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Positive Solutions of Elliptic Systems with Superlinear Terms on the Critical Hyperbola

Mabel Cuesta, Rosa Pardo , and Angela Pistoia

Abstract. We consider a slightly subcritical elliptic system with Dirichlet boundary conditions and a non-power nonlinearity in a bounded smooth domain. For this problem, standard compact embeddings cannot be used to guarantee the existence of solutions as in the case of power-type nonlinearities. Instead, we use the dual method on Orlicz spaces, showing that our problem possesses a mountain pass type solution.

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Keywords. Mountain pass solutions, Critical Sobolev exponent, Dual method.

1. Introduction

Let us consider the system

$$\begin{cases} -\Delta v = \frac{u^p}{(\ln(e+u))^\alpha} & \text{in } \Omega \\ -\Delta u = \frac{v^q}{(\ln(e+v))^\beta} & \text{in } \Omega \\ u > 0, v > 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain of class C^2 , $p, q > 0$, $\alpha \leq p$, $\beta \leq q$, and (p, q) may belong to the critical hyperbola; specifically, either

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \quad (1.2)$$

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or

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}, \quad \frac{\alpha}{p+1} + \frac{\beta}{q+1} > 0. \quad (1.3)$$

We want to find a solution (u, v) of (1.1), positive in both components. Problems of type (1.1) has been considered by several authors, we refer to [3, 9, 11, 17]. In [3] the authors study the case (1.2) and $\alpha = \beta = 0$. Whereas in [9, Theorem 2.7] the authors study related nonlinearities when the pair of exponents (p, q) lies below the critical Sobolev hyperbola (1.2), using variational approaches. In [11, Theorem 1.3], using an Orlicz-space approach, the authors study the existence of solutions of (1.1) when the pairs of exponents (p, q) , (α, β) satisfy (1.3), in particular (p, q) lies on the critical Sobolev hyperbola, with $p, q > 1$; they do not cover the case $p, q > 0$. While in [17] the authors establish a-priori L^∞ bounds when the pair of exponents (p, q) lies on the critical Sobolev hyperbola (1.3), $1 < p, q < \infty$, and $\alpha, \beta > 2/(N-2)$. Concerning one single equation, we can mention [5] establishing L^∞ a priori bounds when $\alpha > 2/(N-2)$, [8] analyzing the asymptotic behavior of the solutions as $\alpha \rightarrow 0$, and [10] including a changing sign weight.

Our main result is the following:

Theorem 1.1. *Assume $p, q > 0$, $\alpha \leq p$, $\beta \leq q$, and either (p, q) satisfy (1.2) or (p, q) , (α, β) satisfy (1.3). Then Problem (1.1) possesses a solution of mountain pass type.*

The proof relies on the dual method which allows to reduce the existence of solutions to the problem (1.1) to finding critical points of an energy functional defined on a suitable Orlicz space, whose accurate choice is the main novelty of the present paper.

Remark 1.2. We would like to comment on condition (1.3). At this aim we point out that Theorem 1.1 also applies to the case of the single equation, namely $p = q = \frac{N+2}{N-2}$ and $\alpha = \beta > 0$. In particular, the problem

$$\begin{cases} -\Delta u = \frac{u^p}{(\ln(e+u))^\alpha} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.4)$$

has always a positive solution. On the other hand, it is known that Pohozaev's identity [21] ensures the non-existence of positive solutions on star-shaped domains to Eq. (1.4) whenever $\alpha \leq 0$, see Remark 4.3.

In a similar way, it could be interesting to prove the *criticality* of our condition 1.3. Indeed, a question naturally arises: if p and q lie on the critical hyperbola and

$$\frac{\alpha}{p+1} + \frac{\beta}{q+1} \leq 0$$

does the system (1.1) have any positive solutions on a star-shaped domain? In Theorem 4.2 we give a partial answer when both α and β are not positive. The general case remains open.

The paper is organized as follows. In Sect. 2 we describe the dual method which allows us to consider $p, q > 0$. Roughly speaking, this method consists in taking the inverse of the Laplace operator, and defining the inverse of the nonlinearities. In Sect. 3 we prove our main result, that the associated energy functional, defined in a suitable product of Orlicz spaces, has a mountain pass geometry. Section 4 is devoted to analyze the criticality of the condition (1.3). Finally, Sect. 5 summarize the theory of Orlicz spaces needed for our purposes.

2. The Dual Method

2.1. The Variational Formulation

Given any $s \in (1, \infty)$, we introduce the operator $K := (-\Delta)^{-1} : L^s(\Omega) \rightarrow W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega)$ which is defined as

$$K(f) := u \quad \text{if and only if} \quad -\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Next, we set

$$a(t) := \frac{t^p}{(\ln(e+t))^\alpha}, \quad \text{and} \quad b(t) := \frac{t^q}{(\ln(e+t))^\beta} \quad \text{with } t \geq 0, \quad (2.1)$$

and consider their primitive functions:

$$A(t) := \int_0^t a(s) ds \quad \text{and} \quad B(t) := \int_0^t b(s) ds \quad \text{for all } t \geq 0. \quad (2.2)$$

If $\alpha \leq p$ and $\beta \leq q$, the functions a, b are strictly increasing and so invertible in $[0, +\infty)$. Let us denote by

$$\tilde{a} := a^{-1} \quad \text{and} \quad \tilde{b} := b^{-1} \quad \text{the inverse functions.} \quad (2.3)$$

If we set $f = a(u)$ and $g = b(v)$ or equivalently $u = \tilde{a}(f)$ and $v = \tilde{b}(g)$, then problem (1.1) can be rewritten as

$$\begin{cases} \tilde{b}(g) = K(f) & \text{in } \Omega \\ \tilde{a}(f) = K(g) & \text{in } \Omega \\ f > 0, g > 0 & \text{in } \Omega. \end{cases} \quad (2.4)$$

Let us introduce their primitive functions:

$$\tilde{A}(t) := \int_0^t \tilde{a}(s) ds \quad \text{and} \quad \tilde{B}(t) := \int_0^t \tilde{b}(s) ds, \quad (2.5)$$

for all $t \geq 0$, and denote $L^{\tilde{A}}(\Omega)$ and $L^{\tilde{B}}(\Omega)$ the Orlicz spaces associated to the functions \tilde{A} and \tilde{B} respectively, cf. Definition 5.3 and Remark 5.7. Observe that, since \tilde{a} and \tilde{b} are continuous and increasing the functions \tilde{A} and \tilde{B} are \mathcal{N} -functions, cf. definition 5.1. We endowed the space $L^{\tilde{A}}(\Omega)$ (resp. $L^{\tilde{B}}(\Omega)$) with either the Luxembourg norm $\|\cdot\|_{(\tilde{A})}$ or the Orlicz norm $\|\cdot\|_{\tilde{A}}$ (resp. $\|\cdot\|_{(\tilde{B})}$ and $\|\cdot\|_{\tilde{B}}$), see definitions (5.3) and (5.4) for those norms. From Lemma 5.5 and Proposition 5.10(i), the spaces $L^{\tilde{A}}(\Omega)$ and $L^{\tilde{B}}(\Omega)$ are complete with both the Orlicz or the Luxemburg norm.

Problem (2.4) has a *variational structure*. Indeed, let us denote by X the Banach space

$$X = L^{\tilde{A}}(\Omega) \times L^{\tilde{B}}(\Omega)$$

endowed with the norm $\|(f, g)\|_X := \|f\|_{\tilde{A}} + \|g\|_{\tilde{B}}$.

By the regularity result Theorem 5.29, the operator K sends $L^{\tilde{A}}(\Omega)$ into $W_0^{1, \tilde{A}}(\Omega) \cap W^{2, \tilde{A}}(\Omega)$, cf. Definition 5.19 of the *Orlicz–Sobolev spaces* $W_0^{1, H}(\Omega)$ and $W^{2, H}(\Omega)$, when H is equal to either \tilde{A} or \tilde{B} . Thus for any $(f, g) \in X$, $(K(f), K(g)) \in W^{2, \tilde{A}}(\Omega) \times W^{2, \tilde{B}}(\Omega)$. By a *solution* (u, v) of (1.1) we mean a pair $(u, v) \in (W_0^{1, \tilde{A}}(\Omega) \cap W^{2, \tilde{A}}(\Omega)) \times (W_0^{1, \tilde{B}}(\Omega) \cap W^{2, \tilde{B}}(\Omega))$ satisfying (1.1) in the weak sense.

Moreover, for (p, q) , (α, β) satisfying either (1.2), or (1.3), we have that the embeddings $W^{2, \tilde{A}}(\Omega) \hookrightarrow L^B(\Omega)$ and $W^{2, \tilde{B}}(\Omega) \hookrightarrow L^A(\Omega)$, are continuous and compact, see Lemma 2.8. Consequently, $fK(g) \in L^1(\Omega)$, $gK(f) \in L^1(\Omega)$, see the 2nd Holder's inequality (5.5) in Proposition 5.10. Furthermore, it follows from the definition of K , that

$$\int_{\Omega} fK(g) \, dx = \int_{\Omega} \nabla K(f) \cdot \nabla K(g) \, dx = \int_{\Omega} gK(f) \, dx. \quad (2.6)$$

For convenience, let us extend a , \tilde{a} , b , \tilde{b} in the whole \mathbb{R} as odd functions, and extend A , \tilde{A} , B and \tilde{B} as even functions. We next introduce the C^1 -functional $J : X \rightarrow \mathbb{R}$ defined by

$$J(f, g) := \int_{\Omega} \tilde{A}(f) \, dx + \int_{\Omega} \tilde{B}(g) \, dx - \frac{1}{2} \int_{\Omega} (fK(g) + gK(f)) \, dx \quad (2.7)$$

whose derivative at $(f, g) \in X$ is equal to

$$J'(f, g)[\psi_1, \psi_2] = \left(\int_{\Omega} \tilde{a}(f)\psi_1 \, dx + \int_{\Omega} \tilde{b}(g)\psi_2 \, dx \right) - \left(\int_{\Omega} \psi_1 K(g) + \int_{\Omega} \psi_2 K(f) \, dx \right),$$

for all $(\psi_1, \psi_2) \in X$.

Trivially, the Eq. (2.4) are the Euler–Lagrange equations associated to the action functional J .

2.2. On the Nonlinearities

Lemma 2.1. (i) *There exists two constants $c_1, c_2 > 0$, only dependent of p , and α , such that*

$$c_1 t^{\frac{1}{p}} (\ln(e+t))^{\frac{\alpha}{p}} \leq \tilde{a}(t) \leq c_2 t^{\frac{1}{p}} (\ln(e+t))^{\frac{\alpha}{p}}, \quad \text{for all } t \geq 0. \quad (2.8)$$

(ii) *There exists two constants $C_1, C_2 > 0$ only dependent of p , and α , such that*

$$C_1 t^{\frac{p+1}{p}} (\ln(e+t))^{\frac{\alpha}{p}} \leq \tilde{A}(t) \leq C_2 t^{\frac{p+1}{p}} ((\ln(e+t))^{\frac{\alpha}{p}}), \quad \text{for all } t \geq 0.$$

Proof of Lemma 2.1. (i) By definition,

$$a\left(t(\ln(e+t))^{\frac{\alpha}{p}}\right) = t^p (\ln(e+t))^{\alpha} \left[\ln\left(e + t(\ln(e+t))^{\frac{\alpha}{p}}\right) \right]^{-\alpha}.$$

Now, checking that

$$\lim_{t \rightarrow 0} \frac{\ln \left(e + t (\ln(e+t))^{\frac{\alpha}{p}} \right)}{\ln(e+t)} = 1, \quad \lim_{t \rightarrow \infty} \frac{\ln \left(e + t (\ln(e+t))^{\frac{\alpha}{p}} \right)}{\ln(e+t)} = 1,$$

and since that quotient of logarithms is a continuous, non vanishing function for $t \geq 0$, we can conclude that there exists two constants $d_1, d_2 > 0$ such that

$$d_1 t^p \leq a \left(t (\ln(e+t))^{\frac{\alpha}{p}} \right) \leq d_2 t^p, \quad \text{for all } t \geq 0,$$

and since \tilde{a} is increasing,

$$\tilde{a}(d_1 t^p) \leq t (\ln(e+t))^{\frac{\alpha}{p}} \leq \tilde{a}(d_2 t^p), \quad \text{for all } t \geq 0.$$

Now, firstly denoting by $s = d_1 t^p$, secondly using that

$$\lim_{t \rightarrow 0} \frac{\ln \left(e + t^{1/p} \right)}{\ln(e+t)} = 1, \quad \lim_{t \rightarrow \infty} \frac{\ln \left(e + t^{1/p} \right)}{\ln(e+t)} = 1,$$

and thirdly, denoting by $s = d_2 t^p$, we obtain (2.8).

(ii) Given $\gamma > 0$, and $\nu \geq -\gamma$, set

$$\sigma_{\gamma, \nu}(t) = \sigma(t) := t^\gamma (\ln(e+t))^\nu, \quad \text{for } t > 0,$$

then

$$\sigma'_{\gamma, \nu}(t) = \sigma'(t) = t^{\gamma-1} (\ln(e+t))^\nu \left[\gamma + \nu \frac{1}{\ln(e+t)} \frac{t}{(e+t)} \right].$$

Observe that

$$a(t) = \sigma_{p, -\alpha}(t),$$

and that

$$\lim_{t \rightarrow \infty} \frac{ta'(t)}{a(t)} = p. \quad (2.9)$$

Moreover, if $\nu \geq -\gamma$, then σ is increasing.

Set

$$h(t) := \frac{1}{\ln(e+t)} \frac{t}{(e+t)} > 0, \quad \text{for } t > 0.$$

Trivially, $\lim_{t \rightarrow 0} h(t) = 0$, $\lim_{t \rightarrow \infty} h(t) = 0$, and there exists a unique $t^* > 0$ such that

$$h(t) \leq h(t^*) = \frac{e}{e+t^*} < 1, \quad \forall t \geq 0.$$

Hence

$$\left(\gamma + \frac{|\nu|e}{e+t^*} \right)^{-1} \sigma'_{\gamma, \nu}(t) \leq t^{\gamma-1} (\ln(e+t))^\nu \leq \left(\gamma - \frac{|\nu|e}{e+t^*} \right)^{-1} \sigma'_{\gamma, \nu}(t), \quad (2.10)$$

and observe that whenever $\gamma > 1$, $t^{\gamma-1} (\ln(e+t))^\nu = \sigma_{\gamma-1, \nu}(t)$, so

$$\left(\gamma + \frac{|\nu|e}{e+t^*}\right)^{-1} \sigma'_{\gamma,\nu}(t) \leq \sigma_{\gamma-1,\nu}(t) \leq \left(\gamma - \frac{|\nu|e}{e+t^*}\right)^{-1} \sigma'_{\gamma,\nu}(t).$$

By definition of \tilde{A} (see (2.2)), using (2.8), and (2.10), we can write

$$\tilde{A}(t) = \int_0^t \tilde{a}(s) ds \leq c_2 \int_0^t s^{\frac{1}{p}} (\ln(e+s))^{\frac{\alpha}{p}} ds \leq C_2 \sigma_{\frac{p+1}{p}, \frac{\alpha}{p}}(t) = C_2 t^{\frac{p+1}{p}} (\ln(e+t))^{\frac{\alpha}{p}}.$$

Likewise is obtained the reverse inequality. \square

Remark 2.2. Likewise, there exists two constants $C'_1, C'_2 > 0$ only dependent of p , and α , such that

$$C'_1 t^{p+1} (\ln(e+t))^{-\alpha} \leq A(t) \leq C'_2 t^{p+1} ((\ln(e+t))^{-\alpha}), \quad \text{for all } t \geq 0.$$

In fact, from (2.10), for all $t \geq 0$,

$$\left(p+1 + \frac{|\alpha|e}{e+t^*}\right)^{-1} t a(t) \leq A(t) \leq \left(p+1 - \frac{|\alpha|e}{e+t^*}\right)^{-1} t a(t).$$

Lemma 2.3. (i) Assume that $\alpha \geq 0$. Then, for all $f \in L^{\tilde{A}}(\Omega)$, and for the constants $C_1, C_2 > 0$ provided by Lemma 2.1, the following hold

$$\int_{\Omega} \tilde{A}(f) dx \geq \frac{C_1}{C_2} \|f\|_{(\tilde{A})}^{\frac{p+1}{p}}, \quad \text{whenever } \|f\|_{(\tilde{A})} \geq 1, \quad (2.11)$$

$$\int_{\Omega} \tilde{A}(f) dx \leq \frac{C_2}{C_1} \|f\|_{(\tilde{A})}^{\frac{p+1}{p}}, \quad \text{whenever } \|f\|_{(\tilde{A})} \leq 1, \quad (2.12)$$

and

$$\int_{\Omega} \tilde{A}(f) dx \geq C_1 \|f\|_{\frac{p+1}{p}}^{\frac{p+1}{p}}. \quad (2.13)$$

(ii) Assume now that $\alpha < 0$. Then,

$$\int_{\Omega} \tilde{A}(f) dx \leq \frac{C_2}{C_1} \|f\|_{(\tilde{A})}^{\frac{p+1}{p}}, \quad \text{whenever } \|f\|_{(\tilde{A})} \geq 1, \quad (2.14)$$

$$\int_{\Omega} \tilde{A}(f) dx \geq \frac{C_1}{C_2} \|f\|_{(\tilde{A})}^{\frac{p+1}{p}}, \quad \text{whenever } \|f\|_{(\tilde{A})} \leq 1, \quad (2.15)$$

and if $f \in L^{\frac{p+1}{p}}(\Omega)$, then

$$\int_{\Omega} \tilde{A}(f) dx \leq C_2 \|f\|_{\frac{p+1}{p}}^{\frac{p+1}{p}}.$$

Remark 2.4. Likewise, for all $f \in L^A(\Omega)$, and for the constants $C'_1, C'_2 > 0$ provided by Remark 2.2, the following hold

(i) Assume that $\alpha \geq 0$. Then,

$$\int_{\Omega} A(f) dx \leq \frac{C'_2}{C'_1} \|f\|_{(A)}^{p+1}, \quad \text{whenever } \|f\|_{(A)} \geq 1,$$

and

$$\int_{\Omega} A(f) dx \geq \frac{C'_1}{C'_2} \|f\|_{(A)}^{p+1}, \quad \text{whenever } \|f\|_{(A)} \leq 1.$$

and if $f \in L^{p+1}(\Omega)$, then

$$\int_{\Omega} A(f) dx \leq C'_2 \|f\|_{p+1}^{p+1}. \quad (2.16)$$

(ii) Assume now that $\alpha < 0$. Then

$$\begin{aligned} \int_{\Omega} A(f) dx &\geq \frac{C'_1}{C'_2} \|f\|_{(A)}^{p+1}, & \text{whenever } \|f\|_{(A)} &\geq 1, \\ \int_{\Omega} A(f) dx &\leq \frac{C'_2}{C'_1} \|f\|_{(A)}^{p+1}, & \text{whenever } \|f\|_{(A)} &\leq 1. \end{aligned} \quad (2.17)$$

and

$$\int_{\Omega} A(f) dx \geq C'_1 \|f\|_{p+1}^{p+1}.$$

Proof of Lemma 2.3. Let

$$\tilde{A}_1(s) := s^{\frac{p+1}{p}} (\ln(e+s))^{\frac{\alpha}{p}}.$$

Since Lemma 2.1

$$C_1 \int_{\Omega} \tilde{A}_1(f) dx \leq \int_{\Omega} \tilde{A}(f) dx \leq C_2 \int_{\Omega} \tilde{A}_1(f) dx.$$

(i) Assume that $\alpha > 0$. If $\|f\|_{(\tilde{A})} \geq 1$, then by Lemma 5.5(iv)

$$\begin{aligned} 1 &= \int_{\Omega} \tilde{A} \left(\frac{|f|}{\|f\|_{(\tilde{A})}} \right) dx \leq C_2 \int_{\Omega} \left(\frac{|f|}{\|f\|_{(\tilde{A})}} \right)^{\frac{p+1}{p}} (\ln(e+|f|))^{\frac{\alpha}{p}} \left(\frac{\ln \left(e + \frac{|f|}{\|f\|_{(\tilde{A})}} \right)}{\ln(e+|f|)} \right)^{\frac{\alpha}{p}} \\ &\leq \frac{C_2}{\|f\|_{(\tilde{A})}^{\frac{p+1}{p}}} \int_{\Omega} |f|^{\frac{p+1}{p}} (\ln(e+|f|))^{\frac{\alpha}{p}} = \frac{C_2}{\|f\|_{(\tilde{A})}^{\frac{p+1}{p}}} \int_{\Omega} \tilde{A}_1(f) dx \leq \frac{C_2}{C_1} \frac{1}{\|f\|_{(\tilde{A})}^{\frac{p+1}{p}}} \int_{\Omega} \tilde{A}(f) dx. \end{aligned}$$

Consequently, (2.11) holds.

Moreover, for all $f \in L^{\tilde{A}}(\Omega)$, with $\|f\|_{(\tilde{A})} \leq 1$,

$$\begin{aligned} 1 &= \int_{\Omega} \tilde{A} \left(\frac{|f|}{\|f\|_{(\tilde{A})}} \right) dx \geq C_1 \int_{\Omega} \left(\frac{|f|}{\|f\|_{(\tilde{A})}} \right)^{\frac{p+1}{p}} (\ln(e+|f|))^{\frac{\alpha}{p}} \left(\frac{\ln \left(e + \frac{|f|}{\|f\|_{(\tilde{A})}} \right)}{\ln(e+|f|)} \right)^{\frac{\alpha}{p}} \\ &\geq \frac{C_1}{\|f\|_{(\tilde{A})}^{\frac{p+1}{p}}} \int_{\Omega} |f|^{\frac{p+1}{p}} (\ln(e+|f|))^{\frac{\alpha}{p}} = \frac{C_1}{\|f\|_{(\tilde{A})}^{\frac{p+1}{p}}} \int_{\Omega} \tilde{A}_1(f) dx \geq \frac{C_1}{C_2} \frac{1}{\|f\|_{(\tilde{A})}^{\frac{p+1}{p}}} \int_{\Omega} \tilde{A}(f) dx. \end{aligned}$$

So (2.12) holds.

Besides, for all $f \in L^{\tilde{A}}(\Omega)$,

$$\int_{\Omega} \tilde{A}(f) dx \geq C_1 \int_{\Omega} \tilde{A}_1(f) dx = C_1 \int_{\Omega} |f|^{\frac{p+1}{p}} (\ln(e+|f|))^{\frac{\alpha}{p}} dx \geq C_1 \|f\|_{\frac{p+1}{p}}^{\frac{p+1}{p}}.$$

(ii) Assume now that $\alpha < 0$, and let us write now $\tilde{A}_2(s) := s^{\frac{p+1}{p}} (\ln(e+s))^{-\frac{|\alpha|}{p}}$. Assume now $\|f\|_{(\tilde{A})} \leq 1$, then

$$\begin{aligned} 1 &= \int_{\Omega} \tilde{A} \left(\frac{|f|}{\|f\|_{(\tilde{A})}} \right) dx \leq C_2 \int_{\Omega} \left(\frac{|f|}{\|f\|_{(\tilde{A})}} \right)^{\frac{p+1}{p}} (\ln(e+|f|))^{-\frac{|\alpha|}{p}} \left(\frac{\ln(e+|f|)}{\ln\left(e + \frac{|f|}{\|f\|_{(\tilde{A})}}\right)} \right)^{\frac{|\alpha|}{p}} \\ &\leq \frac{C_2}{\|f\|_{(\tilde{A})}^{\frac{p+1}{p}}} \int_{\Omega} |f|^{\frac{p+1}{p}} (\ln(e+|f|))^{-\frac{|\alpha|}{p}} = \frac{C_2}{\|f\|_{(\tilde{A})}^{\frac{p+1}{p}}} \int_{\Omega} \tilde{A}_1(f) dx \leq \frac{C_2}{C_1} \frac{1}{\|f\|_{(\tilde{A})}^{\frac{p+1}{p}}} \int_{\Omega} \tilde{A}(f) dx. \end{aligned}$$

So (2.15) holds.

Moreover, for all $f \in L^{\tilde{A}}(\Omega)$, with $\|f\|_{(\tilde{A})} \geq 1$,

$$\begin{aligned} 1 &= \int_{\Omega} \tilde{A} \left(\frac{|f|}{\|f\|_{(\tilde{A})}} \right) dx \geq C_1 \int_{\Omega} \left(\frac{|f|}{\|f\|_{(\tilde{A})}} \right)^{\frac{p+1}{p}} (\ln(e+|f|))^{-\frac{|\alpha|}{p}} \left(\frac{\ln(e+|f|)}{\ln\left(e + \frac{|f|}{\|f\|_{(\tilde{A})}}\right)} \right)^{\frac{|\alpha|}{p}} \\ &\geq \frac{C_1}{\|f\|_{(\tilde{A})}^{\frac{p+1}{p}}} \int_{\Omega} |f|^{\frac{p+1}{p}} (\ln(e+|f|))^{-\frac{|\alpha|}{p}} = \frac{C_1}{\|f\|_{(\tilde{A})}^{\frac{p+1}{p}}} \int_{\Omega} \tilde{A}_1(f) dx \geq \frac{C_1}{C_2} \frac{1}{\|f\|_{(\tilde{A})}^{\frac{p+1}{p}}} \int_{\Omega} \tilde{A}(f) dx. \end{aligned}$$

Consequently, (2.14) holds.

Finally, Lemma 2.1(ii) ends the proof. \square

2.3. On the Space $X = L^{\tilde{A}}(\Omega) \times L^{\tilde{B}}(\Omega)$

We prove in this section that X is reflexive.

To do so, in Proposition 5.10(v) is stated that we only need to check that the \mathcal{N} -functions satisfy the Δ_2 -condition at infinity (see (5.1) for a definition of the Δ_2 -condition at infinity).

Proposition 2.5. *$A, B, \tilde{A}, \tilde{B}$ satisfy the Δ_2 -condition, and X is a reflexive Banach space. Moreover, the dual space*

$$X' := \left(L^{\tilde{A}}(\Omega) \times L^{\tilde{B}}(\Omega) \right)' = \left(L^{\tilde{A}}(\Omega) \right)' \times \left(L^{\tilde{B}}(\Omega) \right)' = L^A(\Omega) \times L^B(\Omega) \quad (2.18)$$

Proof. From Lemma 5.5 and Proposition 5.10(i), the spaces $L^{\tilde{A}}(\Omega)$ and $L^{\tilde{B}}(\Omega)$ are complete with both the Orlicz or the Luxemburg norm. From Proposition 5.10(v) it is enough to check that $A, B, \tilde{A}, \tilde{B}$ satisfy the Δ_2 -condition. To do that, we apply Lemma 5.4 (iii), see condition (5.2).

Indeed, notice that by l'Hôpital rule

$$\lim_{t \rightarrow +\infty} \frac{ta(t)}{A(t)} = \lim_{t \rightarrow +\infty} \frac{ta'(t)}{a(t)} + 1 = p + 1, \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{tb(t)}{B(t)} = q + 1,$$

see (2.9) in the proof of Lemma 2.1. Now, changing the the variable $s = \tilde{a}(t)$, using Young equality (3.1), and l'Hôpital rule

$$\lim_{t \rightarrow +\infty} \frac{t\tilde{a}(t)}{\tilde{A}(t)} = \lim_{s \rightarrow +\infty} \frac{sa(s)}{\tilde{A}(a(s))} = \lim_{s \rightarrow +\infty} \frac{sa(s)}{sa(s) - A(s)} = \lim_{s \rightarrow +\infty} \frac{a(s) + sa'(s)}{sa'(s)} = \frac{1}{p} + 1,$$

see also (2.9) in the proof of Lemma 2.1. Likewise,

$$\lim_{t \rightarrow +\infty} \frac{t\tilde{b}(t)}{\tilde{B}(t)} = \lim_{s \rightarrow +\infty} \frac{sb(s)}{\tilde{B}(b(s))} = \lim_{s \rightarrow +\infty} \frac{sb(s)}{sb(s) - B(s)} = \lim_{s \rightarrow +\infty} \frac{b(s) + sb'(s)}{sb'(s)} = \frac{1}{q} + 1.$$

Finally, Proposition 5.10 (iv) and the fact that $\widetilde{(\tilde{A})} = A$, and $\widetilde{(\tilde{A})} = A$ ends the proof. \square

in order to prove some geometric properties of the functional J , we will made use of the following continuous embeddings:

Lemma 2.6. *If $\alpha, \beta \geq 0$,*

$$L^{\tilde{A}}(\Omega) \hookrightarrow L^{\frac{p+1}{p}}(\Omega) \text{ and } L^{\tilde{B}}(\Omega) \hookrightarrow L^{\frac{q+1}{q}}(\Omega), \quad (2.19)$$

and if $\alpha < 0$, (respectively, $\beta < 0$), for all $\varepsilon > 0$ small enough,

$$L^{\tilde{A}}(\Omega) \hookrightarrow L^{\frac{p+1}{p}-\varepsilon}(\Omega), \quad (\text{respectively, } L^{\tilde{B}}(\Omega) \hookrightarrow L^{\frac{q+1}{q}-\varepsilon}(\Omega)). \quad (2.20)$$

Likewise, if $\alpha, \beta \geq 0$, for all $\varepsilon > 0$ small enough,

$$L^A(\Omega) \hookrightarrow L^{p+1-\varepsilon}(\Omega) \text{ and } L^B(\Omega) \hookrightarrow L^{q+1-\varepsilon}(\Omega), \quad (2.21)$$

and if $\alpha < 0$, (respectively, $\beta < 0$),

$$L^A(\Omega) \hookrightarrow L^{p+1}(\Omega), \quad (\text{respectively, } L^B(\Omega) \hookrightarrow L^{q+1}(\Omega)). \quad (2.22)$$

Proof. We remark that whenever $\alpha, \beta \geq 0$, (2.19) holds using that

$$a(t) \leq t^p \implies \tilde{a}(t) \geq t^{1/p} \implies \tilde{A}(t) \geq \frac{p}{p+1} t^{\frac{p+1}{p}}. \quad (2.23)$$

Moreover, as for all $u \in L^{\tilde{A}}(\Omega)$, $u \neq 0$, we have from (5.6) in Proposition 5.10(iii),

$$1 \geq \int_{\Omega} \tilde{A}\left(\frac{u}{\|u\|_{\tilde{A}}}\right) dx \geq \frac{p}{p+1} \int_{\Omega} \left| \frac{u}{\|u\|_{\tilde{A}}} \right|^{\frac{p+1}{p}} dx = \frac{p}{p+1} \frac{\|u\|_{\tilde{A}}^{\frac{p+1}{p}}}{\|u\|_{\tilde{A}}^{\frac{p+1}{p}}},$$

so, the embedding (2.20) is proved.

Besides, if $\alpha < 0$, (respectively, $\beta < 0$), for all $\varepsilon > 0$ small enough, (2.20) holds. Indeed, fix $\varepsilon > 0$ small, there exists $c = c_{\varepsilon} > 0$ such that

$$a(t) \leq \begin{cases} ct^p & \text{if } t \leq 1 \\ ct^{p+\varepsilon} & \text{if } t \geq 1, \end{cases}$$

then

$$\tilde{a}(s) \geq \begin{cases} c^{-1/p} s^{1/p} & \text{if } s \leq c \\ c^{-1/(p+\varepsilon)} s^{1/(p+\varepsilon)} & \text{if } s \geq c, \end{cases}$$

and

$$\tilde{A}(s) \geq \begin{cases} \frac{p}{p+1} c^{-1/p} s^{(p+1)/p} & \text{if } s \leq c \\ \frac{p+\varepsilon}{p+\varepsilon+1} c^{-1/(p+\varepsilon)} s^{(p+\varepsilon+1)/(p+\varepsilon)} + \left(\frac{p}{p+1} - \frac{p+\varepsilon}{p+\varepsilon+1}\right)c & \text{if } s \geq c. \end{cases}$$

In particular

$$\tilde{A}(s) \geq C_1 s^{\frac{p+1}{p}-\varepsilon'} - C_2,$$

for

$$C_1 := \frac{p + \varepsilon}{p + \varepsilon + 1} c^{-1/(p+\varepsilon)}, \quad C_2 := \max \left\{ \frac{p + \varepsilon}{p + \varepsilon + 1} c, \frac{\varepsilon c}{(p + \varepsilon + 1)(p + 1)} \right\},$$

$$\varepsilon' = \frac{\varepsilon}{p(p + \varepsilon)}.$$

with $C_1, C_2 > 0$. From the above and by definition, if $u \in L^{\tilde{A}}(\Omega)$, $u \neq 0$, then putting $v = \frac{u}{\|u\|_{\tilde{A}}}$

$$1 \geq \int_{\Omega} \tilde{A}(v(x)) dx \geq \int_{\Omega} C_1 |v|^{\frac{p+1}{p}-\varepsilon'} dx - C_2 |\Omega|,$$

so,

$$(1 + C_2 |\Omega|) \|u\|_{\tilde{A}}^{\frac{p+1}{p}-\varepsilon'} \geq C_1 \|u\|_{\tilde{A}}^{\frac{p+1}{p}-\varepsilon'}$$

and the embedding (2.20) is proved.

The proofs of (2.21) and (2.22) are similar. \square

The following Lemma is a technical one, that will be useful to prove Proposition 2.10.

Lemma 2.7. *Let a , A , \tilde{a} , \tilde{A} be defined by (2.1), (2.2), (2.3), and (2.5) respectively. Here \sim means equivalent at infinity, see Remark 5.15. Then*

- (i) *A is equivalent to $\tilde{A} \circ a$ at infinity, ($A \sim \tilde{A} \circ a$), and $L^A(\Omega)$ is isomorphic to $L^{\tilde{A} \circ a}(\Omega)$.*
- (ii) *\tilde{A} is equivalent to $A \circ \tilde{a}$ at infinity, ($\tilde{A} \sim A \circ \tilde{a}$), and $L^{\tilde{A}}(\Omega)$ is isomorphic to $L^{A \circ \tilde{a}}(\Omega)$.*

Proof. (i) Clearly, since Lemma 2.1, and Proposition 5.17, for any $\psi \in L^A(\Omega)$,

$$A(\psi) = A(|\psi|) \sim |\psi|^{p+1} (\ln |\psi|)^{-\alpha}, \quad \text{and} \quad \int_{\Omega} |\psi|^{p+1} (\ln |\psi|)^{-\alpha} dx \leq C.$$

Moreover,

$$a(\psi) \sim |\psi|^p (\ln |\psi|)^{-\alpha},$$

$$\tilde{A}(a(\psi)) \sim (|\psi|^p (\ln |\psi|)^{-\alpha})^{\frac{p+1}{p}} (\ln |\psi|)^{\frac{\alpha}{p}} = |\psi|^{p+1} (\ln |\psi|)^{-\alpha},$$

so $A \sim \tilde{A} \circ a$ at infinity, and

$$\int_{\Omega} \tilde{A}(a(\psi)) dx \leq C, \quad \text{consequently} \quad a(\psi) \in L^{\tilde{A}}(\Omega).$$

(ii) Likewise, for any $\psi \in L^{\tilde{A}}(\Omega)$,

$$\tilde{A}(\psi) = \tilde{A}(|\psi|) \sim |\psi|^{\frac{p+1}{p}} (\ln |\psi|)^{\frac{\alpha}{p}}, \quad \text{and} \quad \int_{\Omega} |\psi|^{\frac{p+1}{p}} (\ln |\psi|)^{\frac{\alpha}{p}} dx \leq C.$$

Moreover,

$$\tilde{a}(\psi) \sim |\psi|^{\frac{1}{p}} (\ln |\psi|)^{\frac{\alpha}{p}},$$

and

$$A(\tilde{a}(\psi)) \sim \left(|\psi|^{\frac{1}{p}} (\ln |\psi|)^{\frac{\alpha}{p}} \right)^{p+1} (\ln |\psi|)^{-\alpha} = |\psi|^{\frac{p+1}{p}} (\ln |\psi|)^{\frac{\alpha}{p}},$$

so $\tilde{A} \sim A \circ \tilde{a}$ at infinity, and

$$\int_{\Omega} A(\tilde{a}(\psi)) dx \leq C, \quad \text{consequently } \tilde{a}(\psi) \in L^A(\Omega).$$

□

2.4. On the Functional J

Let us write $J' = \Phi - \Upsilon$, where Φ and Υ are defined, for all $(f, g) \in X$ and for all $(\psi_1, \psi_2) \in X$, as

$$\Phi(f, g)[\psi_1, \psi_2] := \int_{\Omega} \tilde{a}(f)\psi_1 dx + \int_{\Omega} \tilde{b}(g)\psi_2 dx, \quad (2.24)$$

$$\Upsilon(f, g)[\psi_1, \psi_2] := \int_{\Omega} \psi_1 K(g) + \int_{\Omega} \psi_2 K(f) dx. \quad (2.25)$$

We will see that Υ is a continuous compact operator, see Proposition 2.9, and that Φ is an homeomorphism, see Proposition 2.10.

We start by proving that Υ is a compact operator. Let us recall that, by using the regularity result Theorem 5.29, $(K(f), K(g)) \in W^{2, \tilde{A}}(\Omega) \times W^{2, \tilde{B}}(\Omega)$ for all $(f, g) \in X$. Furthermore, we have

Lemma 2.8. *Let $p, q > 0$, $\alpha, \beta \in \mathbb{R}$ satisfying either (1.2) or (1.3). Then, the embeddings $W^{2, \tilde{A}}(\Omega) \hookrightarrow L^B(\Omega)$ and $W^{2, \tilde{B}}(\Omega) \hookrightarrow L^A(\Omega)$ are compact.*

Proof. We use Theorem 5.27. Assume first that $\frac{p+1}{p} \geq \frac{N}{2}$. In particular, we observe that if $\frac{p+1}{p} > \frac{N}{2}$, or $\frac{p+1}{p} = \frac{N}{2}$ and $\alpha > (1 - \frac{1}{N})p \left(N - \frac{p+1}{p} \right)$, then $W^{2, \tilde{A}}(\Omega) \hookrightarrow C_b(\Omega)$ is compact, where $C_b(\Omega) = C(\Omega) \cap L^\infty(\Omega)$. Hence the embedding

$$W^{2, \tilde{A}}(\Omega) \hookrightarrow L^B(\Omega)$$

is also compact.

Moreover, if $\frac{p+1}{p} = \frac{N}{2}$ and $\alpha \leq (1 - \frac{1}{N})p \left(N - \frac{p+1}{p} \right)$, then

$$W^{2, \tilde{A}}(\Omega) \hookrightarrow L^{A_1}(\Omega)$$

is compact for all \mathcal{N} -function A_1 such that

$$A_1 \ll (\tilde{A}^*)^*,$$

where \ll is defined in Definition 5.14(iii), and $(\tilde{A}^*)^*$, at least of exponential type, is defined by (5.13). Let us verify that $B \ll (\tilde{A}^*)^*$ using that $B(s) \sim s^{q+1} (\ln s)^{-\beta}$, see Lemma 2.1 and Example 5.18. Indeed, for all $c > 0$ fixed, we have that

$$\lim_{s \rightarrow +\infty} \frac{B(s)}{(\tilde{A}^*)^*(cs)} = 0,$$

holds trivially.

Likewise, if $\frac{q+1}{p} \geq \frac{N}{2}$, then the embedding $W^{2,\tilde{B}}(\Omega) \hookrightarrow L^A(\Omega)$, where $A(s) \sim s^{p+1}(\ln s)^{-\alpha}$, is compact.

Let us now study the situation appearing when $\frac{p+1}{p} < \frac{N}{2}$. We now remark (see Lemma 2.1 and Example 5.25) that for $\frac{p+1}{p} < \frac{N}{2}$,

$$\begin{aligned} \tilde{A}(s) \sim s^{\frac{p+1}{p}} (\ln s)^{\frac{\alpha}{p}}, \quad \text{so then } W^{2,\tilde{A}}(\Omega) \hookrightarrow L^{A_1}(\Omega) \quad \forall A_1 \ll ((\tilde{A}^*)^*)^*, \\ \text{with } ((\tilde{A}^*)^*)^* \sim s^{\frac{N(p+1)}{Np-2(p+1)}} [\log(s)]^{\frac{\alpha N}{Np-2(p+1)}}. \end{aligned}$$

Let us verify that $B \ll ((\tilde{A}^*)^*)^*$, in other words, that for all $\delta > 0$,

$$\lim_{s \rightarrow +\infty} \frac{((\tilde{A}^*)^*)^*(\delta s)}{B(s)} = +\infty.$$

Indeed,

$$\frac{((\tilde{A}^*)^*)^*(\delta s)}{B(s)} \sim s^{\frac{N(p+1)}{Np-2(p+1)} - (q+1)} [\log(s)]^{\frac{\alpha N}{Np-2(p+1)} + \beta} \rightarrow \infty,$$

if

$$\frac{N(p+1)}{Np-2(p+1)} > q+1, \quad \text{or} \quad \frac{N(p+1)}{Np-2(p+1)} = q+1, \quad \text{and} \quad \frac{\alpha N}{Np-2(p+1)} + \beta > 0,$$

i.e. if

$$\frac{p}{p+1} - \frac{2}{N} < \frac{1}{q+1}, \quad \text{or} \quad \frac{N}{Np-2(p+1)} = \frac{q+1}{p+1}, \quad \text{and} \quad \frac{\alpha}{p+1} + \frac{\beta}{q+1} > 0.$$

This last inequalities are equivalent to

$$\frac{N-2}{N} - \frac{1}{p+1} < \frac{1}{q+1}, \quad \text{or} \quad \frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}, \quad \text{and} \quad \frac{\alpha}{p+1} + \frac{\beta}{q+1} > 0.$$

The first inequality holds under hypothesis (1.2), the second one holds by hypothesis (1.3).

Likewise if $\frac{q+1}{q} < \frac{N}{2}$, then the embedding $W^{2,\tilde{B}}(\Omega) \hookrightarrow L^A(\Omega)$ is compact, and the proof is achieved. \square

Proposition 2.9. *The linear operator $\Upsilon : X \rightarrow X'$ defined by*

$$\Upsilon(f, g)[\psi_1, \psi_2] := \int_{\Omega} \psi_1 \mathbb{K}(g) dx + \int_{\Omega} \psi_2 \mathbb{K}(f) dx, \quad \forall (\psi_1, \psi_2) \in X$$

is a continuous compact map.

Proof. Let us prove the continuity. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in $L^{\tilde{A}}(\Omega)$ converging to some $f \in L^{\tilde{A}}(\Omega)$ and $\{g_n\}_{n \in \mathbb{N}}$ a sequence in $L^{\tilde{B}}(\Omega)$ converging to some $g \in L^{\tilde{B}}(\Omega)$. Let $\psi_1 \in L^{\tilde{A}}(\Omega)$. We have

$$\begin{aligned} & |\Upsilon(f_n, g_n)[\psi_1, \psi_2] - \Upsilon(f, g)[\psi_1, \psi_2]| \\ & \leq \int_{\Omega} \left(|\mathbb{K}(f_n)\psi_2 - \mathbb{K}(f)\psi_2| + |\mathbb{K}(g_n)\psi_1 - \mathbb{K}(g)\psi_1| \right) dx \\ & \leq C \left(\|\mathbb{K}(f_n) - \mathbb{K}(f)\|_B \|\psi_2\|_{\tilde{B}} + \|\mathbb{K}(g_n) - \mathbb{K}(g)\|_A \|\psi_1\|_{\tilde{A}} \right) \end{aligned}$$

We only have to prove that $\|\mathbb{K}(f_n) - \mathbb{K}(f)\|_B \rightarrow 0$, the proof of $\|\mathbb{K}(g_n) - \mathbb{K}(g)\|_A \rightarrow 0$ is analogous. This result follows from the regularity results of Theorem 5.29 and the above Lemma 2.8.

Let us now prove the compactness. Let $\{f_n\}_{n \in \mathbb{N}}$ be a bounded sequence in $L^{\tilde{A}}(\Omega)$ and $\{g_n\}_{n \in \mathbb{N}}$ a bounded sequence in $L^{\tilde{B}}(\Omega)$. From (5.17) we have

$$\|\mathbb{K}(f_n)\|_{W^{2, \tilde{A}}} \leq C, \quad \|\mathbb{K}(g_n)\|_{W^{2, \tilde{B}}} \leq C$$

for some constant C independent of n . Then, using Lemma 2.8 and the fact that X is reflexive (cf. Proposition 2.5, and Proposition 5.10) we infer the existence of $g \in L^{\tilde{B}}(\Omega)$ and $f \in L^{\tilde{A}}(\Omega)$ such that, up to a subsequence, $\|\mathbb{K}(f_n) - g\|_B \rightarrow 0$, $\|\mathbb{K}(g_n) - f\|_A \rightarrow 0$ as $n \rightarrow \infty$. Hence, using the 2nd Hölder's inequality, for any $(\psi_1, \psi_2) \in X$, defining $\mathcal{I}_{f,g}[\psi_1, \psi_2] := \int_{\Omega} (f\psi_1 + g\psi_2) dx$,

$$\begin{aligned} |\Upsilon(f_n, g_n)[\psi_1, \psi_2] - \mathcal{I}_{f,g}[\psi_1, \psi_2]| & \leq \int_{\Omega} \left(|\mathbb{K}(f_n)\psi_2 - g\psi_2| + |\mathbb{K}(g_n)\psi_1 - f\psi_1| \right) dx \\ & \leq C \left(\|\mathbb{K}(f_n) - g\|_B \|\psi_2\|_{\tilde{B}} + \|\mathbb{K}(g_n) - f\|_A \|\psi_1\|_{\tilde{A}} \right) \end{aligned}$$

so then $\|\Upsilon(f_n, g_n) - \mathcal{I}_{f,g}\|_{X'} \rightarrow 0$ as $n \rightarrow \infty$. \square

We end this section, proving that Φ is an homeomorphism.

Proposition 2.10. *The non-linear operator $\Phi : X \rightarrow X'$ defined by*

$$\Phi(f, g)[\psi_1, \psi_2] := \int_{\Omega} \tilde{a}(f)\psi_1 dx + \int_{\Omega} \tilde{b}(g)\psi_2 dx, \quad \text{for all } (\psi_1, \psi_2) \in X.$$

is an homeomorphism.

Proof. To prove that $\Phi : X \rightarrow X'$ is an homeomorphism, we use Proposition 5.10(iii)-(iv), Remark 5.7, and Lemma 2.1 and Example 5.18.

Step 1. Φ is injective.

Assume that $\Phi(f_1, g_1) = \Phi(f_2, g_2)$, so for any $(\psi_1, \psi_2) \in X$

$$\begin{aligned} 0 & = (\Phi(f_1, g_1) - \Phi(f_2, g_2))[\psi_1, \psi_2] \\ & = \int_{\Omega} [\tilde{a}(f_1) - \tilde{a}(f_2)]\psi_1 dx + \int_{\Omega} [\tilde{b}(g_1) - \tilde{b}(g_2)]\psi_2 dx, \end{aligned}$$

then

$$\tilde{a}(f_1) = \tilde{a}(f_2), \quad \tilde{b}(g_1) = \tilde{b}(g_2) \quad \text{a.e. } x \in \Omega.$$

or equivalently, since $\tilde{a} := a^{-1}$ and $\tilde{b} := b^{-1}$,

$$f_1 = f_2, \quad g_1 = g_2 \quad \text{a.e. } x \in \Omega.$$

Step 2. Φ is surjective.

Let us check that for any $(\psi_1, \psi_2) \in X' = L^A(\Omega) \times L^B(\Omega)$ (see (2.18)), $(a(\psi_1), b(\psi_2)) \in X$, and

$$\Phi(a(\psi_1), b(\psi_2)) = (\psi_1, \psi_2). \quad (2.26)$$

Since Lemma 2.7, $a(\psi_1) \in L^{\tilde{A}}(\Omega)$. Likewise $b(\psi_2) \in L^{\tilde{B}}(\Omega)$, and $(a(\psi_1), b(\psi_2)) \in X$.

Since $\tilde{a} := a^{-1}$ and $\tilde{b} := b^{-1}$, it is straightforward to check that (2.26) holds.

Consequently,

$$\Phi^{-1}(\psi_1, \psi_2) = (a(\psi_1), b(\psi_2)).$$

Step 3. Φ is continuous.

Let $(f_n, g_n) \rightarrow (f, g)$ be a convergent sequence in $L^{\tilde{A}}(\Omega) \times L^{\tilde{B}}(\Omega)$, in other words

$$\|f_n - f\|_{\tilde{A}} + \|g_n - g\|_{\tilde{B}} \rightarrow 0.$$

Claim.

$$\|f_n - f\|_{\tilde{A}} \rightarrow 0 \implies \|\tilde{a}(f_n) - \tilde{a}(f)\|_A \rightarrow 0.$$

Once proved the claim

$$\|\tilde{a}(f_n) - \tilde{a}(f)\|_A + \|\tilde{b}(g_n) - \tilde{b}(g)\|_B \rightarrow 0.$$

Then, since Proposition 5.10(i)-(ii), and Remark 5.11, for all $(\psi_1, \psi_2) \in X$,

$$\begin{aligned} & \left| (\Phi(f_n, g_n) - \Phi(f, g))[\psi_1, \psi_2] \right| \\ &= \left| \int_{\Omega} [\tilde{a}(f_n) - \tilde{a}(f)] \psi_1 \, dx + \int_{\Omega} [\tilde{b}(g_n) - \tilde{b}(g)] \psi_2 \, dx \right| \\ &\leq \|\tilde{a}(f_n) - \tilde{a}(f)\|_A \|\psi_1\|_{\tilde{A}} + \|\tilde{b}(g_n) - \tilde{b}(g)\|_B \|\psi_2\|_{\tilde{B}} \rightarrow 0, \end{aligned}$$

and the continuity is achieved.

Proof of the Claim.

Since Theorem 5.13, to conclude the claim, we need to check that $\tilde{a}(f_n) \rightarrow \tilde{a}(f)$ in A -mean, in other words

$$\int_{\Omega} A(|\tilde{a}(f_n) - \tilde{a}(f)|) \, dx \rightarrow 0.$$

On the one hand since Lemma 2.7, $\tilde{A} \sim A \circ \tilde{a}$ at infinity, and $L^{\tilde{A}}(\Omega)$ is isomorphic to $L^A \circ \tilde{a}(\Omega)$, so, for any $f \in L^{\tilde{A}}(\Omega)$

$$\int_{\Omega} A(\tilde{a}(f)) \, dx \leq C, \quad \text{and consequently } \tilde{a}(f) \in L^A(\Omega).$$

Likewise, $\{\tilde{a}(f_n)\} \subset L^A(\Omega)$.

On the other hand since Theorem 5.13, $f_n \rightarrow f$ in \tilde{A} -mean, so

$$\int_{\Omega} \tilde{A}(f_n - f) dx = \int_{\Omega} \tilde{A}(|f_n - f|) dx \rightarrow 0. \quad (2.27)$$

Assume by contradiction that the claim do not hold. Then, there exist $\varepsilon_0 > 0$ and a subsequence $\{f_{n_k}\}_k$ satisfying

$$\int_{\Omega} A(\tilde{a}(f_{n_k}) - \tilde{a}(f)) dx > \varepsilon_0 \quad \text{for all } k \in \mathbb{N}. \quad (2.28)$$

Put $z_k := \tilde{A}(|f_{n_k} - f|)$. Since (2.27), $z_k \rightarrow 0$ in $L^1(\Omega)$, so by the Lebesgue reverse dominated converge theorem, there exists a subsequence $\{z_{k_j}\}_j$ and there exists $h \in L^1(\Omega)$ such that

$$z_{k_j} \leq h, \quad z_{k_j} \rightarrow 0 \text{ a.e.}, \text{ and also } \tilde{A}^{-1}z_{k_j} = f_{n_{k_j}} - f \rightarrow 0 \text{ a.e.}$$

We have

$$|f_{n_{k_j}}| \leq \tilde{A}^{-1}(h) + |f| \in L^{\tilde{A}}(\Omega).$$

By monotonicity of \tilde{a} , and since Lemma 2.7, $\tilde{A} \sim A \circ \tilde{a}$ at infinity, so

$$\tilde{a}(f_{n_{k_j}}) \leq \tilde{a}\left(\tilde{A}^{-1}(h) + |f|\right) := \ell \in L^A(\Omega),$$

moreover

$$A\left(|\tilde{a}(f_{n_{k_j}}) - \tilde{a}(f)|\right) \leq A(\ell + \tilde{a}(f)) \in L^1(\Omega).$$

Since pointwise convergence $A\left(|\tilde{a}(f_{n_{k_j}}) - \tilde{a}(f)|\right) \rightarrow 0$ a.e., and from Lebesgue dominate convergence theorem we conclude that

$$\int_{\Omega} A\left(|\tilde{a}(f_{n_{k_j}}) - \tilde{a}(f)|\right) dx \rightarrow 0,$$

which contradicts (2.28). This concludes the proof of the claim. \square

3. Proof of Theorem 1.1

First we star by proving that J satisfies the (PS) condition. To this aim we will use of the following (trivial) lemma:

Lemma 3.1. [13, Lemma 3.1] *Let X be a reflexive Banach space and $F \in C^1(X)$ be such that*

- (1) *any Palais–Smale sequence of F is bounded;*
- (2) *for all $u \in X$,*

$$F'(u) = L(u) - T(u),$$

where $L : X \rightarrow X'$ is a homeomorphism and $T : X \rightarrow X'$ is a continuous compact map.

Then, F satisfies the Palais–Smale condition.

Proposition 3.2. *J satisfies the (PS) condition.*

Proof. Let us first check (2) for the space $X = L^{\tilde{A}}(\Omega) \times L^{\tilde{B}}(\Omega)$ and the functional $F = J$ defined (2.7). Observe that X is a reflexive Banach space by Proposition 2.5 and $J \in C^1(X; \mathbb{R})$ is such that $J' = \Phi - \Upsilon$ where Φ is defined in (2.24) and Υ is defined in (2.25). Thus, (2) follows from Propositions 2.5 and 2.9.

Now we check (1). Let $\{(f_n, g_n)\}_{n \in \mathbb{N}}$ be a Palais–Smale sequence for the functional J , that is, we have for some $C > 0$ and some sequence of positive real numbers $\{\varepsilon_n\}_{n \in \mathbb{N}} \rightarrow 0$,

$$(PS1): \quad J(f_n, g_n) \leq C \text{ for all } n \in \mathbb{N},$$

$$(PS2): \quad |J'(f_n, g_n)[\psi_1, \psi_2]| \leq \varepsilon_n \|(\psi_1, \psi_2)\|_X \quad \forall (\psi_1, \psi_2) \in X.$$

Since (2.6), for every $(f, g) \in X$, $\theta \in \mathbb{R}$ we have

$$\frac{1}{2} \Upsilon(f, g)[f, g] = \frac{1}{2} \left[\int_{\Omega} f K(g) + \int_{\Omega} g K(f) dx \right] = \Upsilon(f, g) [\theta f, (1 - \theta)g],$$

and since the identity

$$\begin{aligned} & J(f_n, g_n) - J'(f_n, g_n) [\theta f, (1 - \theta)g] \\ &= \int_{\Omega} (\tilde{A}(f_n) + \tilde{B}(g_n) - \theta f_n \tilde{a}(f_n) - (1 - \theta)g_n \tilde{b}(g_n)) dx \end{aligned}$$

it follows from (PS1)-(PS2),

$$\int_{\Omega} (\tilde{A}(f_n) - \theta f_n \tilde{a}(f_n)) dx + \int_{\Omega} (\tilde{B}(g_n) - (1 - \theta)g_n \tilde{b}(g_n)) dx \leq \varepsilon_n \|(f_n, g_n)\|_X + 2C.$$

Assume by contradiction that $\|(f_n, g_n)\|_X \rightarrow +\infty$. Using the Young equality, see Proposition 5.8 (iii),

$$\tilde{A}(t) = t\tilde{a}(t) - A(\tilde{a}(t)) \tag{3.1}$$

Integrating by parts $A(t) = \int_0^t \frac{s^p}{(\ln(e+s))^\alpha} ds$, we have that

$$A(t) = \frac{t a(t)}{p+1} + \frac{\alpha}{p+1} \int_0^t \frac{s^{p+1}}{(\ln(e+s))^{\alpha+1}} \frac{ds}{e+s},$$

and consequently,

$$A(\tilde{a}(t)) = \frac{1}{p+1} t \tilde{a}(t) + \frac{\alpha}{p+1} \int_0^{\tilde{a}(t)} \frac{s^{p+1}}{(\ln(e+s))^{\alpha+1}} \frac{ds}{e+s}.$$

Likewise

$$B(t) = \frac{1}{q+1} t b(t) + \frac{\beta}{q+1} \int_0^t \frac{s^{q+1}}{(\ln(e+s))^{\beta+1}} \frac{ds}{e+s},$$

and

$$B(\tilde{b}(t)) = \frac{1}{q+1} t \tilde{b}(t) + \frac{\beta}{q+1} \int_0^{\tilde{b}(t)} \frac{s^{q+1}}{(\ln(e+s))^{\beta+1}} \frac{ds}{e+s}.$$

Hence

$$\tilde{A}(t) - \theta t \tilde{a}(t) = \left(\frac{p}{p+1} - \theta \right) t \tilde{a}(t) - \frac{\alpha}{p+1} \int_0^{\tilde{a}(t)} \frac{s^{p+1}}{(\ln(e+s))^{\alpha+1}} \frac{ds}{e+s},$$

and

$$\tilde{B}(t) - (1-\theta)t \tilde{b}(t) = \left(\frac{q}{q+1} - (1-\theta) \right) t \tilde{b}(t) - \frac{\beta}{q+1} \int_0^{\tilde{b}(t)} \frac{s^{q+1}}{(\ln(e+s))^{\beta+1}} \frac{ds}{e+s}.$$

Since l'Hospital rule,

$$\lim_{t \rightarrow \infty} \frac{\int_0^t \frac{s^{p+1}}{(\ln(e+s))^{\alpha+1}} \frac{ds}{e+s}}{t^{p+1}} = \frac{1}{p+1}, \quad \text{moreover} \quad \lim_{t \rightarrow \infty} \frac{t^{p+1}}{(\ln(e+t))^{\alpha+1}} = 0,$$

then, for each $\varepsilon > 0$ there exists a C_ε such that

$$\frac{\alpha}{p+1} \int_0^t \frac{s^{p+1}}{(\ln(e+s))^{\alpha+1}} \frac{ds}{e+s} \leq \varepsilon t a(t) + C_\varepsilon, \quad \text{for all } t \geq 0,$$

and

$$\int_0^{\tilde{a}(t)} \frac{s^{p+1}}{(\ln(e+s))^{\alpha+1}} \frac{ds}{e+s} \leq \varepsilon t \tilde{a}(t) + C_\varepsilon, \quad \text{for all } t \geq 0.$$

Choosing

$$\theta = \frac{1}{2} \left(1 - \frac{1}{p+1} + \frac{1}{q+1} \right) = \frac{1}{2} \left(\frac{p}{p+1} + \frac{1}{q+1} \right) \in [0, 1],$$

we get that

$$\frac{p}{p+1} - \theta = \frac{q}{q+1} - (1-\theta) = \frac{1}{2} \left(1 - \frac{1}{p+1} - \frac{1}{q+1} \right) > 0.$$

Consequently

$$\tilde{A}(t) - \theta t \tilde{a}(t) \geq \left[\frac{1}{2} \left(1 - \frac{1}{p+1} - \frac{1}{q+1} \right) - \varepsilon \right] t \tilde{a}(t) - C_\varepsilon,$$

and

$$\tilde{B}(t) - (1-\theta)t \tilde{b}(t) \geq \left[\frac{1}{2} \left(1 - \frac{1}{p+1} - \frac{1}{q+1} \right) - \varepsilon \right] t \tilde{b}(t) - C_\varepsilon.$$

Hence

$$\int_{\Omega} f_n \tilde{a}(f_n) dx + \int_{\Omega} g_n \tilde{b}(g_n) dx \leq \varepsilon_n \|(f_n, g_n)\|_X + C,$$

and as $t \tilde{a}(t) > 0$ for all t , we then get

$$\frac{1}{\|(f_n, g_n)\|_X} \int_{\Omega} f_n \tilde{a}(f_n) dx \rightarrow 0, \quad \frac{1}{\|(f_n, g_n)\|_X} \int_{\Omega} g_n \tilde{b}(g_n) dx \rightarrow 0.$$

Thus, using that for all $u \in L^H(\Omega)$

$$\|u\|_H \leq 2\|u\|_{(H)} \leq 2 \max \left\{ \int_{\Omega} H(u) dx, 1 \right\}$$

(cf. Lemma 5.5(iii) and Proposition 5.10(i)) for $H = \tilde{A}$ and $H = \tilde{B}$, we reach a contradiction. \square

Proof of Theorem 1.1. We wish to apply a mountain pass type theorem. We divide the proof in 2 steps. Recall that the functional J is defined in (2.7).

Step 1. We prove that $(0, 0)$ is a local minimum of J . By Young's inequality for both A and B implies

$$\begin{aligned} J(f, g) &= \int_{\Omega} \tilde{A}(f) dx + \int_{\Omega} \tilde{B}(g) dx - \frac{1}{2} \int_{\Omega} (f \mathbb{K}(g) + g \mathbb{K}(f)) dx \\ &\geq \int_{\Omega} \tilde{A}(f) dx + \int_{\Omega} \tilde{B}(g) dx - \frac{1}{2} \int_{\Omega} (\tilde{A}(f) + A(\mathbb{K}(g))) dx \\ &\quad - \frac{1}{2} \int_{\Omega} (\tilde{B}(g) + B(\mathbb{K}(f))) dx \\ &= \frac{1}{2} \int_{\Omega} \tilde{A}(f) dx + \frac{1}{2} \int_{\Omega} \tilde{B}(g) dx - \frac{1}{2} \int_{\Omega} A(\mathbb{K}(g)) dx - \frac{1}{2} \int_{\Omega} B(\mathbb{K}(f)) dx. \end{aligned}$$

Let us distinguish 3 cases according to the signs of α and β .

Case 1: Assume that $\alpha \geq 0$ and $\beta \geq 0$. First observe that, since

$$\tilde{A}(t) \geq \frac{p}{p+1} t^{\frac{p+1}{p}}, \quad \tilde{B}(t) \geq \frac{q}{q+1} t^{\frac{q+1}{q}}$$

for all $t \geq 0$ (see (2.23)), then $X \subset L^{\frac{p+1}{p}}(\Omega) \times L^{\frac{q+1}{q}}(\Omega)$. Moreover, using that

$$A(t) \leq \frac{1}{p+1} t^{p+1}, \quad B(t) \leq \frac{1}{q+1} t^{q+1}$$

for all $t \geq 0$, we get

$$\int_{\Omega} A(\mathbb{K}(g)) dx \leq \frac{1}{p+1} \|\mathbb{K}(g)\|_{p+1}^{p+1}, \quad \int_{\Omega} B(\mathbb{K}(f)) dx \leq \frac{1}{q+1} \|\mathbb{K}(f)\|_{q+1}^{q+1}.$$

Besides, using Sobolev embeddings, either the second inequality of (1.2) or the equality in (1.3) and the regularity of \mathbb{K} , there exists a constant $C > 0$ independent of f and g such that

$$\|\mathbb{K}(g)\|_{p+1} \leq C \|\mathbb{K}(g)\|_{W^{2, \frac{q+1}{q}}} \leq C \|g\|_{\frac{q+1}{q}}, \quad \|\mathbb{K}(f)\|_{q+1} \leq C \|\mathbb{K}(f)\|_{W^{2, \frac{p+1}{p}}} \leq C \|f\|_{\frac{p+1}{p}}.$$

Thus

$$2J(f, g) \geq \left(\frac{p}{p+1} \|f\|_{\frac{p+1}{p}}^{\frac{p+1}{p}} - C^{q+1} \|f\|_{\frac{p+1}{p}}^{q+1} \right) + \left(\frac{q}{q+1} \|g\|_{\frac{q+1}{q}}^{\frac{q+1}{q}} - C^{p+1} \|g\|_{\frac{q+1}{q}}^{p+1} \right).$$

Finally, using the first inequality of (1.2) and (1.3) (i.e. $\frac{p+1}{p} < q+1$, or equivalently, $\frac{q+1}{q} < p+1$) and that $\|f\|_{\frac{p+1}{p}} \leq \left(\frac{p+1}{p}\right)^{\frac{p}{p+1}} \|f\|_{\tilde{A}}$, $\|g\|_{\frac{q+1}{q}} \leq \left(\frac{q+1}{q}\right)^{\frac{q}{q+1}} \|g\|_{\tilde{B}}$, we conclude that $J(f, g) > 0$ for all $(f, g) \in X$ with $0 \neq \|(f, g)\|_X$ sufficiently small.

Case 2: Assume that $\alpha \geq 0$ and $\beta < 0$ (the case $\alpha < 0$ and $\beta \geq 0$ is analogous).

Using Sobolev embeddings, the regularity of \mathbb{K} , and since $q+1 \leq \left(\frac{N-2}{N} - \frac{1}{p+1}\right)^{-1} = \left(\frac{p}{p+1} - \frac{2}{N}\right)^{-1}$, we have that

$$\|\mathbb{K}(f)\|_{q+1} \leq C\|\mathbb{K}(f)\|_{\left(\frac{p}{p+1} - \frac{2}{N}\right)^{-1}} \leq C\|\mathbb{K}(f)\|_{W^{2, \frac{p+1}{p}}} \leq C\|f\|_{\frac{p+1}{p}}.$$

On the one hand, since (2.16), and the above

$$\int_{\Omega} B(\mathbb{K}(f)) dx \leq \frac{1}{q+1} \|\mathbb{K}(f)\|_{q+1}^{q+1} \leq C\|f\|_{\frac{p+1}{p}}^{q+1}.$$

On the other hand, since (2.13)

$$\int_{\Omega} \tilde{A}(f) dx \geq C\|f\|_{\frac{p+1}{p}}^{\frac{p+1}{p}}.$$

Hence, using the Young inequality

$$\frac{1}{2} \int_{\Omega} (f\mathbb{K}(g) + g\mathbb{K}(f)) dx = \int_{\Omega} f\mathbb{K}(g) dx \leq \int_{\Omega} \tilde{B}(g) dx + \int_{\Omega} B(\mathbb{K}(f)) dx$$

so

$$J(f, g) \geq \int_{\Omega} \tilde{A}(f) dx - \int_{\Omega} B(\mathbb{K}(f)) dx$$

and using (2.41)

$$J(f, g) \geq C\|f\|_{\frac{p+1}{p}}^{\frac{p+1}{p}} - D\|f\|_{\frac{p+1}{p}}^{q+1} > 0,$$

for $0 \neq \|f\|_{\tilde{A}}$ small enough. Observe that if $f = 0$, then $J(f, g) = \int_{\Omega} \tilde{B}(g) dx > 0$ for $\|g\|_{\tilde{B}} \neq 0$.

Case 3: Assume that $\alpha < 0$ and $\beta < 0$.

Notice that if $\alpha < 0$, $\beta < 0$ then necessarily (1.2) holds. Using Orlicz–Sobolev embeddings (see Lemma 2.8), and the regularity of \mathbb{K} (see Theorem 5.29), we have that

$$\|\mathbb{K}(g)\|_A \leq C\|\mathbb{K}(g)\|_{W^{2, \tilde{B}}} \leq C\|g\|_{\tilde{B}}, \quad \|\mathbb{K}(f)\|_B \leq C\|\mathbb{K}(f)\|_{W^{2, \tilde{A}}} \leq C\|f\|_{\tilde{A}}.$$

On the one hand since (2.17), and the above

$$\int_{\Omega} A(\mathbb{K}(g)) dx \leq C\|\mathbb{K}(g)\|_A^{p+1} \leq C\|g\|_{\tilde{B}}^{p+1}, \quad \int_{\Omega} B(\mathbb{K}(f)) dx \leq C\|\mathbb{K}(f)\|_B^{q+1} \leq C\|f\|_{\tilde{A}}^{q+1}.$$

whenever $\|\mathbb{K}(g)\|_A \leq 1$, $\|\mathbb{K}(f)\|_B \leq 1$.

On the other hand since (2.15) in Lemma 2.3, and Lemma 2.6,

$$\int_{\Omega} \tilde{A}(f) dx \geq C\|f\|_{(\tilde{A})}^{\frac{p+1}{p}}, \quad \int_{\Omega} \tilde{B}(g) dx \geq C\|g\|_{\tilde{B}}^{\frac{q+1}{q}},$$

whenever $\|f\|_{(\tilde{A})} \leq 1$, $\|g\|_{\tilde{B}} \leq 1$.

Then, for all $(f, g) \in X$ with $0 \neq \|(f, g)\|_X$ sufficiently small

$$2J(f, g) \geq d_1\|f\|_{(\tilde{A})}^{\frac{p+1}{p}} - d_2\|f\|_{\tilde{A}}^{q+1} + \left(d_3\|g\|_{\tilde{B}}^{\frac{q+1}{q}} - d_4\|g\|_{\tilde{B}}^{p+1}\right) > 0.$$

Step 2. There exists $(f_1, g_1) \in X$ such that $J(f_1, g_1) < 0$. Indeed, choose $s \in (1/p, q)$ (this is possible as $qp > 1$, which follows from (1.2) or (1.3)). We shall show that there exists $t \in \mathbb{R}$ big enough such that $J(t\phi_1, t^s\phi_1) < 0$. Indeed, since $K(\phi_1) = \phi_1/\lambda_1$, we can write

$$J(t\phi_1, t^s\phi_1) = \int_{\Omega} \tilde{A}(t\phi_1) dx + \int_{\Omega} \tilde{B}(t^s\phi_1) dx - \frac{t^{1+s}}{\lambda_1} \int_{\Omega} (\phi_1)^2 dx.$$

Since Lemma 2.1 and Example 5.18, $\tilde{A}(t\phi_1) \sim (t\phi_1)^{\frac{p+1}{p}} (\ln(t\phi_1))^{\frac{\alpha}{p}}$, and $\tilde{B}(t^s\phi_1) \sim (t^s\phi_1)^{\frac{q+1}{q}} (\ln(t^s\phi_1))^{\frac{\beta}{q}}$. Since $\frac{p+1}{p} < 1+s$, and also $s\frac{q+1}{q} < 1+s$, hence $J(t\phi_1, t^s\phi_1) \rightarrow -\infty$ as $t \rightarrow \infty$.

4. A Non-existence Result

The following lemma provides a Rellich–Pohozaev–Mitidieri type identity, see [18, 21, 22].

Lemma 4.1. (Rellich–Pohozaev–Mitidieri type identity) *Let u and v be in $C^2(\bar{\Omega})$, where Ω is a C^1 domain in \mathbb{R}^N , and $u = v = 0$ on $\partial\Omega$. Then*

$$\begin{aligned} \int_{\Omega} \Delta u (x \cdot \nabla v) + \Delta v (x \cdot \nabla u) &= (N-2) \int_{\Omega} (\nabla u \cdot \nabla v) + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} (x \cdot \nabla v) \\ &\quad + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} (x \cdot \nabla u) - \int_{\partial\Omega} (\nabla u, \nabla v) (x \cdot \nu), \end{aligned}$$

where ν denotes the exterior normal, and $(x \cdot \nu)$ denotes the inner product.

For the proof we refer to [18].

By definition, a solution (u, v) of (1.1) belongs to $(W_0^{1, \tilde{A}}(\Omega) \cap W^{2, \tilde{A}}(\Omega)) \times (W_0^{1, \tilde{B}}(\Omega) \cap W^{2, \tilde{B}}(\Omega))$. Since an estimate of Brezis–Kato [4], based on Moser iteration techniques [19], $u, v \in C^{1, \nu}(\bar{\Omega}) \cap W^{2, s}(\Omega)$ for any $\nu < 1$, $s < +\infty$, see for instance [20, Lemma 2.1 (iv)]. If, in particular Ω is $C^{2, \mu}$, then $u, v \in C^{2, \mu}(\bar{\Omega})$.

Theorem 4.2. Non existence of non-negative non-trivial classical solutions *Let $(u, v) \in (C^2(\bar{\Omega}))^2$ be a pair of non-negative solutions to the problem (1.1).*

Assume that

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}. \quad (4.1)$$

Assume also that Ω is strongly star-shaped with respect to 0 (the inner product $(x \cdot \nu(x)) \geq 0$ for all $x \in \partial\Omega$, and $(x \cdot \nu(x)) \not\equiv 0$) and that $\partial\Omega$ is C^1 . If $\alpha, \beta \leq 0$, then

$$u \equiv 0, \quad v \equiv 0, \quad \text{in } \Omega.$$

Assume now that

$$\frac{1}{p+1} + \frac{1}{q+1} < \frac{N-2}{N}. \quad (4.2)$$

and that Ω is star-shaped with respect to 0 ($(x \cdot \nu(x)) \geq 0$ for all $x \in \partial\Omega$) and that $\partial\Omega$ is C^1 . If $\alpha, \beta \leq 0$, then

$$u \equiv 0, \quad v \equiv 0, \quad \text{in } \Omega.$$

Remark 4.3. In particular, for the single equation, i.e. $p = q$ and $\alpha = \beta \leq 0$, we have the following:

Assume that $p = 2^* - 1$, and also that Ω is strongly star-shaped with respect to 0. If $\alpha \leq 0$, then

$$u \equiv 0, \quad \text{in } \Omega.$$

Assume now that $p > 2^* - 1$, and that Ω is star-shaped with respect to 0. If $\alpha \leq 0$, then

$$u \equiv 0, \quad \text{in } \Omega.$$

Proof of Theorem 4.2. If we set $W(u, v) := A(u) + B(v)$ then $W_u = a(u)$ and $W_v = b(v)$. Therefore, for solutions $u > 0$ and $v > 0$ of (1.1),

$$\begin{aligned} & \int_{\Omega} -[\Delta u (x \cdot \nabla v) + \Delta v (x \cdot \nabla u)] \\ &= \int_{\Omega} \sum_j x_j \left(\frac{\partial W}{\partial v} \frac{\partial v}{\partial x_j} + \frac{\partial W}{\partial u} \frac{\partial u}{\partial x_j} \right) = \int_{\Omega} \sum_j x_j \frac{\partial W}{\partial x_j} \\ &= -N \int_{\Omega} W + \int_{\Omega} \operatorname{div}(W x) = -N \int_{\Omega} [A(u) + B(v)] + \int_{\partial\Omega} (x \cdot \nu) W(u, v), \end{aligned}$$

and

$$\int_{\Omega} \nabla u \nabla v = \int_{\Omega} u a(u) = \int_{\Omega} v b(v). \quad (4.3)$$

Applying Lemma 4.1 (Pohozaev-Rellich-Mitidieri type identity) we get that

$$\begin{aligned} & N \int_{\Omega} [A(u) + B(v)] - (N-2) \int_{\Omega} u a(u) \\ &= \int_{\partial\Omega} (x \cdot \nu) W(u, v) - \int_{\partial\Omega} (\nabla u \cdot \nabla v) (x \cdot \nu) + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} (x \cdot \nabla v) + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} (x \cdot \nabla u). \quad (4.4) \end{aligned}$$

On the one hand, $u = v = 0$ on the boundary, so the first integral in the r.h.s. vanishes. Moreover, since $u = 0$ on $\partial\Omega$, the tangential component of ∇u vanishes and ∇u is parallel to the normal vector $\nu(x)$ at each point $x \in \Omega$, in other words $\nabla u(x) = \pm |\nabla u(x)| \nu(x)$.

On the other hand, since Hopf's Lemma, $\frac{\partial u}{\partial \nu}(x) < 0$ for all $x \in \partial\Omega$, so $\nabla u(x) \cdot \nu = \frac{\partial u}{\partial \nu}(x) = -|\nabla u(x)|$, and consequently, $\nabla u(x) = -|\nabla u(x)| \nu(x)$. Likewise, $\frac{\partial v}{\partial \nu}(x) = -|\nabla v(x)|$, and $\nabla v(x) = -|\nabla v(x)| \nu(x)$. Consequently, the r.h.s. of (4.4) is reduced to

$$\begin{aligned} & \left(-|\nabla u(x)| |\nabla v(x)| + |\nabla u(x)| |\nabla v(x)| + |\nabla u(x)| |\nabla v(x)| \right) (x \cdot \nu) \\ &= |\nabla u(x)| |\nabla v(x)| (x \cdot \nu) = \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} (x \cdot \nu), \end{aligned}$$

hence

$$N \int_{\Omega} [A(u) + B(v)] - (N - 2) \int_{\Omega} u a(u) = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} (x \cdot \nu). \quad (4.5)$$

Integrating by parts $A(t) = \int_0^t \frac{s^p}{(\ln(e+s))^\alpha} ds$, we have that

$$A(t) - \frac{1}{p+1} t a(t) = \frac{\alpha}{p+1} \int_0^t \frac{s^{p+1}}{(\ln(e+s))^{\alpha+1}} \frac{ds}{e+s} = \frac{\alpha}{p+1} \int_0^t \frac{a(s)}{\ln(e+s)} \frac{s}{e+s} ds. \quad (4.6)$$

Likewise

$$B(t) - \frac{1}{q+1} t b(t) = \frac{\beta}{q+1} \int_0^t \frac{b(s)}{\ln(e+s)} \frac{s}{e+s} ds. \quad (4.7)$$

Substituting (4.6)–(4.7) into (4.5), and since (4.3),

$$\begin{aligned} & \left[N \left(\frac{1}{p+1} + \frac{1}{q+1} \right) - (N - 2) \right] \int_{\Omega} u a(u) + \frac{\alpha N}{p+1} \int_{\Omega} \int_0^{u(x)} \frac{a(s)}{\ln(e+s)} \frac{s}{e+s} ds dx \\ & + \frac{\beta N}{q+1} \int_{\Omega} \int_0^{v(x)} \frac{b(s)}{\ln(e+s)} \frac{s}{e+s} ds dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} (x \cdot \nu). \end{aligned} \quad (4.8)$$

Introducing (4.1) in (4.8), we deduce

$$\begin{aligned} & \frac{\alpha N}{p+1} \int_{\Omega} \int_0^{u(x)} \frac{a(s)}{\ln(e+s)} \frac{s}{e+s} ds dx \\ & + \frac{\beta N}{q+1} \int_{\Omega} \int_0^{v(x)} \frac{b(s)}{\ln(e+s)} \frac{s}{e+s} ds dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} (x \cdot \nu). \end{aligned} \quad (4.9)$$

Assume $\alpha, \beta \leq 0$, if $u \not\equiv 0$, then $v \not\equiv 0$, and the l.h.s. of (4.9) is non-positive. On the other hand, if Ω is strongly star-shaped with respect to 0, the r.h.s. of (4.9) is strictly positive, giving a contradiction, and necessarily, $u \equiv 0$, and $v \equiv 0$.

Assume $u \not\equiv 0$, then $v \not\equiv 0$, and introducing (4.2) in (4.8), we deduce

$$\begin{aligned} & \frac{\alpha N}{p+1} \int_{\Omega} \int_0^{u(x)} \frac{a(s)}{\ln(e+s)} \frac{s}{e+s} ds dx \\ & + \frac{\beta N}{q+1} \int_{\Omega} \int_0^{v(x)} \frac{b(s)}{\ln(e+s)} \frac{s}{e+s} ds dx > \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} (x \cdot \nu). \end{aligned} \quad (4.10)$$

Assume $\alpha, \beta \leq 0$, then the l.h.s. of (4.10) is non-positive. On the other hand, if Ω is star-shaped with respect to 0, the r.h.s. of (4.10) is strictly positive, giving a contradiction, and necessarily, $u \equiv 0$, and $v \equiv 0$.

□

We point out that, in general if p and q lie on the critical hyperbola, it would be interesting to check if the condition

$$\frac{\alpha}{p+1} + \frac{\beta}{q+1} \leq 0$$

ensures that

$$\alpha \left(\int_{\Omega} \left[\int_0^{u(x)} \frac{a(s)}{\ln(e+s)} \frac{s}{e+s} ds - \int_0^{v(x)} \frac{b(s)}{\ln(e+s)} \frac{s}{e+s} ds \right] dx \right) \leq 0$$

or equivalently

$$N \int_{\Omega} [A(u) + B(v)] - (N-2) \int_{\Omega} [\theta u a(u) + (1-\theta) v b(v)] \leq 0.$$

5. On Orlicz Spaces

Let us summarise in this section some basics results on Orlicz spaces (cf. [1], [15] and [23]).

5.1. \mathcal{N} -Functions

Definition 5.1. (*\mathcal{N} -functions*) A function $H : [0, \infty) \rightarrow [0, \infty)$ is said to be a \mathcal{N} -function if and only if H is continuous, convex, $H(t) = 0$ if and only if $t = 0$ and

$$\lim_{t \rightarrow 0} H(t)/t = 0, \quad \lim_{t \rightarrow +\infty} H(t)/t = +\infty.$$

Any \mathcal{N} -function is extended to \mathbb{R} as an even function. We denote by h the left derivative of H .

5.2. Orlicz Classes, Orlicz Spaces and the Luxembourg Norm

Associated to the \mathcal{N} -function H we have the following class of functions. Let $\Omega \subset \mathbb{R}^N$ be here an arbitrary open set.

For our application, $\Omega \subset \mathbb{R}^N$ is a **bounded** set.

Definition 5.2. (*Orlicz class*). The *Orlicz class* $K^H(\Omega)$ is defined by

$$K^H(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} : u \text{ is measurable and } \int_{\Omega} H(u(x)) dx < +\infty \right\}.$$

Orlicz classes are convex sets, but in general not linear spaces.

Definition 5.3. (*Orlicz space*). We say that $u \in L^H(\Omega)$ if and only if there exists a constant $c > 0$ such that $cu \in K^H(\Omega)$. $L^H(\Omega)$ is a vector space, and it is called the *Orlicz space associated to H* .

Let us recall the following result:

Lemma 5.4. *Let H be an \mathcal{N} -function. Then,*

(i) $L^H(\Omega) = K^H(\Omega)$ if and only if H satisfies the so called Δ_2 -condition:

$$\exists k \geq 1, \quad H(2t) \leq kH(t) \quad \forall t \geq 0.$$

(ii) If Ω is bounded then $L^H(\Omega) = K^H(\Omega)$ if and only if H satisfies the so called Δ_2 -condition at infinity:

$$\exists k \geq 1, \exists t_0 \geq 0, \quad H(2t) \leq kH(t) \quad \forall t \geq t_0. \quad (5.1)$$

(iii) Assume H is derivable and its derivative h is continuous and strictly increasing. If

$$\lim_{t \rightarrow +\infty} \frac{th(t)}{H(t)} = \alpha \in (1, +\infty) \quad (5.2)$$

then both H and \tilde{H} satisfy the Δ_2 -condition at infinity, where \tilde{H} is defined below in Definition 5.6.¹

Proof. (i) It follows from [15, Theorem 8.2], (ii) from [23, Theorem 2] and (iii) from [15, Theorem 4.1 and Theorem 4.3]. \square

We can provide the Orlicz space $L^H(\Omega)$ with the *Luxemburg norm*:

$$\|u\|_{(H)} := \inf \left\{ \lambda > 0 \int_{\Omega} H\left(\frac{u(x)}{\lambda}\right) dx \leq 1 \right\}. \quad (5.3)$$

Lemma 5.5. (i) $(L^H(\Omega), \|\cdot\|_{(H)})$ is a Banach space.

(ii) If $u \in L^H(\Omega)$, $u \neq 0$, then $\int_{\Omega} H\left(\frac{u}{\|u\|_{(H)}}\right) dx \leq 1$.

(iii) $\|u\|_{(H)} \leq \max\{\int_{\Omega} H(u) dx, 1\}$.

(iv) If H satisfies the Δ_2 -condition then, for all $u \neq 0$, $u \in L^H(\Omega)$, it holds

$$\int_{\Omega} H\left(\frac{u}{\|u\|_{(H)}}\right) dx = 1.$$

If Ω is bounded, the conclusion holds if H satisfies the Δ_2 -condition at infinity.

Proof. The proof of (i) can be found in [23, Theorem 10, p. 67]. The proof of (ii) and (iii) follow trivially from the definition of the Luxemburg norm. The result of (iv) is proved in [23, Proposition 6, p. 77]. \square

5.3. The Young-Conjugate of a \mathcal{N} -Function

Definition 5.6. Let H be an \mathcal{N} -function and denote \tilde{H} its Legendre transform, i.e.

$$\tilde{H}(s) := \sup_{\sigma \in \mathbb{R}} \{s\sigma - H(\sigma)\}.$$

Then, \tilde{H} is called the *Young-conjugate of H* .

Some authors call \tilde{H} the complementary function of H , see [15, p. 11–13]. We denote by \tilde{h} the right derivative of \tilde{H} .

Remark 5.7. If h is strictly increasing, then $\tilde{h}(t) = h^{-1}(t)$.

The following properties are trivial to prove.

Proposition 5.8. Let H be an \mathcal{N} -function.

¹[15, Theorem 4.2] clarify that H and \tilde{H} satisfies the Δ_2 -condition if and only if H satisfies the Δ_2 -condition and the ∇_2 -condition, in other words, there exist numbers $h > 1$ and $t_1 \geq 0$ such that

$$H(t) \leq \frac{1}{2h} H(ht) \quad \text{for } t \geq t_1.$$

- (i) $\widetilde{H} = H$.
- (ii) \widetilde{H} is an \mathcal{N} -function.
- (iii) H and \widetilde{H} satisfy the Young inequality:

$$\forall s, t \in (0, +\infty), \quad st \leq H(s) + \widetilde{H}(t)$$

and the equality holds if and only if $t = h(s)$ or $s = \tilde{h}(t)$.

- (iv) The following 1st Hölder's inequality holds:

$$\forall f \in L^H(\Omega), \forall g \in L^{\tilde{H}}(\Omega), \quad \int_{\Omega} |f(x)g(x)| dx \leq 2\|f\|_{(H)}\|g\|_{(\tilde{H})}.$$

Proof. The proof of (i)–(ii) can be found for instance in [15, Chapter 1]. For the proof of (iv), see [23, Proposition 1 and Remark in p. 58]. \square

5.4. Dual Norm, the Dual of an Orlicz Space and Reflexivity

Definition 5.9. Let H be an \mathcal{N} -function. For all $u \in L^H(\Omega)$ we define the dual norm of u or Orlicz norm of u as

$$\|u\|_H := \sup \left\{ \int_{\Omega} uv dx : \|v\|_{(\tilde{H})} \leq 1 \right\}. \quad (5.4)$$

We have

Proposition 5.10. (i) For all $u \in L^H(\Omega)$

$$\|u\|_{(H)} \leq \|u\|_H \leq 2\|u\|_{(H)}.$$

- (ii) 2nd Hölder's inequality:

$$\forall u \in L^H(\Omega), \forall v \in L^{\tilde{H}}(\Omega), \quad \int_{\Omega} |u(x)v(x)| dx \leq \|u\|_H \|v\|_{(\tilde{H})}. \quad (5.5)$$

- (iii) For all $u \in L^H(\Omega)$, $u \neq 0$,

$$\int_{\Omega} H\left(\frac{u}{\|u\|_H}\right) dx \leq 1. \quad (5.6)$$

- (iv) If \tilde{H} satisfies the Δ_2 -condition then the dual space $(L^H(\Omega), \|\cdot\|_H)'$ coincides with $(L^{\tilde{H}}(\Omega), \|\cdot\|_{(\tilde{H})})$.

- (v) $L^H(\Omega)$ is a reflexive Banach space if and only if H and \tilde{H} satisfy the Δ_2 -condition.

If Ω is bounded, the conclusions (iv) and (v) hold if H and \tilde{H} satisfy the Δ_2 -condition at infinity.

Proof. The proof of (i), (ii), (iii) follows from [15, Lemma 9.2 and Theorem 9.3, p. 74]. For the proof of (iv) and (v) see [23, Theorem 10, p. 113]. \square

Remark 5.11. Since Proposition 5.10 (ii) and (i), 2nd Hölder's inequality can be written:

$$\forall u \in L^H(\Omega), \forall v \in L^{\tilde{H}}(\Omega), \quad \int_{\Omega} |u(x)v(x)| dx \leq \|u\|_H \|v\|_{\tilde{H}}.$$

Observe that Proposition 5.10 (iv) is a Riesz representation theorem for Orlicz spaces, whenever \tilde{H} satisfies the Δ_2 -condition.

Definition 5.12. We say that the sequence $\{u_n\}_{n \in \mathbb{N}}$ converges in H -mean to u whenever

$$\lim_{n \rightarrow \infty} \int_{\Omega} H(|u_n(x) - u(x)|) dx = 0.$$

The following Theorem states that convergence in H -mean is equivalent to the convergence with respect to the Orlicz-norm, provided that the Δ_2 -condition is satisfied, see [15, Theorem 9.4].

Theorem 5.13. Let H be an \mathcal{N} -function satisfying the Δ_2 -condition. Then, the convergence in H -mean is equivalent to the convergence with respect to the $\|\cdot\|_H$ norm.

5.5. Comparison of \mathcal{N} -Functions

It is possible to consider different partial ordering relations between \mathcal{N} -functions, and they imply continuous embedding into Orlicz spaces.

First let us introduce the following ordering relations. The following partial ordering relation between functions is involved in embedding theorems between Orlicz space associated with different \mathcal{N} -functions:

Definition 5.14. Let H and H_1 be two \mathcal{N} -functions.

- (i) The function H_1 is said to *dominate the function H* globally (respectively near infinity), denoted by $H \prec H_1$, if there exists a positive constant c such that

$$H(s) \leq H_1(cs), \quad \text{for } s \geq 0 \quad (s \geq s_0).$$

- (ii) The functions H and H_1 are called *equivalent globally* (near infinity), denoted by $H \sim H_1$, if each dominates the other globally (near infinity).
- (iii) If for every $c > 0$, there exists a number $s_c \geq 0$ such that

$$H(s) \leq H_1(cs), \quad \text{for } s \geq 0 \quad (s \geq s_c),$$

then H is said to *increase essentially more slowly* than H_1 (at infinity), and in this case we write $H \ll H_1$.

Remark 5.15. There are different partial ordering relations between \mathcal{N} -functions.

- (i) It is said that H *grows more slowly* than H_1 at infinity, and it is written $H \ll H_1$, if and only if

$$\forall r > 0, \quad \lim_{s \rightarrow +\infty} \frac{H(rs)}{H_1(s)} = 0.$$

In [23, p. 15, Definition 1, (ii)] is said that H_1 is *essentially stronger* than H . This definition is equivalent to Remark 5.15 (i). See [23, p. 16, Theorem 2.b,

(i) \iff (v)].

- (ii) In particular, if

$$\exists c > 0, \quad \lim_{s \rightarrow +\infty} \frac{H(s)}{H_1(s)} = c,$$

then H is *equivalent* to H_1 at infinity.

Proposition 5.16. *Let H and H_1 be two \mathcal{N} -functions.*

The continuous embedding $L^{H_1}(\Omega) \rightarrow L^H(\Omega)$ holds if either H_1 dominates H globally, or $|\Omega| < \infty$ and H_1 dominates H near infinity.

Moreover, $L^H(\Omega) \subset L^{H_1}(\Omega)$ if and only if there exists a constant $C > 0$ such that $\|u\|_{H_1} \leq C\|u\|_H$ for all $u \in L^H(\Omega)$.

Proof. The proof of the first statement, follows directly from definition 5.3 of Orlicz space, and from Definition 5.14(i).

For the proof of the second statement, see [16, Theorem 4, p. 51]. \square

Proposition 5.17. *Let H and H_1 be two \mathcal{N} -functions. If $H \sim H_1$ then $L^H(\Omega)$ is isomorphic to $L^{H_1}(\Omega)$.*

Example 5.18. (Some useful facts) Let $p > 0$, $\alpha < p$, and

$$a(t) := \frac{t^p}{(\ln(e+t))^\alpha}, \quad t \geq 0.$$

Then, we have

- (i) $A(s) \sim s^{p+1}(\ln s)^{-\alpha}$,
- (ii) $\tilde{a}(s) := a^{-1}(s) \sim s^{\frac{1}{p}}(\ln s)^{\frac{\alpha}{p}}$,
- (iii) $\tilde{A}(s) \sim s^{\frac{p+1}{p}}(\ln s)^{\frac{\alpha}{p}}$,
- (iv) $\tilde{A}^{-1}(s) \sim s^{\frac{p}{p+1}}(\ln s)^{-\frac{\alpha}{p+1}}$.

See Lemma 2.1.

5.6. The Orlicz–Sobolev Spaces $W_0^{1,H}(\Omega)$ and $W^{m,H}(\Omega)$

Let H be an \mathcal{N} -function and let $m \in \mathbb{N}^*$.

Definition 5.19. (i) The space $W^{m,H}(\Omega)$ is defined as

$$W^{m,H}(\Omega) := \{u : \Omega \rightarrow \mathbb{R} : D^\alpha u \in L^H(\Omega) \quad \forall |\alpha| \in \{0, 1, \dots, m\}\}$$

where D^α stands for the weak partial derivative of u .

(ii) The Luxemburg norm for the elements u of this space is

$$\|u\|_{m,(H)} := \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{(H)}$$

and $(W^{m,H}(\Omega), \|\cdot\|_{m,(H)})$ is a Banach space.

(iii) The space $W_0^{1,H}(\Omega)$ is defined as

$$W_0^{1,H}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,(H)}}.$$

(iv) The Orlicz norm for $u \in W_0^{1,H}(\Omega)$ is defined by

$$\|u\|_{1,H} := \sup \left\{ \int_\Omega uv \, dx : v \in W_0^{1,H}(\Omega), \|v\|_{1,(H)} \leq 1 \right\}.$$

The Orlicz norm is equivalent to the Luxemburg norm, see Proposition 5.10(i).

5.7. Optimal Embedding Theorems for Orlicz–Sobolev Spaces

We present in this section some known results concerning the *optimal Orlicz–Sobolev embedding*. For that purpose, we define for any any \mathcal{N} -function H , the *auxiliary function*

$$\Phi_H(s) := \int_0^s \frac{\tilde{H}(\tau)}{\tau^{1+N'}} d\tau \quad (5.7)$$

where $N' = \frac{N}{N-1}$. We will denote by \rightarrow a continuous embedding, and by \hookrightarrow a compact embedding.

Remark 5.20. We can always assume that Φ_H is well defined. Indeed, since Ω is of finite measure (it is bounded), we can assume without loss of generality that

$$\int_0^\infty \frac{\tilde{H}(\tau)}{\tau^{1+N'}} d\tau < \infty$$

otherwise H can be replaced by any \mathcal{N} -function which is equivalent to the original one near infinity and makes the previous integral converge. Such a replacement does not affect the result since the corresponding Orlicz–Sobolev norm is equivalent to the original one.

It turns out that one finds different optimal embeddings depending on whatever

$$i_H := \int_0^\infty \frac{\tilde{H}(\tau)}{\tau^{1+N'}} d\tau \quad (5.8)$$

is finite or not. Notice that if $i_H = +\infty$, in particular Φ_H is strictly increasing (at least for s large) and Φ_H^{-1} is well defined. Consequently we distinguish two cases:

Case 1: $i_H < +\infty$. The following Theorem is a compact Orlicz–Sobolev embedding for the spaces $W_0^{1,H}(\Omega)$ and $W^{1,H}(\Omega)$, in terms of continuous bounded functions, see [12], and [6, Corollary 1]. Recall that a bounded open set Ω is called *strongly Lipschitz* if, for each $x \in \partial\Omega$, there exist a neighbourhood U_x of x , a coordinate system (y_1, \dots, y_N) centered at x and a Lipschitz continuous function ϕ of (y_1, \dots, y_{N-1}) such that

$$\Omega \cap U_x = \{(y_1, \dots, y_N) : y_N > \phi(y_1, \dots, y_{N-1})\}.$$

Theorem 5.21. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded open set. Let H be an \mathcal{N} -function satisfying $i_H < +\infty$, where i_H is defined in (5.8). Let $C_b(\Omega)$ be the space of continuous bounded functions on Ω .*

1. *Then, the embedding*

$$W_0^{1,H}(\Omega) \hookrightarrow C_b(\Omega)$$

is compact.

2. *If in addition Ω has the strong Lipschitz property then, the embedding*

$$W^{1,H}(\Omega) \hookrightarrow C_b(\Omega)$$

is compact.

Case 2: $i_H = +\infty$. The following theorem gives the *optimal Orlicz–Sobolev continuous embedding* for the space $W_0^{1,H}(\Omega)$, in terms of Orlicz spaces, see [7, Theorem 1].

Theorem 5.22. *Let $N \geq 2$. Let H be any \mathcal{N} -function and let Φ_H be the auxiliary function defined in (5.7) Assume that $i_H = +\infty$, where i_H is defined in (5.8). Define for all $s \geq 0$*

$$H^*(s) = H_N^*(s) = \int_0^s t^{N'-1} \left(\Phi_H^{-1}(t^{N'}) \right)^{N'} dt \quad (5.9)$$

Then H^* is an \mathcal{N} -function, and the following continuous embedding holds:

$$W_0^{1,H}(\Omega) \rightarrow L^{H^*}(\Omega). \quad (5.10)$$

Furthermore $L^{H^*}(\Omega)$ is the smallest Orlicz space that renders (5.10) true.

The following theorem is also a continuous Orlicz–Sobolev embedding, this time for the space $W^{1,H}(\Omega)$, in terms of Orlicz spaces, see [7, Theorem 2]. We will say that $\Omega \subset \mathbb{R}^N$ satisfies the *cone property*, if there exists a cone Σ such that for any $x \in \Omega$, the set Ω contains a cone congruent with Σ and whose vertex is x .

Theorem 5.23. *Let $N \geq 2$. Assume that $\Omega \subset \mathbb{R}^N$ is any open bounded connected set, satisfying the cone property. Let H be any \mathcal{N} -function, and let H^* be the function defined by (5.9). Assume that $i_H = +\infty$, where i_H is defined in (5.8). Then, the following holds:*

- (i) *There exists a constant C , depending only on H , $|\Omega|$ and N , such that*

$$\|u - u_\Omega\|_{L^{H^*}(\Omega)} \leq C \|\nabla u\|_{(H)} \quad \text{for all } u \in W^{1,H}(\Omega).$$

Here,

$$u_\Omega := \frac{1}{|\Omega|} \int_\Omega u(x) dx$$

is the mean value of u over Ω .

- (ii) *The continuous embedding*

$$W^{1,H}(\Omega) \rightarrow L^{H^*}(\Omega)$$

holds, where possibly H^* is replaced at zero, in the sens of Remark 5.20.

Finally we give in Theorem 5.24 below a compact Orlicz–Sobolev embedding for the space $W^{1,H}(\Omega)$, this time in terms of Orlicz spaces, see [7, Theorem 3].

Theorem 5.24. *Let $N \geq 2$ and let Ω be any open, bounded, connected set, and satisfying the cone property. Let H be any \mathcal{N} -function. Assume that $i_H = +\infty$, where i_H is defined in (5.8) and H_1 a \mathcal{N} -function increasing essentially more slowly near infinity than H^* defined by (5.9). Then the embedding*

$$W^{1,H}(\Omega) \hookrightarrow L^{H_1}(\Omega)$$

is compact.

Example 5.25. Some useful continuous and compact embeddings
Consider any \mathcal{N} -function H such that

$$H(s) \sim s^p [\log(s)]^\alpha$$

where either $p > 1$ and $\alpha \in \mathbb{R}$ or $p = 1$ and $\alpha > 0$.

Let $\Omega \subset \mathbb{R}^N$ be an bounded open set.

Case 1. The following compact embedding holds:

$$W_0^{1,H}(\Omega) \hookrightarrow C_b(\Omega) \quad \text{if either} \quad \begin{cases} p > N & \text{and } \alpha \in \mathbb{R}, \\ \text{or } p = N & \text{and } \alpha > N - 1. \end{cases}$$

Case 2. The following continuous embedding holds:

$$W_0^{1,H}(\Omega) \rightarrow L^{H^*}(\Omega) \quad \text{if } p \leq N,$$

see Theorem 5.22, where

$$H^*(s) \sim \begin{cases} \left(s^p [\log(s)]^\alpha \right)^{\frac{N}{N-p}} & \text{if } 1 \leq p < N, \\ e^{s^{N/(N-1-\alpha)}} & \text{if } p = N, \alpha < N - 1, \\ e^{e^{s^{N'}}} & \text{if } p = N, \alpha = N - 1. \end{cases} \quad (5.11)$$

By Theorem 5.23, the same embeddings are true with $W^{1,H}(\Omega)$ replacing $W_0^{1,H}(\Omega)$, provided that $\Omega \subset \mathbb{R}^N$ has finite measure and satisfies the cone property.

Moreover, if $p \leq N$, since Theorem 5.24

$$W^{1,H}(\Omega) \hookrightarrow L^{H_1}(\Omega) \quad \text{for any } H_1 \ll H^*.$$

Notice that if $\alpha = 0$, (5.11) agrees with Sobolev's theorem when $p \neq N$, and with Trudinger's theorem when $p = N$.

Example 5.26. Consider a \mathcal{N} -functions $H(s)$ which are equivalent near infinity to

$$s^p \left(\log [\log(s)] \right)^\alpha, \quad \text{where either} \quad \begin{cases} p > 1 & \text{and } \alpha \in \mathbb{R}, \\ \text{or } p = 1 & \text{and } \alpha > 0. \end{cases}$$

Then, Theorem 5.22, and Theorem 5.21 imply that

$$W^{1,H}(\Omega) \hookrightarrow C_b(\Omega) \quad \text{if } p > N,$$

and

$$W^{1,H}(\Omega) \rightarrow L^{H^*}(\Omega) \quad \text{if } p \leq N,$$

where

$$H^*(s) \sim \begin{cases} \left(s^p [\log \log(s)]^\alpha \right)^{\frac{N}{N-p}} & \text{if } 1 \leq p < N, \\ e^{(s^N [\log(s)]^\alpha)^{\frac{1}{N-1}}} & \text{if } p = N. \end{cases}$$

Theorem 5.27. *Assume that Ω has the strong Lipschitz property. Let $C_b^1(\Omega)$ be the set of C^1 functions such that their derivatives are bounded in $L^\infty(\Omega)$. Let \tilde{A} the \mathcal{N} -function defined in (2.5). The following Orlicz–Sobolev continuous embeddings hold:*

$$W^{2,\tilde{A}}(\Omega) \subset \begin{cases} L^{(\tilde{A}^*)^*}(\Omega) & \text{if } \frac{p+1}{p} \leq \frac{N}{2}, \alpha \leq \left(1 - \frac{1}{N}\right)(Np - (p+1)) \\ C_b(\Omega) & \text{if } \frac{p+1}{p} = \frac{N}{2}, \alpha > \left(1 - \frac{1}{N}\right)(Np - (p+1)), \\ & \text{or } \frac{p+1}{p} > \frac{N}{2}, \alpha \in \mathbb{R}; \\ C_b^1(\Omega) & \text{if } \frac{p+1}{p} = N, \alpha > p(N-1), \text{ or } \frac{p+1}{p} > N, \alpha \in \mathbb{R}; \end{cases} \quad (5.12)$$

with $(\tilde{A}^*)^*$ defined by

$$(\tilde{A}^*)^* \sim \begin{cases} s^{\frac{N(p+1)}{Np-2(p+1)}} [\log(s)]^{\frac{\alpha N}{Np-2(p+1)}} & \text{if } \frac{p+1}{p} < \frac{N}{2}, \\ e^{s^{N/(N-1-\alpha \frac{N}{Np-(p+1)})}} & \text{if } \frac{p+1}{p} = \frac{N}{2}, \alpha < \left(1 - \frac{1}{N}\right)(Np - (p+1)) \\ e^{e^{s^{N'}}} & \text{if } \frac{p+1}{p} = \frac{N}{2}, \alpha = \left(1 - \frac{1}{N}\right)(Np - (p+1)). \end{cases} \quad (5.13)$$

Moreover, the following Orlicz–Sobolev compact embeddings hold

$$W^{2,\tilde{A}}(\Omega) \hookrightarrow \begin{cases} L^{A_1}(\Omega) & \forall A_1 \ll (\tilde{A}^*)^*, \text{ if } \frac{p+1}{p} \leq \frac{N}{2}, \\ & \text{and } \alpha \leq \left(1 - \frac{1}{N}\right)(Np - (p+1)); \\ C_b(\Omega) & \text{if } \frac{p+1}{p} = \frac{N}{2}, \alpha > \left(1 - \frac{1}{N}\right)(Np - (p+1)), \\ & \text{or } \frac{p+1}{p} > \frac{N}{2}, \alpha \in \mathbb{R}; \\ C_b^1(\Omega) & \text{if } \frac{p+1}{p} = N, \alpha > p(N-1), \text{ or } \frac{p+1}{p} > N, \alpha \in \mathbb{R}; \end{cases} \quad (5.14)$$

for any A_1 increasing essentially more slowly than $((\tilde{A}^*)^*)^*$, denoted by $A_1 \ll ((\tilde{A}^*)^*)^*$, see Definition 5.14(iii).

Proof of Theorem 5.27 Since Lemma 2.1 and Example 5.18,

$$\tilde{A}(s) \sim s^{\frac{p+1}{p}} (\ln s)^{\frac{\alpha}{p}}.$$

Moreover, by Orlicz–Sobolev continuous embeddings,

$$W^{2,\tilde{A}}(\Omega) \rightarrow \begin{cases} W^{1,(\tilde{A})^*}(\Omega) & \text{if } \frac{p+1}{p} \leq N, \alpha \leq p(N-1) \\ C_b^1(\Omega) & \text{if } \frac{p+1}{p} = N, \alpha > p(N-1), \text{ or } \frac{p+1}{p} > N, \alpha \in \mathbb{R}; \end{cases}$$

with $(\tilde{A})^*$ defined specifically by

$$(\tilde{A})^* \sim \begin{cases} \left(s^{\frac{p+1}{p}} [\log(s)]^{\frac{\alpha}{p}} \right)^{\frac{N}{N-\frac{p+1}{p}}}, & \text{if } \frac{p+1}{p} < N, \\ e^{s^{N/(N-1-\frac{\alpha}{p})}}, & \text{if } \frac{p+1}{p} = N, \alpha < p(N-1) \\ e^{e^{s^{N'}}}, & \text{if } \frac{p+1}{p} = N, \alpha = p(N-1), \end{cases}$$

cf. Theorem 5.23, Theorem 5.21, Example 5.25, and definition (5.11). Observe that $\frac{N}{N-\frac{p+1}{p}} \frac{p+1}{p} = \frac{N(p+1)}{Np-(p+1)} = \left(\frac{p}{p+1} - \frac{1}{N}\right)^{-1}$.

Iterating the above procedure we obtain (5.12), with $(\tilde{A}^*)^*$ defined, by

$$(\tilde{A}^*)^* \sim \begin{cases} \left(s^{\frac{N(p+1)}{Np-(p+1)}} [\log(s)]^{\frac{\alpha N}{Np-(p+1)}} \right)^{\frac{Np-(p+1)}{Np-2(p+1)}} & \text{if } \frac{p+1}{p} < \frac{N}{2}, \\ e^{s^{N/(N-1-\alpha \frac{N}{Np-(p+1)})}} & \text{if } \frac{p+1}{p} = \frac{N}{2}, \alpha < (1 - \frac{1}{N})(Np - (p+1)) \\ e^{e^{s^{N'}}} & \text{if } \frac{p+1}{p} = \frac{N}{2}, \alpha = (1 - \frac{1}{N})(Np - (p+1)), \end{cases}$$

which is obtained iterating twice (5.11). It can be equivalently rewritten as (5.13).

Observe that $\frac{N(p+1)}{Np-(p+1)} \frac{Np-(p+1)}{Np-2(p+1)} = \frac{N(p+1)}{Np-2(p+1)} = \left(\frac{p}{p+1} - \frac{2}{N}\right)^{-1}$.

Moreover, from Theorem 5.24, the Orlicz–Sobolev compact embeddings described in (5.14) hold, see Definition 5.14(iii) for A_1 increasing essentially more slowly than $((\tilde{A}^*)^*)^*$, denoted by $A_1 \ll ((\tilde{A}^*)^*)^*$. \square

Corollary 5.28. *In particular, under the conditions of Theorem 5.27, assume that one of the following two conditions holds:*

- (i) *either (1.2) is satisfied, for any $\alpha \in \mathbb{R}$*
- (ii) *either (1.3) is satisfied, and $\alpha > 0$.*

Then, the following Orlicz–Sobolev compact embedding holds

$$W^{2,\tilde{A}}(\Omega) \hookrightarrow L^{q+1}(\Omega). \quad (5.15)$$

If (1.3) is satisfied, and $\alpha = 0$, then the above embedding is continuous.

Proof. Using Theorem 5.27, we only have to realize that $A_1(s) = s^{q+1}$ increases essentially more slowly than $(\tilde{A}^*)^*$, i.e., $A_1 \ll (\tilde{A}^*)^*$, see Definition 5.14(iii).

We first check that $L^{(\tilde{A}^*)^*}(\Omega) \subset L^{q+1}(\Omega)$. Indeed,

$$\frac{N(p+1)}{Np-2(p+1)} \geq q+1 \iff \frac{p}{p+1} - \frac{2}{N} \leq \frac{1}{q+1} \iff \frac{N-2}{N} \leq \frac{1}{p+1} + \frac{1}{q+1}.$$

Moreover, since $\alpha > 0$ in case (ii), then $A_1 \ll ((\tilde{A}^*)^*)^*$, and so (5.15) holds.

If (1.3) is satisfied, and $\alpha = 0$, then $A_1 = (\tilde{A}^*)^*$, and the embedding is continuous. \square

5.8. The Dirichlet Problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^2 , H a \mathcal{N} -function and $f \in L^H(\Omega)$. Let us consider the Dirichlet problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \quad (5.16)$$

Solutions are understood in the weak sense, i.e.

$$\forall \varphi \in C_0^1(\Omega), \quad \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx.$$

We have the following regularity result (cf. [2, Theorem XI.8], and [14, Theorem 4]).

Theorem 5.29. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^2 . Assume that H and \tilde{H} satisfies the Δ_2 -condition. Then, the unique solution $u = K(f)$ of problem (5.16) belongs to $W^{2,H}(\Omega) \cap W_0^{1,H}(\Omega)$, $-\Delta u = f$ a.e. and*

$$\|u\|_H + \sum_i \|\partial_{x_i} u\|_H + \sum_{ij} \|\partial_{x_i x_j}^2 u\|_H \leq C \|f\|_H \quad (5.17)$$

for some $C = C(N, H, \Omega)$.

See Lemma 5.4 (iii) for having a sufficient condition guarantying that H and \tilde{H} satisfies the Δ_2 -condition.

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Mabel Cuesta

Department of Mathematics

Université du Littoral Côte d’Opale (ULCO), Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville (LMPA)

62100 Calais

France

e-mail: mabel.Cuesta@univ-littoral.fr

Rosa Pardo

Departamento de Análisis Matemático y Matemática Aplicada

Universidad Complutense de Madrid

28040 Madrid

Spain

e-mail: rpardo@ucm.es

Angela Pistoia

Dipartimento SBAI

Sapienza Università di Roma

via Antonio Scarpa 16

00161 Rome

Italy

e-mail: angela.pistoia@uniroma1.it

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