Optimal Control of a Nonsmooth Dynamical System using Measure Relaxation

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Introduction

- **Problem Statement**: Optimal control of a nonsmooth dynamical system is modeled using an evolution variational inequality.
- **Approach**: Both discrete and continuous time formulations are considered, with relaxation in the space of measures. No relaxation gap exists between the original finite-dimensional problem and the relaxed version.
- **Solution Approach**: Approximate solutions via the converging Moment Sum of squares hierarchy.

Measure Formulation

Young measure relaxation: A weakly converging minimizing sequence $(x(\cdot), u(\cdot))$ generates Young measure $\delta_{x(\cdot)} \otimes \omega(du|t)$ such that

$$\dot{x}_r(t) = -\int_{\mathcal{N}_S(x(t))} \zeta \; d\eta(\zeta|x(t)) + \int_U f(x(t),u) d\omega(u|t).$$

Occupation measure relaxation: We use the superposition principle to obtain a linear equation in measures

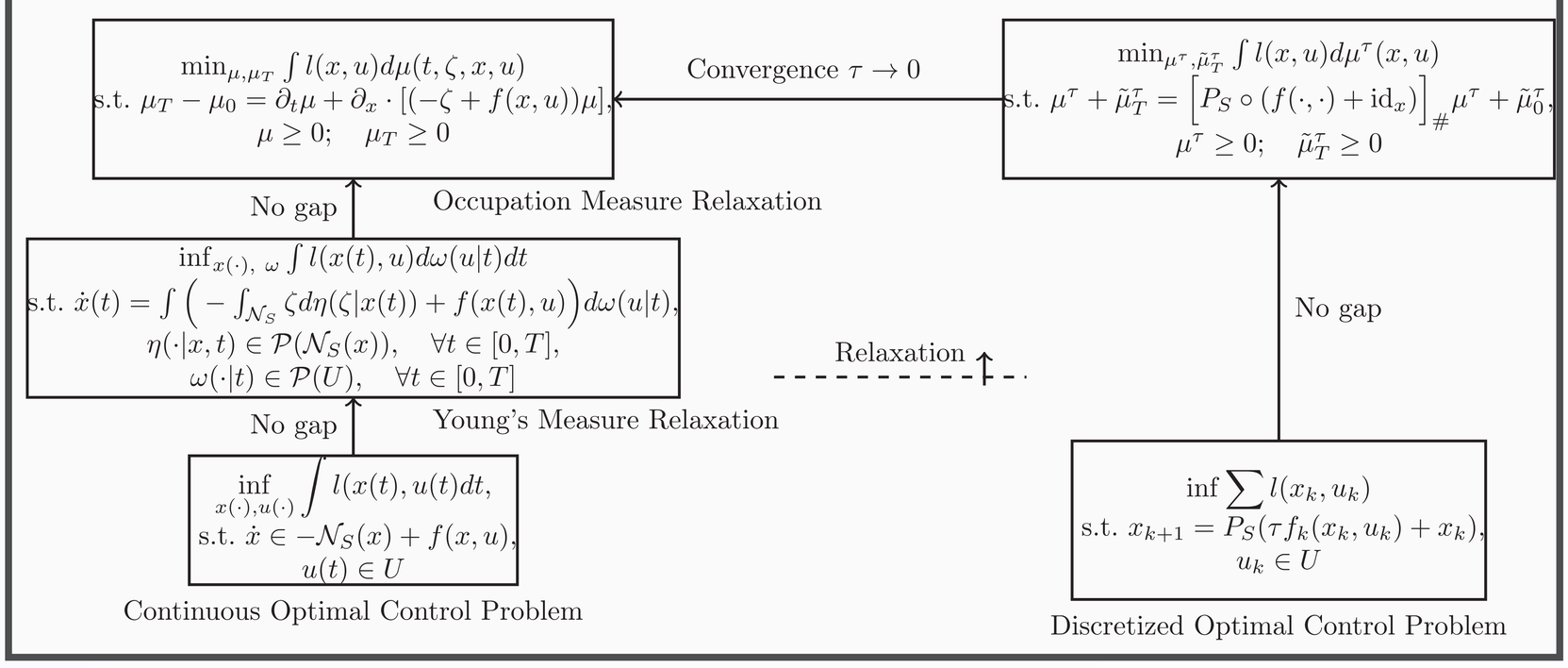
$$\int_{[0,T]\times S(t)} \int_U \int_{\mathcal{N}_{S(t)}(x)} \Big[\frac{\partial \phi(t,x)}{\partial t} + \frac{\partial \phi(t,x)}{\partial x} \cdot (-\zeta + f(x,u))\Big] d\mu(t,x,\zeta,u) = \int_{S(T)} \phi d\mu_T - \phi(t_0,x_0).$$

Discrete-time to continuous-time convergence:

$$\tilde{\mu}^\tau_{k+1} = G^\tau_{k\#}\tilde{\mu}^\tau_k, \quad v^\tau_{k+1} = \frac{G^\tau_k(x) - x}{\tau}, \quad G^\tau_k(x) := P_S\Big(\int_U f(x,u) d\omega^\tau(u|k\tau,x) + x\Big).$$

We show that interpolated measures curves converge uniformly (w.r.t. Wasserstein-2 metric) to the solution to the continuity equation.

Schematic Diagram

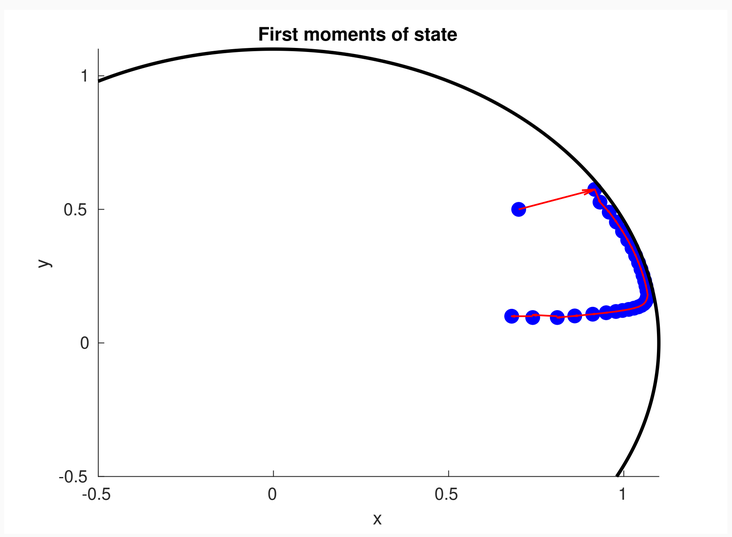


Numerical Results

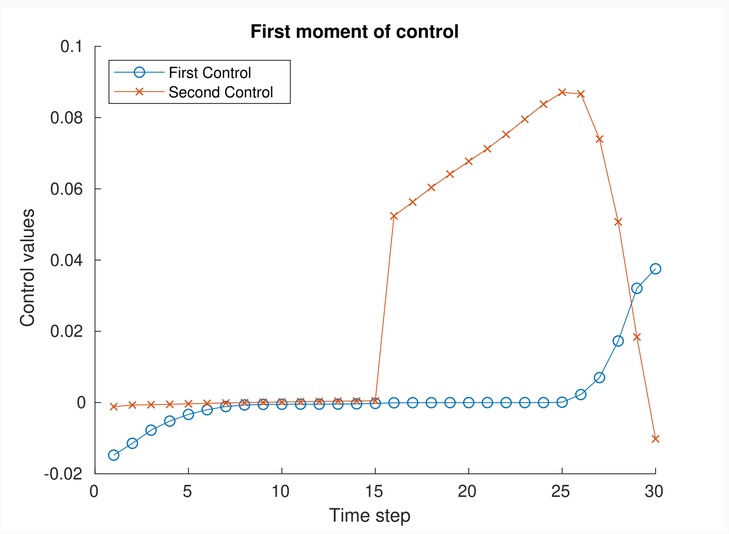
A particle moving in a disc.

$$\dot{x}(t) \in -\mathcal{N}_S(x(t)) + (1 + u_1(t), u_2(t)) \quad \text{where} \quad S := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1.1^2\}$$

$$x_0 = (0.7, 0.5); \; x_T = (0.6, 0.1)$$



Moment SOS approximations of the first-degree moments



Control signal

Nonsmooth Systems

We consider state-constrained systems:

$$\dot{x}(t) \in f(x(t),u(t)) - \mathcal{N}_{S(t)}(x(t))$$

where $f(\cdot,\cdot)$ is a Lipschitz continuous drift, $u(\cdot)$ is the control input, and $\mathcal{N}_{S(t)}(x)$ denotes the outward normal cone to the convex set $S(t)$ at $x \in S(t)$. The model can be expressed as an evolution variational inequality

$$\langle \dot{x}(t) - f(x(t),u(t)), y - x(t)\rangle \geq 0$$

whenever $x(t) \in S(t)$.

Applications:
Elastoplastic Mechanical Systems, Modeling congestion and vehicle interactions in traffic flow, Control systems with state Constraints, Economics and Game Theory

LMI relaxation

The primal problem in the space of measures is written as

$$\begin{aligned} p(x_0)^* = \; & \min_{\mu,\mu_T} \langle \mathrm{l}, \mu\rangle + \langle \mathrm{l_T}, \mu_T\rangle \\ & \text{s.t. } \langle v(t,x), \mu_T\rangle + \langle v(0,x), \mu_0\rangle \\ & \quad = \langle \frac{\partial v}{\partial t} + \sum_{i=1}^n \frac{\partial v}{\partial x_i}(-\zeta + f)_i, \mu\rangle \\ & \mu \in \mathcal{M}_+([0,T]\times B) \quad \mu_T \in \mathcal{M}_+(X_T) \end{aligned}$$

where $B := \{(x,v,u) | x \in \Omega \subset \mathbb{R}^n, v \in \mathcal{N}_{S(t)}(x), u \in U\}$. LMI relaxation based on the Moment SoS hierarchy of the optimal control problem can be derived as follows

$$\begin{aligned} p_d^* = \min_{m,m^T} & \sum_{(a,b)\in\mathbb{N}^{1+2}_{2d}} c_{a,b} m_{a,b} \\ \text{s.t.} \quad & m^T_{a,b} - m^0_{a,b} = a m^S_{a-1,b} + a m^B_{a-1,b} \\ & + \sum_{i=1}^n \int_{X_B} b_i t^a \mathbf{x}^{b-e_i} v_i d\mu_B(t,x,v,u) \\ & + \sum_{i=1}^n \int_{X_S} b_i t^a \mathbf{x}^{b-e_i} v_i d\mu_S(t,x,v,u), \\ & M_d(g_i m) \geq 0. \end{aligned}$$

This finite-dimensional LMI produces a non-decreasing sequence of lower bounds that is solved using off the shelf solver like Mosek. It can be shown that the lower bounds converge

$$p_d^* \leq p_{d+1}^* \leq \ldots \leq p_\infty^* = p^*.$$

Future Works

- Continuity equation for sweeping processes with bounded variation, and related optimal control problems
- Exploiting sparsity to increase computational efficiency

Bibliography

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- Evolution of measures in the nonsmooth dynamical system (hal-04570185v1)