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ORIGINAL ARTICLE

Bifurcation for indefinite-weighted p-Laplacian problems with slightly subcritical nonlinearity

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Abstract

We study a superlinear elliptic boundary value problem involving the p-Laplacian operator, with changing sign weights. The problem has positive solutions bifurcating from the trivial solution set at the two principal eigenvalues of the corresponding linear weighted boundary value problem.

Drabek's bifurcation result applies when the nonlinearity is of power growth. We extend Drabek's bifurcation result to *slightly subcritical* nonlinearities. Compactness in this setting is a delicate issue obtained via Orlicz spaces.

KEYWORDS

bifurcation to positive solutions, changing sign weight, Orlicz spaces, p-Laplacian, slightly subcritical nonlinearity

1 | INTRODUCTION

Our aim is to prove the existence of positive solutions to a Dirichlet problem for a class of quasilinear elliptic equations whose nonlinear term has non-power growth, and involves indefinite nonlinearities. More precisely, we consider

$$-\Delta_p u = \lambda V(x)u^{p-1} + m(x)f(u), \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{1.1}$$

where $p > 1$, $\Omega \subset \mathbb{R}^N$, with $N > p$, is a bounded, connected open set, with $C^{1,\alpha}$ boundary $\partial\Omega$, $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $1 < p < \infty$, $\lambda \in \mathbb{R}$ is a real parameter. The weights $V \in L^\infty(\Omega)$ and $m \in C^1(\overline{\Omega})$ both changing sign in Ω, and $f \in C([0, +\infty))$ is *slightly subcritical* (see $\mathbf{(f1)}_\infty$). Prototype nonlinearities are the following ones:

$$f(s) := \frac{|s|^{p^*-2}s}{[\ln(1+|s|)]^\beta}, \quad f(s) := \frac{|s|^{p^*-2}s}{[\ln(1+\ln(1+|s|))]^\beta}, \qquad \beta < p^* - 1, \tag{1.2}$$

where $p^* := \frac{Np}{N-p}$ is the critical Sobolev exponent, and $\beta > 0$ is a fixed exponent, or even having a smaller perturbation of order $O(|s|^{q-2}s)$ at infinity, with $p < q < p^*$.

There is a huge amount of literature when $V \equiv 1$, and $f(s)$ is p-superlinear and grows at infinity as, say, s^{q-1} with $q < p^*$. The case $p = 2$ and m changing sign was first studied by [2] for a power-like nonlinearity, next by [4], among others, and for the case $p \neq 2$ see, for instance, [5, 25, 26].

We focus in widening the solvability of the quasilinear problem (1.1) to nonlinearities f slightly subcritical. The literature is more scarce when the nonlinearity is not of subcritical power type at infinity. We extend bifurcation results for the p-Laplacian case and for those nonlinearities. Specifically, we will assume the following hypothesis on f:

(f1)$_\infty$ f is slightly subcritical: $\lim_{s\to+\infty} \frac{f(s)}{|s|^{p^*-1}} = 0$;

(f2)$_\infty$ f increasing at infinity and there exists two constant $s_0 > 0$ and $c_0 > 1$ such that

$$\frac{sf(s)}{F(s)} \geq c_0, \qquad \forall s > s_0, \tag{1.3}$$

where F is the primitive of f satisfying $F(0) = 0$.

Using local bifurcation techniques, we prove the existence of positive solutions to problem (1.1). Some hypothesis on the behavior of f close to 0 are needed. We will assume the following classical condition:

(f3)$_0$ $f(0) = 0$, there exists $\delta_0 > 0$ such that $f(s) > 0$ for $s \in (0, \delta_0)$, and f is p-sublinear at zero: $\lim_{s\to 0} \frac{f(s)}{|s|^{p-1}} = 0$.

To characterize a principal eigenvalue of a changing sign weight $V \in L^\infty(\Omega)$, let us consider the following eigenvalue problem:

$$\begin{cases} -\Delta_p \varphi &= \lambda V(x)|\varphi|^{p-2}\varphi \quad \text{in } \Omega, \\ \varphi &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

It is known that Equation (1.4) possesses exactly two principal eigenvalues, denoted by $\lambda_1(V)$ and $\lambda_{-1}(V)$, characterized by

$$\lambda_1(V) = \inf_{u\in S^+} \int_\Omega |\nabla u|^p dx, \quad \lambda_{-1}(V) = -\inf_{u\in S^-} \int_\Omega |\nabla u|^p dx, \tag{1.5}$$

where

$$S^\pm = \left\{ u \in W_0^{1,p}(\Omega) \ : \ \int_\Omega V(x)|u|^p dx = \pm 1 \right\}.$$

Those are the only eigenvalues associated with a non-negative eigenfunction and they are simple and isolated (see [14], and references therein).

Let us denote by $\varphi_1(V)$ ($\varphi_{-1}(V)$) the positive eigenfunction associated with $\lambda_1(V)$ ($\lambda_{-1}(V)$) of L^∞-norm equal to 1, and introduce the following hypothesis on f near 0 and on the weight m:

(f4)$_0$ there exists a constant $C_1 > 0$ and a continuous function $g_0 : [0, +\infty) \to [0, +\infty)$ such that for all $\tau > 0$,

$$\left| \frac{f(\tau s)}{f(s)} \right| \leq C_1(1 + \tau^{p^*-1}) \qquad \text{for all} \quad 0 < |s| < 1, \tag{1.6}$$

and $\lim_{s\to 0} \frac{f(\tau s)}{f(s)} = g_0(\tau)$, uniformly for τ on compact intervals.

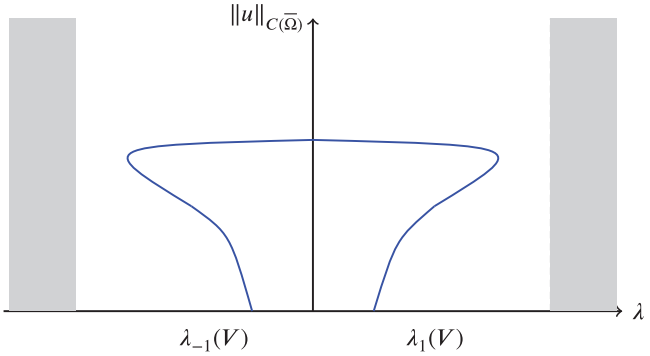

FIGURE 1 Possible global bifurcation diagram representing a *mushroom–shaped* compact component.

Moreover, $g(0) = 0$, $g_0(\tau) > 0$ for all $\tau > 0$, and

$$\int_\Omega m(x) g_0(\varphi_{\pm 1}(V))\, \varphi_{\pm 1}(V)\, dx < 0. \tag{1.7}$$

We prove the existence of two continuum of positive solutions bifurcating from the two principal eigenvalues of Equation (1.4). Drabek [22, Theorem 14.18, p. 189] and Takáč–Girg [23, Proposition 3.5] used Browder–Petryshyn topological degree [7, 36], an extension of the Leray-Schauder degree for monotone mappings. Del Pino–Manasevich [20, Theorem 1.1] used a homotopy to the case $p = 2$. Roughly speaking, bifurcation theorems for the p-Laplacian case apply for subcritical power type nonlinearities, but do not treat slightly subcritical nonlinearities such as Equation (1.2).

When $p = 2$, for a subcritical power type nonlinearity, in [8] the author works on indefinite weights so that Equation (1.1) has a bounded, *mushroom–shaped* compact continuum of positive solutions connecting the two principal eigenvalues of Equation (1.4), see Figure 1. A uniform $L^\infty(\Omega)$ bound for the solutions is guaranteed through the blow up method. Although we prove some L^∞ estimates for the solutions to Equation (1.1), a uniform $L^\infty(\Omega)$ bound is an open problem.

Let us define

$$\Omega^\pm := \{x \in \Omega : \pm m(x) > 0\}, \qquad \Omega^0 := \{x \in \Omega : m(x) = 0\},$$

and assume that:

(m1) $\sup m^+ > 0$ and $\sup m^- > 0$, so both $\Omega^\pm \neq \emptyset$;
(m2) let $\omega^{+,0} := \operatorname{int}\left(\Omega^+ \cup \Omega^0\right)$, $\omega^{+,0}$ have a finite number of connected components, and $\left|\left\{x \in \omega^{+,0} : \pm V(x) > 0\right\}\right| \neq 0$;
(m3) let $\omega^0 := \operatorname{int}\left(\Omega^0\right)$, $\omega^0 \neq \emptyset$ have a finite number of connected components, and $\left|\left\{x \in \omega^0 : \pm V(x) > 0\right\}\right| \neq 0$.

The pair $(\lambda, 0)$ is a solution of Equation (1.1) for every $\lambda \in \mathbb{R}$, whenever $f(0) = 0$. Pairs of this form will be designated as the trivial solutions of Equation (1.1). We say that $(\lambda, 0)$ is a *bifurcation point* of Equation (1.1) if in any neighborhood of $(\lambda, 0)$ there exists a nontrivial solution of Equation (1.1). It is well known that if $(\lambda, 0)$ is a bifurcation point of Equation (1.1) then λ is an eigenvalue of Equation (1.4), see, for instance, [20].

Next, we state a bifurcation theorem for problem (1.1).

Theorem 1.1. *Assume that f satisfies hypothesis **(f1)**$_\infty$–**(f2)**$_\infty$ and also **(f3)**$_0$–**(f4)**$_0$. Assume further that $m \in C^1(\overline{\Omega})$ changes sign in Ω and satisfies hypothesis **(m1)**. Then*

(i) there exists two closed and connected sets $\mathcal{C}_1^+$ and $\mathcal{C}_{-1}^+$ of positive solutions to Equation (1.1), bifurcating from the trivial solution set at the bifurcation point $(\lambda_1(V), 0)$ and $(\lambda_{-1}(V), 0)$, respectively;
(ii) either $\mathcal{C}_1^+$ is unbounded, or $(\lambda_{-1}(V), 0) \in \mathcal{C}_1^+$. Likewise, either $\mathcal{C}_{-1}^+$ is unbounded, or $(\lambda_1(V), 0) \in \mathcal{C}_{-1}^+$;

(iii) $\mathscr{C}_1^+$ *bifurcates to the right and* $\mathscr{C}_{-1}^+$ *to the left. Precisely, there exists* $\delta > 0$ *such that*

$$(\lambda, u) \in \mathscr{C}_1^+, 0 < |\lambda - \lambda_1(V)| < \delta, 0 \neq \|u\| < \delta \Longrightarrow \lambda > \lambda_1(V),$$

and also

$$(\lambda, u) \in \mathscr{C}_1^-, 0 < |\lambda - \lambda_{-1}(V)| < \delta, 0 \neq \|u\| < \delta \Longrightarrow \lambda < \lambda_{-1}(V);$$

(iv) if we assume either ***(m2)*** *or* ***(m3)****, there exist* $\Lambda_1, \Lambda_{-1} \in \mathbb{R}$*, with*

$$\Lambda_{-1} < \lambda_{-1}(V) < 0 < \lambda_1(V) < \Lambda_1,$$

such that if problem $(1.1)_\lambda$ *has non-negative nontrivial solutions, then* $\Lambda_{-1} \leq \lambda \leq \Lambda_1$.

We will give in Proposition 4.1 an upper (resp. lower) bound of Λ_1 (resp. of Λ_{-1}).
The main novelties of this paper are the following:

(i) We extend bifurcation theorems for p-Laplacian problem to slightly subcritical nonlinearities, see Theorem 3.2. Compact embedding of Sobolev spaces into Lebesgue spaces (Rellich–Kondrachov theorem [6, Theorem 9.16]) do not apply for slightly subcritical functions with non-power-like growth. Instead, the compactness is achieved via Orlicz spaces under hypothesis $\mathbf{(f1)}_\infty$, see Proposition 3.1 and Theorem A.2.
(ii) We state sufficient conditions for the existence of a continuum of positive solutions in a quasilinear bifurcation problem, see Theorem 3.4, which complements the results in [20, 23]. The branch of positive solutions satisfies Rabinowitz's alternative.
(iii) In Theorem 1.1, we generalize partially results of [8] on changing sign weights, for $p \neq 2$ and non-power-like nonlinearities.
(iv) In Theorem 2.1, we prove some explicit L^∞ estimates for quasilinear problems with a slightly subcritical nonlinearity.

The paper is organized in the following way. In Section 2, we prove some explicit L^∞ estimates for quasilinear problems with a slightly subcritical nonlinearity. A Drabek's-type bifurcation result is developed in Section 3 followed by a result on the existence of two branches of positive solutions bifurcating from $\lambda_1(V)$ and $\lambda_{-1}(V)$, respectively, see Theorem 3.4. In Section 4.2, we obtain a non-existence result. We end this section with the proof of Theorem 1.1. We have tried to make a summary on tools of Orlicz spaces as simple as possible and self-contained, see Appendix A.

2 | L^∞ BOUNDS AND CLASSICAL C^1- REGULARITY

An important issue for weak solutions u (positive or not) of our problem is if there are uniform a priori bounds. When $\lambda = 0$, $m(x) \equiv 1$, and f is the first case of Equation (1.2), the relationship between L^∞, Sobolev and Lebesgue norms has been studied in [9–11, 15, 29, 31] for the semi-linear case, and in [17, 29, 30] for the p-Laplacian case. A uniform L^∞ a priori bound for positive weak solution, is known when $\beta > \frac{p}{N-p}$. When $p = 2$, it is also known that there is a positive solution blowing up as $\beta \to 0$, see [13] for details. From a recent result of [32], first we derive an estimate of the L^∞ norm of weak solutions to problem (1.1) in terms of the $\|\cdot\|_{p^*}$, see Theorem 2.1.

Throughout this paper, for any $u \in W_0^{1,p}(\Omega)$ we will denote

$$\|u\| := \left(\int_\Omega |\nabla u|^p dx \right)^{1/p}.$$

2.1 | L^∞ bounds

Let us consider the quasilinear problem

$$-\Delta_p u = f(x,u) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{2.1}$$

and assume that Ω is a bounded domain of $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

We recall that $u \in W^{1,p}(\Omega)$ is a *weak solution* to Equation (2.1) if for each $\psi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi \, dx = \int_\Omega f(x,u)\psi \, dx.$$

If $1 < p < N$ and there exist $0 \le q \le p^*$ and $c > 0$ such that

$$|f(x,s)| \le c(1 + |s|^{q-1}) \qquad \text{a.e. } x \in \Omega, \quad \text{and for all } s \in \mathbb{R}, \tag{2.2}$$

then any $u \in W_0^{1,p}(\Omega)$ solving weakly problem (2.1), belongs to $L^\infty(\Omega)$, cf., for example, [24].

The following result is a consequence of [32, Theorem 1.6], and gives an estimate of the L^∞ norm of a solution in terms of its L^{p^*} norm, for nonlinearities f that are slightly subcritical in the sense of $\mathbf{(f1)}_\infty$.

Theorem 2.1 (L^∞ estimates for slightly subcritical problems). *Assume that there exists a continuous function $\tilde f : [0,+\infty) \to [0,+\infty)$ satisfying*

$$|f(x,s)| \le \tilde f(|s|), \quad \forall s \in \mathbb{R}; \quad \tilde f(s) > 0, \quad \forall s > 0. \tag{2.3}$$

where $\tilde f$ satisfies $\mathbf{(f1)}_\infty$. Let $h(s) := \frac{s^{p^*-1}}{\tilde f(s)}$, for $s > 0$, $h(0) = 0$. Then, for any $\varepsilon > 0$ there exists $C = C_\varepsilon > 0$ such that the following holds:

$$h(\|u\|_\infty) \le C\left[\left(1 + \|u\|_{p^*}^{\frac{1}{p-1}}\right)\|u\|_{p^*}\right]^{(p^*-1)\frac{p}{N}+\varepsilon}, \tag{2.4}$$

for every nontrivial weak solution of Equation (2.1), where $C = C(\varepsilon, \tilde f, p, N, |\Omega|)$, and it is independent of u.

Remark 2.2.

1. Note that we can always redefine $\tilde f$ in order to be strictly increasing for $s > 0$, for instance $\max_{0\le t\le s} \tilde f(t) + s^q$, with $p - 1 < q < p^* - 1$.
2. This result also holds for changing sign solutions.

Proof of Theorem 2.1. The proof is adapted from the proof of [32, Theorem 1.6], where it is assumed that

$$|f(x,s)| \le |a(x)|\tilde f(|s|), \text{ a.e } x \in \Omega, \ \forall s \in \mathbb{R};$$

$$a \in L^r(\Omega) \text{ with } N/p < r \le \infty, \ \tilde f(s) > 0, \ \forall s \ne 0.$$

Let u be a solution to Equation (2.1). On one hand, we notice that whenever $\tilde f$ can be chosen strictly increasing for $s > 0$, then we can always follow the second possibility (ii) in [32, Theorem 1.6]. On the other hand, we point out that in pp. 13–14

the following estimate can be read as:

$$h(\|u\|_\infty) \le C\|a\|_r^\theta \left(\|\tilde f(|u|)\|_{\frac{p^*}{p^*_{N/r}-1}} \right)^{\theta-1} \|u\|_{p^*}^{\vartheta},$$

for $u = u_k$, where $\{u_k\}$ is a sequence of solutions to Equation (2.1), $\theta = \frac{p^*_{N/r}-1}{p^*_{N/q}-1}$, $\vartheta = \theta\,(p^*_{N/q} - p)$, $\forall q \in (N/p, N)$, and $p^*_{N/s} := \frac{p^*}{s'} = p^*\left(1 - \frac{1}{s}\right)$, for $1 \le s \le \infty$, where s' is the conjugate exponent of s, $1/s + 1/s' = 1$, and $C = C(p, N, q, |\Omega|)$.

Hypothesis (2.3) implies that $r = \infty$, and $p^*_{N/r} = p^*$, so

$$h(\|u\|_\infty) \le C \left(\|\tilde f(|u|)\|_{\frac{p^*}{p^*-1}} \right)^{\theta-1} \|u\|_{p^*}^{\vartheta},$$

with

$$\theta = \theta(q) = \frac{p^*-1}{p^*_{N/q}-1}, \qquad \vartheta = \vartheta(q) = \theta\,(p^*_{N/q} - p). \tag{2.5}$$

Moreover, from subcriticality (see $\mathbf{(f1)}_\infty$),

$$\|\tilde f(|u|)\|_{\frac{p^*}{p^*-1}} \le C\left(1 + \|u\|_{p^*}^{p^*-1}\right).$$

Consequently

$$h(\|u\|_\infty) \le C\left(1 + \|u\|_{p^*}^{p^*-1}\right)^{\theta-1} \|u\|_{p^*}^{\vartheta}.$$

Let $\Theta(q,t) := \left(1 + t^{p^*-1}\right)^{\theta-1}\, t^{\vartheta}$. Fixed $t = t_0 > 0$, the function $q \to \Theta(q, t_0)$ for $q \in (N/p, N)$, satisfies the following:

$$\ln\left(\Theta(q,t_0)\right) := (\theta - 1)\ln\left(1 + t_0^{p^*-1}\right) + \vartheta \ln t_0.$$

Derivating with respect to q, and since $\frac{d}{dq}\left(p^*_{N/q}\right) = \frac{p^*}{q^2}$, we can write

$$\begin{aligned}
\frac{\frac{d}{dq}(\Theta(q,t_0))}{\Theta(q,t_0)} &= \theta'(q)\ln\left(1 + t_0^{p^*-1}\right) + \vartheta'(q)\ln t_0 \\
&= -\frac{(p^*-1)}{\left(p^*_{N/q}-1\right)^2}\frac{p^*}{q^2}\left[\ln\left(1 + t_0^{p^*-1}\right) + \left[\left(p^*_{N/q} - p\right) - \left(p^*_{N/q} - 1\right)\right]\ln(t_0)\right] \\
&= -\frac{(p^*-1)}{\left(p^*_{N/q}-1\right)^2}\frac{p^*}{q^2}\left[\ln\left(1 + t_0^{p^*-1}\right) - (p-1)\ln(t_0)\right].
\end{aligned}$$

Consequently,

$$\frac{d}{dq}(\Theta(q,t_0)) < 0 \iff \frac{1}{p-1} > \frac{\ln(t_0)}{\ln\left(1 + t_0^{p^*-1}\right)}.$$

It is easy to see that the function $t \to \frac{\ln(t)}{\ln(1+t)}$ is an increasing function for $t > 0$, and so

$$\sup_{t\in\mathbb{R}^+} \frac{\ln(t)}{\ln\left(1+t^{p^*-1}\right)} = \lim_{t\to\infty} \frac{\ln(t)}{\ln\left(1+t^{p^*-1}\right)} = \frac{1}{p^*-1} < \frac{1}{p-1},$$

then $\Theta(q, t_0)$ is decreasing in q for any $t = t_0 \in \mathbb{R}^+$, so, for any t fixed

$$\inf_{q\in(\frac{N}{p},N)} \Theta(q,t) = \Theta(N,t) = \left(1+t^{p^*-1}\right)^{\frac{p'}{N}} t^{\frac{p}{N}(p^*-1)} \le C\left[(1+t)^{\frac{1}{p-1}} t\right]^{(p^*-1)\frac{p}{N}},$$

since Equation (2.5), and then

$$\begin{aligned}\theta(N) - 1 &= \frac{p^*-1}{p^*(1-1/N)-1} - 1 = \frac{p^*/N}{p^*(1-1/N)-1} = \frac{1/N}{1-1/p} = \frac{p'}{N}\\ \vartheta(N) &= \left(1+\frac{p'}{N}\right)p^*\left(1-\frac{1}{N}-1+\frac{p}{N}\right) = p^*\left(\frac{p-1}{N}+\frac{p}{N^2}\right)\\ &= \frac{p}{N}p^*\left(1-\frac{1}{p}+\frac{1}{N}\right) = \frac{p}{N}p^*\left(1-\frac{1}{p^*}\right) = \frac{p}{N}(p^*-1).\end{aligned}$$

Finally, and since the infimum is not attained in $(N/p, N)$, for any $\varepsilon > 0$, there exists a constant $C = C_\varepsilon > 0$ such that

$$h(\|u\|_\infty) \le C\left[\left(1+\|u\|_{p^*}^{\frac{1}{p-1}}\right)\|u\|_{p^*}\right]^{(p^*-1)\frac{p}{N}+\varepsilon},$$

where C is independent of u, ending the proof. □

2.2 | Regularity of weak solutions

Since weak solutions to problem (2.1) under condition (2.2) belong to $L^\infty(\Omega)$, the classical results of [21, 28, 37] for a regular domain Ω of class $C^{1,\alpha}$ provide that u belongs to $C^{1,\mu}(\overline{\Omega})$ for some $\mu = \mu(N, p, \alpha) \in (0, 1)$. Precisely, we have:

Proposition 2.3 ($C^{1,\mu}(\overline{\Omega})$-regularity for Problem (2.1)). *Assume hypothesis (2.2). Then, for any weak solution $u \in W_0^{1,p}(\Omega)$ of Equation (2.1),*

(i) $u \in L^\infty(\Omega)$ and there exists a constant $c_0 > 0$ depending only on p, N, c and $\|u\|_{p^*}$ such that

$$\|u\|_\infty \le c_0.$$

(ii) There exists $\mu = \mu(p, N, \alpha) \in (0, 1)$ for which $u \in C^{1,\mu}(\overline{\Omega})$. Moreover, there exists $C = C(p, N, \alpha, c, \|u\|_\infty)$ such that

$$\|u\|_{C^{1,\mu}(\overline{\Omega})} \le C.$$

In particular, a $W_0^{1,p}(\Omega)$- bounded sequence (or $L^{p^*}(\Omega)$-bounded) of solutions to problem $(1.1)_{\lambda_n}$, with λ_n varying in a bounded interval, is uniformly bounded in $C^{1,\mu}(\overline{\Omega})$.

Proof. Parts (i) and (ii) are well known, see [21, 28, 37]. We prove the last assertion. Thanks to Theorem 2.1 and Proposition 2.3, a sequence of solutions to problem (1.1) $W_0^{1,p}(\Omega)$- bounded sequence or $L^{p^*}(\Omega)$-bounded, remains also bounded

in $C^{1,\mu}(\overline{\Omega})$. Indeed, since Theorem 2.1, $h(\|u_n\|_\infty) \le C$, and due to $h(s) \to \infty$ as $s \to \infty$, hence $\|u_n\|_\infty \le C$. Finally, by part (ii), $\|u_n\|_{C^{1,\mu}(\overline{\Omega})}$ remains also bounded. □

Remark 2.4. In the critical case $q = p^*$, the $\|\cdot\|_\infty$-norm of weak solutions cannot be bounded in terms of the Sobolev norm, as was observed by [19, p. 725]. Consequently, a sequence of solutions to problem (2.1) with $q = p^*$, bounded in $W_0^{1,p}(\Omega)$, not necessarily remains bounded in $C^{1,\mu}(\overline{\Omega})$.

3 | A BIFURCATION RESULT FOR A QUASILINEAR PROBLEM WITH A SLIGHTLY SUBCRITICAL NONLINEARITY

The first main result in this section, Theorem 3.2, proves a slight generalization of a Drabek's bifurcation theorem for quasilinear equations, see [22, Theorem 14.18]. These bifurcation results are based on the well-known works of [35, Theorem 1.3] and [18, Lemma 1 and Theorem 2] for the semilinear case. The second main result, Theorem 3.4, proves Rabinowitz's alternative for a branch of positive solutions.

Consider the quasilinear elliptic problem with parameter λ

$$\begin{cases} -\Delta_p u &= \lambda V(x)|u|^{p-2}u + g(\lambda, x, u), & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $g : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that, for any bounded set $J \subset \mathbb{R}$ there exists some continuous function $\bar{f} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|g(\lambda, x, s)| \le \bar{f}(|s|) \quad a.e.\, x \in \Omega, \quad \forall (\lambda, s) \in J \times \mathbb{R}, \tag{3.1}$$

where $\bar{f}$ is slightly subcritical. We denote by $\bar{F}$ the primitive of $\bar{f}$ vanishing at 0.

Let us define the following operators $N, S : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ and $G : \mathbb{R} \times W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$:

$$\begin{aligned} \langle N(u), \psi\rangle &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi \, dx, \\ \langle S(u), \psi\rangle &= \int_\Omega V(x)|u|^{p-2}u\psi \, dx, \\ \langle G(\lambda, u), \psi\rangle &= \int_\Omega g(\lambda, x, u)\psi \, dx. \end{aligned} \tag{3.2}$$

Define the product space $E := \mathbb{R} \times W_0^{1,p}(\Omega)$ endowed with the norm $\|(\lambda, u)\|_E := |\lambda| + \|u\|$.

In order to reformulate Drabek's bifurcation theorem for the equation $N - \lambda S - G(\lambda, \cdot) = 0$ we need to assure that G is a *compact* operator.

Proposition 3.1. *Assume that g satisfies the growth condition (3.1) with $\bar{f}(0) = 0$, $\bar{f}$ continuous, strictly increasing for $s \ge 0$, and satisfies **(f1)**$_\infty$–**(f2)**$_\infty$ with $f = \bar{f}$.*

Then for any bounded closed interval $I \subset \mathbb{R}$, the operator $G : I \times W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is compact.

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence in I, and $\{u_n\}_{n\in\mathbb{N}}$ a bounded sequence of $W_0^{1,p}(\Omega)$. Then, up to a subsequence, there exist $\lambda_0 \in I$ and u such that $\lambda_n \to \lambda_0$, $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, strongly in L^{p^*-1}, a.e. and in measure. Let us show that the sequence $\{G(\lambda_n, u_n)\}_{n\in\mathbb{N}}$ converges in $W^{-1,p'}(\Omega)$. For any $\psi \in W_0^{1,p}(\Omega)$,

$$\left| \int_\Omega (g(\lambda_n, x, u_n) - g(\lambda_0, x, u))\psi \, dx \right| \le \|\psi\|_{p^*} \left\| g(\lambda_n, \cdot, u_n) - g(\lambda_0, \cdot, u) \right\|_{(p^*)'}$$

Let $z_n := g(\lambda_n, \cdot, u_n)$ and $z := g(\lambda_0, \cdot, u)$ and let us apply Theorem A.2 of Appendix A to estimate $\|z_n - z\|_{(p^*)'}$.

We choose $a = \bar{f}^{-1}$ and $b(t) = \frac{p^*}{p^*-1} t^{\frac{1}{p^*-1}}$. Since $\bar{f}$ is strictly increasing, $a^*(t) = a^{-1}(t) = \bar{f}(t)$, see Definition (A.7), (A.8), moreover $B(t) := t^{(p^*)'} = t^{\frac{p^*}{p^*-1}}$, for $t \geq 0$. Remark that, by $\mathbf{(f1)}_\infty$,

$$a^*(s) = \bar{f}(s) \leq cs^{p^*-1} + d, \qquad A^*(s) = \bar{F}(s) \leq \frac{c}{p^*} s^{p^*} + ds$$

for some $c, d > 0$.

Now, we check the hypotheses of Theorem A.2.

1. The sequence $\{z_n\}_{n\in\mathbb{N}}$ is bounded in $K_A(\Omega)$, that is, for all $n \in \mathbb{N}$, $\int_\Omega A\left(|z_n(x)|\right)\, dx \leq C$ for some $C > 0$. Indeed, using Equation (A.10) we can write

$$\begin{aligned}\int_\Omega A\big(|z_n(x)|\big)\, dx &\leq \int_\Omega A(a^*(|u_n(x)|))\, dx \\ &= \int_\Omega (|u_n(x)|a^*(|u_n(x)|) + A^*(|u_n|))dx \leq \int_\Omega (c'|u_n|^{p^*} + d')dx \leq C.\end{aligned}$$

   One proves similarly that $z \in K_A(\Omega)$.
2. A satisfies the Δ_2-condition at infinity if there exist constants $k_0 > 1$ and $t_0 > 0$ such that,

$$\frac{ta(t)}{A(t)} \leq k_0, \qquad \text{for} \quad t > t_0,$$

   see Proposition A.1.

   Through the change of variable $s = \bar{f}^{-1}(t) = a(t)$, since Equation (A.10) and by hypothesis $\mathbf{(f2)}_\infty$, we can write

$$\frac{ta(t)}{A(t)} = \frac{s\bar{f}(s)}{\bar{f}(s)s - \bar{F}(s)} \leq \frac{c_0}{c_0 - 1} =: k_0, \qquad \text{for all} \quad t \geq \bar{f}(s_0).$$

3. It remains to prove that B increases essentially more slowly than A at $+\infty$. It is equivalent to prove that A^* increases essentially more slowly than B^*, see [27, Lemma 13.1]. That is to check, that $\bar{F}$ increases essentially more slowly than $t \to t^{p^*}$, which is a consequence of hypothesis $\mathbf{(f1)}_\infty$. This concludes the proof. □

We define $\mathcal{S}$ as the closure in E of the nontrivial weak solution set of (P_λ):

$$\mathcal{S} := \overline{\big\{(\lambda, u) \in E : \ u \neq 0 \ \text{and} \ N(u) - \lambda S(u) - G(\lambda, u) = 0\big\}}^{E}. \tag{3.3}$$

Now, we can prove the following bifurcation theorem:

Theorem 3.2. *Let g satisfies the growth condition (3.1) with $\bar{f}(0) = 0$, $\bar{f}$ continuous, strictly increasing, slightly subcritical at infinity, see $\mathbf{(f1)}_\infty$, satisfying also $\mathbf{(f2)}_\infty$ and $\mathbf{(f3)}_0$ with $f = \bar{f}$. Let μ_1 be either $\lambda_1(V)$ or $\lambda_{-1}(V)$.*

Then, there exist a maximal closed connected set $\mathcal{C}$ of $\mathcal{S}$ containing $(\mu_1, 0)$ such that either

(i) $\mathcal{C}$ is unbounded on E, or else
(ii) $\mathcal{C}$ contains a point $(\tilde{\lambda}, 0)$ where $\tilde{\lambda} \neq \mu_1$ is an eigenvalue of problem (1.4).

Proof. We adapt the proofs of [22, Theorem 14.18 and Theorem 14.20], and of [23, Proposition 3.5], based on the well-known result of Rabinowitz [35, Theorem 1.3] to the case of a nonlinearity g satisfying Equation (3.1). Note that the operator G is compact according to Proposition 3.1.

Let deg denotes the Browder–Petryshyn topological degree [7, 36] defined for monotone mappings generalizing the Leray–Schauder degree, see also [22, Section 14.6]. Since μ_1 is an isolated eigenvalue, reasoning as is [22, Theorem 14.18],

for any $r > 0$, and $\delta > 0$ small enough, the following degree is well defined in any open neighborhood B_r of 0, and

$$\deg(N - (\mu_1 + \delta)S, B_r, 0) \neq \deg(N - (\mu_1 - \delta)S, B_r, 0).$$

Assume for a moment that

$$\lim_{\|u\|\to 0} \frac{\|G(\lambda, u)\|_{W^{-1,p'}(\Omega)}}{\|u\|^{p-1}} = 0, \tag{3.4}$$

uniformly for λ on bounded sets. Then, the invariance for homotopy yields that for some $r,\ \delta > 0$ small enough,

$$\deg(N - (\mu_1 + \delta)S - G(\mu_1 + \delta, \cdot), B_r, 0) \neq \deg(N - (\mu_1 - \delta)S - G(\mu_1 - \delta, \cdot), B_r, 0), \tag{3.5}$$

ending the proof.

To prove Equation(3.4), let $0 \neq \{u_n\}_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,p}(\Omega)$ such that $\|u_n\| \to 0$. Then, for λ on a bounded set $J \subset \mathbb{R}$

$$\lim_{n\to\infty} \frac{\|G(\lambda, u_n)\|_{W^{-1,p'}(\Omega)}}{\|u_n\|^{p-1}} = \lim_{n\to\infty} \sup_{\|\psi\|\leq 1} \int_\Omega \frac{|g(\lambda, x, u_n)\psi|}{\|u_n\|^{p-1}}\, dx$$

Let us denote $w_n = \frac{u_n}{\|u_n\|}$, fix $\varepsilon > 0$ and $\psi \in W_0^{1,p}(\Omega)$ with $\|\psi\| = 1$. Since $\mathbf{(f3)}_0$, there exists $\delta_1 > 0$ such that, for $\lambda \in J$,

$$|g(\lambda, x, s)| \leq \varepsilon |s|^{p-1}, \quad \text{a.e. } x \in \Omega,\ \forall |s| \leq \delta_1.$$

Denote for all $\delta > 0$,

$$\Omega_n^\delta := \{x \in \Omega :\ |u_n(x)| \geq \delta\}.$$

Since we can assume that $u_n \to 0$ in measure, for any $\delta > 0$ fixed $|\Omega_n^\delta| \to 0$. So, there exists an $n_0 = n_0(\varepsilon)\,(= n_0(\varepsilon, \delta_1))$ such that $|\Omega_n^{\delta_1}| \leq \varepsilon$ for all $n \geq n_0$.

Here and thereafter C stands for a constant independent of n. By Holder's inequality,

$$\int_{\Omega\setminus\Omega_n^{\delta_1}} \frac{|g(\lambda, x, u_n)\psi|}{\|u_n\|^{p-1}}\, dx \leq \varepsilon \|\psi\|_p \|w_n\|_p^{p-1} \leq C\varepsilon. \tag{3.6}$$

Using that $\bar{f}$ satisfies $\mathbf{(f1)}_\infty$, there exists $\delta_2 > \delta_1$ such that,

$$|g(\lambda, x, s)| \leq \varepsilon |s|^{p^*-1}, \quad \text{a.e. } x \in \Omega,\ \forall |s| \geq \delta_2.$$

Besides, let $c(\delta_1, \delta_2) > 0$ be such that

$$\bar{f}(|s|) \leq c(\delta_1, \delta_2)|s|^{p-1}, \quad \forall \delta_1 \leq |s| \leq \delta_2$$

and let $n_1 = n_1(\varepsilon)\,(= n_1(\varepsilon, \delta_1, \delta_2)) \in \mathbb{N}$ be such that

$$|\Omega_n^{\delta_1}|^{\frac{p^*-p}{p^*}} \leq \frac{\varepsilon}{c(\delta_1, \delta_2)}, \quad \forall n \geq n_1.$$

Then, since $W_0^{1,p}(\Omega)$ is embedded into $L^{p^*}(\Omega)$, using Holder's inequality, we have

$$\begin{aligned}\int_{\Omega_n^{\delta_1}} \frac{|g(\lambda,x,u_n)\psi|}{\|u_n\|^{p-1}}\,dx &\le \varepsilon \int_{\Omega_n^{\delta_2}} \frac{|u_n|^{p^*-1}|\psi|}{\|u_n\|^{p-1}}\,dx + c(\delta_1,\delta_2)\int_{\Omega_n^{\delta_1}\setminus\Omega_n^{\delta_2}} |w_n|^{p-1}\,|\psi|\,dx\\ &\le C\,\varepsilon\|u_n\|^{p^*-p}\|\psi\|_{p^*} + c(\delta_1,\delta_2)\|w_n\|_{p^*}^{p-1}\|\psi\|_{p^*}|\Omega_n^{\delta_1}|^{\frac{p^*-p}{p^*}} \qquad (3.7)\\ &\le C\varepsilon + Cc(\delta_1,\delta_2)|\Omega_n^{\delta_1}|^{\frac{p^*-p}{p^*}} \le C\varepsilon, \quad \forall n \ge n_1.\end{aligned}$$

Inequalities (3.6) and (3.7) give Equation (3.4). □

We complete this section by a regularity result for solutions bifurcating from zero, that we will use later.

Proposition 3.3. *Assume the hypotheses of Theorem 3.2. Let $\{(\lambda_n,u_n)\}_{n\in\mathbb{N}}$ be a sequence of solutions to (P_λ) for $\lambda=\lambda_n$, such that $\lim_{n\to\infty}\lambda_n=\lambda^*$. Then, the following three statements are equivalent, as $n\to\infty$:*

(i) $\|u_n\|_{p^*}\to 0$;
(ii) $\|u_n\|_\infty\to 0$;
(iii) $\|u_n\|_{C^{1,\mu}(\overline{\Omega})}\to 0$.

Moreover, the following hold

(1) if $\lim_{n\to\infty}\lambda_n=\lambda_{\pm1}(V)$, and $\|u_n\|_{p^*}\to 0$, then the sequence $w_n:=\frac{u_n}{\|u_n\|_\infty}$ is the union of two disjoint sub-sequences $\{w_n'\}_{n\in\mathbb{N}}$ and $\{w_n''\}_{n\in\mathbb{N}}$, one of them possibly empty, such that, if nonempty, they satisfy $w_n'\to\varphi_{\pm1}(V)$, and/or $w_n''\to-\varphi_{\pm1}(V)$ in $C^{1,\mu'}(\overline{\Omega})$ as $n\to\infty$. Here, $\mu'\in(0,\mu)$ is arbitrary and μ is given in Proposition 2.3.
(2) If $\|u_n\|_{p^*}\to 0$, then $\lim_{n\to\infty}\lambda_n=\lambda_{\pm1}(V)$.

Proof. Clearly, (iii) implies (i) and (ii). We will prove that (i)$\Rightarrow$ (ii) and (ii)$\Longrightarrow$ (iii).

(i)$\Longrightarrow$ (ii). Our hypothesis on g guarantees that, for all $n\in\mathbb{N}$, (P_λ) for $\lambda=\lambda_n$ satisfies the hypothesis of Theorem 2.1. So, using Equation (3.1), we apply Theorem 2.1 with $\tilde f(s):=(|\lambda_{\pm1}(V)|+\delta)\|V\|_\infty s^{p-1}+\bar f(s)$ for $s>0$. Fix $\varepsilon>0$, by Theorem 2.1 and their conclusion (2.4), $h(\|u_n\|_\infty)\to 0$ as $n\to\infty$ with $h(s):=\frac{s^{p^*-1}}{\tilde f(s)}$ for $s>0$. Since h is a continuous function, $h(s)>0$, $\forall s>0$, and

$$\lim_{s\to0^+}\frac{1}{h(s)}=\lim_{s\to0^+}(|\lambda_{\pm1}(V)|+\delta)\|V\|_\infty s^{p-p^*}+\frac{\bar f(s)}{s^{p^*-1}}=+\infty,$$

then $h(0)=0$ and $\|u_n\|_\infty\to0$ as $n\to\infty$, and (ii) holds.

(ii)$\Longrightarrow$ (iii). The function $w_n:=u_n/\|u_n\|_\infty$ satisfies

$$\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\cdot\nabla\psi\,dx=\int_\Omega \kappa_n(x)\psi\,dx \qquad (3.8)$$

for all $\psi\in W_0^{1,p}(\Omega)$, with

$$\kappa_n(x):=\lambda_n V(x)|w_n|^{p-2}w_n+\frac{g(\lambda_n,x,u_n)}{\|u_n\|_\infty^{p-1}}.$$

Since $\bar f$ satisfies $\mathbf{(f1)}_\infty$ and $\mathbf{(f3)}_0$, and $\|w_n\|_\infty=1$, there exists $D>0$ such that, for all n large enough,

$$|\kappa_n(x)|\le D \quad \text{a.e. } x\in\Omega,$$

and from Proposition 2.3, there exists a constant $C > 0$

$$\|w_n\|_{C^{1,\mu}(\overline{\Omega})} \le C = C(p, N, D)$$

and (iii) follows.

Now, we prove the last two parts of the lemma.

(1) The embedding $C^{1,\mu}(\overline{\Omega}) \to C^{1,\mu'}(\overline{\Omega})$ being compact, by Arzelà–Ascoli's theorem, the sequence $\{w_n\}_{n\in\mathbb{N}}$ contains a subsequence that converges into $C^{1,\mu'}(\overline{\Omega})$ to some w; we denote it again by $w_n \to w$. Note that $\|w\|_\infty = 1$. We let $n \to \infty$ in Equation (3.8), to conclude that $w \in C^{1,\mu'}(\overline{\Omega})$ must satisfy

$$\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla\psi \, dx = \lambda_{\pm 1}(V) \int_\Omega V(x)|w|^{p-2}w\psi \, dx.$$

By the simplicity of $\lambda_{\pm 1}(V)$, we have $w = \pm\varphi_{\pm 1}(V)$.

(2) Assume that $\lambda_n \to \lambda^*$. Then, reasoning as in part (1), $0 < w_n \to w$ in $C^{1,\mu'}(\overline{\Omega})$ and $w \ge 0$, $w \ne 0$, satisfy

$$\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla\psi \, dx = \lambda^* \int_\Omega V(x)|w|^{p-2}w\psi \, dx,$$

so w is a non-negative eigenfunction, and necessarily $\lambda^* = \lambda_{\pm 1}(V)$. □

Theorem 3.4. *Let g satisfies the growth condition (3.1) with $\bar{f}(0) = 0$, $\bar{f}$ continuous, strictly increasing, slightly subcritical at infinity, see $(f1)_\infty$, satisfying also $(f2)_\infty$ and $(f3)_0$ with $f = \bar{f}$.*

Denote $E = \mathbb{R} \times W_0^{1,p}(\Omega)$ and

$$\mathcal{P}^+ := \left\{ u \in C_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega \right\}.$$

Then,

(i) there exists a closed connected set $\mathcal{C}_1^+ \subset E$ of positive weak solutions to (P_λ) containing the bifurcation point $(\lambda_1(V), 0)$;
(ii) there exists a closed connected set $\mathcal{C}_{-1}^+ \subset E$ of positive weak solutions to (P_λ) containing the bifurcation point $(\lambda_{-1}(V), 0)$;
(iii) either $\mathcal{C}_1^+$ is unbounded, or $(\lambda_{-1}(V), 0) \in \mathcal{C}_1^+$. Likewise, either $\mathcal{C}_{-1}^+$ is unbounded, or $(\lambda_1(V), 0) \in \mathcal{C}_{-1}^+$;

Proof. Let us check that all the hypothesis of bifurcation Theorem 3.2 are accomplished, for $\hat{g}(\lambda, x, s)$, where $\hat{g}(\lambda, x, \cdot)$ is the odd extension of $g(\lambda, x, \cdot)$.

Consider the modified problem

$$-\Delta_p u = \lambda V(x)|u|^{p-2}u + \hat{g}(\lambda, x, u) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{3.9}$$

Trivially, if $u \le 0$ is a solution of Equation (3.9) then $-u$ is a positive solution of Equation (3.9), and positive solutions to Equation (3.9) are positive solutions to (P_λ).

Denote μ_1 either $\lambda_1(V)$ or $\lambda_{-1}(V)$ and let $G : E \to W^{-1,p}(\Omega)$ be the map

$$\langle G(\lambda, u), \psi\rangle = \int_\Omega \hat{g}(\lambda, x, u)\psi \, dx.$$

Theorem 3.2 applied to the principal eigenvalue $\mu_1 = \lambda_1(V)$, implies the existence of a maximal closed connected set $\hat{\mathcal{C}}_1$ (with respect to the closure of nontrivial solutions $\mathcal{S}$) for Equation (3.9), containing $(\lambda_1(V), 0)$. Likewise, we get the existence of another maximal closed connected set $\hat{\mathcal{C}}_{-1}$ containing $(\lambda_{-1}(V), 0)$. Both sets $\hat{\mathcal{C}}_{\pm 1}$ satisfy properties (i) and (ii) of Theorem 3.2. Moreover, by oddity of $\hat{g}$, if $(\lambda, u) \in \hat{\mathcal{C}}_1$, also $(\lambda, -u) \in \hat{\mathcal{C}}_1$, in other words $-\hat{\mathcal{C}}_1 = \hat{\mathcal{C}}_1$ and $-\hat{\mathcal{C}}_{-1} = \hat{\mathcal{C}}_{-1}$.

(i) Let us first prove that there exists $\delta > 0$ such that

$$\hat{\mathcal{C}}_1 \cap B_\delta((\lambda_1(V),0)) \setminus \{(\lambda_1(V),0)\} \subset (\lambda_1(V)-\delta, \lambda_1(V)+\delta) \times \left(\mathscr{P}^+ \cup -\mathscr{P}^+\right). \tag{3.10}$$

Assume by contradiction that there exists a sequence of solutions $(\lambda_n, u_n) \in \mathbb{R} \times (W_0^{1,p}(\Omega) \setminus \{0\})$, $u_n \notin \mathscr{P}^+ \cup -\mathscr{P}^+$, such that

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla \psi \, dx = \lambda_n \int_\Omega V(x)|u_n|^{p-2}u_n\psi \, dx + \int_\Omega \hat{g}(\lambda, x, u_n)\psi \, dx$$

for all $\psi \in W_0^{1,p}(\Omega)$ and $(\lambda_n, u_n) \to (\lambda_1(V), 0)$. Denote by $w_n := \frac{u_n}{\|u_n\|_\infty}$. Due to Proposition 3.3,

$$w_n := \frac{u_n}{\|u_n\|_\infty} \to \pm\varphi_1(V), \qquad \text{in} \quad C^{1,\mu'}(\overline{\Omega}). \tag{3.11}$$

Since $\varphi_1(V) \in \mathscr{P}^+$ and $\mathscr{P}^+$ is an open set in the C_0^1-topology, there exists a neighborhood $\mathscr{V} \subset \mathscr{P}^+$ such that $w_n \in \mathscr{V}$, and we conclude that also $u_n \in \mathscr{P}^+$, contradicting that $\{u_n\}$ is not in $\mathscr{P}^+ \cup -\mathscr{P}^+$. If $w_n \to -\varphi_1(V)$, we take instead the sequence of solutions $\{-w_n\}_{n\in\mathbb{N}}$ and reason as in the previous case. We have proved Equation (3.10).

Note that $\hat{\mathcal{C}}_1 \cap \left(\mathbb{R} \times \mathscr{P}^+\right) \neq \emptyset$ as a result of Equation (3.10) and the fact that $\hat{\mathcal{C}}_1 = -\hat{\mathcal{C}}_1$. Let $\hat{\mathcal{C}}_1^{\,+}$ be the maximal subcontinuum of $\mathcal{S}$ containing the branch of positive solutions bifurcating from $(\lambda_1(V), 0)$ contained in $\hat{\mathcal{C}}_1$. By a subcontinuum of $\mathcal{S}$ we mean a subset of $\mathcal{S}$ which is closed and connected in the Banach space E.

Taking $\mathcal{C}_1^+ := \hat{\mathcal{C}}_1^{\,+} \neq \emptyset$, let us prove that

$$\mathcal{C}_1^+ \subset \mathbb{R} \times \mathscr{P}^+ \cup \{(\lambda_{\pm 1}(V), 0)\}. \tag{3.12}$$

Assume by contradiction that there exits $(\lambda^*, u^*) \in \mathcal{C}_1^+$ with $(\lambda^*, u^*) \neq (\lambda_{\pm 1}(V), 0)$ and that there exits $\{(\lambda_n, u_n)\}_n$ in $\mathcal{C}_1^+ \cap \mathbb{R} \times \mathscr{P}^+$ such that $\lambda_n \to \lambda^*, u_n \to u^*$ and $W_0^{1,p}(\Omega)$. Since $u_n > 0$ then $u^* \geq 0$. If $u^* = 0$ then $(\lambda^*, 0)$ is a bifurcation point of positive solution and therefore $\lambda^* = \lambda_{\pm 1}(V)$, a contradiction. Since $u^* \geq 0, u^* \neq 0$ then u^* is a nontrivial non-negative weak solution to problem (P_λ) for $\lambda = \lambda^*$, and therefore, by the regularity results of Proposition 2.3 and the standard strong maximum principles for quasilinear elliptic equations [33, 34, 38] we deduce that $u^* > 0$ in Ω and $u^* \in C^{1,\mu}(\overline{\Omega})$, so the claim (3.12) is proved.

(ii) The proof is similar to the case (i) with minor changes.

(iii) To prove that either $\mathcal{C}_1^+$ is unbounded, or $(\lambda_{-1}(V), 0) \in \mathcal{C}_1^+$, we use $\hat{\mathcal{C}}_1^{\,+}$. We claim that

$$\hat{\mathcal{C}}_1 \subset \hat{\mathcal{C}}_1^{\,+} \cup \left(-\hat{\mathcal{C}}_1^{\,+}\right) \cup \{(\lambda_{-1}(V), 0)\}. \tag{3.13}$$

Since Equation (3.12), the branch $\hat{\mathcal{C}}_1^{\,+} \setminus \{(\lambda_{\pm 1}(V), 0)\}$ only contains strictly positive solutions.

In particular, a pair $(\lambda^*, u^*) \in \mathcal{C}_1^+$ cannot be approached by changing sign solutions, and the claim (3.13) is proved. Consequently, recalling that $\hat{\mathcal{C}}_1$ satisfies Rabinowitz's alternative, either $\mathcal{C}_1^+ = \hat{\mathcal{C}}_1^{\,+}$ is unbounded or else $(\lambda_{-1}(V), 0) \in \mathcal{C}_1^+$. □

4 | NON-EXISTENCE AND PROOF OF THEOREM 1.1

We will made use of the well-known Picone's identity [3], that we recall here. Let us denote

$$L(v_1, v_2) := |\nabla v_1|^p + (p-1)\frac{v_1^p}{v_2^p}|\nabla v_2|^p - p\frac{v_1^{p-1}}{v_2^{p-1}}\nabla v_1 \cdot |\nabla v_2|^{p-2}\nabla v_2, \tag{4.1}$$

$$R(v_1, v_2) := |\nabla v_1|^p - |\nabla v_2|^{p-2}\nabla v_2 \cdot \nabla\left(\frac{v_1^p}{v_2^{p-1}}\right),$$

for any couple of *a.e.* differentiable functions v_1 and v_2 defined on an open set $\mathcal{O} \subset \mathbb{R}^N$, with $v_2 > 0$. Then one has:

1. $0 \le L(v_1, v_2) = R(v_1, v_2)$.
2. If $v_1/v_2 \in W^{1,1}_{loc}(\mathcal{O})$ then $L(v_1, v_2) = 0$ a.e. in $\mathcal{O}$ implies that $v_1 = cv_2$ for some constant $c \in \mathbb{R}$ in each connected component of $\mathcal{O}$.

4.1 | A non-existence result

Let $\omega \subset \Omega$ be an open bounded set, possibly non-connected, with $\{\omega_i\}_{i\in I}$ its family of open connected components. We recall that if $\Omega' \subsetneq \Omega$ in an open domain, then $\lambda_{\pm1}(V, \Omega') > \lambda_{\pm1}(V, \Omega)$, see [14]. Let us denote by

$$\lambda_1(V, \omega) = \inf_{i\in I} \lambda_1(V, \omega_i), \quad \lambda_{-1}(V, \omega) := \sup_{i\in I} \lambda_{-1}(V, \omega_i)$$

where $\lambda_{\pm1}(V, \omega_i)$ are resp. the smallest positive eigenvalue and largest negative eigenvalue of the Dirichlet eigenvalue problem

$$-\Delta_p \varphi = \lambda V(x)|\varphi|^{p-2}\varphi \quad \text{in } \omega_i, \qquad \varphi = 0 \quad \text{on } \partial\omega_i,$$

see Equation (1.5). Clearly, if I is finite, then $\lambda_{\pm1}(V, \omega) = \lambda_{\pm1}(V, \omega_i)$ for some i (not necessarily unique). Observe that if

$$\big|\{x \in \omega_i \,:\, V(x) > 0\}\big| > 0, \qquad \text{then} \quad \lambda_1(V, \omega_i) < +\infty.$$

Likewise, if

$$\big|\{x \in \omega_i \,:\, V(x) < 0\}\big| > 0, \qquad \text{then} \quad \lambda_{-1}(V, \omega_i) > -\infty.$$

Let us agree that $\lambda_1(V, \omega_i) = +\infty$ if $V \le 0$ a.e. in ω_i (resp. $\lambda_{-1}(V, \omega_i) = -\infty$ if $V \ge 0$ a.e. in ω_i). Finally, when $V \equiv 1$ we will write $\lambda_1(\omega) = \lambda_1(1, \omega)$, and $\lambda_{\pm1}(V) = \lambda_{\pm1}(V, \Omega)$, when $\omega = \Omega$.

Proposition 4.1. *Let f satisfy hypothesis **(f1)**$_\infty$ and **(f3)**$_0$.*

*(i) Assume that m satisfies **(m1)** and **(m2)**. Set $\omega^{+,0} := \operatorname{int}\left(\Omega^+ \cup \Omega^0\right)$. Let us denote:*

$$\alpha_{+,0} := 1 + \frac{C_0 \sup m^+}{\lambda_1(\omega^{+,0})}, \qquad \text{and} \quad C_0 := \inf\{C \ge 0 \,:\, f(s) + Cs^{p-1} > 0 \text{ for all } s > 0\} < +\infty.$$

If problem (1.1)$_\lambda$ has a non-negative nontrivial solution, then

$$-\infty < \alpha_{+,0}\lambda_{-1}\left(V, \omega^{+,0}\right) < \lambda < \alpha_{+,0}\lambda_1\left(V, \omega^{+,0}\right) < +\infty.$$

*(ii) Assume hypothesis **(m3)**. Set $\omega^0 := \operatorname{int}\left(\Omega^0\right)$. If problem (1.1)$_\lambda$ has a non-negative nontrivial solution, then*

$$-\infty < \lambda_{-1}\left(V, \omega^0\right) < \lambda < \lambda_1\left(V, \omega^0\right) < +\infty.$$

Proof.

(i) Let $\lambda > \alpha_{+,0}\lambda_1\left(V, \omega^{+,0}\right)$ and assume by contradiction that there exists a non-negative nontrivial solution $u \in W^{1,p}_0(\Omega)$ to Equation (1.1)$_\lambda$. By standard maximum principles for quasilinear elliptic equations (see [33, 34, 38]) we deduce that $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$.

Let us write $\omega^{+,0} = \cup_i^m w_i$ the decomposition of $\omega^{+,0}$ as the disjoint union of its open connected components. Then, $\lambda_1(V, \Omega^{0,+}) = \lambda_1(V, \omega_i)$ for some connected component ω_i of $\omega^{+,0}$. Let $\varphi_1^{+,0} := \varphi_1(V, \omega_i)$ be a positive eigenfunction

associated with $\lambda_1(V,\omega^{+,0})$ of L^∞-norm equal to 1. For simplicity, we will also denote by $\varphi_1^{+,0}$ the extension by 0 of $\varphi_1^{+,0}$ in all Ω.

Fix $\varepsilon>0$ and put $\xi := \frac{(\varphi_1^{+,0})^p}{(u+\varepsilon)^{p-1}}$. If we multiply Equation $(1.1)_\lambda$ by ξ and integrate along Ω we find,

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\left(\frac{(\varphi_1^{+,0})^p}{(u+\varepsilon)^{p-1}}\right)dx=\lambda\int_\Omega V(x)(\varphi_1^{+,0})^p\frac{u^{p-1}}{(u+\varepsilon)^{p-1}}+\int_\Omega m^+(x)f(u)\frac{(\varphi_1^{+,0})^p}{(u+\varepsilon)^{p-1}}\,dx.$$

Using Picone's identity, we deduce

$$\begin{aligned}0\le\int_\Omega L\left(\varphi_1^{+,0},u+\varepsilon\right)dx&=\int_\Omega R\left(\varphi_1^{+,0},u+\varepsilon\right)dx\int_\Omega\left|\nabla(\varphi_1^{+,0})\right|^p dx-\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla\left(\frac{(\varphi_1^{+,0})^p}{(u+\varepsilon)^{p-1}}\right)dx\\&=\int_\Omega\left|\nabla(\varphi_1^{+,0})\right|^p dx-\lambda\int_\Omega V(x)(\varphi_1^{+,0})^p\left(\frac{u}{u+\varepsilon}\right)^{p-1}-\int_{\omega^{+,0}}m^+(x)f(u)\frac{(\varphi_1^{+,0})^p}{(u+\varepsilon)^{p-1}}\,dx\\&\le\int_{\omega^{+,0}}\left|\nabla(\varphi_1^{+,0})\right|^p dx-\lambda\int_{\omega^{+,0}}V(x)(\varphi_1^{+,0})^p\left(\frac{u}{u+\varepsilon}\right)^{p-1}+C_0\sup m^+\int_{\omega^{+,0}}\left(\frac{u}{u+\varepsilon}\right)^{p-1}(\varphi_1^{+,0})^p\,dx.\end{aligned}$$

Letting $\varepsilon\to 0$, we obtain

$$0\le\lim_{\varepsilon\to0}\int_\Omega L(\varphi_1^{+,0},u+\varepsilon)\,dx\le\left(\alpha_{0,+}-\frac{\lambda}{\lambda_1(V,\omega^{+,0})}\right)\int_{\omega^{+,0}}|\nabla(\varphi_1^{+,0})|^p\,dx$$

which is clearly a contradiction whenever $\lambda>\alpha_{+,0}\lambda_1(V,\omega^{+,0})$. In the case $\lambda=\alpha_{+,0}\lambda_1(V,\omega^{+,0})$, we have

$$\lim_{\varepsilon\to0}\int_\Omega L(\varphi_1^{+,0},u+\varepsilon)\,dx=0$$

and using Fatou's lemma

$$\int_\Omega\liminf_{\varepsilon\to0}L(\varphi_1^{+,0},u+\varepsilon)\,dx\le0$$

Since $L(\varphi_1^{+,0},u+\varepsilon)\ge0$ for all $\varepsilon>0$ then we get,

$$L(\varphi_1^{+,0},u)=\liminf_{\varepsilon\to0}L(\varphi_1^{+,0},u+\varepsilon)=0\quad a.e.\ \text{in}\ \Omega.$$

Since trivially $\frac{\varphi_1^{+,0}}{u}\in W^{1,1}_{\mathrm{loc}}(\Omega)$, from the results of Picone's identity mentioned above we infer that, in ω_i, $\varphi_1^{+,0}=cu$ for some $c\in\mathbb{R}$. But $\varphi_1^{+,0}$ vanishes on $\partial(\omega_i)\cap\Omega$ although $u>0$ in Ω, a contradiction.

(ii) The proof of non-existence for $\lambda\le\lambda_{-1}\left(V,\omega^{+,0}\right)$ is similar to the previous case and we omit it. □

4.2 | Proof of Theorem 1.1

Since hypotheses on f, choosing $\bar f$ strictly increasing as in Remark 2.2, hypotheses of Theorem 3.4 are satisfied. Hence, parts (i)–(ii) are accomplished.

(iii) We have to prove that there exists a neighborhood $\mathcal{U}\subset\mathbb{R}\times W^{1,p}_0(\Omega)$ of $(\lambda_1(V),0)$ such that for all (λ,u_λ) in $\mathscr{C}^+\cap\mathcal{U}$, $\lambda>\lambda_1(V)$. Assume that there is a sequence (λ_n,u_n) of bifurcated positive solutions to Equation (1.1) in a neighborhood $\mathcal{U}\subset\mathbb{R}\times W^{1,p}_0(\Omega)$ of $(\lambda_1(V),0)$, with $(\lambda_n,u_n)\to(\lambda_1(V),0)$. Let $\varepsilon>0$ be fixed. Using Picone's identity, see

Definition (4.1), and that $(\lambda_1(V), \varphi_1(V))$ is an eigenpair, we deduce that for all $\varepsilon > 0$

$$\begin{aligned} 0 \le \int_\Omega L(u_n, \varphi_1(V)+\varepsilon)\,dx &= \int_\Omega R(u_n, \varphi_1(V)+\varepsilon)\,dx \\ &\le \int_\Omega |\nabla u_n|^p\,dx - \int_\Omega |\nabla \varphi_1(V)|^{p-2}\nabla\varphi_1(V)\cdot\nabla\left(\frac{u_n^p}{(\varphi_1(V)+\varepsilon)^{p-1}}\right)dx \\ &= \int_\Omega m(x) f(u_n)u_n\,dx + \int_\Omega V(x)u_n^p\left[\lambda_n - \lambda_1(V)\left(\frac{\varphi_1(V)}{\varphi_1(V)+\varepsilon}\right)^{p-1}\right]dx. \end{aligned} \tag{4.2}$$

Since hypothesis $\mathbf{(f4)}_0$ on f and Equation (3.11), we deduce for $w_n = \frac{u_n}{\|u_n\|_\infty}$ that

$$\frac{f(w_n(x)\|u_n\|_\infty)}{f(\|u_n\|_\infty)} \to g_0(\varphi_1(V)(x)) \qquad \text{as} \quad n\to\infty, \qquad \text{a.e.} \quad x\in\Omega.$$

Indeed, fix $\delta \in (0,1)$. Hypothesis $\mathbf{(f4)}_0$ on f, implies that for all $\varepsilon_1 > 0$ there exists a $n_1 > 0$ such that

$$\left|\frac{f(\mu\|u_n\|_\infty)}{f(\|u_n\|_\infty)} - g_0(\mu)\right| \le \varepsilon_1,\ \forall n > n_1,\ \forall \mu \in [(1-\delta)\varphi_1(V), (1+\delta)\varphi_1(V)].$$

Clearly, $(1-\delta)\varphi_1(V) \le w_n \le (1+\delta)\varphi_1(V)$ for n big enough, and so

$$\frac{f(w_n(x)\|u_n\|_\infty)}{f(\|u_n\|_\infty)} - g_0(w_n(x)) \to 0 \qquad \text{as} \quad n\to\infty, \qquad \text{a.e.} \quad x\in\Omega.$$

Hence, using Equation (3.11), we deduce

$$\int_\Omega m(x)\frac{f(u_n)}{f(\|u_n\|_\infty)}\frac{u_n}{\|u_n\|_\infty}\,dx \underset{n\to\infty}{\to} \int_\Omega m(x) g_0(\varphi_1(V))\,\varphi_1(V)\,dx =: I_1 < 0,$$

by hypothesis $\mathbf{(f4)}_0$.

Now, we prove that $\lambda_n > \lambda_1(V)$. Indeed, on one hand

$$\lim_{n\to\infty}\frac{1}{\|u_n\|_p^p}\int_\Omega V(x)u_n^p\,dx = \int_\Omega V(x)\varphi_1(V)^p\,dx = \frac{\|\varphi_1(V)\|^p}{\lambda_1(V)} > 0$$

and then $\int_\Omega V(x)u_n^p dx > 0$ for n large enough. On the other hand, letting $\varepsilon \to 0$ in Equation (4.2), dividing by $\|u_n\|_\infty f(\|u_n\|_\infty)$ and using Equation (3.11)

$$\frac{(\lambda_1(V)-\lambda_n)}{\|u_n\|_\infty f(\|u_n\|_\infty)}\int_\Omega V(x)u_n^p \le \int_\Omega m(x)\frac{f(u_n)}{f(\|u_n\|_\infty)}\frac{u_n}{\|u_n\|_\infty}\,dx \underset{n\to\infty}{\to} I_1 < 0.$$

We have proved the first part of (iii). The second part is identical to that one.

(iv) Let us finally define

$$\Lambda_1 := \sup\left\{\lambda > 0 :\ (1.1)_\lambda \text{ admits a positive solution}\right\},$$

$$\Lambda_{-1} := \inf\left\{\lambda < 0 :\ (1.1)_\lambda \text{ admits a positive solution}\right\}.$$

Since the above, $\Lambda_{-1} < \lambda_{-1}(V) < \lambda_1(V) < \Lambda_1$. Moreover, as a consequence of Proposition 4.1, Λ_1, Λ_{-1} are finite. We have proved (iv).

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ENDNOTES

[1]Note that the continuity of the function g_0 and the fact that $g_0(st) = g_0(s)g_0(t)$ for all s, t positive, imply that $g_0(s) = s^{q-1}$ for some $q \le p^*$.
[2]The map $u \to \|u\|_A$ is the Minkowski functional of the convex set $C = \{u \in K_A(\Omega) : \int_\Omega A(|u(x)|)\,dx \le 1\}$.

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APPENDIX A: THE CONVEX SET $K_A(\Omega)$ AND ITS GAUGE $\|\cdot\|_A$

Let us summarize some definitions and results on convex sets and its Minkowski functional (gauge). We refer the reader to [1, 16, 27].

A.1 | The convex set $K_A(\Omega)$

Let $a : \mathbb{R}^+ \to \mathbb{R}^+$ be increasing (that is, $0 \le s \le t$ implies $a(s) \le a(t)$) and right continuous. Let $A : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as

$$A(t) = \int_0^t a(s)\,ds. \tag{A.1}$$

Then, A is a continuous, strictly increasing and convex function such that $A(0) = 0$.

If furthermore $a(0) = 0$ and $\lim_{s\to+\infty} a(s) = +\infty$, we will say that A is a N-function.

Let Ω be an open subset of $\mathbb{R}^N$. The convex set $K_A(\Omega)$ is defined as the set of real-valued measurable functions $u : \Omega \to \mathbb{R}$ such that:

$$\int_\Omega A(|u(x)|)\,dx < +\infty.$$

For all $u \in K_A(\Omega)$, one defines the *gauge*

$$\|u\|_A := \inf\left\{\lambda > 0 \ : \ \int_\Omega A\left(\frac{|u(x)|}{\lambda}\right) dx \le 1\right\}. \tag{A.2}$$

Observe that $\|su\|_A = |s|\|u\|_A$ for all $s \in [-1,1]$, $\|u+v\|_A \le \|u\|_A + \|v\|_A$ for all $u, v \in K_A(\Omega)$ and $\|u\|_A = 0$ if and only if $u = 0$ a.e. Note also that $K_A(\Omega)$ is a convex set which is not a linear space in general.[2] From the convexity of A, it follows that

$$\forall u \in K_A(\Omega), \quad \|u\|_A \le \max\left\{\int_\Omega A(|u(x)|)\, dx,\ 1\right\}. \tag{A.3}$$

Furthermore, one have

$$\forall u \in K_A(\Omega), u \ne 0, \quad \int_\Omega A\left(\frac{|u(x)|}{\|u\|_A}\right) dx \le 1. \tag{A.4}$$

A.2 | Conjugate of a N- function, generalization of Young's and Holder's inequalities

Let A be an N-function. The conjugate A^* of A is defined by

$$A^*(t) := \max\{st - A(s),\ s > 0\}, \quad \forall t \ge 0 \tag{A.5}$$

or, equivalently,

$$A^*(t) := \int_0^t a^*(s)\, ds, \quad \forall t \ge 0, \tag{A.6}$$

where $a^* : \mathbb{R}^+ \to \mathbb{R}^+$ is given by

$$a^*(t) := \sup\{s \ : \ a(s) \le t\}. \tag{A.7}$$

Observe that for any $t \ge 0$, $a^*(a(t)) \ge t$ and, whenever a is strictly increasing,

$$a^*(t) = a^{-1}(t). \tag{A.8}$$

If follows directly from Definition (A.6) that A^* is an N-function as well. It follows also directly from Definition (A.5), the following Young's inequality:

$$\forall s, t \in \mathbb{R}^+, \quad st \le A(s) + A^*(t). \tag{A.9}$$

Equality holds in Equation (A.9) if and only if $t = a(s)$ or $s = a^*(t)$. In particular $\forall s, t \in \mathbb{R}^+$,

$$sa(s) = A(s) + A^*(a(s)), \quad ta^*(t) = A(a^*(t)) + A^*(t). \tag{A.10}$$

Let $\Omega \subset \mathbb{R}^N$ be open. Using Equations (A.9) and (A.4), the following generalization of Holder's inequality holds:

$$\forall u \in K_A(\Omega),\ \forall v \in K_{A^*}(\Omega), \quad \left|\int_\Omega uv\, dx\right| \le 2\|u\|_A\, \|v\|_{A^*}. \tag{A.11}$$

A.3 | Δ_2-condition and comparison of convex functions

Let A be an N-function, A is said to satisfy the Δ_2-condition if for all $r > 1$ there exists $k = k(r) > 0$ such that:

$$A(rt) \le kA(t) \quad \forall t \ge 0. \tag{A.12}$$

When Equation (A.12) holds only for all $t \geq t_0$, for some $t_0 \geq 0$, then A is said to satisfy the Δ_2-condition *near infinity*.

Proposition A.1. *Let A be an N-function. A necessary and sufficient condition for A to satisfy the Δ_2-condition at infinity is that there exist constants $k_0 > 1$ and $t_0 > 0$ such that*

$$\frac{ta(t)}{A(t)} \leq k_0, \qquad for \quad t > t_0.$$

See [27, Theorem 4.1].

Let A and B be two N-functions. We say that B increases essentially more slowly than A *near infinity* if

$$\forall \delta > 1 \qquad \lim_{t \to +\infty} \frac{B(\delta t)}{A(t)} = 0. \tag{A.13}$$

We have the following result:

Theorem A.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set with finite volume and let A and B be two N-functions. Assume also that*

(1) A satisfies the Δ_2-condition near infinity,
(2) B increases essentially more slowly than A near infinity.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a $\|\cdot\|_A$-bounded sequence of $K_A(\Omega)$ converging in measure to some $u \in K_A(\Omega)$. Then

$$\lim_{n\to\infty} \|u_n - u\|_B = 0.$$

Starting from boundedness and convergence in measure, this theorem is a simplified approach to a certain strong convergence. Alternatively, one could use compact embedding in Orlicz–Sobolev spaces [12].

Proof of Theorem A.2. Let $0 < \varepsilon < 2\sup_j \|u_j\|_A$ be fixed and put $v_n := \frac{|u_n - u|}{\varepsilon}$. Let us show that there exists $K = K(\varepsilon) > 0$ such that

$$\forall n \in \mathbb{N}, \quad \|v_n\|_A \leq K. \tag{A.14}$$

Indeed, let us choose $r := \max\left\{\frac{2\sup_j \|u_j\|_A}{\varepsilon}, \frac{2}{\varepsilon}\right\}$, and $k = k(r)$, $t_0 > 0$ as given in the Δ_2-condition, then

$$\forall t \geq 0, \quad A(rt) \leq A(rt_0) + kA(t) \tag{A.15}$$

Since A is increasing, convex, and satisfies Equation (A.15), we have

$$A(v_n) \leq \frac{1}{2}A\left(\frac{2\|u_n\|_A}{\varepsilon}\frac{|u_n|}{\|u_n\|_A}\right) + \frac{1}{2}A\left(\frac{2|u|}{\varepsilon}\right) \leq \frac{1}{2}\left(2A(rt_0) + kA\left(\frac{|u_n|}{\|u_n\|_A}\right) + kA(|u|)\right).$$

Hence, using Equations (A.3) and (A.4),

$$\|v_n\|_A \leq \int_\Omega A(|v_n(x)|)\,dx + 1 \leq \frac{1}{2}\left(2A(rt_0)|\Omega| + k + k\int_\Omega A(|u(x)|)\,dx\right) + 1 := K,$$

and so Equation (A.14) is satisfied. Since Equation (A.13), let us choose $t_1 > 0$ such that

$$B(t) \leq \frac{1}{4}A(t/K) \quad \forall t \geq t_1.$$

Set

$$\Omega_n := \left\{ x \in \Omega \ : \ v_n(x) > B^{-1}\left(\frac{1}{2|\Omega|}\right)\right\},\ \Omega_n' = \{x \in \Omega_n \ : \ v_n(x) \geq t_1\}.$$

Since the sequence $\{u_n\}_n$ converges in measure to u, and ε is fixed, $\{v_n\}_n$ converges in measure to 0 for $\delta = \frac{1}{4B(t_1)}$ there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0,\ |\Omega_n| \leq \delta$. Thus

$$\begin{aligned}\int_\Omega B(v_n(x))\,dx &= \int_{\Omega\setminus\Omega_n} B(v_n(x))\,dx + \int_{\Omega_n'} B(v_n(x))\,dx + \int_{\Omega_n\setminus\Omega_n'} B(v_n(x))\,dx \\ &\leq \frac{|\Omega|}{2|\Omega|} + \frac{1}{4}\int_{\Omega_n'} A\left(\frac{v_n(x)}{K}\right)dx + \delta B(t_1) \leq 1,\end{aligned}$$

thanks to Equation (A.14). Consequently, $\int_\Omega B\left(\frac{|u_n(x)-u(x)|}{\varepsilon}\right)dx \leq 1$, and so

$$\forall n \geq n_0, \quad \|u_n - u\|_B \leq \varepsilon.$$

□