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Transverse vibration midpoint control of an elastic string via an autoregressive control

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Abstract: This paper focuses on the transverse vibration midpoint control of a string. More precisely, we explore the use of a control block that includes delays in both input and output signals under the assumption that the memory-window length is also a parameter of the control law. Relatively simple to construct, such a control law was successfully used in the boundary control of the transport and wave equations. The effectiveness of the proposed method is illustrated through numerical simulations and the corresponding finite-difference scheme is detailed.

Keywords: time-delay, hyperbolic PDE, pointwise control.

1. INTRODUCTION

In recent years, a new *partial pole placement* method has been developed for the control of infinite-dimensional systems. It is essentially based on two properties called, respectively *multiplicity-induced-dominancy* (MID) and *coexistent-real-roots-inducing-dominancy* (CRRID), see for example Boussaada et al. (2022, 2020); Mazanti et al. (2021); Ramírez et al. (2016); Schmoderer et al. (2023, 2024). Although the MID property has been widely studied and has recently emerged as closely related to the spectral abscissa minimization problem, as pointed out in Michiels et al. (2023); Pinney (1958), the CRRID property seems more complex to characterize but offers a greater degree of robustness, see for instance Schmoderer et al. (2023).

In other respects, conventional control methods often rely on instantaneous feedback information, which may be limited in capturing the nuanced dynamics of infinite-dimensional systems. In this note, we present a new methodology that exploits past data to improve the stability of wave systems through autoregressive control, as suggested in Ammari et al. (2023, 2024b) in the boundary control as well as in Ammari et al. (2024a) in the pointwise control of some hyperbolic PDEs. By integrating past information into the control framework, our approach offers a proactive strategy for mitigating instabilities and improving overall system behavior. The main benefit is a new perspective that exploits past data to build the control signal, thus improving system stability and performance.

The contribution of the paper is twofold: first, to explore the potential of the autoregressive control law in pointwise stabilization of an elastic string by using the delay as a control parameter, as proposed in Ammari et al. (2024a). We note that the pointwise stabilization of elastic structures (strings or beams) without delay, or, equivalently,

stabilization of serially connected stings or beams with dissipative joints, has been widely studied in recent literature, see Ammari and Tucsnak (2000), Ammari et al. (2002), Ammari et al. (2001), Ammari and Mercier (2015) and references therein. Such a control block is of low-complexity type since it includes a reduced number of parameters, and it offers the advantage of being simpler to implement. Furthermore, the idea to leave the delay as a parameter to tune allows capturing dynamics property that improves the behaviour of the closed-loop system. Second, the corresponding numerical framework and related simulations are presented in some extent (see, for instance, Ammari et al. (2023)).

The remaining of the paper is organized as follows: the problem formulation and main results are presented in Section 2. The material for the numerical implementation is given in Section 3, where simulations are performed for various cases. Some concluding remarks end the paper. The notations are standard and explained when used first time.

2. PROBLEM FORMULATION AND MAIN RESULTS

Given a perfectly elastic string of length $\ell > 0$ with one fixed end that can pivot and the other one that can slide while not pivoting, under the assumption of constant density and elasticity parameters, when there are no body forces, the motion equation is described by the following Partial Differential Equation (PDE) of hyperbolic type:

$$\begin{cases} U_{tt}(\mathbf{x}, \mathbf{t}) - c^2 U_{xx}(\mathbf{x}, \mathbf{t}) = 0, \\ \text{with } (\mathbf{x}, \mathbf{t}) \in (0, \ell) \times (0, +\infty) \\ U(0, \mathbf{t}) = 0, \quad U_x(\ell, \mathbf{t}) = 0, \quad \mathbf{t} > 0, \\ U(\mathbf{x}, 0) = f(\mathbf{x}), \quad U_t(\mathbf{x}, 0) = g(\mathbf{x}), \quad \mathbf{x} \in (0, \ell), \end{cases} \quad (\tilde{\mathcal{P}})$$

where $U(\mathbf{x}, \mathbf{t})$ denotes the transversal displacement of a point belonging to the string, located at position \mathbf{x} and given at time \mathbf{t} ; the constant $c > 0$ is the wave celerity; finally, f and g are smooth univariate functions that are specified further below.

Remark 2.1. The wave equation problem modeled by equation $(\tilde{\mathcal{P}})$ can be normalized, *i.e.* by using dimensionless time t and position x variables, in such a way that their corresponding relations to the considered true variables are $\mathbf{x} = \ell x$ and $\mathbf{t} = dt$ with $d := \frac{\ell}{c}$. In that case, the normalized transversal displacement denoted $u(x, t)$ is related to the original one by the relation

$$u(x, t) := U(\mathbf{x}, \mathbf{t}) = U(\ell x, dt). \quad (1)$$

The control structure is defined by the following delay-based autoregressive control law applied in a pointwise manner at the midpoint $\mathbf{x} = \xi$, $\xi := \frac{\ell}{2}$, in $(\tilde{\mathcal{P}})$ where now $(\mathbf{x}, \mathbf{t}) \in (0, \ell) \setminus \{\xi := \frac{\ell}{2}\} \times (0, +\infty)$. It writes

$$\begin{cases} U(\xi^-, \mathbf{t}) = U(\xi^+, \mathbf{t}), & \xi := \frac{\ell}{2}, \quad \mathbf{t} > 0, \\ U_{\mathbf{x}}(\xi^+, \mathbf{t}) - U_{\mathbf{x}}(\xi^-, \mathbf{t}) + \alpha (U_{\mathbf{x}}(\xi^+, \mathbf{t} - \tau)) \\ - U_{\mathbf{x}}(\xi^-, \mathbf{t} - \tau) = \frac{\beta}{c} U_{\mathbf{t}}(\xi, \mathbf{t}) + \frac{\gamma}{c} U_{\mathbf{t}}(\xi, \mathbf{t} - \tau), & (\tilde{\mathcal{C}}) \\ \mathbf{t} > 0, \\ U_{\mathbf{t}}(\xi, \mathbf{t} - \tau) = 0, & \mathbf{t} \in (0, \tau), \\ U_{\mathbf{x}}(\xi^+, \mathbf{t} - \tau) - U_{\mathbf{x}}(\xi^-, \mathbf{t} - \tau) = 0, & \mathbf{t} \in (0, \tau), \end{cases}$$

where $\tau := d\tau$ is the delay used here as a control parameter, and τ is the corresponding normalized delay.

It has been shown in Ammari et al. (2024a) that the normalized problem derived from $(\tilde{\mathcal{P}})$ by using (1), in closed-loop with the output feedback control law in $(\tilde{\mathcal{C}})$ leads to the characteristic function in the Laplace variable s , denoted $\Gamma(s, \tau)$, that is depending on the control law's parameters α , β , γ and τ , and given by $\Gamma(s, \tau) =$

$$1 + \frac{2 - \beta}{2 + \beta} e^{-2s} + \frac{2\alpha + \gamma}{2 + \beta} e^{-\tau s} + \frac{2\alpha - \gamma}{2 + \beta} e^{-(\tau+2)s}, \quad (2)$$

for $\tau \neq 2$, and by

$$1 + \frac{(2 + 2\alpha - \beta + \gamma) e^{-2s}}{2 + \beta} + \frac{(2\alpha - \gamma) e^{-4s}}{2 + \beta}, \quad (3)$$

for $\tau = 2$. In both cases, by exploiting the Multiplicated Induced Dominancy (MID) paradigm first proposed in Boussaada et al. (2016), the partial pole placement of the roots of either (2) or (3) can be handled in the following way.

Theorem 2.2. Given a real number $s_0 < 0$ and consider the output feedback stabilization of the problem $(\tilde{\mathcal{P}})$ with the control law in $(\tilde{\mathcal{C}})$ associated with an arbitrary but given positive delay τ . Then the following assertions hold:

- If $\tau = 2$, and α , β , γ are such that $\gamma \neq 2\alpha$ and $\gamma \neq \beta - 2\alpha - 2$, with

$$\begin{cases} \alpha = \frac{(1 - e^{2s_0})^2}{4} \beta - e^{2s_0} + \frac{e^{4s_0}}{2} - \frac{1}{2}, \\ \gamma = \left(\frac{1}{2} - e^{2s_0} - \frac{e^{4s_0}}{2} \right) \beta - (1 + e^{2s_0})^2, \end{cases} \quad (4)$$

then the solution's exponential decay rate is assigned at an arbitrary $-s_0$; s_0 is a double root of (3) and is

its spectral abscissa. Furthermore, all zeros of (3) are double and lie on the vertical axis $\Re\{s\} = s_0$.

- If $\tau = 2k$ where k is an integer greater than one, and α , β , γ are given by

$$\begin{cases} \alpha = \frac{(\tau - 2) e^{2s_0} - (\tau + 2)}{(\tau + 2) e^{2s_0} - (\tau - 2)} e^{\tau s_0}, \\ \beta = -2 \frac{(\tau + 2) e^{2s_0} + (\tau - 2)}{(\tau + 2) e^{2s_0} - (\tau - 2)}, \\ \gamma = -2 \frac{(\tau - 2) e^{2s_0} + (\tau + 2)}{(\tau + 2) e^{2s_0} - (\tau - 2)} e^{\tau s_0}, \end{cases} \quad (5)$$

then the solution's exponential decay rate at an arbitrary $-s_0$; s_0 is a triple root of (2) and is its spectral abscissa.

- If $\tau \neq 2k$, then the control parameter tuning prescribed in (5) allows a closed-loop solution decaying exponentially faster than $-s_0 - \hat{\rho}(\tau)$, where $\hat{\rho}$ is defined by

$$\hat{\rho}(\tau) := \frac{1}{\min\{\tau, 2\}} \ln \left(1 + 2 \frac{\tau + 2}{|\tau - 2|} \right). \quad (6)$$

The proof of this result is proposed in Ammari et al. (2024a).

Remark 2.3. Notice that the total mechanical energy of the string is given, up to a multiplicative coefficient, by

$$E(\mathbf{t}) := \frac{1}{2} \int_0^1 (|U_{\mathbf{t}}(\mathbf{x}, \mathbf{t})|^2 + |U_{\mathbf{x}}(\mathbf{x}, \mathbf{t})|^2) dx. \quad (7)$$

The well-posedness of the closed-loop system $\{(\tilde{\mathcal{P}}), (\tilde{\mathcal{C}})\}$ has been established in Ammari et al. (2024a) by using this expression. Moreover, with this last, some sufficient conditions have also been proposed in the same reference for the dissipativity of a differential operator related to the proposed problem.

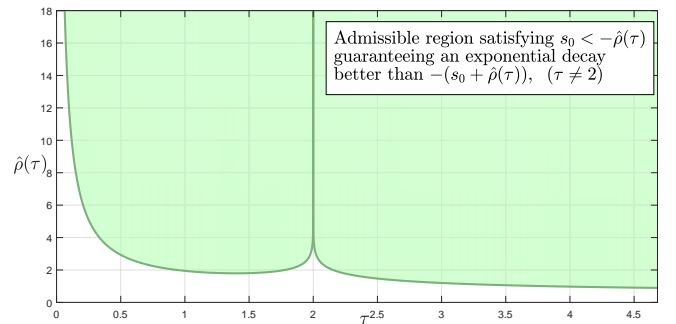


Fig. 1. Plot of the function $\hat{\rho}(\tau)$ given by expression (6).

3. NUMERICAL IMPLEMENTATION

Simulating the time response of PDE-based system can raise several issues related to the selected numerical scheme. This is particularly acute when an output-to-input mathematical relationship must be considered for control purposes. The presence of delays in the control law strengthens the difficulty. To validate numerically the performances of the proposed control scheme, we first propose performing a finite difference discretization of the problem $\{(\tilde{\mathcal{P}}), (\tilde{\mathcal{C}})\}$ using the following approximations, suited to the control structure $(\tilde{\mathcal{C}})$.

3.1 Finite difference scheme

Let us define the constant space step $\Delta x > 0$, the constant time step $\Delta t > 0$ and the numerical sequences

$$\begin{aligned} \mathbf{x}_i &:= i \Delta x \quad (i = 0, 1, 2, \dots, N), \\ \mathbf{t}_j &:= j \Delta t \quad (j = 0, 1, 2, \dots, n) \\ \text{and } U_{i,j} &:= U(\mathbf{x}_i, \mathbf{t}_j), \end{aligned} \quad (8)$$

where $M, N, n \in \mathbb{N}$ are such that

$$M \Delta x = \frac{l}{2}, \quad N = 2M \quad \text{and} \quad n \Delta t = T_f \quad (9)$$

where T_f stands for the final time for the simulation. Let us

$$\begin{aligned} \text{denote } U_{\mathbf{t}^i, j} &:= \left. \frac{\partial U(\mathbf{x}, \mathbf{t})}{\partial \mathbf{t}} \right|_{\substack{\mathbf{x}=\mathbf{x}_i, \\ \mathbf{t}=\mathbf{t}_j}}, \quad U_{\mathbf{x}^i, j} := \left. \frac{\partial U(\mathbf{x}, \mathbf{t})}{\partial \mathbf{x}} \right|_{\substack{\mathbf{x}=\mathbf{x}_i, \\ \mathbf{t}=\mathbf{t}_j}}, \\ U_{\mathbf{t}\mathbf{t}^i, j} &:= \left. \frac{\partial^2 U(\mathbf{x}, \mathbf{t})}{\partial \mathbf{t}^2} \right|_{\substack{\mathbf{x}=\mathbf{x}_i, \\ \mathbf{t}=\mathbf{t}_j}}, \quad \text{and } U_{\mathbf{x}\mathbf{x}^i, j} := \left. \frac{\partial^2 U(\mathbf{x}, \mathbf{t})}{\partial \mathbf{x}^2} \right|_{\substack{\mathbf{x}=\mathbf{x}_i, \\ \mathbf{t}=\mathbf{t}_j}}. \end{aligned}$$

For all $\mathbf{t} > 0$, the boundary conditions in $(\tilde{\mathcal{P}})$ are transcribed as follows with $j \in \mathbb{N}^*$:

$$\begin{aligned} U(0, \mathbf{t}) = 0 &\longrightarrow U_{0,j} = 0, \\ U_{\mathbf{t}}(0, \mathbf{t}) = 0 &\longrightarrow U_{\mathbf{t}^0, j} = 0, \\ U_{\mathbf{t}\mathbf{t}}(0, \mathbf{t}) = 0 &\longrightarrow U_{\mathbf{t}\mathbf{t}^0, j} = 0. \end{aligned}$$

For all $\forall \mathbf{x} \in (0, l)$, the initial conditions read, for $i = 0, 1, 2, \dots, N$,

$$\begin{aligned} U(\mathbf{x}, 0) = f(\mathbf{x}) &\longrightarrow U_{i,0} = f(i \Delta x), \\ U_{\mathbf{t}}(\mathbf{x}, 0) = g(\mathbf{x}) &\longrightarrow U_{\mathbf{t}^i, 0} = g(i \Delta x), \\ U_{\mathbf{x}}(\mathbf{x}, 0) = f'(\mathbf{x}) &\longrightarrow U_{\mathbf{x}^i, 0} = f'(i \Delta x), \\ U_{\mathbf{x}\mathbf{x}}(\mathbf{x}, 0) = f''(\mathbf{x}) &\longrightarrow U_{\mathbf{x}\mathbf{x}^i, 0} = f''(i \Delta x). \end{aligned}$$

By continuity of $U(\mathbf{x}, \mathbf{t})$ in the variable \mathbf{t} , the initial conditions are then

$$\begin{aligned} U(0, 0) = 0 &\Rightarrow f(0) = 0 \longrightarrow U_{0,0} = 0, \\ U_{\mathbf{x}}(l, \mathbf{t}) = 0 &\Rightarrow U_{\mathbf{x}}(l, 0) = 0 \Rightarrow f'(l) = 0 \longrightarrow U_{\mathbf{x}^N, j} = 0, \\ U_{\mathbf{t}}(0, \mathbf{t}) = 0 &\Rightarrow U_{\mathbf{t}}(0, 0) = 0 \Rightarrow g(0) = 0 \longrightarrow U_{\mathbf{t}^0, 0} = 0. \end{aligned}$$

Moreover, at $i = 0$ and $j = 0$, it turns out that $f''(0) = 0$. To summarize, at this stage, the functions f and g should satisfy

$$\begin{aligned} f(0) = 0, \quad f''(0) = 0, \quad f'(l) = 0, \\ g(0) = 0. \end{aligned} \quad (10)$$

Consider the following approximations of the first-order partial derivatives

$$U_{\mathbf{t}^i, j} = \frac{1}{2 \Delta t} (3 U_{i,j} - 4 U_{i,j-1} + U_{i,j-2}), \quad (11)$$

$$U_{\mathbf{x}^i, j} = \frac{1}{2 \Delta x} (U_{i+1,j} - U_{i-1,j}), \quad (12)$$

and the ones for the second-order partial derivatives

$$U_{\mathbf{t}\mathbf{t}^i, j} = \frac{1}{\Delta t^2} (U_{i,j} - 2 U_{i,j-1} + U_{i,j-2}), \quad (13)$$

$$U_{\mathbf{x}\mathbf{x}^i, j} = \frac{1}{\Delta x^2} (U_{i+1,j} - 2 U_{i,j} + U_{i-1,j}), \quad (14)$$

defined for $i = 1, 2, \dots, N$ and $j = 2, 3, \dots, n$. For $j = 1$, we set

$$\begin{aligned} U_{i,1} &= U_{i,0} + \Delta t U_{\mathbf{t}^i, 0}, \\ &= f(i \Delta x) + \Delta t g(i \Delta x). \end{aligned} \quad (15)$$

The delay-based autoregressive control law $(\tilde{\mathcal{C}})$ is turned into a numerical sequence given by

$$\begin{cases} U_{\mathbf{x}^{M+1}, j} - U_{\mathbf{x}^{M-1}, j} = \frac{\beta}{c} U_{\mathbf{t}^M, j}, \\ \quad \text{for } j = 0, 1, \dots, \delta - 1, \\ U_{\mathbf{x}^{M+1}, j} - U_{\mathbf{x}^{M-1}, j} = -\alpha (U_{\mathbf{x}^{M+1}, j-\delta} - U_{\mathbf{x}^{M-1}, j-\delta}) \\ \quad + \frac{\beta}{c} U_{\mathbf{t}^M, j} + \frac{\gamma}{c} U_{\mathbf{t}^M, j-\delta}, \text{ for } j = \delta, \delta + 1, \dots, n \end{cases} \quad (16)$$

where $\delta := \lfloor \tau / \Delta t \rfloor$ denotes the integer part of $\tau / \Delta t$ close to $-\infty$, and α, β, γ are real scalar parameters that set the control law as indicated by Theorem 2.2. Here, $U_{M-1, j}$ and $U_{M+1, j}$ stands for $U(\xi^-, \mathbf{t}_j)$ and $U(\xi^+, \mathbf{t}_j)$ respectively. Therefore, it comes

$$U_{M+1, j} = U_{M-1, j} = U_{M, j}, \quad (17)$$

for the sake of consistency when $\Delta x \rightarrow 0$. The autoregressive control law carries on the gradient jump term $U_{\mathbf{x}}(\xi^+, \mathbf{t}) - U_{\mathbf{x}}(\xi^-, \mathbf{t})$ leading to the following finite difference approximation

$$\begin{aligned} U_{\mathbf{x}}(\xi^+, \mathbf{t}) - U_{\mathbf{x}}(\xi^-, \mathbf{t}) &\longrightarrow U_{\mathbf{x}^{M+1}, j} - U_{\mathbf{x}^{M-1}, j}, \\ U_{\mathbf{x}^{M+1}, j} - U_{\mathbf{x}^{M-1}, j} &= \frac{U_{M+2, j} - 2 U_{M, j} + U_{M-2, j}}{2 \Delta x}, \quad (18) \\ &\forall j \in \mathbb{N}^*. \end{aligned}$$

Using (17), one should pay attention to the following fact

$$U_{\mathbf{x}}(\xi, \mathbf{t}_j) \longrightarrow U_{\mathbf{x}^M, j} = \frac{U_{M+1, j} - U_{M-1, j}}{2 \Delta x} = 0, \quad (19)$$

$$\forall j \in \mathbb{N}.$$

As a consequence, in addition to (10), the following additive constraints should be taken into account for f

$$f'(M \Delta x) = 0, \quad f''(M \Delta x) = 0. \quad (20)$$

Now we are ready to write the finite difference scheme for both the uncontrolled wave equation in $(\tilde{\mathcal{P}})$, and the closed-loop system combining $(\tilde{\mathcal{P}})$ and $(\tilde{\mathcal{C}})$.

3.2 Open-loop case

Let $\mu := \frac{c \Delta t}{\Delta x}$. When there is no output feedback control applied to $(\tilde{\mathcal{P}})$, the finite difference scheme writes

$$\begin{aligned} U_{\mathbf{t}\mathbf{t}^i, j} &= c^2 U_{\mathbf{x}\mathbf{x}^i, j}, \quad i = 0, 1, 2, \dots, N-1, \quad \forall j \in \mathbb{N} \setminus \{0, 1\}, \\ U_{N, j} &= U_{N-1, j}, \quad \forall j \in \mathbb{N}, \end{aligned}$$

leading to the following implicit scheme

$$-\mu^2 U_{i+1, j} + (1 + 2\mu^2) U_{i, j} - \mu^2 U_{i-1, j} = 2 U_{i, j-1} - U_{i, j-2}. \quad (21)$$

3.3 Closed-loop case

When the output feedback control given by $(\tilde{\mathcal{C}})$ is applied to $(\tilde{\mathcal{P}})$, the finite difference scheme becomes

$$\begin{aligned} U_{\mathbf{t}\mathbf{t}^i, j} &= c^2 U_{\mathbf{x}\mathbf{x}^i, j}, \\ i &= 0, 1, 2, \dots, M-2, M+2, \dots, N-1, \quad \forall j \in \mathbb{N} \setminus \{0, 1\}, \\ U_{N, j} &= U_{N-1, j}, \quad \forall j \in \mathbb{N}, \end{aligned}$$

$$\begin{aligned} & \frac{U_{M+2,j} - 2U_{M,j} + U_{M-2,j}}{2\Delta x} = \\ & -\alpha \frac{U_{M+2,j-\delta} - 2U_{M,j-\delta} + U_{M-2,j-\delta}}{2\Delta x} \\ & + \frac{\beta}{c} \frac{3U_{M,j} - 4U_{M,j-1} + U_{M,j-2}}{2\Delta t} \\ & + \frac{\gamma}{c} \frac{3U_{M,j-\delta} - 4U_{M,j-1-\delta} + U_{M,j-2-\delta}}{2\Delta t}, \\ & \forall j \in \mathbb{N} \setminus \{0, 1\}, \end{aligned}$$

where, for all $i, j \in \mathbb{N}$, $U_{i,j-k} = 0$ for any $k \in \mathbb{N}$ such that $j - k < 0$. It comes,

$$\left\{ \begin{array}{l} -\mu^2 U_{i+1,j} + (1 + 2\mu^2) U_{i,j} - \mu^2 U_{i-1,j} = \\ \quad 2U_{i,j-1} - U_{i,j-2}, \quad \forall j \in \mathbb{N} \setminus \{0, 1\}, \\ U_{N,j} = U_{N-1,j}, \quad \forall j \in \mathbb{N}, \\ U_{M+2,j} - \left(2 + 3\beta \frac{\Delta x}{\Delta t}\right) U_{M,j} + U_{M-2,j} = \\ \quad -4\beta \frac{\Delta x}{\Delta t} U_{M,j-1} + \beta \frac{\Delta x}{\Delta t} U_{M,j-2} \\ \quad + \left(2\alpha + 3\gamma \frac{\Delta x}{\Delta t}\right) U_{M,j-\delta} \\ \quad - \alpha (U_{M+2,j-\delta} - U_{M-2,j-\delta}) \\ \quad - 4\gamma \frac{\Delta x}{\Delta t} U_{M,j-1-\delta} + \gamma \frac{\Delta x}{\Delta t} U_{M,j-2-\delta}, \end{array} \right. \quad (22)$$

where, for all $i, j \in \mathbb{N}$, $U_{i,j-k} = 0$ for any $k \in \mathbb{N}$ such that $j - k < 0$.

Even if naturally implicit, the open-loop and the closed-loop numerical schemes can easily be transformed into explicit numerical schemes and written in an appropriate matricial form. For the sake of brevity, the study of stability, consistency, and convergence of both these numerical schemes is left in this paper.

3.4 Numerical simulations

In the sequel, we use the numerical values of Table 1 for the considered string of problem $(\tilde{\mathcal{P}})$.

Length (m)	ℓ	10
Wave propagation speed (m/s)	c	1.118
Time scaling factor (s)	d	8.944
Wave magnitude	A	0.5

Table 1. Features of the string.

To guarantee stability and convergence of both numerical schemes (21) and (22), we set $\Delta x = 0.05$, $\Delta t = 0.005$ and $T_f = 100$.

Finally, to cope with all the requirements described in subsection 3.1, the initial conditions are chosen as follows:

$$\begin{aligned} f(x) &= A \left(3 \frac{x}{\ell} - 17 \left(\frac{x}{\ell} \right)^3 + 27 \left(\frac{x}{\ell} \right)^4 - 12 \left(\frac{x}{\ell} \right)^5 \right), \\ g(x) &= 0. \end{aligned}$$

For illustration purposes, we precise the computation of the Energy function in (7) using the previous numerical scheme for both the open-loop and the closed-loop cases. More precisely, for $j \geq 2$, this function is computed as

$$\begin{aligned} E(j\Delta t) &= \frac{\Delta x}{2} \left(\left(\frac{U_{1,j} - U_{0,j}}{\Delta x} \right)^2 + \sum_{i=2}^{N-1} \left(\frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x} \right)^2 \right. \\ & \quad + \left(\frac{U_{N,j} - U_{N-1,j}}{\Delta x} \right)^2 \\ & \quad \left. + \sum_{i=0}^N \left(\frac{3U_{i,j} - 4U_{i,j-1} + U_{i,j-2}}{2\Delta t} \right)^2 \right), \end{aligned}$$

which for $j \geq 2$, given the boundary conditions in $(\tilde{\mathcal{P}})$, reduces to

$$\begin{aligned} E(j\Delta t) &= \frac{\Delta x}{2} \left(\frac{U_{1,j}^2}{\Delta x^2} + \sum_{i=2}^{N-1} \left(\frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x} \right)^2 \right. \\ & \quad \left. + \sum_{i=0}^N \left(\frac{3U_{i,j} - 4U_{i,j-1} + U_{i,j-2}}{2\Delta t} \right)^2 \right). \end{aligned} \quad (23)$$

Moreover, for $j = 0$ and $j = 1$ and by considering (15)

$$\begin{aligned} E(0) = E(1\Delta t) &= \frac{\Delta x}{2} \left(\left(\frac{f(\Delta x)}{\Delta x} \right)^2 \right. \\ & \quad \left. + \sum_{i=2}^{N-1} \left(\frac{f((i+1)x) - f((i-1)x)}{2\Delta x} \right)^2 \right). \end{aligned} \quad (24)$$

Several simulation cases have been performed. They are summarized in Table 2.

s_0	-2			$-1.005\hat{\rho}(\tau)$ $\simeq -14.481$	$-2.01\hat{\rho}(\tau)$ $\simeq -28.961$
	2	3	4	0.08	
τ	2	3	4	0.08	
α	0.2046	0.01359	0.001058	-0.3401	-0.1068
β	3.0	2.150	1.997	1.789	1.789
γ	0.40758	0.02450	0.001917	-0.6085	-0.1910
MID	Met	Unmet	Met	Unmet	Unmet
Time plot	Fig. 2d	Fig. 2e	Fig. 2f	Fig. 2b	Fig. 2c

Table 2. Feedback parameters, various cases.

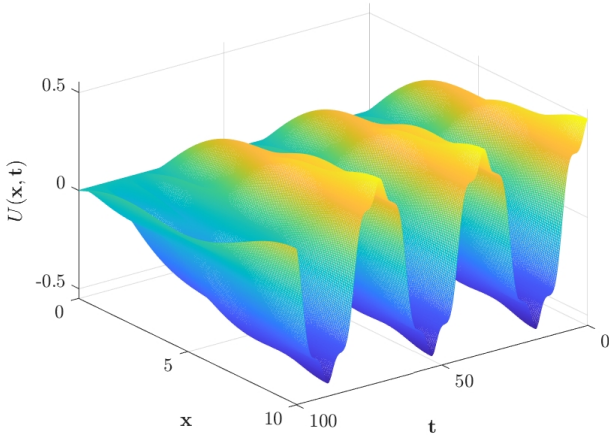
Figure 2a shows the free time response of the wave equation of problem $(\tilde{\mathcal{P}})$, *i.e.* only subjected to initial conditions and without any control. Figures 2b, 2c, 2d, 2e and 2f show the closed-loop time response to the same initial conditions for the various cases described in Table 2.

In Figure 3, the total energy logarithm is plotted for the various closed-loop cases, where the energy is the one defined in (7) and computed through (23). The blue curve illustrates the case where there is no feedback control.

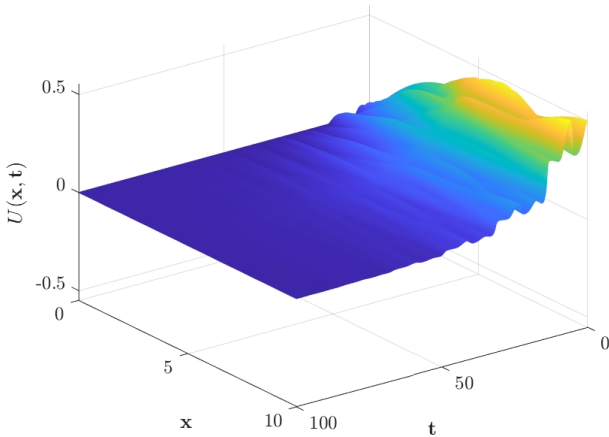
In Figure 4, the time response of the closed-loop control signals for the various cases is plotted. One can notice the periodic occurrence of impulsive peaks on the control signal, whose magnitude seems to be lower as the delay parameter increases.

In Figure 3, one can observe the exponential decay in closed-loop, in particular its controlling by the assignment of s_0 . Moreover, it can also be noticed that for the same $s_0 = -2$ and for different values of τ , the best decay rate is achieved with the MID case described in Theorem 2.2, say for a triple multiplicity of s_0 with $\tau = 4$. One also notices that this case provides a similar exponential decay as the one with $s_0 \simeq -28.961$ and $\tau = 0.08$, but gives a closed-

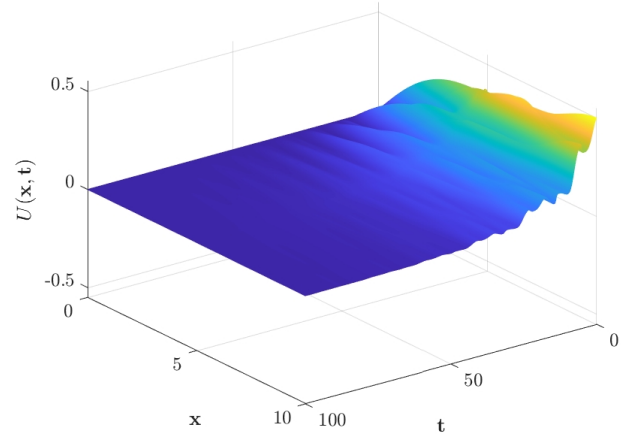
loop control signal with a lower magnitude in comparison with the other cases.



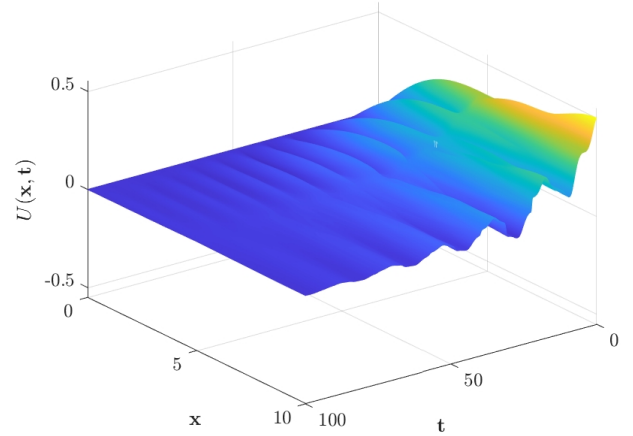
(a) Without control.



(b) Case $s_0 \simeq -14.481$ triple and $\tau = 0.08$, without MID property.



(c) Case $s_0 \simeq -28.961$ triple and $\tau = 0.08$, without MID property.



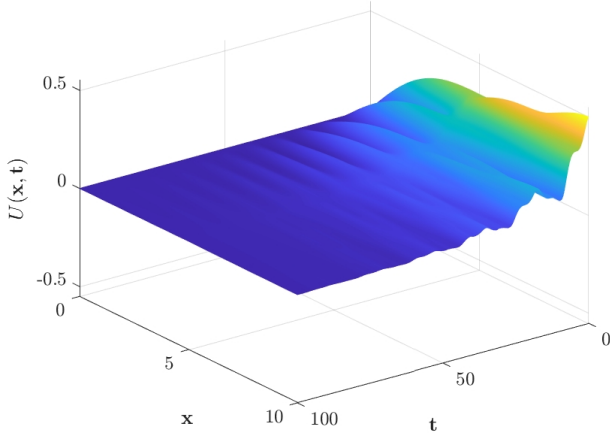
(d) Case $s_0 = -2$ double and $\tau = 2$, with MID property.

4. CONCLUSION

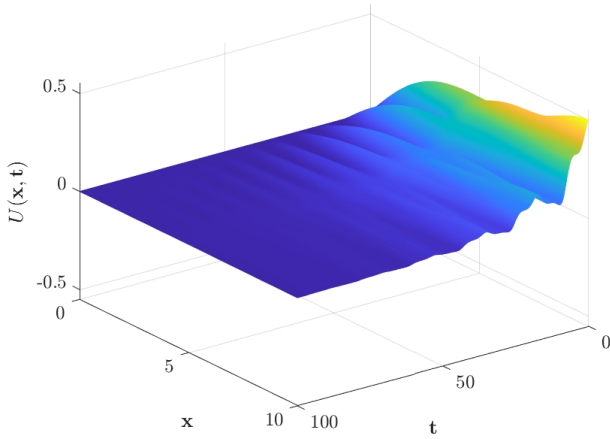
This note discusses controlling the wave equation by applying an autoregressive output feedback control law at the midpoint. The aim is to ensure the exponential stability of the system while dictating the decay of the solution. Conditions on the control parameters are derived to achieve this stability. Numerical results demonstrate the effectiveness of this approach, particularly in scenarios like a vibrating string with specific boundary and initial conditions.

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(e) Case $s_0 = -2$ triple and $\tau = 3$, without MID property.



(f) Case $s_0 = -2$ triple and $\tau = 4$, with MID property.

Fig. 2. Time simulations, various cases.

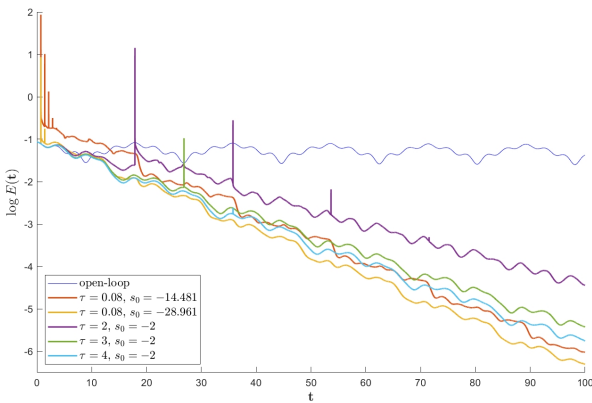


Fig. 3. Time simulation of the energy logarithm defined in (7) for the various cases reported in Tab. 2.

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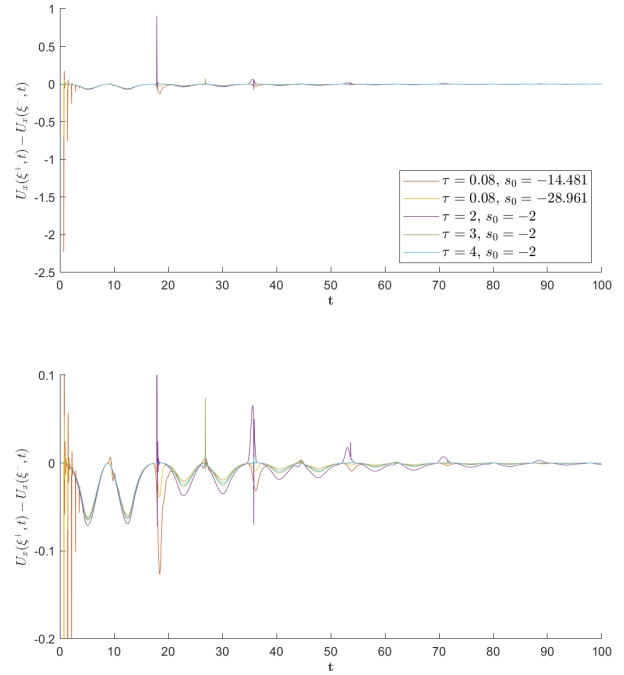


Fig. 4. (Top) Time simulation of the closed-loop control signal for the various cases reported in Tab. 2. (Bottom) Focus on the control signals around zero.

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