
39 granular temperature, they obtained the exponential decay of the particle concentration and distributions
40 of particle and fluid mean horizontal velocities that agreed reasonably well with experiments (Creysseis *et al.*
41 2009). The model was later extended to deal with unsteadiness and inhomogeneities (Jenkins and Valance
42 2018). Interestingly, the constitutive relation for the particle shear stress of Jenkins *et al.* (2010) coincides
43 with the dilute and collisionless limit of the expression derived by Garzó *et al.* (2012) by solving the Enskog
44 kinetic theory for monodisperse gas-solid flows. This is an indication that both constitutive relations share
45 the same assumption, namely that they only account for the influence of the fluid drag and not gravity nor
46 buoyancy on the particle path.

1 Here, we formulate a collisionless kinetic theory for saltating particles over an horizontal bed. We
 2 distinguish between ascending and descending particles and assume that the velocity distribution function
 3 of the ascending particles is an anisotropic Maxwellian, as in Creyssels et al. (2009). Then, we employ
 4 approximate analytical expressions for the particle trajectories in viscous and turbulent shearing flows to
 5 determine the transfers of horizontal and vertical momentum, and kinetic energy associated with the particle
 6 vertical motion across an horizontal surface. Once averaged over all possible particle trajectories using the
 7 velocity distribution function, we obtain approximate analytical constitutive relations for the particle stresses
 8 and the energy flux of fluctuation kinetic energy.

9 As suggested by Pasini and Jenkins (2005), collisions above the base can be ignored if the mean free
 10 path of kinetic theory is larger than the length of the ballistic trajectory. Given that the mean free path
 11 strongly decreases if the particle concentration increases (Chapman and Cowling 1970), and that the particle
 12 concentration near erodible beds (composed of particles at rest identical to those in saltation) is large (Tholen
 13 *et al.* 2023), we argue that, strictly speaking, collisionless saltation is possible only if it takes place over rigid
 14 beds.

15 We phrase and solve the balances of fluid and particle momenta and particle fluctuation energy for
 16 a steady, fully-developed saltation over an horizontal, rigid, bumpy bed, driven by either a viscous or a
 17 turbulent shearing flow, in the boundary layer approximation, with appropriate boundary conditions. As in
 18 all previous models of saltation, we assume that the fluid velocity is solely in the horizontal direction. This
 19 permits to uncouple the determination of quantities associated with the particle vertical motion, such as
 20 intensity of the vertical velocity fluctuations, particle concentration and normal stress, from those that
 21 involve the particle horizontal motion and are, therefore affected by the flow regime of the fluid, such as
 22 particle shear stress and particle and fluid mean horizontal velocity. Depending on the flow regime of the
 23 fluid, only one or two differential equations must be solved numerically to determine the vertical profiles of
 24 the corresponding quantities. The rest of the profiles, including the depth of the saltation layer, are obtained
 25 analytically.

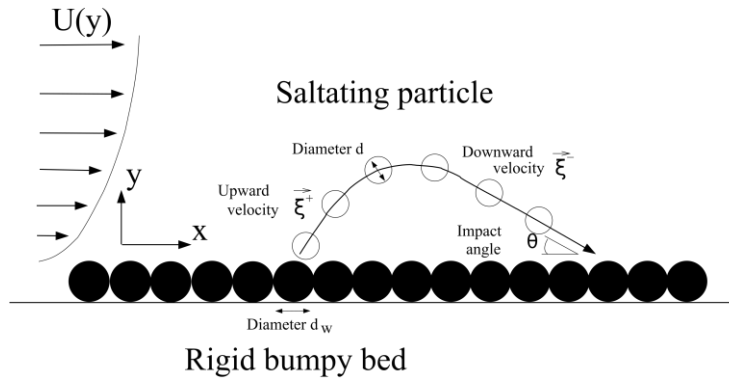
26 After deriving scaling laws for, e.g., the particle mass flux in some special limits, we successfully test
 27 the results of the theory against quasi-2D DC simulations of saltation in viscous and turbulent shearing flows
 28 that were carried out suppressing the possibility of mid-trajectory collisions for a wide range of particle-to-
 29 fluid mass density ratio, ranging from Mars to Earth environments, at different values of the intensity of the
 30 shearing flows, the amount of particles in the system and the viscosity of the fluid. The relation between the
 31 particle mass flux and the amount of particles in the system in wind-tunnel experiments (Ho 2012) is also
 32 satisfactorily captured.

33 In Section II, we present the constitutive relations, the balance equations and the boundary
 34 conditions that we employ to obtain semi-analytical solutions to steady and fully developed saltation over
 35 rigid, bumpy beds. In Section III, we show how to obtain scaling laws for various quantities in the limit of
 36 rarefied saltation in both viscous and turbulent flows, at least when the drag coefficient reduces to its
 37 asymptotic expressions. Comparisons against DC simulations and experiments are detailed in Section IV.
 38 Finally, in Section V, we conclude with a summary of the main findings and an outline of future endeavors.

39 II. Governing equations and semi-analytical solutions

40 The saltating particles are identical spheres of diameter d and mass density ρ^s . A shearing flow of a fluid of
 41 mass density ρ^f and molecular viscosity μ^f drives the flow in the presence of gravity, with g the gravitational
 42 acceleration. The mean horizontal velocities of the particles and the fluid are u and U . We assume that the
 43 flow is steady and uniform, so that the velocities are only functions of the vertical distance from the bed y .
 44 We imagine that the bed is made bumpy by gluing identical particles of diameter d_w , in close contact to each

1 other, over a flat plate. Hence, the ratio of d_w to d is a natural measure of the bed roughness. We characterize
 2 the particles through the fall particle Reynolds number $R = \rho^f \sqrt{g(r-1)/r} d^{3/2} / \mu^f$, where $r = \rho^s / \rho^f$ is the
 3 density ratio. A sketch of the flow configuration is shown in Figure 1. In what follows, all quantities are made
 4 dimensionless using the diameter and mass density of the particles and the reduced gravitational
 5 acceleration, $g(r-1)/r$. Then, lengths, velocities and stresses are expressed in units of d , $[g(r-1)d/r]^{1/2}$ and $\rho^s g(r-1)d/r$, respectively.



8 Fig. 1. Sketch of a particle driven into saltation over a rigid, bumpy bed by a shearing flow.

10 The balance equations for the x and y -momenta of the particles, under steady and fully-developed
 12 conditions, are:

$$13 \quad \frac{d\sigma_y}{dy} = -c; \quad (1)$$

$$14 \quad \frac{ds}{dy} = -cC_D(U - u), \quad (2)$$

15 where σ_y and s are the particle normal stress in the y -direction and the particle shear stress, respectively;
 16 c is the particle volume concentration; and C_D is the drag coefficient that, for a single particle, has a
 17 component independent of the relative velocity between the single particle and the fluid and a component
 18 proportional to the absolute value of that velocity difference. Here, we assume that C_D is independent of y
 19 and equal to its average over all the spherical particles evaluated at the bed,

$$20 \quad C_D = \frac{18}{St} + \frac{0.3}{r} |U_0 - u_0|, \quad (3)$$

21 where $St = rR$ is the fall Stokes number, and U_0 and u_0 are the fluid and particle horizontal velocities evaluated
 22 at the bed (here and in what follows, the subscript 0 refers to quantities evaluated at the bed). We employ
 23 the term *Stokes drag* to refer to situations in which $C_D \approx 18/St$, *form drag* for situations in which
 24 $C_D \approx 0.3|U_0 - u_0|/r$ and *nonlinear drag* for the generic case in which both Stokes and form drag must be
 25 accounted for. A more appropriate expression for the average C_D should also involve the strength of the
 26 particle velocity fluctuations (Jenkins and Hanes 1998). However, including it would have only a small
 27 quantitative effect on the results. Also permitting the drag coefficient to vary along y does not significantly
 28 alter the solution to the continuum model.

29 For flows over rigid beds, we take the no-slip boundary condition for the fluid,

$$30 \quad U_0 = 0. \quad (4)$$

1 If saltating particles with a large value of the coefficient of sliding friction impact a rigid bed, made bumpy by
 2 gluing identical spheres in close contact on a flat plate, at an average small angle θ with respect to the
 3 horizontal (Figure 1), then, using the analysis of Lämmel et al. (2017) in Appendix C,

$$4 \quad u_0 = \alpha_u \frac{1}{C_D \theta}, \quad (5)$$

5 where α_u is a strictly positive coefficient of order unity that is weakly dependent on the rebound properties
 6 of the particles and the flow regime of the fluid.

7 In the boundary layer approximation, the sum of the fluid and particle shear stress, S and s
 8 respectively, is constant and equal to the far-field fluid shear stress, which, in dimensionless terms, is the
 9 Shields parameter:

$$10 \quad s + S = Sh. \quad (6)$$

11 To close the problem, we require constitutive relations for the particle and fluid stresses.

12 In the absence of particle collisions above the bed, the only mechanism responsible for the particle
 13 stresses is the transfer of momentum associated with the particles crossing a reference surface. Hence, the
 14 particle normal stress in the y -direction is simply given by the average vertical flux of y -momentum,

$$15 \quad \sigma_y = c T_y, \quad (7)$$

16 where T_y is the mean square of the vertical velocity fluctuations of the particles.

17 The distribution of the additional hydrodynamic field, T_y , along y is governed by the balance of kinetic
 18 energy associated with the particle vertical motion, that is, the yy -component of the particle second-moment
 19 tensor, which, in a steady and fully-developed flow, and in the absence of mid-trajectory collisions, reduces
 20 to (Saha and Alam 2016, 2017)

$$21 \quad -\frac{dQ_{yyy}}{dy} - 2C_D \sigma_y = 0. \quad (8)$$

22 Here, Q_{yyy} is the yyy -element of the flux of second moment, \mathbf{Q} , a third rank tensor, and $2C_D \sigma_y$ is the
 23 energy dissipation due to the fluid drag.

24 Assuming that the fluid motion is only horizontal and that the drag coefficient is independent of y ,
 25 we can obtain approximate analytical expressions for the trajectories of the saltating particles even in the
 26 turbulent case (Appendix A). In doing so, we distinguish between ascending and descending particles. With
 27 the further assumptions that the velocity distribution of the ascending particles is an anisotropic Maxwellian
 28 (Creysse et al. 2009) and that the vertical velocity of the ascending particles is much larger than $1/C_D$ (the
 29 settling velocity), in Appendix B we derive a simple expression for Q_{yyy} :

$$30 \quad Q_{yyy} = \sqrt{\frac{8}{\pi}} c^+ (T_y^+)^{3/2}. \quad (9)$$

31 where c^+ and T_y^+ are the volume concentration and the mean square of the vertical velocity fluctuations of
 32 the ascending particles, respectively. At least under steady state conditions, c^+ and T_y^+ are related to c and T_y
 33 through (see Appendix B):

$$c = c^+ \left(2 + \frac{3}{4} C_D \sqrt{\frac{2T_y^+}{\pi}} \right); \quad (10)$$

2 and

$$T_y = \frac{4\sqrt{\pi} C_D T_y^+ + 4\sqrt{2T_y^+}}{8\sqrt{\pi} C_D + 3C_D^2 \sqrt{2T_y^+}}. \quad (11)$$

4 Using equations (7), (10) and (11) into equation (1) leads to

$$\frac{dc^+}{dy} = -c^+ \frac{8C_D + 3C_D^2 \sqrt{2T_y^+/\pi}}{4C_D T_y^+ + 4\sqrt{2T_y^+/\pi}} - c^+ \frac{2C_D + \sqrt{2/\pi T_y^+}}{2C_D T_y^+ + 2\sqrt{2T_y^+/\pi}} \frac{dT_y^+}{dy}, \quad (12)$$

6 that, combined with equations (8) and (9), permits to obtain an ordinary differential equation for T_y^+ :

$$\left[\sqrt{8\pi} C_D (T_y^+)^{3/2} + 8T_y^+ \right] \frac{dT_y^+}{dy} = (6 - 4\pi) C_D^2 (T_y^+)^2 - 8T_y^+, \quad (13)$$

8 whose analytical solution, with the boundary condition $T_y^+(y=0) = T_{y,0}^+$, is:

$$\begin{aligned} & 4\sqrt{2\pi(2\pi-3)} \tan^{-1} \left(\frac{C_D}{2} \sqrt{(2\pi-3)T_y^+} \right) \\ & - 2(2\pi-3) \left[\sqrt{2\pi T_y^+} C_D + 2 \log \left(2 - iC_D \sqrt{(2\pi-3)T_y^+} \right) + 2 \log \left(2 + iC_D \sqrt{(2\pi-3)T_y^+} \right) \right] \\ & = (3-2\pi)^2 C_D^2 y + 4\sqrt{2\pi(2\pi-3)} \tan^{-1} \left(\frac{C_D}{2} \sqrt{(2\pi-3)T_{y,0}^+} \right) \\ & - 2(2\pi-3) \left[\sqrt{2\pi T_{y,0}^+} C_D + 2 \log \left(2 - iC_D \sqrt{(2\pi-3)T_{y,0}^+} \right) + 2 \log \left(2 + iC_D \sqrt{(2\pi-3)T_{y,0}^+} \right) \right]. \end{aligned} \quad (14)$$

11 Under the same assumptions that we employ to derive equation (5), a simple dependence of $T_{y,0}^+$
12 calculated at the bed on the impact angle θ can be obtained (Appendix C) in the form:

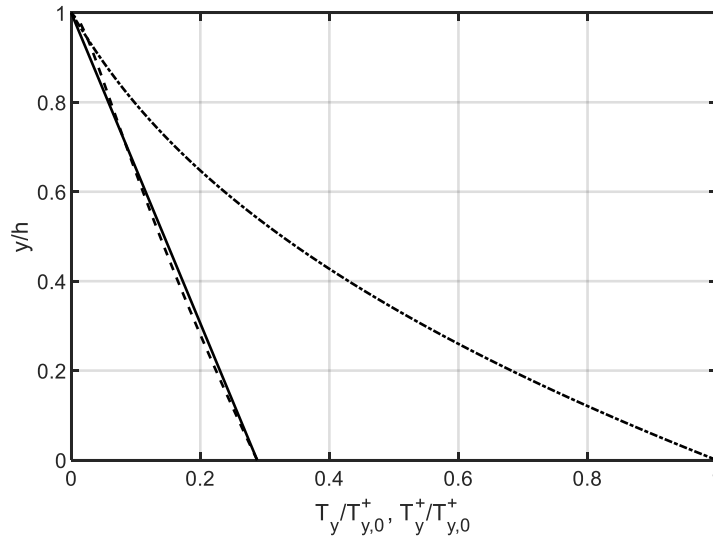
$$T_{y,0}^+ = \alpha_T \frac{1}{C_D^2 \theta} \frac{2d_w}{1+d_w}, \quad (15)$$

14 where α_T is another strictly positive coefficient of order unity that is weakly dependent on the rebound
15 properties of the particles and the flow regime of the fluid. We emphasize that the impact angle, and
16 consequently $T_{y,0}^+$, remains an unknown at this stage of the analysis. We will show how to determine it in
17 the next two sub-sections, when we describe quantities associated with the horizontal motion of the
18 particles, and, therefore, influenced by the flow regime of the fluid.

19 Equation (14) provides the analytical distribution of T_y^+ along y that, inserted into equation (11),
20 would also give the analytical distribution of T_y . As shown in Figure 2, the latter is roughly linear and well
21 approximated as

$$T_y = \frac{h-y}{h} T_{y,0}^+, \quad (16)$$

1 where the depth of the saltation layer, h , is to be determined. The mean square of the particle velocity
 2 fluctuations at the bed, $T_{y,0}$, can be obtained from equation (11), if $T_{y,0}^+$ is known.



3
 4 Fig. 2. Normalized profiles of T_y^+ (dot-dashed line) and T_y (dashed line) obtained from equation (14) and (11),
 5 respectively, when $C_D = 0.18$. The solid line represents the approximate linear distribution of T_y (equation 16).
 6

7 Inserting equation (7) into equation (1), with equation (16), and integrating gives the following
 8 power-law distribution of the particle concentration,

$$9 \quad c = c_0 \left(\frac{h-y}{h} \right)^{\frac{h-T_{y,0}}{T_{y,0}}}, \quad (17)$$

10 where c_0 is the particle concentration at the bed. Integrating equation (1) with equation (17) also gives

$$11 \quad \sigma_y = c_0 T_{y,0} \left(\frac{h-y}{h} \right)^{\frac{h}{T_{y,0}}}. \quad (18)$$

12 After introducing the particle mass hold-up, $M = \int_0^h c dy$, i.e., the particle mass per unit basal area, by
 13 integrating equation (1) over the saltation layer, with $\sigma_y = 0$ at $y = h$, we obtain

$$14 \quad c_0 = \frac{M}{T_{y,0}}. \quad (19)$$

15 Integrating the energy balance (equation 8) with equations (18) and (19), provides

$$16 \quad Q_{yyy} = 2C_D M T_{y,0} \frac{h}{h+T_{y,0}} \left[\left(\frac{h-y}{h} \right)^{\frac{h+T_{y,0}}{T_{y,0}}} - 1 \right] + Q_{yyy,0}, \quad (20)$$

17 where the value of Q_{yyy} at the bed can be obtained from the ratio of equations (9) and (7), with equations
 18 (10), (11) and (19), as

$$Q_{yyy,0} = \frac{4C_D T_{y,0}^+}{C_D \sqrt{2\pi T_{y,0}^+} + 2} M. \quad (21)$$

Given that Q_{yyy} must vanish at the top of the saltation layer, the depth h is determined from equation (20) as

$$h = \frac{Q_{yyy,0}}{2C_D M T_{y,0} - Q_{yyy,0}} T_{y,0}. \quad (22)$$

The governing equations, constitutive relations and analytical results described so far apply, in general, to both saltation in viscous and in turbulent shearing flows and can be calculated only after the determination of the impact angle θ . To proceed, we now must distinguish between the two regimes of the shearing flow.

Before doing that, we support the assumption regarding particle collisions above the bed. As suggested by Pasini and Jenkins (2005), chances of mid-trajectory collisions are low if the mean free path, λ , the average distance travelled by a particle in between two successive collisions predicted by the kinetic theory of granular gases, is less than twice the height of the particle trajectory, here determined by gravity and fluid drag. If we use the expression for the mean free path in a dilute gas of Chapman and Cowling (1970), evaluate this at the bed, and take h as the average height of the particle trajectories, we obtain

$$\lambda_0 = \frac{1}{6c_0 \sqrt{2}} \geq 2h. \quad (23)$$

Given that, from equation (22), $h \sim T_{y,0}$, equation (23) implies that there is a maximum mass hold-up M (for given values of r , St and Sh) above which mid-trajectory collisions play a role. The mean free path is a decreasing function of the particle concentration. If the saltation process were to take place over an erodible—a densely packed assembly of particles identical to those in saltation that can be mobilized due to impacts—rather than a rigid bed, and if the volume concentration is continuous across the interface with the erodible bed, then the theoretical mean free path calculated at the top of the erodible bed, where $c \approx 0.6$, would be much less than the mean free path evaluated at the rigid bed, where $c \ll 1$. Hence, we expect that the mass hold-up at which saltating particles over a rigid bed start to deposit and form an erodible substrate is larger than the maximum mass hold-up for which mid-trajectory collisions can be ignored.

Saltation in viscous shearing flows

In the absence of turbulence, the expression for the fluid shear stress is (Valance and Berzi 2022)

$$S = \frac{1-c}{St} \frac{dU}{dy} \approx \frac{1}{St} \frac{dU}{dy}, \quad (24)$$

where we have neglected the particle concentration with respect to unity.

Equation (24) implies that, in the absence of particles and in the boundary layer approximation that we have employed, the fluid velocity distribution would be linear. To solve for the particle trajectory, we assume in Appendix A that the fluid velocity profile encountered by an ascending particle during its ballistic trajectory above a certain position y is at least locally linear, in the sense that is linear in the region comprised between y and the top of the trajectory. However, we let the slope of the linear profile to change with y . This assumption and the assumptions that we have already employed in the derivation of the expression for Q_{yyy} in equation (9) permit the derivation in Appendix B of a simple expression for the particle shear stress,

$$s = \frac{3}{5} C_D \sigma_y \left(U - u + \frac{2}{3} \frac{1}{C_D^2} \frac{dU}{dy} \right). \quad (25)$$

Interestingly, the particle shear stress does not depend on the particle shear rate, as in Jenkins et al. (2010), but only on the velocity difference and on the fluid shear rate. The physical reason is that the particles do not interact with each other, but only with the surrounding fluid.

As explained in Appendix C, the angle θ between the velocity of the particles before the impact with the bed and the horizontal for saltation in viscous flows is estimated as

$$\theta = \frac{C_D}{St(Sh - s_0)}, \quad (26)$$

where s_0 is the particle shear stress at the bed.

Using equations (4) and (5) in equation (3), with equation (26), we obtain a relationship between the drag coefficient and the particle shear stress at the bed,

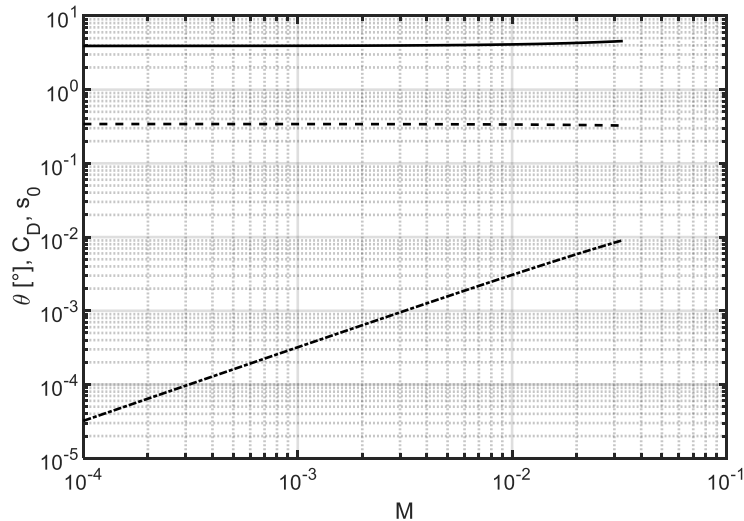
$$C_D^3 = \frac{18}{St} C_D^2 + \frac{0.3}{r} \alpha_u St (Sh - s_0). \quad (27)$$

The particle shear stress at the bed is obtained from equation (25), with equations (4), (5), (6), (19) and (24), as

$$s_0 = \frac{(2 - 3\alpha_u) M St Sh}{5C_D + (2 - 3\alpha_u) M St}, \quad (28)$$

The system of equations (27) and (28) can be solved to determine the drag coefficient and the particle shear stress at the bed once the values of d_w , r , St , Sh and M are given. Then, the impact angle (equation 26), the values of all the variables at the bed (equations 5, 15 and 21), the depth of the saltation layer (equation 22), and the analytical distributions of T_y^+ , T_y , c , σ_y and Q_{yyy} (equations 14, 16-18 and 20) can also be calculated for saltation in viscous shearing flows.

Figure 3 shows the variation of the impact angle, the drag coefficient and the particle shear stress at the bed with the particle mass hold-up, as predicted by equations (26-28), for, e.g., saltation of 100 μm basalt grains in viscous shearing flows on Venus, with $\alpha_u = 0.6$ (see Section IV for more details about the choice of the parameters). The range of the mass hold-up in Figure 3 is that for which equation (23) is satisfied. Notice that for almost the entire range of mass hold-up, the particle shear stress at the bed linearly increases with M , while θ and C_D are constant.



1
2 Fig. 3. Predicted impact angle (in degrees, solid line), drag coefficient (dashed line) and particle shear stress at the bed
3 (dot-dashed line) as functions of the mass hold-up for saltation in viscous shearing flows when $d_w = 1$, $St = 100$, $r = 50$,
4 and $Sh = 0.05$ (with $\alpha_u = 0.6$).
5

6 Using equation (25) in equation (2), with equations (6), (18) and (24), gives a first order, linear, non-
7 homogeneous, differential equation for the particle shear stress,

$$8 \quad \frac{ds}{dy} = - \left[\frac{5}{3} \frac{1}{T_{y,0}} \left(\frac{h-y}{h} \right)^{-1} + \frac{2}{3} \frac{c_0 St}{C_D} \left(\frac{h-y}{h} \right)^{\frac{h-T_{y,0}}{T_{y,0}}} \right] s + \frac{2}{3} \frac{c_0 St Sh}{C_D} \left(\frac{h-y}{h} \right)^{\frac{h-T_{y,0}}{T_{y,0}}}. \quad (29)$$

9 The analytical solution of equation (29) is:

$$10 \quad s = Sh \left(\frac{2 M St}{3 C_D} \right)^{5/3} \Gamma \left(-\frac{2}{3}, \frac{2 M St}{3 C_D} \left(\frac{h-y}{h} \right)^{\frac{h}{T_{y,0}}} \right) \exp \left[\ln \left[\left(\frac{h-y}{h} \right)^{\frac{5}{3} \frac{h}{T_{y,0}}} \right] + \frac{2 M St}{3 C_D} \left(\frac{h-y}{h} \right)^{\frac{h}{T_{y,0}}} \right] \\ - Sh \left(\frac{2 M St}{3 C_D} \right)^{5/3} \Gamma \left(-\frac{2}{3}, \frac{2 M St}{3 C_D} \left(\frac{h-y}{h} \right)^{\frac{h}{T_{y,0}}} \right) \exp \left[\ln \left[\left(\frac{h-y}{h} \right)^{\frac{5}{3} \frac{h}{T_{y,0}}} \right] + \frac{2 M St}{3 C_D} \left(\frac{h-y}{h} \right)^{\frac{h}{T_{y,0}}} \right] \\ + s_0 \exp \left[\ln \left[\left(\frac{h-y}{h} \right)^{\frac{5}{3} \frac{h}{T_{y,0}}} \right] + \frac{2 M St}{3 C_D} \left(\frac{h-y}{h} \right)^{\frac{h}{T_{y,0}}} - \frac{2 M St}{3 C_D} \right], \quad (30)$$

11 in which Γ is the incomplete gamma function.

12 The fluid velocity profile is determined by integrating the constitutive relation (equation 24)
13 $dU/dy = St(Sh - s)$, with the particle shear stress given by equation (30). Unfortunately, there is no general
14 analytical solution. Hence, we obtain the distribution of U by numerically solving this ordinary differential
15 equation, with the no-slip condition, equation (4), at the bed. Once U is determined, the particle horizontal
16 velocity is given by equation (25) as

$$17 \quad u = U - \frac{5}{3} \frac{s}{C_D \sigma_y} + \frac{2 St (Sh - s)}{3 C_D^2}. \quad (31)$$

1 We can now calculate the particle mass flux q per unit basal area through numerical integration as

$$2 \quad q = \int_0^h cudy. \quad (32)$$

3 In the special case of rarefied saltation, that is for $M \rightarrow 0$ and $Sh \gg s$, the analytical solution of
4 equation (29) simplifies to

$$5 \quad s = s_0 \left(\frac{h-y}{h} \right)^{\frac{5}{3}T_{y,0}} + \frac{MStSh}{C_D} \left[\left(\frac{h-y}{h} \right)^{\frac{h}{T_{y,0}}} - \left(\frac{h-y}{h} \right)^{\frac{5}{3}T_{y,0}} \right], \quad (33)$$

6 and the fluid velocity profile is simply

$$7 \quad U = (StSh)y. \quad (34)$$

8 Then, from equations (28), (31), (33) and (34), and the expression for s_0 , the particle horizontal velocity is

$$9 \quad u = \left[\frac{5-2+3\alpha_u}{3} \left(\frac{h-y}{h} \right)^{\frac{2}{3}T_{y,0}} + C_D^2 y - 1 \right] \frac{StSh}{C_D^2}. \quad (35)$$

10 Notice that in the rarefied limit of saltation in viscous shearing flows, the particle and the fluid velocity
11 profiles are independent of the mass hold-up. Using equations (17), (19) and (35) in equation (32), and
12 integrating, gives

$$13 \quad q = MStShT_{y,0} \frac{h}{h+T_{y,0}}. \quad (36)$$

14 *Saltation in turbulent shearing flows*

15 In the case of saltation in turbulent shearing flows, we model the fluid shear stress using the classical mixing
16 length approach (Jenkins *et al.* 2010),

$$17 \quad S = \frac{1-c}{r} \kappa^2 (y+y_0)^2 \left(\frac{dU}{dy} \right)^2 \simeq \frac{\kappa^2 (y+y_0)^2}{r} \left(\frac{dU}{dy} \right)^2, \quad (37)$$

18 where $\kappa = 0.41$ is Von Kármán's constant and $y_0 = \frac{1}{9St} \sqrt{\frac{r}{Sh}} + \frac{k_s}{30} \left[1 - \exp \left(-k_s \frac{St}{26} \sqrt{\frac{Sh}{r}} \right) \right]$ is an expression

19 for the origin of the logarithmic fluid velocity profile that encompasses hydrodynamically smooth, rough and
20 transitional beds (Guo and Julien 2007). The roughness length scale k_s is taken to be $k_s = d_w$ (Ho 2012).

21 Equation (37) implies that, in the absence of particles and in the boundary layer approximation that
22 we have employed, the fluid velocity distribution would be logarithmic. We assume that the fluid velocity
23 profile encountered by an ascending particle during its ballistic trajectory above a certain position y is uniform
24 and equal to the average of the logarithmic profile in the region comprised between y and the top of the
25 trajectory. This average value of U changes with y . This assumption, and the assumptions that we have
26 already employed in the derivation of the expression for Q_{yyy} , equation (9), permits the derivation in Appendix
27 B of an expression for the particle shear stress,

$$s = \frac{3}{5} C_D \sigma_y \left[U - u + \frac{3}{4} \frac{\sqrt{rS}}{\kappa} \ln \left(\frac{\kappa}{C_D} \sqrt{\frac{T_y^+}{rS}} \frac{dU}{dy} + 1 \right) \right]. \quad (38)$$

As in the viscous case of equation (25), the particle shear stress in turbulent shearing flows does not depend on the particle shear rate. The saltating particles, when treated as a continuous medium, can experience shear stress even if the horizontal particle velocity is uniform in the flow domain.

As explained in Appendix C, the impact angle θ for saltation in turbulent flows is estimated as

$$\theta \approx \left[0.85 C_D \frac{\sqrt{r(\text{Sh} - s_0)}}{\kappa} \ln \left(\frac{1}{C_D} \frac{\sqrt{T_{y,0}^+}}{y_0} \right) \right]^{-1}. \quad (39)$$

Using equation (5) in equation (3) gives the drag coefficient in terms of the impact angle,

$$C_D = \frac{9}{\text{St}} + \frac{1}{2} \sqrt{\left(\frac{18}{\text{St}} \right)^2 + \frac{1.2}{r} \alpha_u \frac{1}{\theta}}. \quad (40)$$

Equation (38) evaluated at the bed, with equations (5) and (39) also gives the particle shear stress at the bed as a function of the impact angle,

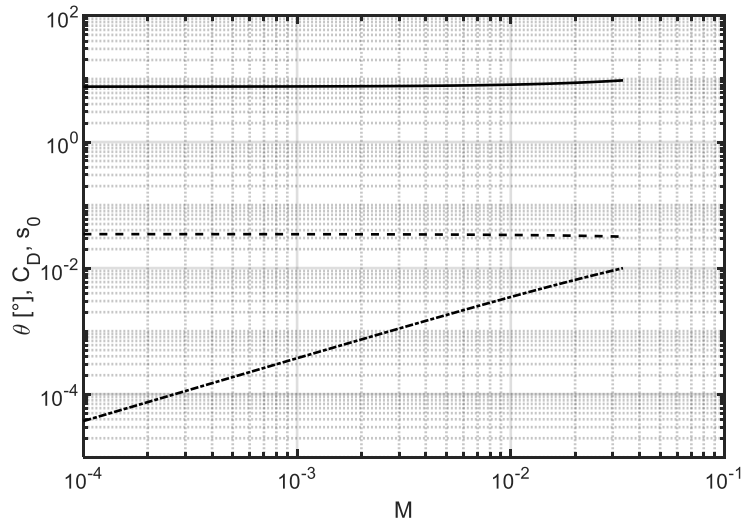
$$s_0 = \frac{3}{5} \frac{M}{\theta} (0.9 - \alpha_u), \quad (41)$$

Then, equation (39) with equations (15), (40) and (41) results in a transcendental equation for θ ,

$$\frac{1}{\theta} \left(\frac{9}{\text{St}} + \frac{1}{2} \sqrt{\left(\frac{18}{\text{St}} \right)^2 + \frac{1.2}{r} \alpha_u \frac{1}{\theta}} \right)^{-1} \approx \frac{0.85}{\kappa} \sqrt{r \left(\text{Sh} - \frac{3}{5} M (0.9 - \alpha_u) \frac{1}{\theta} \right)} \times \ln \left[\frac{1}{y_0} \left(\frac{9}{\text{St}} + \frac{1}{2} \sqrt{\left(\frac{18}{\text{St}} \right)^2 + \frac{1.2}{r} \alpha_u \frac{1}{\theta}} \right)^{-2} \sqrt{\frac{\alpha_\tau}{\theta} \frac{2d_w}{1+d_w}} \right], \quad (42)$$

that can be solved to determine the impact angle. Once θ is known, C_D (equation 40), s_0 (equation 41), the boundary values of the remaining variables at the bed (equations 5, 15 and 21), the depth of the saltation layer (equation 22), and the analytical distributions of T_y^+ , T_y , c , σ_y and Q_{yyy} (equations 14, 16-18 and 20) can also be calculated for saltation in turbulent shearing flows.

Figure 4 shows the variation of the impact angle, the drag coefficient and the particle shear stress at the bed with the particle mass hold-up, as predicted by equations (40-42), for, e.g., saltation of 240 μm sand grains in a turbulent wind on Earth, with $\alpha_u = 0.8$ and $\alpha_\tau = 1$ (see Section IV for more details about the choice of the parameters). The range of the mass hold-up in Figure 4 is that for which equation (23) is satisfied. As for saltation in viscous shearing flows (Figure 3), the particle shear stress at the bed linearly increases with M , while θ and C_D are almost constant.



1
2 Fig. 4. Predicted impact angle (in degrees, solid line), drag coefficient (dashed line) and particle shear stress at the bed
3 (dot-dashed line) as functions of the mass hold-up for saltation in turbulent shearing flows with $d_w = 1.5$, $St = 1681$,
4 $r = 2208$, and $Sh = 0.04$ (with $\alpha_u = 0.8$ and $\alpha_T = 1$).

5
6 Equation (2), with equations (38) and (6), and equation (37) can be written as a system of two
7 ordinary differential equations:

$$8 \quad \frac{ds}{dy} = \frac{3}{4} C_D c \frac{\sqrt{r(Sh-s)}}{\kappa} \ln \left(\frac{1}{C_D} \frac{\sqrt{T_y^+}}{y+y_0} + 1 \right) - \frac{5}{3} \frac{s}{T_y}; \quad (43)$$

9 and

$$10 \quad \frac{dU}{dy} = \frac{\sqrt{r(Sh-s)}}{\kappa(y+y_0)}; \quad (44)$$

11 these can be numerically integrated, with the boundary conditions of equations (4) and (41), to obtain the
12 distributions of the particle shear stress and the fluid horizontal velocity. Then, the particle horizontal velocity
13 is given by equation (38), with equation (6), as

$$14 \quad u = U + \frac{3}{4} \frac{\sqrt{r(Sh-s)}}{\kappa} \ln \left(\frac{1}{C_D} \frac{\sqrt{T_y^+}}{y+y_0} + 1 \right) - \frac{5}{3} \frac{s}{C_D \sigma_y}. \quad (45)$$

15 Finally, the particle mass flux q per unit basal area is determined through numerical integration of
16 equation (32), with the profiles of u and c obtained under turbulent conditions.

17 III. Scaling laws for rarefied saltation

18 The semi-analytical solutions to particle saltation in shearing flows described in the previous section permit
19 simple scalings to be obtained, at least in the extreme cases of rarefied saltation, that is for $M \rightarrow 0$ and
20 $s_0 \ll Sh$, when there is only Stokes drag – $C_D = 18/St$, when the particle Reynolds number based upon the
21 relative velocity between the particles and the fluid at the bed, $u_0 St/r$, is less than unity, or form drag –

1 $C_D = 0.3u_0/r$, when $10^3 < u_0St/r < 10^5$. In the following, we will consider that at leading order, equations
 2 (11) and (22) imply that $T_{y,0} \propto h \propto T_{y,0}^{+1/2} / C_D$.

3 We emphasize that there is no feedback of the particles on the fluid velocity profile, which, in rarefied
 4 saltation, is exactly linear in the viscous case, and logarithmic in the turbulent case. As a consequence, the
 5 particle mass hold-up only affects the particle concentration, and it is linearly proportional to it, and does not
 6 influence the particle velocity. Given that the mass flux involves the product of particle concentration and
 7 horizontal velocity, q must also be linearly related to the mass hold-up.

8 *Rarefied saltation in viscous shearing flows with Stokes drag*

9 In this case, the density ratio plays no role in the equations governing the saltation process. As mentioned,
 10 $C_D \propto St^{-1}$. Then, with equation (26), we obtain that $\theta \propto St^{-2}Sh^{-1}$. With this, equations (5), (15), (19) and
 11 (36) imply the scalings for the various quantities that we report in Table 1. In particular, we notice that the
 12 scaling for the mass flux, $q \propto MSt^4Sh^{3/2}$, was also derived in Valance and Berzi (2022) by using an approach
 13 in which all particles were assumed to follow the same periodic trajectory, and not a distribution of
 14 trajectories as in the present work.

15 *Rarefied saltation in viscous shearing flows with form drag*

16 In the case of form drag, $C_D \propto u_0r^{-1}$. Then, we expect the scaling laws to involve also the density ratio. With
 17 equation (5), we obtain $u_0 \propto r^{1/2}\theta^{-1/2}$, and, with equations (26) and (15), $\theta \propto r^{-1/3}St^{-2/3}Sh^{-2/3}$ and $T_{y,0}^+ \propto r$
 18 . Hence, equations (5), (15), (19) and (36) imply the scalings for the various quantities that we report in
 19 Table 1.

20 *Rarefied saltation in turbulent shearing flows with form drag*

21 In this limiting case, the fall Stokes number, St , plays no role in the equations governing the saltation process
 22 and cannot be involved in the scalings. With $C_D \propto u_0r^{-1}$ and equation (5), we obtain $u_0 \propto r^{1/2}\theta^{-1/2}$, and,
 23 with equations (41) and (15), at leading order, $\theta \propto Sh^{-1}$ and $T_{y,0}^+ \propto r$. Hence, $u_0 \propto r^{1/2}Sh^{1/2}$,
 24 $C_D \propto r^{-1/2}Sh^{1/2}$, and $h \propto T_{y,0} \propto T_{y,0}^{+1/2} / C_D \propto rSh^{-1/2}$. Equation (19), then, provides $c_0 \propto Mr^{-1}Sh^{1/2}$. As
 25 shown later, the concentration and particle velocity profiles in the case of turbulent saltation are
 26 approximately uniform along y . Therefore, we expect $q \propto c_0u_0h \propto MSh^{1/2}r^{1/2}$. We summarize the scaling
 27 laws for rarefied turbulent saltation with form drag in Table 1.

28

29 Table 1. Summary of the scaling laws for rarefied saltation.

	Viscous saltation with Stokes drag	Viscous saltation with form drag	Turbulent saltation with form drag
Drag coefficient, C_D	St^{-1}	$St^{1/3}Sh^{1/3}r^{-1/3}$	$Sh^{1/2}r^{-1/2}$
Impact angle, θ	$St^{-2}Sh^{-1}$	$St^{-2/3}Sh^{-2/3}r^{-1/3}$	Sh^{-1}
Particle slip velocity, u_0	St^3Sh	$St^{1/3}Sh^{1/3}r^{2/3}$	$Sh^{1/2}r^{1/2}$
Depth of saltation layer, h	$St^3Sh^{1/2}$	$St^{-1/3}Sh^{-1/3}r^{5/6}$	$rSh^{-1/2}$
Particle concentration at the bed, c_0	$MSt^{-3}Sh^{-1/2}$	$MSt^{1/3}Sh^{1/3}r^{-5/6}$	$MSh^{1/2}r^{-1}$
Particle mass flux, q	$MSt^4Sh^{3/2}$	$MSt^{2/3}Sh^{2/3}r^{5/6}$	$MSh^{1/2}r^{1/2}$

1 IV. Comparisons with numerical simulations and experiments

2 Here, we make comparisons between the predictions of the present theory and the results of discrete-
 3 continuum numerical simulations of saltation of spheres over a rigid bumpy bed made of a layer of particles
 4 identical to those saltating in close contact. The simulations are performed in a quasi-2D cell of streamwise
 5 length equal to 5120 particle diameters and transverse width equal to one particle diameter, with periodic
 6 boundary conditions in the streamwise direction. The cell is not upper-bounded.

7 We solve Newton's equations of motion for the individual spherical particles under the influence of
 8 fluid drag, buoyancy, gravity and contact forces when they collide with the bed (mid-trajectory collisions are
 9 forbidden), while treating the fluid as either a viscous or a turbulent continuum, depending on the closure
 10 for the fluid shear stress, for which balance equations are phrased. Once changed in sign, the sum over all
 11 particles of drag and buoyancy enter the momentum balance for the fluid, thus ensuring the two-way
 12 coupling between the phases. In the numerical simulations, we assume that the fluid only possesses
 13 horizontal velocity and we suppress the possibility of interparticle collisions above the bed. The individual
 14 particles experience a fluid drag based upon the local difference between the instantaneous velocity of the
 15 particle and the average velocity of the fluid. As in our theoretical treatment, we control the amount of
 16 particles in the simulations –the particle mass hold-up, M , the bumpiness of the rigid bed –the wall-particle
 17 diameter, d_w , the viscosity –the inverse of the fall Stokes number, St , the mass density –the inverse of the
 18 density ratio, r , of the fluid, and the intensity of the shearing flow –the Shields number, Sh . In what follows,
 19 all measurements have been taken once a steady state is attained and, subsequently, time-averaged. This
 20 type of numerical simulations have already been used in the context of saltation in both turbulent (Durán *et al.*
 21 2012; Pähitz *et al.* 2015; Pähitz and Durán 2020a; Ralaiaisoa *et al.* 2020) and viscous (Valance and Berzi
 22 2022) shearing flows. A more detailed description of the numerical simulations, including the contact
 23 parameters that we employ, can be found in Appendix D.

24 In a total of 50 simulations, we have numerically investigated the saltation process of four different
 25 types of solid particles in terrestrial and extra-terrestrial environment, as summarized in Table 2, by changing
 26 the mass hold-up between 0.0004 and about 0.03 that is from the rarefied limit to the maximum mass hold-
 27 up for which the mid-trajectory collisions can be neglected and equation (23) is satisfied. Although the fluid
 28 regime on Mars and Venus is almost certainly turbulent, one can at least imagine performing experiments in
 29 pressurized wind-tunnels, in which the turbulence is somehow suppressed, thus recovering the conditions
 30 reported in the first two rows of Table 2. The conditions reported in the last two rows of Table 2 are, instead,
 31 much closer to actual physical applications.

32 The chosen flow conditions serve the purpose of isolating and testing the assumptions that we have
 33 made in building our theory: (i) anisotropic Maxwellian velocity distribution for the particles, vertical velocity
 34 of the ascending particles much larger than the settling velocity and locally linear velocity for the fluid in the
 35 case of viscous regime and Stokes drag –first row in Table 2; (ii) additional assumption of drag coefficient
 36 uniform and equal to that evaluated at the bed in the case of viscous regime and nonlinear drag –second and
 37 third row in Table 2; and (iii) fluid velocity profile encountered by an ascending particle uniform and equal to
 38 the average of the logarithmic profile in the case of turbulent regime and nonlinear drag –fourth row in
 39 Table 2.

40 The parameters of the turbulent case in Table 2 match those of physical experiments of saltation on
 41 rigid, bumpy beds performed in a wind-tunnel (Ho 2012); for these, measurements of particle mass flux and
 42 profiles of particle concentration and horizontal particle and fluid velocities at different values of the particle
 43 mass hold-up are available. In the experiments, unlike the numerical simulations, the vertical velocity of the
 44 fluid surrounding the particles is nonzero, due to the no-slip condition on the particle surface, and this would
 45 permit a test of its influence on the results. However, as shown later, the depth of the saltation layer in the

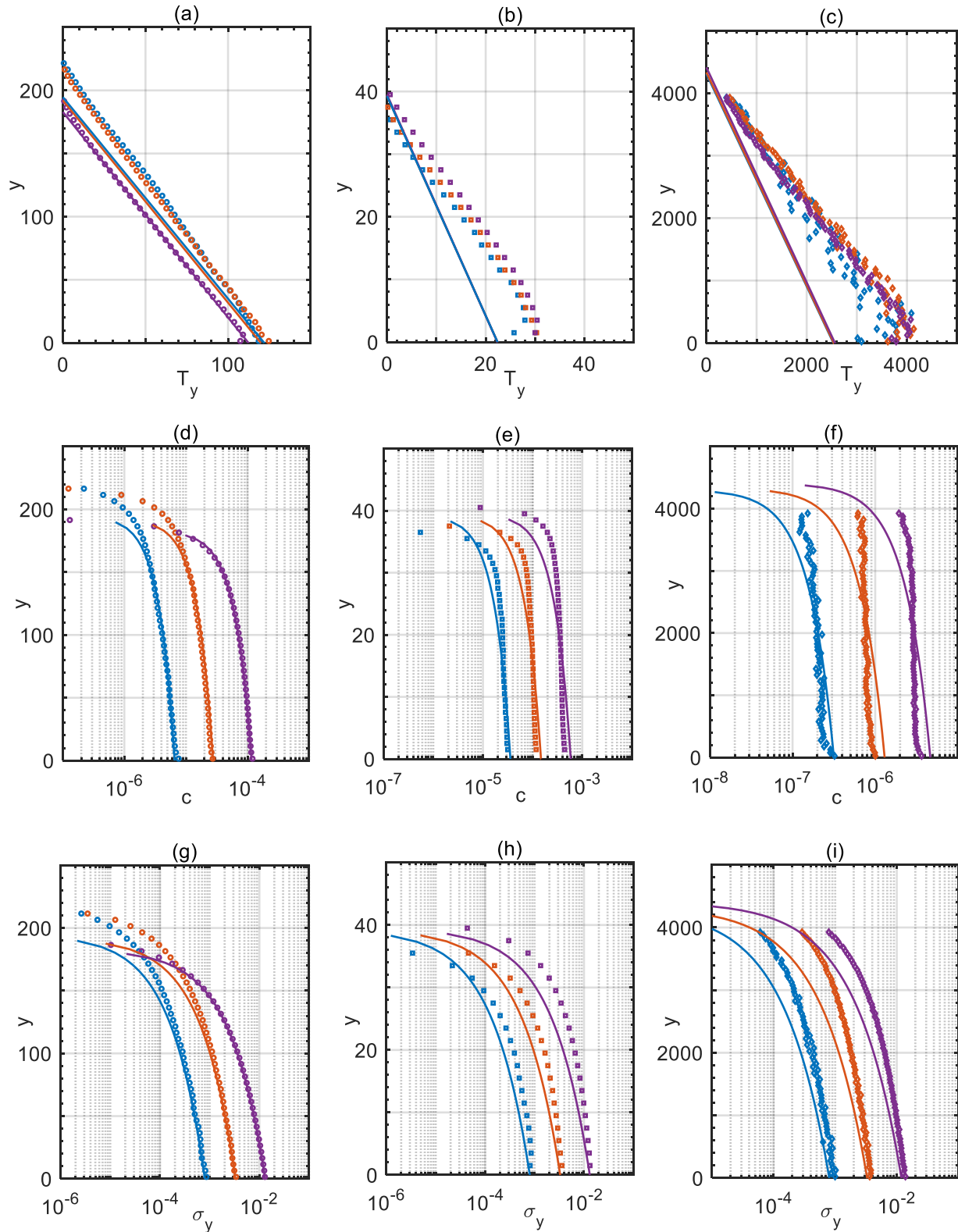
1 absence of an upper bound predicted by our model and measured in the numerical simulations is of the order
 2 of 4000 particle diameters. The experiments were performed in a rectangular closed conduit, with an
 3 horizontal lid placed at about 1200 particle diameters above the rigid base. As a consequence, the top
 4 boundary conditions are different from those of the present numerical simulations and the semi-analytical
 5 treatment. Although the order of magnitude of the profiles measured in the experiments is in good
 6 agreement with our model, their shape reveals the influence of the upper boundary. We postpone to a future
 7 work the solution of the appropriate two-point boundary value problem, with our proposed constitutive
 8 relations for the particle stresses and energy flux, and detailed comparisons against the experimental
 9 measurements.

10
 11 Table 2. Summary of the combination of parameters employed in the numerical simulations.

Description	Regime	St	r	Sh	d_w
60 or 75 μm basalt particles on Mars	viscous (Stokes drag)	60, 100	150000	0.05, 0.075	1
100 μm basalt particles on Venus	viscous (nonlinear drag)	100	50	0.05	1
650 μm sand particles in water on Earth or, equivalently,	viscous (nonlinear drag)	100	2.5	0.05	1
3 mm sand particles in crude oil on Earth					
240 μm sand particles in air on Earth	turbulent (nonlinear drag)	1681	2208	0.025, 0.04	1.5

12
 13 Figures 5 and 6 show the comparisons between profiles of T_y , c , σ_y , s , u and U relative to selected
 14 values of the mass hold-up (similar agreement, not shown here for brevity, is obtained for all admissible
 15 values of M) for saltation in viscous shearing flows with Stokes drag –first column of plots in both figures,
 16 saltation in viscous shearing flows with nonlinear drag –second column, and saltation in turbulent shearing
 17 flows with nonlinear drag –third column. In determining the semi-analytical solution that we have highlighted
 18 in Section II, we have employed $\alpha_u = 0.6$ and $\alpha_T = 0.5$ for saltation in viscous flows, and $\alpha_u = 0.8$ and $\alpha_T = 1$ for
 19 saltation in turbulent flows. Different values of these parameters would not alter the qualitative features of
 20 our semi-analytical solution, but only slightly impact the values of T_y , c and u at the bed, therefore causing a
 21 shift to the right or to the left in the relative profiles.

22 Figures 5(a-c) confirm that the intensity of the velocity fluctuations of the particles decreases linearly
 23 with the distance from the rigid bed, and the model captures both the weak dependence of T_y on M for
 24 viscous saltation and Stokes drag (Figure 5a), and its independence for nonlinear drag (Figures 5b and c). The
 25 depth of the saltation layer is well predicted by the continuum model, but the slope of the linear decrease is
 26 underestimated with respect to the numerical simulations for nonlinear drag. Also, the nonmonotonic
 27 behaviour of T_y near the rigid bed is not captured. These are the consequences of neglecting the dependence
 28 of the drag coefficient on the vertical direction. The agreement with the measurements in the numerical
 29 simulations would indeed improve if C_D is allowed to vary locally; however, this prevents our obtaining semi-
 30 analytical solutions of the governing equations, with little gain on the results.



1

2

3

4 Fig. 5. Profiles of (a-c) mean square of particle velocity fluctuations in the vertical direction, (d-f) particle concentration
 5 and (g-i) particle normal stress along y measured in numerical simulations of: saltation in viscous shearing flows with
 6 $d_w = 1$, $St = 100$, $r = 150000$, $Sh = 0.05$ and $M = 0.0008$ (blue circles), $M = 0.0033$ (red circles), $M = 0.0131$ (purple circles);
 7 saltation in viscous shearing flows with $d_w = 1$, $St = 100$, $r = 50$, $Sh = 0.05$ and $M = 0.0008$ (blue squares), $M = 0.0033$
 8 (squares), $M = 0.0131$ (purple squares); saltation in turbulent shearing flows with $d_w = 1.5$, $St = 1681$, $r = 2208$, $Sh = 0.04$
 9 and $M = 0.0008$ (blue diamonds), $M = 0.0033$ (red diamonds), $M = 0.0119$ (purple diamonds). The solid lines are the
 10 predictions of the present theory.

11

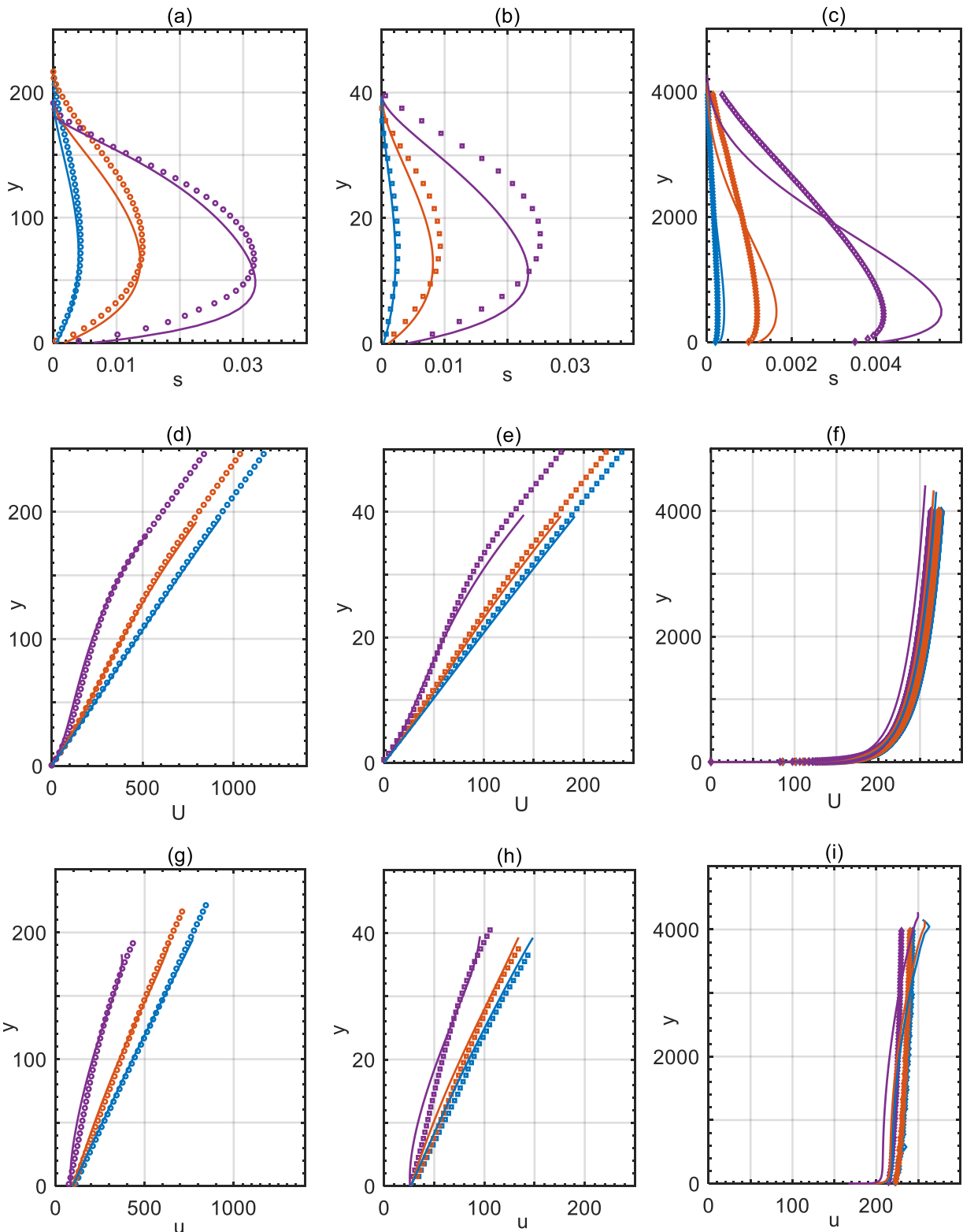


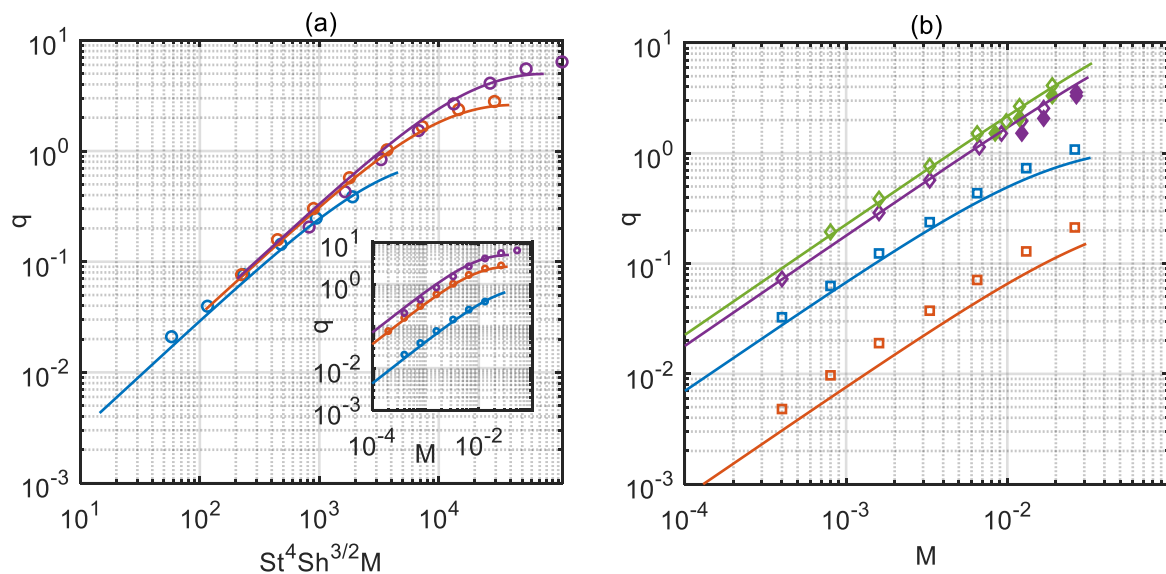
Fig. 6. Profiles of (a-c) particle shear stress, (d-f) fluid and (g-i) particle mean horizontal velocities measured in numerical simulations of: saltation in viscous shearing flows with $d_w = 1$, $St = 100$, $r = 150000$, $Sh = 0.05$ and $M = 0.0008$ (blue circles), $M = 0.0033$ (red circles), $M = 0.0131$ (purple circles); saltation in viscous shearing flows with $d_w = 1$, $St = 100$, $r = 50$, $Sh = 0.05$ and $M = 0.0008$ (blue squares), $M = 0.0033$ (squares), $M = 0.0131$ (purple squares); saltation in turbulent shearing flows with $d_w = 1.5$, $St = 1681$, $r = 2208$, $Sh = 0.04$ and $M = 0.0008$ (blue diamonds), $M = 0.0033$ (red diamonds), $M = 0.0119$ (purple diamonds). The solid lines are the predictions of the present theory.

1 In all cases, the continuum model satisfactorily reproduces the profiles of particle concentration and
 2 normal stress in the y -direction (Figures 5d-i), which indeed follow the power-law distributions of equations
 3 (17) and (18). In the turbulent case, as anticipated, the particle concentration is rather uniform along y
 4 (Figure 5f). The values of the particle concentration are also much smaller in the case of turbulent shearing
 5 flows (Figure 5f), as a consequence of the greater agitation of the particles (Figure 5c).

6 Figure 5 shows only profiles of quantities associated with the vertical motion of the saltating
 7 particles, which is entirely uncoupled from the horizontal motion of the fluid in our continuum model and in
 8 the numerical simulations, were it not for the dependence of the drag coefficient on U . The horizontal motion
 9 of the fluid, and in particular its flow regime, strongly affects the distribution of particle shear stress and
 10 particle and fluid horizontal velocity shown in Figure 6.

11 Our proposed constitutive relations for the particle shear stress (equations 25 and 38) permit the
 12 qualitatively and quantitatively reproduction of the measurements in numerical simulations (Figures 6a-c).
 13 In particular, the position and the magnitude of the peak in the particle shear stress are well captured. The
 14 physical reason for the presence of this maximum value of the particle shear stress above the bed is in the
 15 change of sign of the horizontal force exerted by the fluid on the particles. The particles slip at the rigid bed,
 16 while the fluid does not; hence, the particles drag the fluid near the bed, while the fluid drags the particles
 17 as the top of the saltation layer is approached. Interestingly, the particle shear stress can reach values up to
 18 50-60 % of the total shear stress –the Shields parameter– for saltation in viscous shearing flows, but only up
 19 to 10 % of the total shear stress for saltation in turbulent shearing flows.

20 As the mass hold-up increases, the fluid and particle horizontal velocity profiles become progressively
 21 nonlinear for saltation in viscous shearing flows (Figures 6d, e, g and h), but the continuum model is capable
 22 of capturing this behaviour. For saltation in turbulent shearing flows, the fluid velocity profile is logarithmic,
 23 and the mass hold-up has little influence on it (Figure 6f). The horizontal velocity of the particles is almost
 24 uniform vertically (Figure 6i). This confirms our finding that in the absence of particle interactions, the particle
 25 shear stress does not depend on the particle shear rate. The slight discrepancies between the predictions of
 26 the continuum model and the results of the numerical simulations are a seemingly minor consequence of the
 27 lack of perfect agreement in terms of particle shear stress (Figure 6c).
 28



29 Fig. 7. (a) Particle mass flux against scaled particle mass hold-up measured in discrete-continuum numerical simulations
 30 (open symbols) and predicted from the present theory (lines) in the case of saltation in viscous shearing flows with
 31 $d_w = 1$, $r = 150000$ and: $St = 60$ and $Sh = 0.05$ (blue circles); $St = 100$ and $Sh = 0.05$ (orange circles); $St = 100$ and
 32

1 $Sh = 0.075$ (purple circles). The inset depicts the same data in terms of q against M . (b) Particle mass flux against particle
 2 mass hold-up measured in discrete-continuum numerical simulations (open symbols) and in wind-tunnel experiments
 3 (filled symbols, after Ho 2012) and predicted from the present theory (lines) in the case of saltation in viscous shearing
 4 flows with $d_w = 1.5$, $St = 100$ $Sh = 0.05$, and $r = 2.5$ (orange squares) and $r = 50$ (blue squares); and in the case of saltation
 5 in turbulent shearing flows with $d_w = 1.5$, $St = 1681$, $r = 2208$, and $Sh = 0.025$ (purple diamonds) and $Sh = 0.04$ (green
 6 diamonds).

7

8 Finally, we compare the dependence of the particle mass flux on the particle mass hold-up in
 9 Figure 7. The continuum model notably reproduces the measurements in numerical simulations in both the
 10 viscous and turbulent shearing flows. The experimental results on turbulent saltation in a wind-tunnel (Ho
 11 2012) are in sufficiently good agreement with both the numerical simulations and the continuum model
 12 (Figure 7b). The overestimate of the experimental mass flux is likely due to the additional resistance induced
 13 by the presence of the horizontal lid above the rigid bed, that we do not account for. Figure 7a also assesses
 14 the validity of the scaling for q reported in Table 1 for the case of viscous saltation and Stokes drag, in the
 15 rarefied limit.

16 V. Conclusions

17 We have derived constitutive relations for the particle stresses and flux of particle kinetic energy associated
 18 with the fluctuating vertical velocities that apply to saltation of particles in viscous and turbulent shearing
 19 flows, in the absence of collisions above the bed. To do this, we have employed the averaging methods of
 20 statistical mechanics based on an anisotropic Maxwellian velocity distribution function for the ascending
 21 particles and approximate analytical expressions for the particle trajectories under the influence of fluid drag,
 22 gravity and buoyancy.

23 Given that we neglect the possibility of particle-particle interactions, we have obtained the perhaps
 24 unexpected result that the particle shear stress does depend on the local relative velocity between the grains
 25 and the carried fluid and the fluid shear rate, but not on the particle shear rate. We have combined the
 26 constitutive relations for the particle phase and well-known expressions for the fluid shear stress with
 27 momentum and energy balances to obtain semi-analytical solutions of steady, fully-developed saltation over
 28 horizontal, rigid beds. We have employed boundary conditions appropriated for particles rebounding at a
 29 bumpy base with no upper-bounds, and assumed that the carrier fluid has only mean horizontal velocity.
 30 Hence, the regime of the carrier fluid, viscous or turbulent, does not affect the profiles of particle
 31 concentration and normal stress, which follow a power-law decrease with height, and mean square of the
 32 vertical velocity fluctuations, linearly decreasing with the distance from the bed. These results are in contrast
 33 with previous theoretical and experimental works in which the granular temperature was assumed to be
 34 uniform and the concentration was shown to decrease exponentially with height.

35 We have confirmed our findings through extensive comparisons with discrete-continuum
 36 simulations in both viscous and turbulent regime. Thus, we are inclined to blame mid-fluid collisions for the
 37 above mentioned qualitative difference in the distribution of particle concentration. We have also shown
 38 that the predictions of the theory in terms of profiles of particle shear stress, and particle and fluid mean
 39 horizontal velocity are in excellent agreement with the numerical simulations. In natural units of particle
 40 mass density, diameter and reduced gravity, we have determined that the saltation process is controlled by
 41 the mass of particles in the system, the intensity of the shearing flow, the fluid mass density and viscosity
 42 and the bumpiness of the rigid base.

43 We have successfully tested the dependence of the horizontal mass flux per unit basal area of the
 44 particles as a function of the mass hold-up obtained with our theory against discrete-continuum numerical
 45 simulations and wind-tunnel experiments, for a large range of the control parameters. We have also

1 determined simple scaling laws for the dependence of various quantities of interest on the control variables
 2 in the special limits of rarefied viscous and turbulent saltation, and solely Stokes or form drag on the particles.
 3 In a future, we plan to apply our theory to two-point boundary value problems, to, e.g., mimic
 4 available experiments of saltation in enclosed wind-tunnels. We also wish to extend the present work to deal
 5 with unsteady and /or developing flows. More importantly, the inclusion of mid-fluid collisions, that is the
 6 transition from a collisionless to a collisional kinetic theory of saltation, is a crucial future step, especially in
 7 view of modelling transport phenomena over erodible beds, of more interest for geophysical and planetary
 8 science applications.

9

10 Declaration of interests. The authors report no conflict of interest.

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23
24
25

1 Appendix A. Approximate analytical trajectories for the saltating particles

2 We characterize the drag exerted on the particles through a drag coefficient, C_D , that we take to be
3 independent of y , as in Pasini and Jenkins (2005), to obtain analytical expressions for the particle trajectories.

4 Integrating the vertical momentum balance for the particle:

$$5 \quad \frac{d\xi_y}{dt} + C_D \xi_y + 1 = 0, \quad (\text{A1})$$

6 with ξ_y the vertical component of the particle velocity, we obtain the vertical velocity at any time t after the
7 particle reaches a certain ascending velocity ξ_y^+ at a certain location y_r ,

$$8 \quad \xi_y = \frac{C_D \xi_y^+ + 1 - \exp(-C_D t)}{C_D \exp(-C_D t)}; \quad (\text{A2})$$

9 and, with another integration, the position y as a function of time,

$$10 \quad y - y_r = \frac{(C_D \xi_y^+ + 1)[1 - \exp(-C_D t)] - C_D t}{C_D^2}. \quad (\text{A3})$$

11 From equation (A3) we obtain an implicit expression for the time that the particle spends at $y \geq y_r$

$$12 \quad t_f = \frac{C_D \xi_y^+ + 1}{C_D} [1 - \exp(-C_D t_f)]. \quad (\text{A4})$$

13 Then, from equation (A2), with $t = t_f$, the downward vertical velocity of the particle at $y = y_r$ is

$$14 \quad \xi_y^- = \xi_y^+ - t_f. \quad (\text{A5})$$

15 To permit subsequent analytical integrations, we propose the following explicit expression for the time of
16 flight,

$$17 \quad t_f = \xi_y^+ + \frac{1}{C_D} - \frac{1}{C_D} \exp(-C_D \xi_y^+), \quad (\text{A6})$$

18 which has the right limits at both small $C_D \xi_y^+$, i.e., $t_f = 2\xi_y^+$, and large $C_D \xi_y^+$, i.e., $t_f = \xi_y^+ + C_D^{-1}$. With
19 this, equation (A5) gives

$$20 \quad \xi_y^- = -\frac{1}{C_D} + \frac{1}{C_D} \exp(-C_D \xi_y^+). \quad (\text{A7})$$

21 In the limit of large $C_D \xi_y^+$, equation (A7) reduces to

$$22 \quad \xi_y^- = -\frac{1}{C_D}. \quad (\text{A8})$$

23 We next integrate the horizontal particle momentum balances, and distinguish between saltation in
24 viscous and in turbulent shearing flows.

25 A1. Saltation in viscous shearing flows

1 In the case of saltation in viscous shearing flows, we assume that the fluid velocity profile is locally linear
2 (that is for $y \geq y_r$), so that the horizontal momentum balance for the particles is:

$$3 \quad \frac{d\xi_x}{dt} = C_D \left[U + \frac{dU}{dy} (y - y_r) - \xi_x \right], \quad (\text{A9})$$

4 with ξ_x the horizontal component of the particle velocity, and U and $\frac{dU}{dy}$ the fluid horizontal velocity and
5 shear rate at the reference level. In the following integrations, we treat $\frac{dU}{dy}$ for saltation in viscous shearing
6 flows as if it were constant. Then, after inserting equation (A3) into equation (A9), and integrating with the
7 initial condition $\xi_x(t=0) = \xi_x^+$,

$$8 \quad \xi_x = U + \frac{dU}{dy} \frac{(C_D \xi_y^+ + 1)}{C_D^2} \left[1 - C_D t \exp(-C_D t) - \exp(-C_D t) \right] \\ - \frac{dU}{dy} \frac{C_D t - 1 + \exp(-C_D t)}{C_D^2} + (\xi_x^+ - U) \exp(-C_D t). \quad (\text{A10})$$

9 The horizontal velocity after the time t_f is, then, with equations (A4) and (A6),

$$10 \quad \xi_x^- = U + \frac{1}{C_D^2} \frac{dU}{dy} \frac{1}{C_D \xi_y^+ + 1} \left[C_D \xi_y^+ + 1 - \exp(-C_D \xi_y^+) \right] \left[1 - \exp(-C_D \xi_y^+) - C_D \xi_y^+ \exp(-C_D \xi_y^+) \right] \\ + \frac{\xi_x^+ - U}{C_D \xi_y^+ + 1} \exp(-C_D \xi_y^+), \quad (\text{A11})$$

12 In the limit of large $C_D \xi_y^+$, equation (A11) gives

$$13 \quad \xi_x^- = U + \frac{1}{C_D^2} \frac{dU}{dy}. \quad (\text{A12})$$

14 Equation (A10) can be integrated with the boundary condition $x(t=0) = x_i$ to obtain the horizontal
15 displacement from the initial position at the reference level as a function of time:

$$16 \quad x = x_i + Ut + \frac{(C_D \xi_y^+ + 1)}{C_D^2} \frac{dU}{dy} \left[\exp(-C_D t) \left(\frac{2}{C_D} + t \right) + t - \frac{2}{C_D} \right] \\ - \frac{1}{C_D^2} \frac{dU}{dy} \left[\frac{C_D t^2}{2} - t - \frac{1}{C_D} \exp(-C_D t) + \frac{1}{C_D} \right] + \frac{1}{C_D} (\xi_x^+ - U) [1 - \exp(-C_D t)]. \quad (\text{A13})$$

17 The horizontal displacement Δ after the time t_f is

$$18 \quad \Delta = Ut_f + \frac{(C_D \xi_y^+ + 1)}{C_D^2} \frac{dU}{dy} \left[\exp(-C_D t_f) \left(\frac{2}{C_D} + t_f \right) + t_f - \frac{2}{C_D} \right] \\ - \frac{1}{C_D^2} \frac{dU}{dy} \left[\frac{C_D t_f^2}{2} - t_f - \frac{1}{C_D} \exp(-C_D t_f) + \frac{1}{C_D} \right] + \frac{1}{C_D} (\xi_x^+ - U) [1 - \exp(-C_D t_f)]. \quad (\text{A14})$$

19 In the limit of large $C_D \xi_y^+$, $t_f = \xi_y^+ + C_D^{-1}$ and equation (A14) reduces to

$$\Delta = \frac{1}{2C_D} \frac{dU}{dy} \xi_y^{+2} + U \xi_y^+ + \frac{\xi_x^+ - U}{C_D}. \quad (\text{A15})$$

2 A2. Saltation in turbulent shearing flows

3 In the case of saltation in turbulent shearing flows, we assume that the fluid velocity is equal to its average,
4 \bar{U} , in the region between the reference location y_r and the top of the trajectory, y_M , determined from
5 equation (A3) at the time t for which ξ_y in equation (A2) vanishes. Then, the horizontal momentum balance
6 for the particles reads:

$$7 \quad \frac{d\xi_x}{dt} = C_D (\bar{U} - \xi_x) \quad (\text{A16})$$

8 where, with the expression for the turbulent fluid shear stress,

$$\begin{aligned} \bar{U} &= \frac{1}{y_M - y_r} \int_{y_r}^{y_M} U + \frac{\sqrt{rS}}{\kappa} \ln \left(1 + \frac{y - y_r}{y_r + y_0} \right) dy \\ &= U + \frac{\sqrt{rS}}{\kappa} \left[\left(1 + \frac{y_r + y_0}{y_M - y_r} \right) \ln \left(\frac{y_M - y_r}{y_r + y_0} + 1 \right) - 1 \right] \\ 9 \quad &= U + \frac{\sqrt{rS}}{\kappa} \left[\left(1 + C_D^2 \frac{y_r + y_0}{C_D \xi_y^+ - \ln(C_D \xi_y^+ + 1)} \right) \ln \left(\frac{1}{C_D^2} \frac{C_D \xi_y^+ - \ln(C_D \xi_y^+ + 1)}{y_r + y_0} + 1 \right) - 1 \right] \quad (\text{A17}) \\ &\approx U + 0.85 \frac{\sqrt{rS}}{\kappa} \ln \left(\frac{1}{C_D} \frac{\xi_y^+}{y_r + y_0} + 1 \right) = U + 0.85 \frac{\sqrt{rS}}{\kappa} \ln \left(\frac{\kappa \xi_y^+}{C_D \sqrt{rS}} \frac{dU}{dy} + 1 \right), \end{aligned}$$

10 with U and $\frac{dU}{dy}$ the fluid horizontal velocity and shear rate at the reference level. In equation (A17), we have

11 taken the limit of large $C_D \xi_y^+$ to approximate the average fluid velocity. The factor 0.85 in front of the
12 logarithm allows to better fit the trajectories of discrete-continuum simulations of saltation in turbulent
13 shearing flows.

14 Then, after integrating equation (A16) with the initial condition $\xi_x(t=0) = \xi_x^+$,

$$15 \quad \xi_x = \bar{U} [1 - \exp(-C_D t)] + \xi_x^+ \exp(-C_D t). \quad (\text{A18})$$

16 The horizontal velocity after the time t_f is, then, with equations (A5) and (A7),

$$17 \quad \xi_x^- = \bar{U} \left[1 - \frac{\exp(-C_D \xi_y^+)}{C_D \xi_y^+ + 1} \right] + \frac{\xi_x^+}{C_D \xi_y^+ + 1} \exp(-C_D \xi_y^+). \quad (\text{A19})$$

18 In the limit of large $C_D \xi_y^+$, equation (A19) gives

$$19 \quad \xi_x^- = \bar{U}. \quad (\text{A20})$$

20 Equation (A18) can be integrated with the boundary condition $x(t=0) = x_i$ to obtain the horizontal
21 displacement from the initial position at the reference level as a function of time:

1
$$x = x_i + \bar{U}t + \frac{1}{C_D} [\xi_x^+ - \bar{U}] [1 - \exp(-C_D t)]. \quad (\text{A21})$$

2 The horizontal displacement Δ after the time t_f is

3
$$\Delta = \bar{U}t_f + \frac{1}{C_D} [\xi_x^+ - \bar{U}] [1 - \exp(-C_D t_f)]. \quad (\text{A22})$$

4 In the limit of large $C_D \xi_y^+$, $t_f = \xi_y^+ + C_D^{-1}$ and equation (A22) reduces to

5
$$\Delta = \bar{U} \xi_y^+ + \frac{\xi_x^+}{C_D}. \quad (\text{A23})$$

6

1 Appendix B. Constitutive relations

2 We define the average of a certain quantity φ at a certain distance y from the bed as:

$$3 \quad \langle \varphi \rangle = \frac{\pi}{6c} \int_{all \xi} \varphi f d^2 \xi = \frac{\pi}{6c} \int_{all \xi^+} \varphi^+ f^+ d^2 \xi^+ + \frac{\pi}{6c} \int_{all \xi^-} \varphi^- f^- d^2 \xi^- \quad (B1)$$

4 where f is the velocity distribution of particles at that height and $6c/\pi$ is the number density. As in Creyssels
5 et al. (2009), we distinguish between ascending or descending particles and introduce velocity distributions
6 for both species. We assume that the velocity distribution of the ascending particles is an anisotropic
7 Maxwellian:

$$8 \quad f^+ = \frac{6}{\pi} \frac{c^+}{\pi \sqrt{T_x^+ T_y^+}} \exp \left[-\frac{(\xi_x^+ - u)^2}{2T_x^+} \right] \exp \left(-\frac{\xi_y^{+2}}{2T_y^+} \right), \quad (B2)$$

9 where: $u = \langle \xi_x \rangle$; and $T_x^+ = \frac{\pi}{6c^+} \int_{all \xi^+} (\xi_x^+ - u)^2 f^+ d^2 \xi^+$ and $T_y^+ = \frac{\pi}{6c^+} \int_{all \xi^+} \xi_y^{+2} f^+ d^2 \xi^+$ are the mean
10 squares of the horizontal velocity fluctuations of the ascending particles. The concentration of the ascending
11 particles, c^+ , is equal to $\frac{\pi}{6} \int_{all \xi^+} f^+ d^2 \xi^+$, while that of the descending particle is $c^- = \frac{\pi}{6} \int_{all \xi^-} f^- d^2 \xi^-$. When
12 $\varphi = 1$, equation (B1) gives $c = c^+ + c^-$.

13 Given that we are dealing with steady states, $\langle \xi_y \rangle = 0$, so that

$$14 \quad \langle \xi_y \rangle = \frac{\pi}{6c} \int_{all \xi^+} \xi_y^+ f^+ d^2 \xi^+ + \frac{\pi}{6c} \int_{all \xi^-} \xi_y^- f^- d^2 \xi^- = 0. \quad (B3)$$

15 Then, we take the limit for large $C_D \xi_y^+$ of ξ_y^- (equation A9) and, inserting equation (B2) into equation (B3),
16 and integrating, we obtain

$$17 \quad \frac{c^+}{\pi \sqrt{T_x^+ T_y^+}} \int_0^{+\infty} \xi_y^+ \exp \left(-\frac{\xi_y^{+2}}{2T_y^+} \right) \int_{-\infty}^{+\infty} \exp \left[-\frac{(\xi_x^+ - u)^2}{2T_x^+} \right] d\xi_x^+ d\xi_y^+ = -\frac{\pi}{6} \int_{all \xi^-} \xi_y^- f^- d^2 \xi^-$$

$$\frac{c^+ \sqrt{2}}{\sqrt{\pi T_y^+}} \int_0^{+\infty} \xi_y^+ \exp \left(-\frac{\xi_y^{+2}}{2T_y^+} \right) d\xi_y^+ = \frac{1}{C_D} \frac{\pi}{6} \int_{all \xi^-} f^- d^2 \xi^- \quad (B4)$$

$$c^+ C_D \sqrt{\frac{2T_y^+}{\pi}} = c^-,$$

18 that is a relation between the concentrations of the two species of particles. As shown later, we found that
19 equation (B4) does not correctly capture the right limit as T_y^+ tends to zero, and also slightly underestimates
20 c^- . An almost perfect agreement can be obtained, instead, with

$$21 \quad c^- = c^+ + \frac{3}{4} c^+ C_D \sqrt{\frac{2T_y^+}{\pi}}. \quad (B5)$$

22 From this and the definition of c ,

$$c = c^+ + c^- = c^+ \left(2 + \frac{3}{4} C_D \sqrt{\frac{2T_y^+}{\pi}} \right). \quad (\text{B6})$$

Taking the limit for large $C_D \xi_y^+$ permits equations (B4)-(B6) to be obtained without assuming a velocity distribution for the descending particles. We show the comparisons between equations (B5) and (B6) and the results of numerical simulations on saltation of particles in a viscous shearing flow experiencing Stokes drag in figure B1.

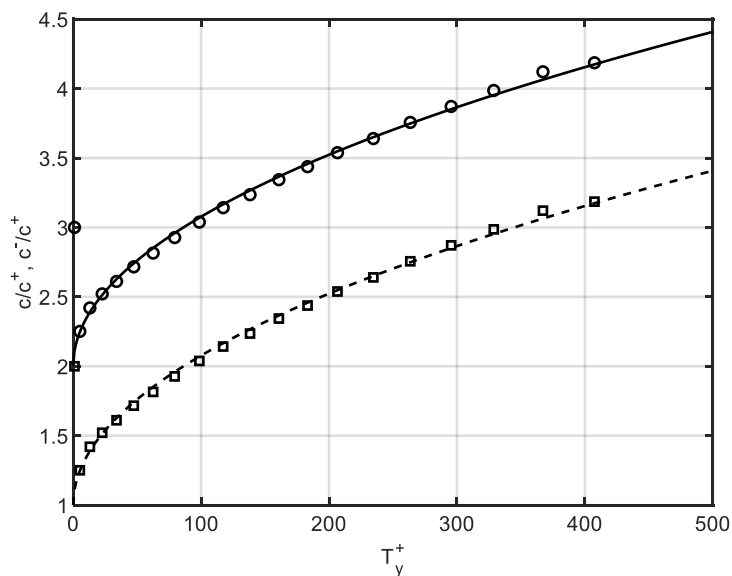


Fig. B1. Concentration ratios c/c^+ (circles) and c^-/c^+ (squares) as functions of T_y^+ measured in numerical simulations of saltation in a viscous shearing flow with $d_w = 1$, $St = 100$, $r = 150000$, $Sh = 0.05$ (so that the Stokes drag is dominant and $C_D = 18/St = 0.18$) and $M = 0.0131$. The dashed and solid lines are the predictions of equations (B5) and (B6), respectively.

Equation (B1) can be rewritten as

$$\begin{aligned} \langle \varphi \rangle &= \frac{\pi}{6c} \int_{all \xi^+} \varphi^+ f^+ d^2 \xi^+ + \frac{\pi}{6c} \int_{all \xi^-} \varphi^- f^- d^2 \xi^- = \frac{\pi}{6c} \int_{all \xi^+} \varphi^+ f^+ d^2 \xi^+ + \frac{\pi}{6c} \int_{all \xi^+} \varphi^- f^- |J^-| d^2 \xi^+ \\ &= \frac{\pi}{6c} \int_{all \xi^+} [\varphi^+ f^+ + \varphi^- f^- |J^-|] d^2 \xi^+, \end{aligned} \quad (\text{B7})$$

where J is the Jacobian of the transformation from ξ^+ into ξ^- . When we calculate the average vertical flux of the generic quantity,

$$\langle \varphi \xi_y \rangle = \frac{\pi}{6c} \int_{all \xi^+} [\varphi^+ \xi_y^+ f^+ + \varphi^- \xi_y^- f^- |J^-|] d^2 \xi^+ = \frac{\pi}{6c} \int_{all \xi^+} [\varphi^+ - \varphi^-] \xi_y^+ f^+ d^2 \xi^+, \quad (\text{B8})$$

where we made use of the fact that, at steady state, $\langle \xi_y \rangle = 0$ and, therefore, from equation (B3), if the trajectories are independent of each other (that is for small mass hold-ups), $\xi_y^+ f^+ = -\xi_y^- f^- |J^-|$ (Pasini and Jenkins 2005).

1 The particle normal stress in the y -direction, in the absence of particle interaction, is simply given by
 2 the average vertical flux of y -momentum, $\sigma_y \equiv c \langle \xi_y \xi_y \rangle = c T_y$, where $T_y = \langle \xi_y^2 \rangle$ is the mean square of the
 3 vertical velocity fluctuation for all the particles. However, using equation (B8),

$$4 \quad \sigma_y = \frac{\pi}{6} \int_0^{+\infty} \int_{-\infty}^{+\infty} (\xi_y^+ - \xi_y^-) \xi_y^+ f^+ d\xi_x^+ d\xi_y^+. \quad (\text{B9})$$

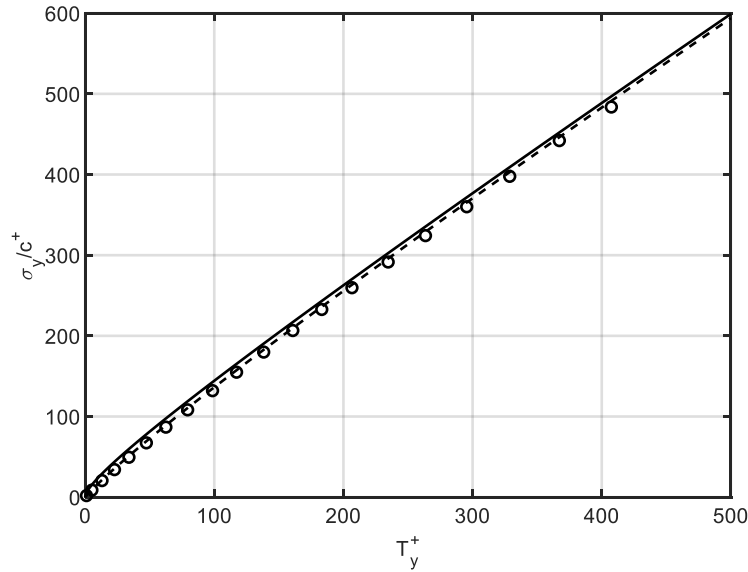
5 Then, upon taking the limit for large $C_D \xi_y^+$ of ξ_y^- (equation A9) and, inserting equation (B2) into equation
 6 (B9), and integrating, we obtain

$$7 \quad \begin{aligned} \sigma_y &= \frac{c^+}{\pi \sqrt{T_x^+ T_y^+}} \int_0^{+\infty} \left(\xi_y^+ + \frac{1}{C_D} \right) \xi_y^+ \exp\left(-\frac{\xi_y^{+2}}{2T_y^+}\right) \int_{-\infty}^{+\infty} \exp\left[-\frac{(\xi_x^+ - u)^2}{2T_x^+}\right] d\xi_x^+ d\xi_y^+ \\ &= \frac{c^+}{\pi \sqrt{T_x^+ T_y^+}} \sqrt{2\pi T_x^+} \int_0^{+\infty} \left(\xi_y^+ + \frac{1}{C_D} \right) \xi_y^+ \exp\left(-\frac{\xi_y^{+2}}{2T_y^+}\right) d\xi_y^+ = c^+ T_y^+ + c^+ \frac{1}{C_D} \sqrt{\frac{2T_y^+}{\pi}}. \end{aligned} \quad (\text{B10})$$

8 The limit for large $C_D \xi_y^+$ permits a simple expression for σ_y (and for the other constitutive relations) to be
 9 obtained. This approximation is valid whenever the Maxwellian vertical velocity distribution of the ascending
 10 particles is wide, that is for large T_y^+ , so that the contribution of the large ξ_y^+ is significant. Conversely, it
 11 becomes less accurate near the top of the saltating layer, where we expect small values of T_y^+ and a
 12 subsequent narrower distribution of ξ_y^+ around zero. If we use the more accurate equation (A7) in equation
 13 (B9), we would obtain the following constitutive relation for the particle normal stress:

$$14 \quad \begin{aligned} \sigma_y &= \frac{c^+}{\pi \sqrt{T_x^+ T_y^+}} \int_0^{+\infty} \left(\xi_y^+ + \frac{1}{C_D} - \frac{1}{C_D} \exp(-C_D \xi_y^+) \right) \xi_y^+ \exp\left(-\frac{\xi_y^{+2}}{2T_y^+}\right) \int_{-\infty}^{+\infty} \exp\left[-\frac{(\xi_x^+ - u)^2}{2T_x^+}\right] d\xi_x^+ d\xi_y^+ \\ &= \frac{c^+}{\pi \sqrt{T_x^+ T_y^+}} \sqrt{2\pi T_x^+} \int_0^{+\infty} \left(\xi_y^+ + \frac{1}{C_D} - \frac{1}{C_D} \exp(-C_D \xi_y^+) \right) \xi_y^+ \exp\left(-\frac{\xi_y^{+2}}{2T_y^+}\right) d\xi_y^+ \\ &= c^+ T_y^+ + c^+ T_y^+ \exp\left(\frac{C_D^2 T_y^+}{2}\right) \operatorname{erfc}\left(\sqrt{\frac{C_D^2 T_y^+}{2}}\right). \end{aligned} \quad (\text{B11})$$

15
16



1
2 Fig. B2. Ratio of particle normal stress over particle concentration of ascending particles as a function of T_y^+ measured
3 in numerical simulations of saltation in a viscous shearing flow with $d_w = 1$, $St = 100$, $r = 150000$, $Sh = 0.05$ (so that the
4 Stokes drag is dominant and $C_D = 18/St = 0.18$) and $M = 0.0131$. The solid and dashed lines are the predictions of
5 equations (B10) and (B11), respectively.

6
7 The difference between the approximate equation (B10) and the more accurate equation (B11) is shown in
8 figure B2 when the drag coefficient is equal to 0.18, and we notice that the mismatch becomes relevant only
9 at small T_y^+ , as expected. The agreement with the measurements in the numerical simulations of saltation of
10 particles in a viscous shearing flow experiencing Stokes drag is remarkable.

11 From $\sigma_y = cT_y$, equations (B10) and (B6), we obtain

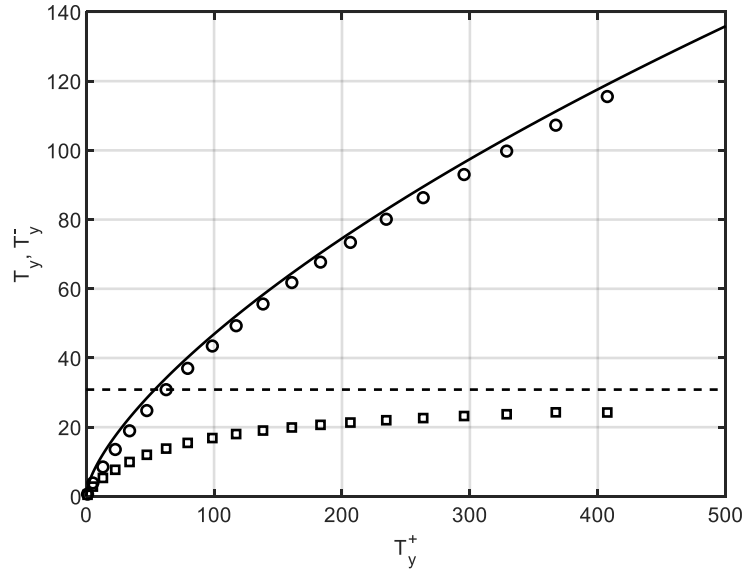
$$12 \quad T_y = \frac{4\sqrt{\pi}C_D T_y^+ + 4\sqrt{2T_y^+}}{8\sqrt{\pi}C_D + 3C_D^2 \sqrt{2T_y^+}}; \quad (B12)$$

13 while, with equation (A10),

$$14 \quad T_y^- = \frac{\pi}{6c^-} \int_{all \xi^-} (\xi_y^-)^2 f^- d^2 \xi^- = \frac{1}{C_D^2}. \quad (B13)$$

15 We show the agreement between equations (B12) and (B13) and the measurements in the numerical
16 simulations of saltation of particles in a viscous shearing flow experiencing Stokes drag in figure B3.

17



1
2 Fig. B3. T_y (circles) and T_y^- (squares) as functions of T_y^+ measured in numerical simulations of saltation in a viscous
3 shearing flow $d_w = 1$, $St = 100$, $r = 150000$, $Sh = 0.05$ (so that the Stokes drag is dominant and $C_D = 18/St = 0.18$) and
4 $M = 0.0131$. The solid and dashed lines are the predictions of equations (B12) and (B13), respectively.

5

6 The yyy -element of the vertical flux of the particle second moment is $Q_{yyy} \equiv c \langle \xi_y \xi_y \xi_y \rangle$, so that,
7 with equation (B8),

$$8 \quad Q_{yyy} = \frac{\pi}{6} \int_0^{+\infty} \int_{-\infty}^{+\infty} (\xi_y^{+2} - \xi_y^{-2}) \xi_y^+ f^+ d\xi_x^+ d\xi_y^+. \quad (\text{B14})$$

9 Then, In the limit of large $C_D \xi_y^+$, upon using equations (A8) and (B2) in equation (B14), and integrating,

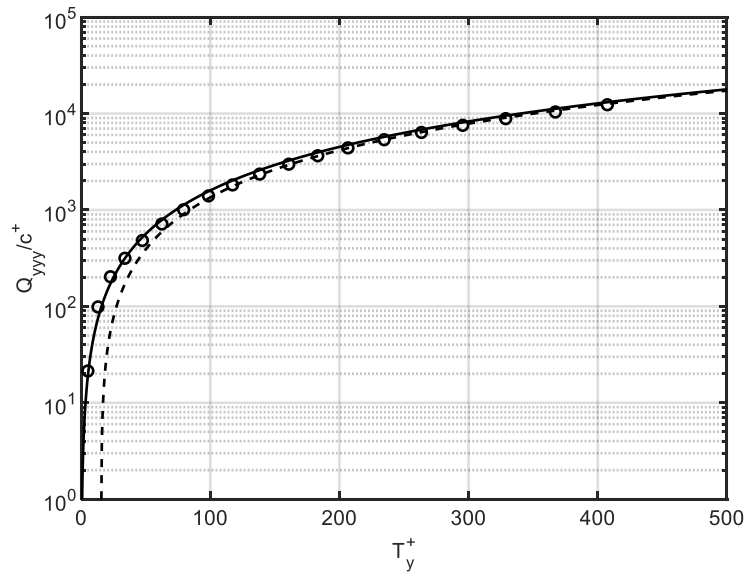
$$10 \quad Q_{yyy} = \frac{c^+}{\pi \sqrt{T_x^+ T_y^+}} \int_0^{+\infty} \int_{-\infty}^{+\infty} \left(\xi_y^{+2} - \frac{1}{C_D^2} \right) \xi_y^+ \exp \left[-\frac{(\xi_x^+ - u)^2}{2T_x^+} \right] \exp \left(-\frac{\xi_y^{+2}}{2T_y^+} \right) d\xi_x^+ d\xi_y^+ \quad (\text{B15})$$

$$= \frac{c^+}{\pi \sqrt{T_x^+ T_y^+}} \sqrt{2\pi T_x^+} \int_0^{+\infty} \left(\xi_y^{+3} - \frac{1}{C_D^2} \xi_y^+ \right) \exp \left(-\frac{\xi_y^{+2}}{2T_y^+} \right) d\xi_y^+ = c^+ \sqrt{\frac{2T_y^+}{\pi}} \left(2T_y^+ - \frac{1}{C_D^2} \right).$$

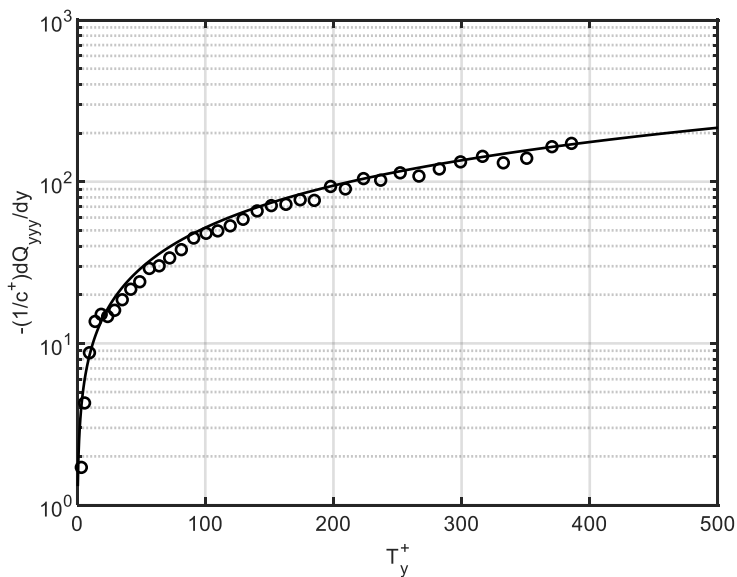
11 An undesirable, non-physical consequence of equation (B15) is that Q_{yyy} vanishes at a finite, nonzero value
12 of T_y^+ . However, in the same limit of large $C_D \xi_y^+$, we can ignore the second term between brackets and
13 obtain

$$14 \quad Q_{yyy} = \sqrt{\frac{8}{\pi}} c^+ T_y^{+3/2}. \quad (\text{B16})$$

15



1
2 Fig. B4. Ratio of energy flux Q_{yyy} over particle concentration of ascending particles as a function of T_y^+ measured in
3 numerical simulations of saltation in a viscous shearing flow with $d_w = 1$, $St = 100$, $r = 150000$, $Sh = 0.05$ (so that the
4 Stokes drag is dominant and $C_D = 18/St = 0.18$) and $M = 0.0131$. The dashed and solid lines are the predictions of
5 equations (B15) and (B16), respectively.
6



7
8 Fig. B5. Negative of the derivative of Q_{yyy} over particle concentration of ascending particles as a function of T_y^+ measured
9 in numerical simulations of saltation in a viscous shearing flow with $d_w = 1$, $St = 100$, $r = 150000$, $Sh = 0.05$ (so that the
10 Stokes drag is dominant and $C_D = 18/St = 0.18$) and $M = 0.0131$. The solid line is the predictions of D_{yy}/c^+ from equations
11 (B17).
12

13 Figure B4 indicates that the predictions of equations (B15) and (B16) are in excellent agreement with
14 the measurements in numerical simulations of saltation of particles in a viscous shearing flow experiencing
15 Stokes drag, but only equation (B16) has the correct limit for vanishing T_y^+ .

16 The yy -element of the dissipation tensor due to fluid drag in the balance of particle second moment
17 is $D_{yy} \equiv 2cC_D \langle \xi_y \xi_y \rangle$ (Saha and Alam 2017), so that, with the definition of σ_y and equation (B10),

$$D_{yy} = 2C_D \sigma_y = 2C_D c^+ T_y^+ + 2c^+ \sqrt{\frac{2T_y^+}{\pi}}. \quad (\text{B17})$$

Equation (9) indicates that D_{yy} should be equal to the negative of the derivative of Q_{yyy} . Figure B5 shows that the values of $-\frac{1}{c^+} \frac{dQ_{yyy}}{dy}$ measured in numerical simulations of saltation of particles in a viscous shearing flow experiencing Stokes drag as a function of T_y^+ do indeed match the predictions of D_{yy} / c^+ from equation (B17).

Finally, the particle shear stress is equal to the negative of the average vertical flux of x -momentum, $s \equiv -c \langle \xi_x \xi_y \rangle$, so that, from equation (B8),

$$s = -\frac{\pi}{6} \int_0^{+\infty} \int_{-\infty}^{+\infty} (\xi_x^+ - \xi_x^-) \xi_y^+ f^+ d\xi_x^+ d\xi_y^+. \quad (\text{B18})$$

To proceed, we must distinguish between saltation in viscous and turbulent shearing flows, because of the influence of the fluid velocity on the particle horizontal velocity.

B1. Saltation in viscous shearing flows

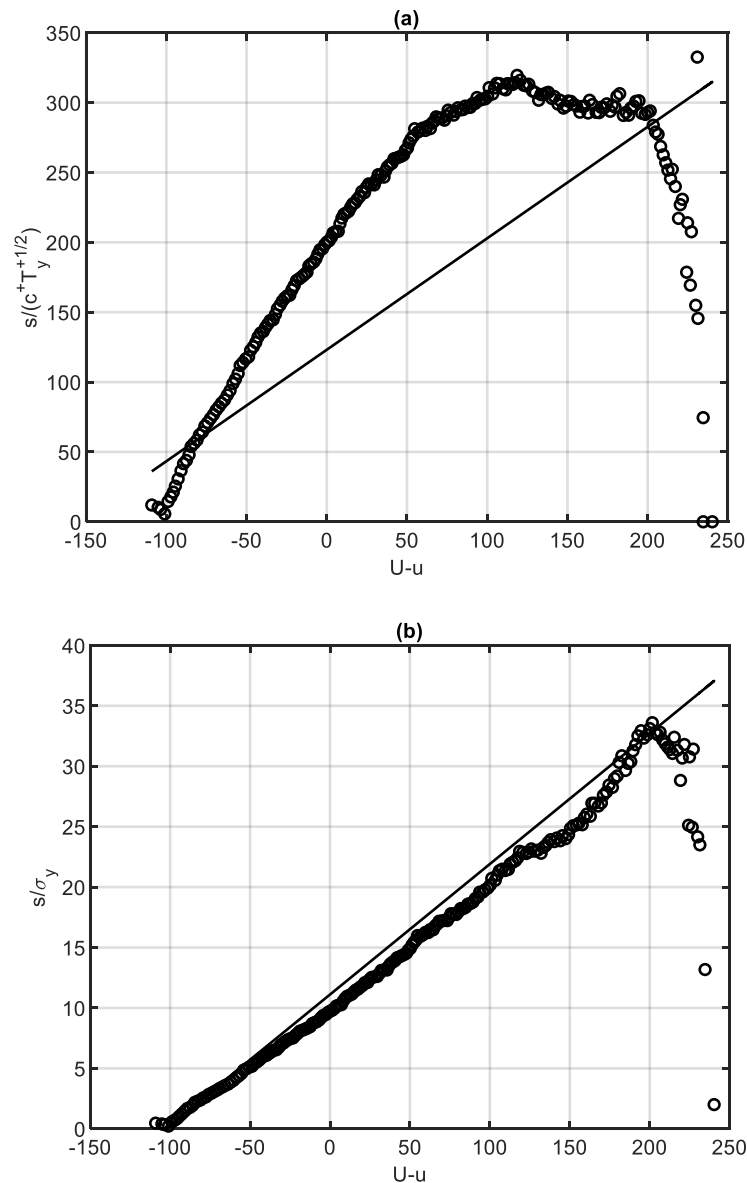
In the limit of large $C_D \xi_y^+$, using equations (A12) and (B2) into equation (B18), and integrating, we obtain the following expression, in the case of saltation in viscous shearing flows,

$$\begin{aligned} s &= -\frac{c^+}{\pi \sqrt{T_x^+ T_y^+}} \int_0^{+\infty} \xi_y^+ \exp\left(-\frac{\xi_y^{+2}}{2T_y^+}\right) \int_{-\infty}^{+\infty} \left[(\xi_x^+ - u) + u - U - \frac{1}{C_D^2} \frac{dU}{dy} \right] \exp\left[-\frac{(\xi_x^+ - u)^2}{2T_x^+}\right] d\xi_x^+ d\xi_y^+ \\ &= \frac{c^+}{\pi \sqrt{T_x^+ T_y^+}} \sqrt{\pi 2T_x^+} \left(U - u + \frac{1}{C_D^2} \frac{dU}{dy} \right) \int_0^{+\infty} \xi_y^+ \exp\left(-\frac{\xi_y^{+2}}{2T_y^+}\right) d\xi_y^+ \\ &= c^+ \sqrt{\frac{2T_y^+}{\pi}} \left(U - u + \frac{1}{C_D^2} \frac{dU}{dy} \right). \end{aligned} \quad (\text{B19})$$

Equation (B19) indicates that there is no influence of T_x^+ on the shear stress, because of the assumed Gaussian distribution of ξ_x^+ that causes the term involving $(\xi_x^+ - u)$ in the integrand of equation (B15) to disappear. However, (Valance and Berzi 2022) showed that the PDF of the downward velocity of the particles impacting the bed (dominated by the horizontal component) is actually non-symmetric. Given the perfect agreement with respect to the other constitutive relations obtained in this Appendix, which do not involve the horizontal velocity of the ascending particles, we are inclined to ascribe to the non-symmetric distribution of ξ_x^+ the fact that the shear stress measured in simulations of saltation particles in a viscous shearing flow experiencing Stokes drag (figure B6a) is not linear in the velocity difference, $U - u$, as implied by equation (B19). At small T_y^+ , equation (B10) in equation (B19) actually indicates that $s \propto C_D \sigma_y \left(U - u + \frac{1}{C_D^2} \frac{dU}{dy} \right)$. We found that

$$s = \frac{3}{5} C_D \sigma_y \left(U - u + \frac{2}{3} \frac{1}{C_D^2} \frac{dU}{dy} \right) \quad (\text{B20})$$

1 permits to reproduce the behaviour of the particle shear stress for saltation in viscous shearing flows better
 2 than equation (B19) in almost the entire flow domain (figure B6b).
 3



4

5
 6 Fig. B6. Ratio of (a) particle shear stress s and the product of particle concentration of ascending particles and the square
 7 root of T_y^+ and (b) particle shear stress and the particle normal stress as functions of the velocity difference, $U-u$,
 8 measured in numerical simulations of saltation in a viscous shearing flow with $d_w = 1$, $St = 100$, $r = 150000$, $Sh = 0.05$ (so
 9 that the Stokes drag is dominant and $C_D = 18/St = 0.18$) and $M = 0.0004$. The solid lines are the predictions of equations
 10 (B19) and (B20), respectively.
 11

12 B2. Saltation in turbulent shearing flows

13 In the limit of large $C_D \xi_y^+$, using equations (A17), (A20) and (B2) into equation (B14), and integrating, we
 14 obtain the following expression, in the case of saltation in turbulent shearing flows,

1

$$s = -\frac{c^+}{\pi\sqrt{T_x^+T_y^+}} \int_0^{+\infty} \xi_y^+ \exp\left(-\frac{\xi_y^{+2}}{2T_y^+}\right) \int_{-\infty}^{+\infty} \left\{ (\xi_x^+ - u) + u - U - 0.85 \frac{\sqrt{rS}}{\kappa} \left[\ln\left(\frac{\kappa\xi_y^+}{C_D\sqrt{rS}} \frac{dU}{dy} + 1\right) \right] \right\} \\ \times \exp\left[-\frac{(\xi_x^+ - u)^2}{2T_x^+}\right] d\xi_x^+ d\xi_y^+$$

2

$$= \frac{c^+}{\pi\sqrt{T_x^+T_y^+}} \sqrt{\pi 2T_x^+} \int_0^{+\infty} \xi_y^+ \left\{ U + 0.85 \frac{\sqrt{rS}}{\kappa} \left[\ln\left(\frac{\kappa\xi_y^+}{C_D\sqrt{rS}} \frac{dU}{dy} + 1\right) \right] - u \right\} \exp\left(-\frac{\xi_y^{+2}}{2T_y^+}\right) d\xi_y^+ \\ \simeq c^+ \sqrt{\frac{2T_y^+}{\pi}} \left[U - u + 0.85 \frac{\sqrt{rS}}{\kappa} \ln\left(\frac{\kappa}{C_D} \sqrt{\frac{T_y^+}{rS}} \frac{dU}{dy} + 1\right) \right] \simeq \frac{3}{5} C_D \sigma_y \left[U - u + \frac{3}{4} \frac{\sqrt{rS}}{\kappa} \ln\left(\frac{\kappa}{C_D} \sqrt{\frac{T_y^+}{rS}} \frac{dU}{dy} + 1\right) \right],$$

3

(B21)

4

where we have taken $\xi_y^+ \approx \sqrt{T_y^+}$ in the argument of the logarithm to carry out the analytical integration,

5

and, we have kept the same dependency on the drag coefficient and the normal stress as in the case of

6

saltation in viscous shearing flows (equation B20). The factor 3/4 in front of the logarithm allows to better fit

7

the results of discrete-continuum simulations of saltation in turbulent shearing flows.

8

1 Appendix C. Boundary conditions for shallow impacts at a rigid, bumpy bed

2 After Lämmel et al. (2017), in the case of angle of impact θ much less than $\pi/2$ (shallow impact) of particles
 3 with large coefficient of friction, the negative of the average ratio of the vertical particle velocity after and
 4 before the impact is

$$5 \left\langle -\frac{\xi_{y,0}^+}{\xi_{y,0}^-} \right\rangle = -e_t + \frac{2}{3}(e_n + e_t) \sqrt{\frac{2}{\theta} \frac{2d_w}{1+d_w}}, \quad (C1)$$

6 where e_n and e_t are the coefficients of normal and tangential restitution, respectively, and, here and in what
 7 follows, we use the subscript 0 to indicate quantities evaluated at the bed.

8 In the limit of large take-off velocity, equations (A8), (A12) and (24) give

$$9 \theta \simeq \tan\left(-\xi_{y,0}^- / \xi_{x,0}^-\right) \simeq \frac{C_D}{\text{St}(\text{Sh} - s_0)}, \quad (C2)$$

10 for saltation in viscous shearing flows.

11 In the case of saltation in turbulent shearing flows, instead, in the limit of large take-off velocity,
 12 equations (A8), (A17) and (A20) give

$$13 \theta \simeq \left[0.85 C_D \frac{\sqrt{r(\text{Sh} - s_0)}}{\kappa} \ln \left(\frac{1}{C_D} \frac{\sqrt{T_{y,0}^+}}{y_0} \right) \right]^{-1}, \quad (C3)$$

14 with $\xi_{y,0}^+ \simeq \sqrt{T_{y,0}^+}$.

15 By inserting equation (A8) into equation (C1), we obtain

$$16 \left\langle \xi_{y,0}^+ \right\rangle = -\frac{e_t}{C_D} + \frac{2\sqrt{2}}{3}(e_n + e_t) \frac{1}{C_D \theta^{1/2}} \sqrt{\frac{2d_w}{1+d_w}}. \quad (C4)$$

17 Then, we can assume that $T_{y,0}^+ = \left\langle (\xi_{y,0}^+)^2 \right\rangle \propto \left\langle \xi_{y,0}^+ \right\rangle^2$, so that at leading order,

$$18 T_{y,0}^+ = \alpha_T \frac{1}{C_D^2 \theta} \frac{2d_w}{1+d_w}, \quad (C5)$$

19 with the coefficient of proportionality α_T weakly dependent on the coefficients of normal and tangential
 20 restitution and, perhaps, on the flow regime of the fluid. Lämmel et al. (2017) also obtained

$$21 \left\langle \frac{(\xi_{y,0}^+)^2 + (\xi_{x,0}^+)^2}{(\xi_{y,0}^-)^2 + (\xi_{x,0}^-)^2} \right\rangle = \left[e_t - \frac{(e_t^2 - e_n^2)}{2e_t} \theta \frac{2d_w}{1+d_w} \right]^2, \quad (C6)$$

22 so that, with equations (A8) and (A12), and $\left\langle (\xi_{x,0}^+)^2 \right\rangle \propto \left\langle \xi_{x,0}^+ \right\rangle^2 = (u_0^+)^2$,

$$23 u_0^+ \propto \sqrt{\frac{1}{C_D^2} \left(1 + \frac{1}{\theta^2}\right) \left[e_t - \frac{(e_t^2 - e_n^2)}{2e_t} \theta \frac{2d_w}{1+d_w} \right]^2} - T_{y,0}^+. \quad (C7)$$

24 Then, we can determine the total particle slip velocity at the bed as

1
$$u_0 = \frac{c_0^+ u_0^+ + c_0^- u_0^-}{c_0}. \quad (\text{C8})$$

2 With equations (B5) and (B6), and u_0^- given by equation (A12), we obtain, at leading order,

3
$$u_0 = \alpha_u \frac{1}{C_D \theta}, \quad (\text{C9})$$

4 where, once again, the coefficient of proportionality α_u is a weak function of the coefficients of restitution
5 and the flow regime of the fluid. Using equation (C5) into equation (C9) gives

6
$$u_0 = \frac{\alpha_u}{\alpha_T} \frac{1+d_w}{2d_w} C_D T_{y,0}^+. \quad (\text{C10})$$

7

1 Appendix D. Discrete-continuum numerical simulations

2 The discrete-continuum (DC) simulations is based on the combination of a discrete element method for the
3 particle dynamics coupled to a continuum description of hydrodynamics, as developed in Durán et al. (2012),
4 Ralaiarisoa *et al.* (2020) and Valance et al. (2022).

5 The particle motion is described by a Lagrangian approach according to which the particle labelled p obeys
6 the following dimensionless equations:

$$7 \quad \frac{d\xi^p}{dt} = -\mathbf{e}_y + \sum_q \mathbf{f}_c^{p,q} + \mathbf{f}_{drag}^p \quad (D1)$$

$$8 \quad I \frac{d\boldsymbol{\omega}^p}{dt} = \frac{1}{2} \sum_q \mathbf{n}^{p,q} \times \mathbf{f}_c^{p,q} \quad (D2)$$

9 where ξ^p and $\boldsymbol{\omega}^p$ are the translational and angular velocity vectors of particle p , respectively; \mathbf{e}_x and \mathbf{e}_y
10 are the horizontal and vertical unit vectors, respectively; $\mathbf{f}_{drag}^p = C_D \left[(U - \xi_x^p) \mathbf{e}_x - \xi_y^p \mathbf{e}_y \right]$ is the
11 dimensionless drag force exerted by the fluid on the p -particle and $\mathbf{f}_c^{p,q}$ is the dimensionless contact force
12 between particles p and q ; $I = 1/10$ is the moment of inertia of a sphere; and $\mathbf{n}^{p,q}$ is the unit vector along the
13 contact direction.

14 The normal component $f_{c,n}$ of the contact force is modelled by a linear spring dashpot, so that
15 $f_{c,n} = (k_n \delta + \gamma_n v_n)$, where k_n is the spring stiffness, δ the overlap between the compliant spheres, γ_n the viscous
16 damping coefficient and v_n the normal component of the relative translational particle velocities. The
17 negative of the ratio between the normal relative velocity before and after the collision is the coefficient of
18 normal restitution e_n . If the values of e_n and k_n are prescribed, γ_n is deduced from the following relation: $\gamma_n =$
19 $(\pi/6) \sqrt{12k_n / (1 + \pi^2 / \ln(e_n)^2)}$. The tangential component $f_{c,t}$ of the contact force is described via a
20 Coulomb friction model regularized through a viscous damping: $f_{c,t} = -\min(\mu f_{c,n}, \gamma_t v_t) \text{sign}(v_t)$, where μ is the
21 Coulomb friction coefficient, v_t the relative slip velocity at contact and γ_t the tangential viscous damping
22 coefficient. The values chosen for the parameters are: $k_n = p/6 \cdot 10^7$, $\gamma_n = \gamma_t$, $e_n = 0.88$, and $\mu = 0.5$.

23 The fluid motion is solved by an Eulerian description based on Eq. (24) or Eq. (37) for laminar and
24 turbulent flows, respectively. The vertical component of the fluid velocity is assumed to be zero, so that only
25 the horizontal momentum balance is required and reads

$$26 \quad \frac{dS}{dy} = F_x, \quad (D3)$$

27 where $F_x = c < \sum_{p \in [y; y+dy]} C_D (U - \xi_x^p) > / < \sum_{p \in [y; y+dy]} 1 >$, with the angular brackets denoting
28 ensemble averaging, represents the x -component of the average volume force exerted on the fluid by the
29 particles whose centers are located in the horizontal slice between y and $y+dy$. The integration of
30 equation (D3) gives

$$31 \quad S = \text{Sh} - \int_0^\infty F_x dy, \quad (D4)$$

32 where the infinite upper bound of the integral means that all moving grains located above y must be
33 accounted for. Once the vertical profile $S(y)$ of the fluid shear stress is determined, the horizontal fluid
34 velocity profile can then be obtained from the integration Eq. (24) or Eq. (37) for laminar and turbulent
35 shearing flows, respectively, with the no-slip boundary condition $U = 0$ at $y = 0$.

1 The simulated system is quasi-two-dimensional with a streamwise length equal to 5120 particle
2 diameters and a transverse length equal to one diameter. We use spherical particles with a polydispersity of
3 $\pm 10\%$ and adjust their number in the system to obtain the desired value of the mass hold-up M . Periodic
4 boundary conditions are employed in the streamwise direction. The domain is not upper bounded, while the
5 lower boundary is composed of a layer of immobile particles of diameter d_w in close contact (rigid, bumpy
6 bed; see Figure 1). As mentioned earlier in the paper, we suppress the possibility of particle–particle collisions
7 above the bed.

8 Operatively, at every time step, we integrate equations (D1) and (D2) for every particle in the system
9 and determine its new velocity and position. We then use this information to update the profiles of fluid
10 shear stress and horizontal velocity via equations (4), and (24) for viscous flows or equation (37) for turbulent
11 flows and proceed with the next time step until we reach a steady state, that is, when the horizontal mass
12 flux averaged over a window of 100 unit time steps is stationary. Initially, the fluid profile is taken to be linear
13 or logarithmic for viscous and turbulent shearing flows, respectively, and corresponds to the unperturbed
14 profile in the absence of particles. The particles are initially displayed on a horizontal row located at ten
15 diameters from the rigid, bumpy bed, with a constant inter-particle distance equal to two diameters and zero
16 initial velocity. Importantly, the final state is independent of the initial conditions as long as the number of
17 particles in the flow does not surpass its transport capacity.

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