Output-feedback stabilization of an underactuated network of N interconnected n + m hyperbolic PDE systems

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***Abstract*—In this article, we detail the design of an output feedback stabilizing control law for an underactuated network of N subsystems of $n+m$ heterodirectional linear first-order hyperbolic Partial Differential Equations interconnected through their boundaries. The network has a chain structure, as only one of the subsystems is actuated. The available measurements are located at the opposite extremity of the chain. The proposed approach introduces a new type of integral transformation to tackle in-domain couplings in the different subsystems while guaranteeing a "clear actuation path" between the control input and the different subsystems. Then, it is possible to state several essential properties of each subsystem: output trajectory tracking, input-to-state stability, and predictability (the possibility of designing a state prediction). We recursively design a stabilizing state-feedback controller by combining these properties. We then design a state-observer that reconstructs delayed values of the states. This observer is combined with the state-feedback control law to obtain an output-feedback controller. Simulations complete the presentation.**

***Index Terms*—backstepping, PDEs networks, difference systems, predictor, tracking**

I. Introduction

THE interconnection of hyperbolic systems (potentially coupled with ODEs) represents a well-established topic, given its inherent occurrence in various industrial contexts (e.g., electric power transmission systems [53], control of after-treatment devices in exhaust lines [28], or traffic networks [65]). Specifically, interconnections characterized by a **cascade chain structure** have garnered notable attention [2]. This particular network configuration holds significance due to its capacity to model intricate industrial phenomena, such as the propagation of torsional waves in drilling systems [2], deepwater construction vessels [55], density-flow systems as lossless electrical lines, frictionless open channels, or gas pipes [17], [16].

Most existing constructive control strategies for interconnected systems are grounded in the backstepping approach. Notably, significant attention has been directed towards cascaded interconnections of hyperbolic PDE-ODE systems, as evidenced in [1], [29], [7], as well as ODE-PDE-ODE configurations [31], [19], [58], where the design of control hinges upon the reformulation of the interconnection as a *time-delay system*. Recent advancements have also emerged for interconnected PDE systems containing non-linear ODEs by designing a modular approach involving tracking controllers [39], [38]. These contributions generally establish stringent rank conditions on the various coupling matrices, among other requisites. In numerous scenarios encompassing underactuated PDEs (such as the simple interconnection of two scalar hyperbolic systems [13], wherein only one of the subsystems is actuated), these conditions remain unmet, despite the existence of stabilizing controllers [13]. This explains why underactuated PDEs have been the source of several contributions these last few years.

While the design of comprehensive control strategies for all types of underactuated systems or network configurations appears to be currently overly ambitious, several existing methodologies in the literature have put forward constructive control designs specifically tailored to networks with well-defined **structural characteristics**. Particularly noteworthy are **chain** configurations featuring a cascade structure, which have garnered significant attention. In such configurations, the network is a straight line, and the actuator/sensor is located at one of its extremities. This class of systems can arise in scenarios like oil production systems comprised of interconnected pipes, where the main conduit is known as the manifold [46]. More precisely, the lower part of the drill string is usually made up of drill collars that can significantly impact global dynamics due to their inertia. These pipes may have distinct lengths, densities, inertia, or Young's modulus. The variations in characteristic line impedance across space may lead to reflections manifesting at junctions. Such simple chain-structured networks can also model ventilation within buildings [60], density-flow systems [35], open canals [27], or traffic systems, as described in [65], [64] in the case of two cascaded freeway segments. Among other examples of interest, we can cite networks of 1-D flexible multi-structures [25] as interconnected Timoshenko beams that can be used to model compliant mechanical structures such as cantilevers or flexible endoscopes.

Recently, several methodologies have emerged to formulate stabilizing controllers for such chain structures. Among these methodologies, PI boundary controllers have been explored in [17], [35] for fully actuated networks (i.e., networks with one control input per set of heterodirectional PDEs). Explicit stability conditions were obtained by utilizing suitable quadratic Lyapunov functions. In [41], the authors consider a flatness-based design of a feedforward control of tree-like transmission networks. Analogous scenarios involving interconnected systems have been scrutinized in [57], where

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a velocity recirculation phenomenon in a wave equation was considered. Furthermore, the assessment of exact boundary controllability for nodal profiles of quasilinear hyperbolic systems with interface conditions in tree-like networks was conducted in [59] using the method of characteristics. In a more recent study, detailed in [56], the authors investigated output feedback control strategies for interconnections of 2×2 semilinear hyperbolic systems using the dynamics on the characteristic lines. Backstepping-based controllers have been developed in [29], [13], [4] in the case of interconnected scalar subsystems. While these advancements have contributed significantly to the field, they exhibited adaptability limitations across different chain structures. For example, integrating an additional PDE subsystem within the chain structure was not feasible in [4]. To overcome this limitation, a novel approach was proposed in [50] through the introduction of a **new recursive dynamics framework**. This framework, which is grounded in innovative prediction-based control laws [3], [18], [21] for difference equations, exhibits **modularity**. This modularity stems from the requirement that the control law only necessitates fundamental properties for each subsystem (controllability, trackability, observability, predictability). However, it should be noted that the techniques proposed in [50] do not lend themselves directly to non-scalar subsystems. In the case of two non-scalar subsystems, a two-step procedure was proposed in [6].

In this paper, we overcome the limitations of [50] (regarding the dimension of the subsystems) and of [6] (regarding the number of subsystems) to design a stabilizing output-feedback controller for an interconnection of an arbitrary number of non-scalar PDE subsystems with a chain structure. Although the methodology we propose uses the same ingredients presented in [6] and [50], extending such results to a chain of non-scalar subsystems is far from trivial. Indeed, when dealing with non-scalar systems, backstepping transformations cannot usually remove all the in-domain coupling terms [23]. Therefore, due to these remaining in-domain coupling terms, the approach presented in [50] does not work as the predictors cannot be adequately defined (causality problem). In the case of two subsystems, this problem was partially solved in [6] using appropriate flatness-based feedforward tracking components in the control input. However, this solution required adding additional terms to the backstepping transformation depending on the downstream subsystems, leading to complex, intricate kernel equations when there are more than two subsystems. We introduce a **new type of integral transformation**, with a time-affine component to overcome these limitations. This new component is used to "clear the actuation path" of each subsystem by removing the local terms initially present in the system and avoiding additional terms coming from the downstream subsystem. To the best of the author's knowledge, such a paper represents a novelty in the literature as it solves a challenging stabilization problem by taking advantage of a new class of time-affine transformations. Finally, it is the first time an observer is designed for a chain of non-scalar subsystems (only state-feedback stabilization was considered in [6]).

The paper is organized as follows. We introduce the class of system under consideration in Section II. In Section III, we present a new class of integral transformations and recursively design a stabilizing state-feedback control law. An observer that estimates delayed values of the states is proposed in Section IV. It is used to obtain an output-feedback stabilizing control law. The results of the paper are illustrated with simulations in Section V. Finally, we give some concluding remarks in Section VI.

A. Notations

In this section, we detail the notations used throughout this paper. For any distinct real numbers a and b, any positive integer n, we denote $L^2([a,b],\mathbb{R}^n)$ the space of real-valued square-integrable functions defined on $[a,b]$ with the standard L^2 norm, *i.e.*, for any $f\in L^2([a,b],\mathbb{R}^n)$, $||f||^2_{L^2([a,b])}=\int_a^b f^T(x)f(x)\mathrm{d}x$. For $n\in\mathbb{N}$ functions u_k in $L^2([a,b],\mathbb{R})$, we define the L^2 norm of the vector $(u_1,\dots,u_n)$ as the sum of the square of the L^2-norm of each function composing the vector: $||(u_1,\dots,u_n)||^2_{L^2}=\sum_{k=1}^n||u_k||^2_{L^2([a,b])}$. The set $L^\infty([0,1],\mathbb{R})$ denotes the space of bounded real-valued functions defined on $[0,1]$ with the standard L^∞ norm, *i.e.*, for any $f\in L^\infty([0,1],\mathbb{R})$, $||f||_{L^\infty}=\operatorname*{ess\,sup}_{x\in[0,1]}|f(x)|$. For any positive integer n, we denote $H^1([a,b],\mathbb{R}^n)$ the one-dimensional Sobolev space. For any integer $m>0$ and any real delay $\tau>0$, we denote $L^2([-\tau,0],\mathbb{R}^m)$ the Banach space of L^2 functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^m$. For a function $\phi:[-\tau,\infty)\mapsto\mathbb{R}^m$, we define its partial trajectory $\phi_{[t]}$ by $\phi_{[t]}:\phi(t+\theta)$, $-\tau\leq\theta\leq0$. This maximum delay τ will be related to the transport velocities of the considered PDE system. The associated norm is denoted $||\phi_{[t]}||_{L^2_\tau}$ where for every $\tau>r$, we define

$$||\phi_{[t]}||_{L^2_r}=\left(|\int_{-r}^0\phi^T(t+\theta)\phi(t+\theta)d\theta|\right)^{\frac{1}{2}}. \tag{1}$$

The identity matrix of dimension n will be denoted Id_n (or Id if no confusion on the dimensions arises).

II. Problem under consideration

A. Interconnection with a cascade structure

In this paper, we consider a system composed of $N>0$ PDE subsystems interconnected through their boundaries in a chain structure, as schematically represented in Figure 1. The control input and the available sensors are located at one extremity of the chain. Each subsystem is composed of an arbitrary number of linear hyperbolic PDEs and is modeled by the following set of equations ($i\in\{1,\dots,N\}$)

$$\begin{aligned}\partial_t u_i(t,x)+\Lambda_i^+\partial_x u_i(t,x)&=\Sigma_i^{++}(x)v_i(t,x)\\&\quad+\Sigma_i^{+-}(x)v_i(t,x),\end{aligned}\tag{2}$$

$$\begin{aligned}\partial_t v_i(t,x)-\Lambda_i^-\partial_x v_i(t,x)&=\Sigma_i^{-+}(x)u_i(t,x)\\&\quad+\Sigma_i^{--}(x)v_i(t,x),\end{aligned}\tag{3}$$

evolving in $\{(t,x)\text{ s.t. }t>0,\ x\in[0,1]\}$, where $u_i=(u_i^1,\dots,u_i^{n_i})^T$ and $v_i=(v_i^1,\dots,v_i^{m_i})^T$, all the n_i and m_i being positive integers. The matrices Λ_i^+ and Λ_i^- are diagonal and represent the **transport velocities** of each subsystem. We

have $\Lambda_i^+ = \text{diag}\ (\lambda_i^j)$ and $\Lambda_i^- = \text{diag}\ (\mu_i^j)$ and we assume that their coefficients satisfy

$$-\mu_i^{m_i} < \cdots < -\mu_i^1 < 0 < \lambda_i^1 < \cdots < \lambda_i^{n_i}.$$

These transport velocities are assumed to be constant. However, all our results can be extended to space-dependent transport velocities at the cost of technical and lengthy computations. The spatially-varying coupling matrices $\Sigma_i^{\cdot\cdot}$ are regular matrices (we assume here that each coefficient of the matrix is a continuous function). Without any loss of generality, we can assume that the diagonal entries of Σ_i^{++} and Σ_i^{--} are equal to zero [24]. The different subsystems are connected through their boundaries in a chain structure. We have

$$u_i(t,0) = Q_{i,i}v_i(t,0) + Q_{i,i-1}u_{i-1}(t,1), \tag{4}$$

$$v_i(t,1) = R_{i,i}u_i(t,1) + R_{i,i+1}v_{i+1}(t,0) \tag{5}$$

where the different coupling and $R_{i,j}, Q_{i,j}$ are constant. By convention we consider that $R_{N,N+1} = 0$ and $Q_{1,0} = \text{Id}$. The function $u_0(t,1)$ corresponds to the **control input**, $U(t) \in \mathbb{R}^{n_1}$. The measured output is denoted as $y(t)$ and verifies $y(t) = u_N(t,1)$. The initial conditions of each subsystem belong to $H^1([0,1],\mathbb{R}^{n_i}) \times H^1([0,1],\mathbb{R}^{m_i})$. They satisfy the appropriate compatibility conditions (as stated in [17]), so that the system (2)-(5) is well-posed [17, Theorem A.1]. Finally, we denote τ_i the maximum transport delay associated to each PDE subsystem: $\tau_i = \frac{1}{\lambda_i^1} + \frac{1}{\mu_i^1}$.

The interconnected system (2)-(5) can be recast under a more condensed form as a general $n+m$ system, using a technique referred to as **folding** (see [4], [6], [26] for details). However, such a system would still be underactuated and classical results from the literature could not be applied. Moreover, such a condensed representation would shadow the cascade structure between the different subsystems. Conversely, the representation (2)-(5) highlights that the interactions between the different subsystems only occur at the boundaries. For a subsystem i, we will call the subsystem $i+1$ the **downstream subsystem** and the subsystem $i-1$ the **upstream subsystem**.

B. Structural assumptions

To design an appropriate stabilizing output feedback controller, we require several assumptions. First, to guarantee the possibility of designing a delay-robust controller, we must avoid having an infinite number of unstable roots in the right-half plan (as shown in [44]) This induces the following assumption (see [12])

Assumption 1: The open-loop system (2)-(5) without the in-domain coupling terms $\Sigma_i^{\cdot\cdot}$ is exponentially stable.

Then, we need the following assumption to stabilize the downstream subsystem states using actuation from the upstream subsystem.

Assumption 2: For all $i \in \{2,\dots,N\}$, the rank of the matrix $Q_{i-1,i}$ is equal to n_i.

This assumption implies that the matrix $Q_{i-1,i}$ admits a right inverse. A possible choice is given by the Moore–Penrose right inverse: $Q_{i-1,i}^\top(Q_{i-1,i}Q_{i-1,i}^\top)^{-1}$. This **conservative** assumption will be used to design a *virtual actuation* for each subsystem. It implies that the dimension of the (virtual) input entering the subsystem i (which corresponds to the effect of the upstream subsystem) is equal to the number of the rightward propagating states n_i of this subsystem. Such a condition is usually required to design stabilizing controllers for hyperbolic systems that do not have a specific structure (see, e.g., [23], [11], [36]). Indeed, to the best of our knowledge, only marginal results currently exist in the literature for stabilizing under-actuated systems (i.e., systems for which the dimension of the control input is smaller than the dimension of the boundary state) with no specific cascade structure (see [8]). All in all, Assumption 2 is required to avoid such an underactuated configuration. Therefore, to stabilize the system (2)-(5), we will design a specific control strategy that takes advantage of the interconnection structure. We are led to a similar assumption to designing a state observer.

Assumption 3: For all $i \in \{2,\dots,N\}$, the rank of the matrix $Q_{i-1,i}$ is equal to n_{i-1}.

This assumption implies that the matrix $Q_{i-1,i}$ admits a left inverse. This condition was not present in [6] (as only state-feedback stabilization was considered) since it is required to recursively design the proposed observer. Again, Assumption 3 is conservative and is used to avoid having under-measured subsystems. Combining Assumption 2 and Assumption 3, we obtain that all the n_i are equal (i.e., all the subsystems have the same number of rightward propagating states) and that the matrices $Q_{i-1,i}$ are invertible. This is related to the fact that we considered anti-collocated measurements. In the case of collocated measurements (i.e., $y(t) = v_1(t,0)$), Assumption 3 would have been expressed as a rank condition on the matrices $R_{i,i+1}$ (as it is the case in [50] and the rightward propagating states would not need to have the same dimensions anymore).

C. Toward a recursive design

The objective of this paper is to design an output-feedback control law that that stabilizes the interconnected system (2)-(5) in the sense of the L^2-norm. From Figure 1, it can be seen that a subsystem i will act on the downstream subsystem $i+1$ through $u_i(t,1)$, and on the upstream subsystem $i-1$ through $v_i(t,0)$. Thus, each subsystem can only be stabilized through its upstream subsystem and estimated through its downstream subsystem. Due to the hyperbolic nature of the different subsystems, the effect of the control input $U(t)$ on the subsystem i will be delayed and modified by the different in-domain coupling terms.

To stabilize the whole chain, we extend the *recursive interconnected dynamics control framework* introduced in [50]. Roughly speaking, the control law is recursively obtained by considering stabilizing virtual inputs for each subsystem and ensuring the output of the upstream subsystem converges to this desired virtual input. The control design becomes more straightforward and is based on simple assumptions that can be independently verified for each subsystem. Therefore, the proposed recursive design is somehow inspired by the classical integrator backstepping. Such connections were, for instance, already mentioned in [34] for an ODE-PDE-ODE interconnection when using an analogous approach. We propose the following control strategy:

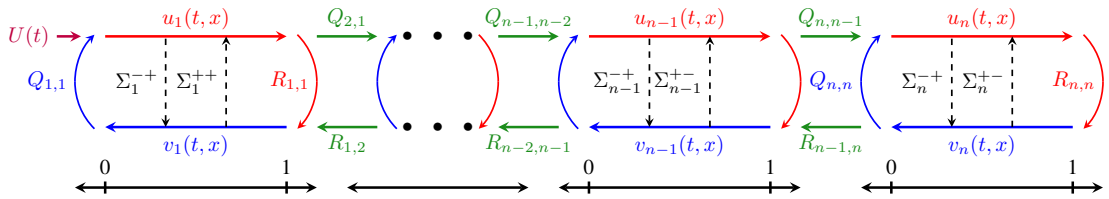

Fig. 1. Schematic representation of the chain of linear PDE subsystem (2)-(3).

1) First, using integral transformations, we simplify the structure of each subsystem to remove the in-domain coupling terms that appear in the u_i-PDEs (equation (2)). The transformations proposed in this paper are new as they include a time-affine component.
2) Then, for each subsystem i, we consider the effect of the upstream subsystem $i-1$ as a delayed virtual input $U_i(t-\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1})$ and the effect of the downstream subsystem $i+1$ as a disturbance term. We combine appropriate **state predictions** with a **flatness-based feedforward tracking controller** to guarantee that the right output of this subsystem converges to the delayed virtual input $U_{i+1}(t-\sum_{j=1}^{i}\frac{1}{\lambda_j^1})$ that will stabilize the downstream subsystem. Iterating such a procedure, it is possible to design a stabilizing control law $U(t)$ for the whole system.
3) The closed-loop stability is shown recursively, using **Input-to-State Stability** (ISS) properties.
4) A similar **recursive approach** is used to design a state observer. Going recursively from one subsystem to the next, we can estimate delayed values of the states.
5) Finally, similarly to what has been done for finite-dimensional systems [40], we can adjust the state predictors to obtain an output-feedback controller.

The proposed framework allows for a "plug-and-play"-like approach to control design since additional subsystems satisfying similar conditions can be added to the network using the same procedure.

III. State-feedback controller

A. Backstepping transformations

The first objective before applying our recursive control strategy is to simplify the structure of the interconnected system (2)-(5) in order to "clear the actuation path" of each subsystem by removing the local terms initially present in equation (2). In the case of two subsystems, this was done in [6] using a specific backstepping transformation adjusted from [37], since due to the interconnection between the i^{th} subsystem and the downstream subsystem $(i+1)$, the backstepping transformation given in [37] displays additional terms (depending on the state $v_{i+1}(t,0)$) that can cause causality issues when designing the control law. Unfortunately, for more than two subsystems, they cannot be straightforwardly removed by adjusting the transformation given in [6] without adding stringent conditions on the boundary coupling terms (more restrictive than Assumption 2 and Assumption 3). To avoid such conservative conditions, we consider a new class of transformations. For each subsystem i, we combine a classical backstepping Volterra transformation (inspired from [11]) with an integral term that depends on delayed values of the downstream state $v_{i+1}(t,0)$. More precisely, for all $i\in\{1,\dots,N\}$, and all $t\geq\frac{1}{\lambda_i^1}$, we consider the integral transformation defined by

$$\alpha_i(t,x)=u_i(t,x)+\int_0^{\frac{x}{\lambda_i^1}}F_i(x,y)v_{i+1}(t-y,0)dy+\int_x^1 K_i^{uu}(x,y)u_i(t,y)+K_i^{uv}(x,y)v_i(t,y)dy, \tag{6}$$

$$\beta_i(t,x)=v_i(t,x)+\int_x^1 K_i^{vu}(x,y)u_i(t,y)dy+\int_x^1 K_i^{vv}(x,y)v_i(t,y)dy, \tag{7}$$

where the kernels $K_i^{\cdot\cdot}$ are piecewise continuous functions defined on $\mathcal{T}_u=\{(x,y)\in[0,1]^2\mid x\leq y\}$, while the kernels F_i are piecewise continuous functions defined on the triangular domain $\{(x,y)\in[0,1]\times[0,\frac{1}{\lambda_i^1}],\ y\leq\frac{x}{\lambda_i^1}\}$. By convention $F_{N+1}=0$. The kernels $K_i^{\cdot\cdot}$ and $K_i^{\cdot\cdot}$ verify the following set of PDEs

$$\Lambda_i\partial_x K_i+\partial_y K_i\Lambda_i=-K_i\Sigma_i(y)+\begin{pmatrix}G_i(x)&0\\0&0\end{pmatrix}K_i, \tag{8}$$

where $\Lambda_i=\text{diag}(\Lambda_i^+,-\Lambda_i^-)$, $\Sigma_i=\begin{pmatrix}\Sigma_i^{++}&\Sigma_i^{+-}\\\Sigma_i^{-+}&\Sigma_i^{--}\end{pmatrix}$ and $K_i=\begin{pmatrix}K_i^{uu}&K_i^{uv}\\K_i^{vu}&K_i^{vv}\end{pmatrix}$ and with the boundary conditions

$$\Lambda_i K_i(x,x)-K_i(x,x)\Lambda_i=\Sigma_i(x)-\begin{pmatrix}G_i(x)&0\\0&0\end{pmatrix}, \tag{9}$$

$$K_i^{uu}(x,1)\Lambda_i^+=K_i^{uv}(x,1)\Lambda_i^-R_{ii}, \tag{10}$$

where $G_i(x)$ is a piecewise continuous **strictly upper-triangular** matrix function defined on $[0,1]$ through the first boundary condition of (9). More precisely, for all $1\leq k,\ell\leq n_i$, the boundary condition $\Lambda_i^+K_i^{uu}(x,x)-K_i^{uu}(x,x)\Lambda_i^+=\Sigma_i^{++}(x)-G_i(x)$ rewrites

$$(G_i(x))_{k\ell}=(\Sigma_i^{++})_{k\ell}+(\lambda_i^\ell-\lambda_i^k)(K_i^{uu}(x,x))_{k\ell},\quad\text{if } k\leq\ell,$$

$$(K_i^{uu}(x,x))_{k\ell}=\frac{(\Sigma_i^{++})_{k\ell}}{\lambda_i^k-\lambda_i^\ell},\quad\text{if } k>\ell. \tag{11}$$

It is important to emphasize that the matrix G_i is strictly upper-triangular since for $k=\ell$, we have $(G_i(x))_{k\ell}=0$ (as the diagonal entries of Σ_i^{++} are equal to zero). To these boundary conditions we add arbitrary boundary conditions for $(K_i^{vv}(0,y))_{k\ell}$ when $k<\ell$, and arbitrary conditions for $(K_i^{vv}(x,1))_{k\ell}$ when $\ell\leq k$. With these additional boundary

conditions, the set of kernel equations (8)-(11) admits a unique piecewise continuous solution [11]. The kernels F_i verify

$$\Lambda_i^+\partial_x F_i(x,y)+\partial_y F_i(x,y)=G_i(x)F_i(x,y), \tag{12}$$

$$F_i(x,0)=-K_i^{uv}(x,0)\Lambda_i^- R_{i,i+1}, \tag{13}$$

$$(F_i(x,\frac{x}{\lambda_i^1}))_{k\ell}=0,\quad 1<k\leq n_i,\ 1\leq \ell\leq m_{i+1}. \tag{14}$$

Applying [32, Theorem 3.2] (on the triangular domain $\{(x,y)\in[0,1]\times[0,\frac{1}{\lambda_i^1}],\ y\leq\frac{x}{\lambda_i^1}\}$), one can show that equations (12)-(14) admit a unique piecewise continuous solution. The transformation (6)-(7) is a Volterra transformation to which an affine term that depends on the state v_{i+1} is added. Consequently, it is invertible [63] and there exist piecewise continuous functions $L_i^{\cdot\cdot}$ defined on $\mathcal{T}_u$ and piecewise continuous functions $H_i^{\cdot}$ defined on the rectangular domain $\{(x,y)\in[0,1]\times[0,\frac{1}{\lambda_i^1}]\}$ such that for all $t\geq\frac{1}{\lambda_i^1}$,

$$\begin{aligned}u_i(t,x)&=\alpha_i(t,x)+\int_0^{\frac{1}{\lambda_i^1}}H_i^{\alpha}(x,y)v_{i+1}(t-y,0)dy\\&+\int_x^1 L_i^{\alpha\alpha}(x,y)\alpha_i(t,y)+L_i^{\alpha\beta}(x,y)\beta_i(t,y)dy,\end{aligned} \tag{15}$$

$$\begin{aligned}v_i(t,x)&=\beta_i(t,x)+\int_0^{\frac{1}{\lambda_i^1}}H_i^{\beta}(x,y)v_{i+1}(t-y,0)dy\\&+\int_x^1 L_i^{\beta\alpha}(x,y)\alpha_i(t,y)+L_i^{\beta\beta}(x,y)\beta_i(t,y)dy.\end{aligned} \tag{16}$$

Compared to (6)-(7), the inverse transformation (15)-(16) has integral components involving $v_{i+1}(t-y,0)$ on both equations. Moreover, the upper limits of these integrals are $\frac{1}{\lambda_i^1}$ instead of $\frac{x}{\lambda_i^1}$. This can be seen by applying the inverse Volterra transformation to the vector $\begin{pmatrix}\int_0^{\frac{x}{\lambda_i^1}}F_i(x,y)v_{i+1}(t-y,0)dy\\0\end{pmatrix}$, and applying Fubini's theorem. However, we emphasize that the kernels H_i^{α} and H_i^{β} may be equal to zero on some parts of the rectangular domains $[0,1]\times[0,\frac{1}{\lambda_i^1}]$.

Remark 1: Interestingly, the transformation (6)-(7) shares several features with the one introduced in [49], as they both combine a classical backstepping Volterra transformation with a component that depends on delayed values of the (downstream) state. However, their nature and purpose are fundamentally different:

- In [49], the objective is to map the initial PDE system to any target system with a similar structure but whose source terms can be arbitrarily chosen. The triangular time-affine component of the integral transformation is used to modify the remaining in-domain coupling terms in the target system;
- Here, the component $\int_0^{\frac{x}{\lambda_i^1}}F_i(x,y)v_{i+1}(t-y,0)$ of the transformation (6) is used to avoid displaying in the actuation path of the i^{th}-subsystem additional terms depending on $v_{i+1}(t,0)$. The kernels F_i do not have a triangular structure, and the kernel equations are completely different from the ones given in [49].

All in all, the transformation (6)-(7) and the one introduced in [49] can be seen as analogous tools applied in different contexts to overcome some limitations of the classical backstepping Volterra transformations.

B. Target system

For all $t\geq\frac{1}{\lambda_i^1}$, the transformation (6)-(7) maps the system (2)-(5) to the target system

$$\partial_t\alpha_i(t,x)+\Lambda_i^+\partial_x\alpha_i(t,x)=G_i(x)\alpha_i(t,x), \tag{17}$$

$$\begin{aligned}\partial_t\beta_i(t,x)-\Lambda_i^-\partial_x\beta_i(t,x)&=\bar{G}_i(x)\alpha_i(t,1)\\&+\bar{f}_i(x)v_{i+1}(t,0),\end{aligned} \tag{18}$$

with the boundary conditions

$$\begin{aligned}\alpha_i(t,0)&=Q_{i,i}v_i(t,0)+Q_{i,i-1}\alpha_{i-1}(t,1)\\&+\int_0^1 K_i^{uu}(0,y)u_i(t,y)+K_i^{uv}(0,y)v_i(t,y)dy\\&-Q_{i,i-1}\int_0^{\frac{1}{\lambda_{i-1}^1}}F_{i-1}(1,y)v_i(t-y,0)dy,\end{aligned} \tag{19}$$

$$\begin{aligned}\beta_i(t,1)&=R_{i,i}\alpha_i(t,1)+R_{i,i+1}v_{i+1}(t,0)\\&-R_{i,i}\int_0^{\frac{1}{\lambda_i^1}}F_i(1,y)v_{i+1}(t-y,0)dy,\end{aligned} \tag{20}$$

where $\bar{G}_i(x)=K_i^{vv}(x,1)\Lambda_i^-R_{i,i}-K_i^{vu}(x,1)\Lambda_i^+$ and $\bar{f}_i(x)=K_i^{vv}(x,1)\Lambda_i^-R_{i,i+1}$. By convention, we have $F_0=0$ and $Q_{1,0}\alpha_0(t,1)=U(t)$. The in-domain coupling terms appearing in equation (2) now have a triangular structure. In equation (5), all the local terms have been replaced by non-local terms that depend on $\alpha_i(t,1)$ and $v_{i+1}(t-\frac{x}{\lambda_i^1},0)$. Using the inverse transformation (15)-(16), it may be possible to substitute the remaining u_i and v_i terms that appear in the target system (17)-(20) by α_i- and β_i-terms. However, this transformation also induces the appearance of $v_{i+1}(t,0)$-terms. Although it is possible to apply the transformation (17) recursively-(20) (till we reach the last subsystem N) to get rid of all these $v_j(t,0)$ terms, the resulting expression would be cumbersome and involve intricate sums depending on delayed version of the state α_j and β_j. Therefore, we decided not to express these terms as functions of α_i and β_i to increase readability. This will not affect the proposed analysis. In the next sections, we state several elementary properties for the system (17)-(20). We will then combine these properties to design our recursive stabilizing controller. For $t>\max_i\frac{1}{\lambda_i^1}$ and all $1\leq i\leq N$, all the integral transformations (6)-(7) are well defined.

C. Output trajectory tracking

Consider the i^{th} subsystem composing the interconnection (17)-(20). Let us define the virtual control input acting on this subsystem as

$$\hat{U}_i(t)=Q_{i,i-1}\alpha_{i-1}(t+\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1},1). \tag{21}$$

This virtual control input represents the action of the upstream subsystem on the subsystem i. The delay $\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1}$ corresponds to the total largest transport time between the control input $U(t)$ and the subsystem i. It reflects the fact that the control input cannot directly act on the subsystem i, but that its effect is subject to this delay. Equation (19) rewrites

$$\alpha_i(t,0)=Q_{i,i}v_i(t,0)+\hat{U}_i(t-\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1})$$

$$+\int_0^1 K_i^{uu}(0,y)u_i(t,y)+K_i^{uv}(0,y)v_i(t,y)dy$$
$$-Q_{i,i-1}\int_0^{\frac{1}{\lambda_{i-1}^1}} F_{i-1}(1,y)v_i(t-y,0)dy. \quad (22)$$

We have the following property that guarantees the possibility for each subsystem to track any arbitrary function as long as predictions of the different states are available

Property 1: Consider the i^{th} subsystem (17)-(20) ($i \in \{1,\dots,N-1\}$) and define ζ_i an arbitrary known $H^1([0,\infty),\mathbb{R}^{n_{i+1}})$ function. Assume that there exists $t_0>0$ such that for all $t>t_0$ and all $x\in[0,1]$, it is possible to obtain a $\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1}$ -ahead of time prediction of the PDE states $u_i(t,x)$, $v_i(t,x)$, $\alpha_i(t,x)$, $\beta_i(t,x)$, i.e. there exist predictor functions P_{u_i}, P_{v_i}, P_{α_i} and P_{β_i} such that for all $t>t_0$ and all $x\in[0,1]$, $P_{u_i}(t,x)=u_i(t+\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1},x)$, $P_{v_i}(t,x)=v_i(t+\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1},x)$, $P_{\alpha_i}(t,x)=\alpha_i(t+\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1},x)$, $P_{\beta_i}(t,x)=\beta_i(t+\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1},x)$. Then, there exists a control law $\hat{U}_i(t)$ such that for any $t>t_0+\frac{1}{\lambda_i^1}$, we have $\alpha_i(t,1)=\zeta_i(t)$. Moreover, if $\zeta_i(t)\equiv 0$, and $v_{i+1}(t)\equiv 0$, then, such a control law exponentially stabilizes the i^{th} subsystem.

Proof 1: The proof is inspired by [36]. We want to find the virtual control input $\hat{U}_i$ such that the function $\alpha_i(t,1)$ (right output of this subsystem i) tracks the reference signal ζ_i. Let us first introduce the intermediate virtual control input $\hat{U}_i^{tr}(t)$ such that for all $t>t_0+\frac{1}{\lambda_i^1}$

$$\hat{U}_i(t)=\hat{U}_i^{tr}(t)-Q_{i,i}P_{v_i}(t,0)$$
$$-\int_0^1 (K_i^{uu}(0,y)P_{u_i}(t,y)dy+K_i^{uv}(0,y)P_{v_i}(t,y)dy$$
$$+Q_{i,i-1}\int_0^{\frac{1}{\lambda_{i-1}^1}} F_{i-1}(1,y)P_{v_i}(t-y,0). \quad (23)$$

This gives $\alpha_i(t,0)=\hat{U}_i^{tr}(t-\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1})$. Due to Lemma 2 (given in Appendix), the control law $(\hat{U}_i^{tr}(t))$ defined for all $1\le j\le n_i$ by

$$(\hat{U}_i^{tr}(t))_j=\alpha_i^j(t+\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1},0)=\zeta_i^j(t+\frac{1}{\lambda_i^j}+\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1})$$
$$+\sum_{\ell=j+1}^{n_i}\int_0^{\frac{1}{\lambda_i^j}}(\check{G}_i(0,\nu))_{j\ell}\zeta_i^\ell(t+\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1}+\nu)d\nu, \quad (24)$$

guarantees $\alpha_i(t,1)=\zeta_i(t)$ for any $t\ge t_0+\frac{1}{\lambda_i^1}$. This expression shows that $\alpha_i(t,1)$ corresponds to a flat output [43] that is used for trajectory planning (similarly to [45]). The controller $\hat{U}_i$ can be seen as a flatness-based feedforward tracking controller. Note that Lemma 2 provides flatness-based parametrization of the PDE state. Let us now consider that $\zeta_i(t)\equiv 0$, and $v_{i+1}(t)\equiv 0$. Then the i^{th} subsystem with the control law (23) is now autonomous, with the boundary conditions $\alpha_i(t,0)=0$, and $\beta_i(t,1)=R_{i,i}\alpha_i(t,1)$. As shown in [11], this target system is exponentially stable (and even finite-time stable). ■

The fact that we need future values of the functions u_j, v_j, α_j, β_j ($j\ge i$) is induced by the presence of the delay $\sum_{j=1}^{i-1}\frac{1}{\lambda_j^1}$ in the virtual control input $\hat{U}_i(t)$. Due to the transport delay to go from the left boundary of the α_i-PDE ($x=0$, where is located the virtual actuation) to its right boundary ($x=1$, where is defined the output we want to track), we also need future values of the reference signal ζ_i. However, one can verify in the proof of Property 1 that only $(t+\sum_{k=1}^{i}\frac{1}{\lambda_k^1})$-ahead of time values of ζ_i are required. Finally, we emphasize that Property 1 does not have to be satisfied for the last subsystem.

D. Input-to-State stability

As explained in Section II-C, the control framework we propose recursively stabilizes each subsystem, starting from the last one. However, to guarantee closed-loop stability of the whole chain, we need the following Input-to-State Stability (ISS) property for each subsystem.

Property 2: Consider the i^{th} subsystem ($i\in\{1,\dots,N-1\}$) and consider that Property 1 holds, where $\hat{U}_i(t)$ is defined by equation (23). Then, there exist two constants $\kappa_i>0$ and $\eta_i>0$ such that for all $t>t_0+\frac{1}{\lambda_i^1}+\frac{1}{\mu_i^1}$, we have

$$||(\alpha_i(t,\cdot),\beta_i(t,\cdot))||_{L^2}^2\le\kappa_i\big(||(\zeta_i)_{[t]}||_{L^2_{\eta_i}}^2+||(\zeta_i)_{[t]}||_{L^2_{-\eta_i}}^2$$
$$+||(v_{i+1}(\cdot,0))_{[t]}||_{L^2_{\eta_i}}^2\big). \quad (25)$$

Proof 2: Due to Property 1, we have for all $t>t_0+\frac{1}{\lambda_i^1}$, $\alpha_i(t,1)=\zeta_i(t)$. Applying the method of characteristics on equation (18), we can express $\beta(t,x)$ as a delayed function of $\zeta_i(t)$ and $v_{i+1}(t,0)$. Using the boundary condition (20), we obtain for all $j\in\{1,\dots,N\}$, for all $t>t_0+\frac{1}{\lambda_i^1}+\frac{1}{\mu_i^1}$

$$\beta_i^j(t,x)=\sum_{k=1}^{n_i}(R_{i,i})_{jk}\zeta_i(t-\frac{1-x}{\mu_i^j})$$
$$+\sum_{k=1}^{m_{i+1}}(R_{i,i+1})_{jk}v_{i+1}^k(t-\frac{1-x}{\mu_i^j},0)-\sum_{k=1}^{n_i}\sum_{\ell=1}^{m_{i+1}}$$
$$\int_0^{\frac{1}{\lambda_i^1}}(R_{i,i})_{jk}(F_i(1,y))_{k\ell}v_{i+1}^\ell(t-y-\frac{1-x}{\mu_i^j},0)dy$$
$$+\sum_{k=1}^{n_i}\int_0^{\frac{1-x}{\mu_i^j}}(\bar{G}_i(x+\mu_i^j\nu))_{jk}\zeta_i^k(t-\nu)d\nu$$
$$+\sum_{k=1}^{m_{i+1}}\int_0^{\frac{1-x}{\mu_i^j}}(\bar{f}_i(x+\mu_i^j\nu))_{jk}v_{i+1}^k(t-\nu,0)d\nu. \quad (26)$$

Since the functions $\bar{f}_i$ and $\bar{G}_i$ are bounded, straightforward (but tedious), computations give the existence of a constant $K_{\beta_i}>0$ such that

$$||\beta_i(t,\cdot)||_{L^2}^2\le K_{\beta_i}\big(||(\zeta_i)_{[t]}||_{L^2_{\frac{1}{\mu_i^1}}}^2+||(v_{i+1}(\cdot,0))_{[t]}||_{L^2_{\frac{1}{\mu_i^1}+\frac{1}{\lambda_i^1}}}^2\big).$$

Similarly, we can show using Lemma 2, that there exists a constant $K_{\alpha_i}>0$ such that $||\alpha_i(t,\cdot)||_{L^2}^2\le K_{\alpha_i}||(\zeta_i)_{[t]}||_{L^2_{-\frac{1}{\lambda_i^1}}}^2$.

This concludes the proof.■

The right-hand side of equation (25) involves past and future values of the functions ζ_i, which is not an issue from a stability perspective. Moreover, due to Property 2, the finite-time convergence to zero of the functions ζ_i and v_{i+1} directly implies the finite-time stability of the state (α_i,β_i).

E. State prediction

The virtual control law given in Property 1 requires the prediction of future values of the functions α_i, β_i, u_i, and v_i. The following property states that it is possible to design such predictors.

Property 3: Consider the i^{th} subsystem (17)-(20) ($i \in \{1,\dots,N\}$) with the virtual input $\hat{U}_i(t)$ defined in equation (21). For $t > \max_r \tau_r + \sum_{k=1}^{i-1} \frac{1}{\lambda_k^1}$, for all $x \in [0,1]$, and all $j \in \{i,\dots,N\}$, it is possible to obtain a $\sum_{k=1}^{i-1} \frac{1}{\lambda_k^1}$-ahead of time prediction of the functions $u_j(t,x)$, $v_j(t,x)$, $\alpha_j(t,x)$, $\beta_j(t,x)$. More precisely there exist predictor functions P_{u_j}, P_{v_j}, P_{α_j}, and P_{β_j} that only depends on past values of the state, such that for all $t > \max_r \tau_r + \sum_{k=1}^{i-1} \frac{1}{\lambda_k^1}$, for all $x \in [0,1]$, $P_{u_j}(t,x) = u_j(t+\sum_{k=1}^{i-1} \frac{1}{\lambda_k^1}, x)$, $P_{v_j}(t,x) = v_j(t+\sum_{k=1}^{i-1} \frac{1}{\lambda_k^1}, x)$, $P_{\alpha_j}(t,x) = \alpha_j(t+\sum_{k=1}^{i-1} \frac{1}{\lambda_k^1}, x)$, $P_{\beta_j}(t,x) = \beta_j(t+\sum_{k=1}^{i-1} \frac{1}{\lambda_k^1}, x)$.

Proof 3: Consider the i^{th} subsystem (17)-(20) ($i \in \{1,\dots,N\}$) with the virtual input $\hat{U}_i(t)$ defined in equation (21). Consider $j \in \{i,\dots,N\}$. For $t > \max_r \tau_r + \sum_{k=1}^{i-1} \frac{1}{\lambda_k^1}$, we will design predictors for the functions $v_j(t,0)$, $\alpha_j(t,1)$ and $\alpha_j(t,0)$. From these predictors, it will be possible to predict the functions $u_j(t,x)$, $v_j(t,x)$, $\alpha_j(t,x)$, $\beta_j(t,x)$ ($x \in [0,1]$). Due to the backstepping transformation (16), we have

$$v_j(t,0) = \beta_j(t,0) + \int_0^{\frac{1}{\lambda_j^1}} H_i^\beta(0,y) v_{j+1}(t-y,0)dy + \int_0^1 L_j^{\beta\alpha}(0,y)\alpha_j(t,y) + L_j^{\beta\beta}(0,y)\beta_j(t,y)dy.$$

Combining equation (40) and equation (26) (where $\zeta_i = \alpha_i(t,1)$), we obtain, for all $1 \le k \le m_j$,

$$\begin{aligned} v_j^k(t,0) = &\sum_{\ell=1}^{n_j} (R_{j,j})_{k\ell}\alpha_j^\ell(t-\frac{1}{\mu_j^k},1) + \sum_{\ell=1}^{m_{j+1}} (R_{j,j+1})_{k\ell} \\ &v_{j+1}^\ell(t-\frac{1}{\mu_j^k},0) + \sum_{\ell=1}^{n_j}\int_0^{\tau_j} (g_j^1)_{k\ell}(\nu)\alpha_j^\ell(t-\nu,0)d\nu \\ &+ \sum_{\ell=1}^{m_{j+1}}\int_0^{\tau_j} (g_j^2)_{k\ell}(\nu) v_{j+1}^\ell(t-\nu,0)d\nu, \end{aligned} \tag{27}$$

where g_i^1 and g_i^2 are piecewise continuous functions. We do not give their explicit expression for the sake of concision. We recall that by convention $v_{N+1}(t,0) \equiv 0$. Consider now equation (22). We can substitute the terms $u_i(t,\cdot)$ and $v_i(t,\cdot)$ that appear in the right-hand side of this equation by their expressions as functions of $\alpha_i(t,\cdot)$, $\beta_i(t,\cdot)$ and $v_{i+1}(t,0)$ using the inverse transformations (15)-(16). Then, applying the method of characteristics (see [12], [10] and equations (40) and (26)), we can express $\alpha(t,0)$ as a delayed function of $\alpha(t,0), \alpha(t,1)$ and $v(t,0)$. We obtain for $j > i$,

$$\begin{aligned} \alpha_j^k(t,0) = &\sum_{\ell=1}^{m_j} (Q_{j,j})_{k\ell} v_j^\ell(t,0) + \sum_{\ell=1}^{n_{j-1}} (Q_{j,j-1})_{k\ell}\alpha_{j-1}^\ell(t,1) \\ &+ \sum_{\ell=1}^{n_j}\int_0^{\tau_j} (h_i^2)_{k\ell}(\nu)\alpha_j^\ell(t-\nu,0)d\nu \\ &+ \sum_{\ell=1}^{m_{j+1}}\int_0^{\tau_j} (h_j^2)_{k\ell}(\nu) v_{j+1}^\ell(t-\nu,0)d\nu \\ &+ \sum_{\ell=1}^{m_j}\int_0^{\tau_{j-1}} (h_j^3)_{k\ell}(\nu) v_j^\ell(t-\nu,0)d\nu, \end{aligned} \tag{28}$$

where h_j^1, h_j^2, and h_j^3 are piecewise continuous functions. We do not give their explicit expression for the sake of concision. If $j = i$, the term $\sum_{\ell=1}^{n_{i-1}} (Q_{i,i-1})_{k\ell}\alpha_{i-1}^\ell(t,1)$ is replaced by $\sum_{\ell=1}^{n_{j-1}} (Q_{i,i-1})_{k\ell}(\hat{U}_i(t-\sum_{r=1}^{i-1}\frac{1}{\lambda_r^1}))_\ell$. Inspired by [18], [20], [8], [6], we respectively define for $t \ge \max_r \tau_r + \sum_{r=1}^{i-1} \frac{1}{\lambda_r^1}$, $k \in \{1,\dots,n_j\}$, $\ell \in \{1,\dots,m_j\}$ and $s \in [t - \max_r \tau_r - \sum_{r=1}^{i-1} \frac{1}{\lambda_r^1}, t]$, the functions $P_{\alpha_j^0}^k(t,s)$, $P_{\alpha_j^1}^k(t,s)$, and $P_{v_j^0}^\ell(t,s)$ as the **state predictions** of $\alpha_j^k(t,0)$, $\alpha_j^k(t,1)$, and $v_\ell^k(t,0)$ ahead a time $\sum_{r=1}^{i-1} \frac{1}{\lambda_r^1}$. They are *explicitly* defined by equations (29)-(31) with the convention $\sum_{\ell=1}^{n_{i-1}} (Q_{i,i-1})_{kq} P_{\alpha_{i-1}^1}^q(t,s) = \hat{U}_i^q(s)$. We write the predictors as functions of two arguments to emphasize that the predictions should be computed by incorporating delayed states available at time t to improve its robustness in practice. It should be noticed that the expressions of these predictors are causal as they only depend on past values of the functions $\alpha_j^k(t,0)$, $\alpha_j^k(t,1)$ and $v_j^k(t,0)$.

From the predictors (29)-(31), it is possible to apply equations (26) and equation (40), to obtain the corresponding state predictions for the states $\alpha_j(t,x)$ and $\beta_j(t,x)$. Finally, using the transformations (15)-(16), we obtain the predictions of the state $u_j(t,x)$ and $v_j(t,x)$. ■

Note that the definitions of the predictors implicitly depend on the initial subsystem i we consider. Indeed, the different time horizons depend on the parameter i. We chose to omit this dependency as we believe the notations are sufficiently heavy. The definitions of the predictors rely on a time-delay representation (inspired from [12]) of the interconnected system (2)-(5). In this context, the functions $\alpha_i(t,1)$, $\alpha_i(t,0)$ and $v_i(t,0)$ can be seen as a (quasi) flatness-based parametrization of (2)-(5). Similar parameterizations have been used in the literature for flatness-based open-loop design, controllability analysis, and closed-loop design (see, e.g. [61] or [48]).

F. Recursive state-feedback stabilization

We now have all the tools to apply our recursive dynamics interconnection framework

Theorem 1: For $i \in \{1,\dots,N\}$, and for $t > \sum_{j=1}^N 2\tau_j$, define the sequences of functions ζ_i, with $\zeta_N(t) = 0$ and for all $i < N$

$$\zeta_i(t) = (Q_{i+1,i}^T(Q_{i+1,i}Q_{i+1,i}^T)^{-1})\hat{U}_{i+1}(t-\sum_{j=1}^i \frac{1}{\lambda_j^1}), \tag{32}$$

where the functions $\hat{U}_i$ are defined by equations (23)-(24), where the different predictors are given in Property 3 (using the function $\hat{U}_i$ and equations (29)-(31)). Then, the interconnected system (2)-(5) with the control law $U(t) = \hat{U}_1(t)$ is exponentially stable. Moreover, the equilibrium is reached in finite time.

$$P^k_{\alpha^0_j}(t,s)=\begin{cases}\alpha^k_j(s+\sum_{r=1}^{i-1}\frac{1}{\lambda^1_r},0) \quad \text{if } s\in[t-\max_r\tau_r-\sum_{r=1}^{i-1}\frac{1}{\lambda^1_r},t-\sum_{r=1}^{i-1}\frac{1}{\lambda^1_r}]\\ \sum_{q=1}^{m_j}(Q_{j,j})_{kq}P^q_{v^0_j}(t,s)+\sum_{q=1}^{n_{j-1}}(Q_{j,j-1})_{kq}P^q_{\alpha^1_{j-1}}(t,s)+\sum_{q=1}^{n_j}\int_0^{\tau_j}(h^1_i)_{kq}(\nu)P^q_{\alpha^0_j}(t,s-\nu)d\nu\\ +\sum_{q=1}^{m_{j+1}}\int_0^{\tau_j}(h^2_j)_{kq}(\nu)P^q_{v^0_{j+1}}(t,s-\nu)d\nu+\sum_{q=1}^{m_j}\int_0^{\tau_{j-1}}(h^3_j)_{kq}(\nu)P^q_{v^0_j}(t,s-\nu)d\nu, \quad \text{otherwise,}\end{cases} \tag{29}$$

$$P^k_{\alpha^1_j}(t,s)=\begin{cases}\alpha^k_j(s+\sum_{r=1}^{i-1}\frac{1}{\lambda^1_r},1) \quad \text{if } s\in[t-\max_r\tau_r-\sum_{r=1}^{i-1}\frac{1}{\lambda^1_r},t-\sum_{r=1}^{i-1}\frac{1}{\lambda^1_r}]\\ P^k_{\alpha^0_j}(t,s-\frac{1}{\lambda^k_j})+\sum_{q=k+1}^{n_j}\int_0^{\frac{1}{\lambda^k_j}}(\tilde{G}_j(1-\lambda^k_j\nu))_{kq}P^q_{\alpha^0_j}(t,s-\nu)d\nu, \quad \text{otherwise,}\end{cases} \tag{30}$$

$$P^\ell_{v^0_j}(t,s)=\begin{cases}v^\ell_j(s+\sum_{r=1}^{i-1}\frac{1}{\lambda^1_r},0) \quad \text{if } s\in[t-\max_r\tau_r-\sum_{r=1}^{i-1}\frac{1}{\lambda^1_r},t-\sum_{r=1}^{i-1}\frac{1}{\lambda^1_r}]\\ \sum_{q=1}^{n_j}(R_{j,j})_{\ell q}P^q_{\alpha^1_j}(t,s-\frac{1}{\mu^\ell_j})+\sum_{q=1}^{m_{j+1}}(R_{j,j+1})_{\ell q}P^q_{v^0_{j+1}}(t,s-\frac{1}{\mu^\ell_j})+\sum_{q=1}^{n_j}\int_0^{\tau_j}(g^1_j)_{\ell q}(\nu)P^q_{\alpha^1_j}(t,s-\nu)d\nu\\ +\sum_{q=1}^{m_{j+1}}\int_0^{\tau_j}(g^2_j)_{\ell q}(\nu)P^q_{v^0_{j+1}}(t,s-\nu)d\nu, \quad \text{otherwise,}\end{cases} \tag{31}$$

Proof 4: First observe that the matrices $Q^T_{i+1,i}(Q^T_{i+1,i}Q^T_{i+1,i})^{-1}$ are well defined due to Assumption 2. The quantity $\zeta_i(t+\sum_{j=1}^{i}\frac{1}{\lambda^j_i})$ that appears in the proof of Property 1 can be explicitly computed from $\hat{U}_{i+1}(t)$. Then, the sequences ζ_i and $\hat{U}_i$ are well defined (since equations (29)-(31) are always well defined). Consequently, the control input $U(t)$ is well-defined and causal.

Next, we briefly show that the closed-loop system (2)-(5) with the control input $U(t)$ is well-posed. This can be done either by considering the admissibility of the control operator [22] (the control law is continuous in time), or by adjusting the proof of Theorem [17, Theorem A.1] (that is based on Lumer-Philipps theorem). Indeed, the different components of the proposed control input $U(t)$ (including the predictors) can be expressed as delayed values of the boundary states of the system (as $v_i(t,0)$) or delayed values of themselves. Such delayed values, could then be expressed using PDEs (after tedious computations).

We now need to prove that the proposed control law stabilizes the system. To ease the computations, the parameter T (that will be overloaded in the rest of the proof) denotes a finite time large enough to guarantee that the different predictors and tracking controllers are well-defined. Consider the first subsystem ($i=1$) with the control law $U(t)=\hat{U}_1(t)$. For $i=1$, equation (23) and equation (24) do not require any state predictions but can be computed using current values of the different functions. Then, using Property 1, we have that $\alpha_1(t,1)=\zeta_1(t)$ for $t>T$. Consequently, $Q_{2,1}\alpha_1(t,1)=\hat{U}_2(t-\frac{1}{\lambda^1_1})$. Therefore the functions defined through equations (29)-(31) corresponds to exact $\frac{1}{\lambda^1_1}$-ahead of time predictions of the real states. Thus, Property 1 implies that $\alpha_2(t,1)=\zeta_2(t)$ after a finite time T. Iterating the procedure, we obtain that after a finite time T, for all $i\in\{1,\dots,N\}$, $\alpha_i(t,1)=\zeta_i$ Consider now the last subsystem ($i=N$). Since $\alpha_N(t,1)=\zeta_N=0$, the functions $\alpha_N(t,x)$ and $\beta_N(t,x)$ converge to zero in finite time. Applying Property 2, we obtain the convergence to zero of the functions $\alpha_{N-1}(t,x)$ and $\beta_{N-1}(t,x)$ in finite time. Iterating the procedure, all the states (α_i,β_i) converge to zero in finite time. Using the inverse backstepping transformations (15)-(16), we obtain that the system (2)-(5) reaches its equilibrium in finite-time. The well-posedness of the closed-loop system implies its exponential stability. ■

One major advantage of the proposed framework and of the recursive design presented in Theorem 1 is that it can easily be extended to different classes of subsystems (as ODEs, for instance), as long as it is possible to derive analogous properties to Property 1, Property 2 and Property 3.

Remark 2: The state-feedback controller designed in Theorem 1 can be easily extended to the case of a delayed control input. Indeed, similarly to what has been done in the case of ODEs [42], one needs to consider an additional upstream subsystem corresponding to a pure transport equation, thus inducing a delay corresponding to the input delay.

G. A remark on robustness and computational aspects

Although the controller designed in Theorem 1 fulfills the control objective and stabilizes the system (2)-(5), it presents several drawbacks that could impact its implementation:

- First, the proposed approach is based on an exact reconstruction of the state using state predictors. The computation of the predictors can be time-consuming (as the numerical complexity increases with the number of subsystems) and troublesome. Indeed, implicit expressions of prediction-based feedback sometimes lead to burdensome numerical procedures and induce poor robustness margins, as shown in [40], [47] for the ODE case. For fully actuated integral difference equations (but with a known delay in the control input), it has been shown in [14] that it is possible to obtain

explicit expressions of such predictors and consequently improve the efficiency and robustness of the proposed control design. We believe it may be possible to obtain such an explicit expression, even if the results from [14] do not directly apply due to the underactuated configuration. Overall, it is essential to envision model reduction strategies in order to apply the proposed control law to real test cases.
- Then, the proposed approach consists of recursively canceling all the boundary reflection terms for each subsystem to track the virtual input of the downstream subsystem. This may lead to vanishing robustness margins, as shown in [5]. Unfortunately, the robustification procedure proposed in [9] cannot be directly applied since, due to the tracking part, our control law does not fit in the framework of the theorems given in [9]. However, using the cascade structure of the problem, it should be possible (under Assumption 1) to extend the results from [9] to the system (2)-(5). This solution was successfully tested in simulations.

IV. State estimation and output-feedback stabilization

To design the recursive stabilizing controller we presented in Section III, we need the knowledge of the states $u_i(t,x)$ and $v_i(t,x)$ all over the spatial domain $[0,1]$. Since the available measurement corresponds to $u_N(t,1)$, we must design a state observer. In this section, we show how to easily obtain estimated **delayed** values of these states. Adjusting the predictors introduced in Section III-E, it is then possible to reconstruct the desired states.

A. Delayed interconnection

Inspired by [40], we consider a delayed version of the interconnected system (2)-(5). Let us consider $\tau > \sum_{j=1}^{N} \frac{1}{\lambda_j^1} > 0$ a fixed, known delay. We define the *τ-delay operator* $\bar{\cdot}$, such that for all functions γ defined on $[0,+\infty)$, $\forall t > \tau, \bar{\gamma}(t) = \gamma(t-\tau)$. Using this operator, we can obtain the τ-delayed version of system (2)-(5). For all $t \geq \tau$, we have:

$$\partial_t \overline{u}_i(t,x) + \Lambda_i^+ \partial_x \overline{u}_i(t,x) = \Sigma_i^{++}(x)\overline{u}_i + \Sigma_i^{+-}(x)\overline{v}_i, \quad (33)$$

$$\partial_t \overline{v}_i(t,x) - \Lambda_i^- \partial_x \overline{v}_i(t,x) = \Sigma_i^{-+}(x)\overline{u}_i + \Sigma_i^{--}(x)\overline{v}_i, \quad (34)$$

with the boundary conditions:

$$\overline{u}_i(t,0) = Q_{i,i}\overline{v}_i(t,0) + Q_{i,i-1}\overline{u}_{i-1}(t,1), \quad (35)$$

$$\overline{v}_i(t,1) = R_{i,i}\overline{u}_i(t,1) + R_{i,i+1}\overline{v}_{i+1}(t,0), \quad (36)$$

where we still use the convention that $Q_{1,0}\bar{u}_0(t,0) = U(t-\tau)$ and $R_{N,N+1} = 0$. The available measurement is now given as $\bar{y}(t) = y(t-\tau)$. It implies that we know τ-ahead future values of the function $\bar{y}$(t). Using the backstepping transformations (6)-(7), we can define the states $\bar{\alpha}_i(t,x)$ and $\bar{\beta}_i(t,x)$. They are solutions of (17)-(20) with a τ-delayed control input.

B. Estimation of the delayed states

We now estimate the delayed state $\bar{u}_i(t,x)$ and $\bar{v}_i(t,x)$ from the available measurements.

Lemma 1: For all $i \in \{1,\dots,N\}$, we can design exact state estimators $\hat{\alpha}_i(\cdot,1)$ and $\hat{v}_{i+1}(\cdot,0)$ that causally depend on the measurement $y(t)$ such that for all $\nu \in [t, t+\sum_{j=1}^{i} \frac{1}{\lambda_j^1}]$, $\hat{\alpha}_i(t+\nu,1) = \bar{\alpha}_i(t+\nu,1)$ and $\hat{v}_{i+1}(t+\nu,0) = \bar{v}_{i+1}(t+\nu,0)$.

Proof 5: The proof relies on an induction argument. Lemma 1 obviously holds for $i = N$ with $\hat{\alpha}_N(t,1) = \bar{y}(t)$ and $\hat{v}_{N+1}(t,0) = 0$. Let us now consider the i^{th} subsystem $i \in \{2,\dots,N\}$, $t > 0$ and assume that we can design exact state estimations $\hat{\alpha}_i(\nu,1)$ and $\hat{v}_{i+1}(\nu,0)$ that causally depend on the measurement $y(t)$ such that for all $\nu \in [t, t+\sum_{j=1}^{i} \frac{1}{\lambda_j^1}]$, $\hat{\alpha}_i(t+\nu,1) = \bar{\alpha}_i(t+\nu,1)$ and $\hat{v}_{i+1}(t+\nu,0) = \bar{v}_{i+1}(t+\nu,0)$. From equation (39), we can define the intermediate estimator $\hat{\alpha}_i(t,0)$ such that for all $1 \leq k \leq n_i$ and all $t > 0$

$$\hat{\alpha}_i^k(t,0) = \hat{\alpha}_i^k(t + \frac{1}{\lambda_i^k}, 1) - \sum_{\ell=k+1}^{n_i} \int_0^{\frac{1}{\lambda_i^k}} (\check{G}_i(\nu))_{k\ell} \hat{\alpha}_i^\ell(t+\nu,1) d\nu. \quad (37)$$

We immediately obtain that for all $\nu \in [t, t+\sum_{j=1}^{i-1} \frac{1}{\lambda_j^1}]$, $\hat{\alpha}_i(t+\nu,0) = \bar{\alpha}_i(t+\nu,0)$. We now define the function $\hat{v}_i(t,0)$, such that for all $1 \leq k \leq m_i$

$$\hat{v}_i^k(t,0) = \sum_{\ell=1}^{n_i} (R_{i,i})_{k\ell} \hat{\alpha}_j^\ell(t - \frac{1}{\mu_i^k}, 1) + \sum_{\ell=1}^{m_{i+1}} (R_{i,i+1})_{k\ell} \hat{v}_{i+1}^\ell(t - \frac{1}{\mu_i^k}, 0) + \sum_{\ell=1}^{n_i} \int_0^{\tau_i} (g_i^1)_{k\ell}(\nu) \hat{\alpha}_i^\ell(t-\nu,0) d\nu + \sum_{\ell=1}^{m_{i+1}} \int_0^{\tau_i} (g_i^2)_{k\ell}(\nu) \hat{v}_{i+1}^\ell(t-\nu,0) d\nu. \quad (38)$$

We have that for all $\nu \in [t, t+\sum_{j=1}^{i-1} \frac{1}{\lambda_j^1}]$, $\hat{v}_i(t+\nu,0) = \bar{v}_i(t+\nu,0)$ due to equation (27). Finally, combining Assumption 3 and equation (28), we can obtain the desired estimations of $\bar{\alpha}_{i-1}(t,1)$. The different estimators are causal as they only require past values of the function y. This concludes the proof. ■

From Lemma 1, we obtain the following property

Property 4: For all $i \in \{1,\dots,N\}$, we can design exact state estimators $\hat{u}(t,x)$ and $\hat{v}(t,x)$ that causally depend on the measurement $y(t)$ such that for all $t > 0$, and all $x \in [0,1]$, $\hat{u}(t,x) = \bar{u}(t,x)$ and $\hat{v}(t,x) = \bar{v}(t,x)$.

Proof 6: Combining the state estimations given in Lemma 1 with the method of characteristics, it is possible to estimate the state $\bar{\alpha}_i$ and $\bar{\beta}_i$. Then, we can compute the estimators $\hat{u}(t,x)$ and $\hat{v}(t,x)$ using the inverse transformations (15)-(16). We do not give the explicit expression of these state estimators for the sake of concision.

C. Stabilizing output-feedback controller

We have designed in Property 4 a state-observer that provides a real-time exact estimation of the delayed states $(\bar{u}_i, \bar{v}_i)$. This state-observer can be combined with the state-feedback controller designed in Theorem 1 to obtain an output-feedback stabilizing controller. Indeed, combining Remark 2 and Theorem 1, we can design a state-feedback controller for the delayed system (33)-(36). This state feedback controller requires the knowledge of the delayed states $(\bar{u}_i, \bar{v}_i)$, provided

by Property 4. Therefore, we can obtain a stabilizing output-feedback controller for the delayed system (33)-(36). The exponential stability of the delayed system (33)-(36) implies the exponential stability of the original system (2)-(5).

V. Simulation results

We now illustrate our results in simulations using Matlab. The PDE system is simulated using a high-resolution explicit scheme similar to the one used in [30] with 101 spatial discretization points. The algorithm we use to compute the different kernels is the following. Using the method of characteristics, we write the integral equations associated to the kernel PDE-systems. These integral equations are solved using a fixed-point algorithm. The predictor is implemented using a backward Euler approximation of the integral. The numerical values used are given below

$$\Lambda_1^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1.4 \end{pmatrix},\ \Lambda_1^- = 1.2,\ \Sigma_1^{++} = \begin{pmatrix} 0 & 0.4 \\ 0.4 & 0 \end{pmatrix},$$
$$\Sigma_1^{--} = 0,\ \Sigma_1^{-+}(x) = \begin{pmatrix} 0.2 & 0.9 + 0.1\sin(x) \end{pmatrix},$$
$$\Sigma_1^{+-} = \begin{pmatrix} 0.8 \\ 0.8 + 0.1\cos(x) \end{pmatrix},\ \Lambda_2^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1.5 \end{pmatrix},$$
$$\Lambda_2^- = 1,\ \Sigma_2^{++} = \begin{pmatrix} 0 & 1 \\ 0.5 & 0 \end{pmatrix},\ \Sigma_2^{-+} = \begin{pmatrix} 0.2 & 0 \end{pmatrix},$$
$$\Sigma_2^{--} = 0,\ \Sigma_2^{+-} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \Lambda_3^+ = \begin{pmatrix} 1.1 & 0 \\ 0 & 1.6 \end{pmatrix},$$
$$\Lambda_3^- = 0.8,\ \Sigma_3^{++} = \begin{pmatrix} 0 & 1 \\ 0.5 & 0 \end{pmatrix},\ \Sigma_3^{+-} = \begin{pmatrix} 0.3 \\ 0 \end{pmatrix},$$
$$\Sigma_3^{--} = 0,\ \Sigma_3^{-+} = \begin{pmatrix} -0.5 & 0.2 \end{pmatrix},$$
$$Q_{11} = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix},\ R_{11} = \begin{pmatrix} 0.7 & 0.1 \end{pmatrix},\ Q_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$Q_{22} = \begin{pmatrix} 0.4 \\ 0.3 \end{pmatrix},\ R_{22} = \begin{pmatrix} 0.2 & 0 \end{pmatrix},\ Q_{32} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$R_{12} = R_{23} = 0.1,\ Q_{33} = \begin{pmatrix} 0.3 \\ 0.6 \end{pmatrix},\ R_{33} = \begin{pmatrix} 0 & 0.4 \end{pmatrix}.$$

These coefficients have been chosen to make the whole interconnection unstable in open-loop. However, the corresponding divergence rate is small to avoid numerical issues (having a greater divergence rate would require more accuracy in the simulations and, consequently, a longer simulation time). The output-feedback control law presented in Section IV-C is combined with a well-tuned low-pass filter (see Section III-G). We used a simple low-pass filter of 4th order with a bandwidth of 125 rad.s^{-1}. We added an input delay of $0.1s$ to show the robustness of the design to small delays in the loop. Some parameters are subject to constant multiplicative uncertainties (up to 5%). We have pictured in Figure 2 the time evolution of the L^2-norm of the open-loop system and the closed-loop system. As expected, the resulting system exponentially converges to zero. We do not have finite-time convergence due to the filter and the different numerical approximations. The corresponding control effort has been plotted in Fig 3.

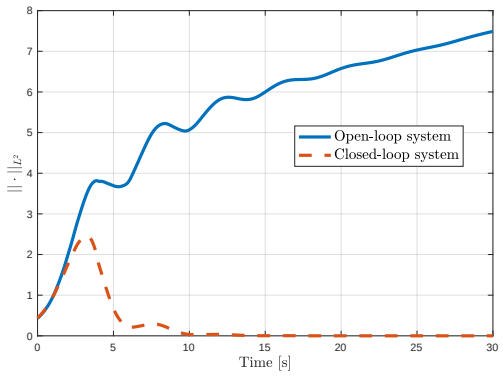

Fig. 2. Evolution of the L^2-norm of the system (2)-(5) in open-loop (blue) and in closed-loop (dashed red), using the (filtered) output-feedback control law presented in Section IV-C.

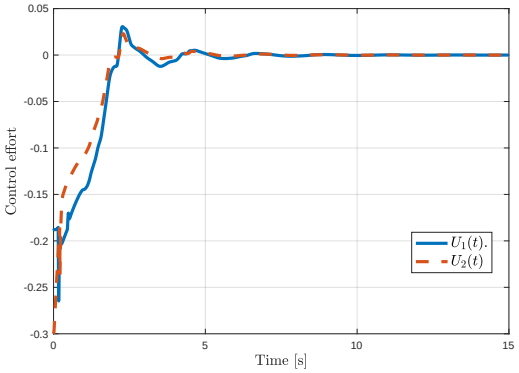

Fig. 3. Evolution of the control effort $U_1(t)$ and $U_2(t)$.

VI. Conclusion and perspectives

In this paper, we have introduced a recursive methodology to design a stabilizing output-feedback controller for a network of N PDE subsystems with a chain structure. The different subsystems are interconnected through their boundaries, and the control input is located at one extremity of the chain. The proposed framework required several fundamental properties for each subsystem: output trajectory tracking, input-to-state stability, predictability (we can design predictors of the different states), and observability. We have shown that these properties were always satisfied for hyperbolic subsystems. The proposed approach is modular in that additional subsystems can easily be included. Moreover, we believe the proposed framework can be extended to different types of subsystems (such as ODEs and parabolic equations) as it has been done in [50] with an ODE at the end of the chain, provided similar properties can still be verified. Recent results have been developed in [62] for parabolic systems using an analogous recursive approach. In [30], the authors have considered interconnections between hyperbolic and parabolic systems. It could be interesting in future works to analyze if the control strategy developed in [30] could be applied to interconnections of hyperbolic systems and compared with our methodology.

One current limitation of our proposed approach is its high complexity and computational burden. We need to compute state predictions for each subsystem composing the interconnection, which may be time-consuming. This numerical burden

may explode with the number of subsystems, thus making any implementation impossible. To leverage the numerical effort induced by these controllers, it may be necessary to approximate them (e.g., by finite-dimensional systems). This emphasizes the necessity of investigating the questions of model reduction using late-lumping techniques [33], [15]. Recently, machine-learning approximations (based on the DeepONet algorithm) have been successfully tested in [54] on simple examples, but there is no general proof of convergence yet. Concerning implementing the proposed recursive control law, we underline that the robustness aspects have been neglected in this paper. However, we believe the results from [9] can be adjusted to cover the proposed control strategy. In future works, we will also consider the case of having the actuator located at one of the intersection nodes of the chain. As shown in [52], [51], this raises challenging **controllability questions**. In most cases, such interconnected systems may not be controllable, and appropriate controllability conditions must be derived.

APPENDIX A
TECHNICAL LEMMA

Lemma 2: Consider the i^{th} subsystem ($1 \leq i \leq N$). There exist matrix functions $\check{G}_i$ and $\tilde{G}_i$ such that for all $1 \leq j \leq n_i$, for all $x \in [0,1]$, for all $t > t_0 + \frac{1}{\lambda_i^1}$

$$\alpha_i^j(t,x) = \alpha_i^j(t + \frac{1-x}{\lambda_i^j}, 1) + \sum_{\ell=j+1}^{n_i} \int_0^{\frac{1-x}{\lambda_i^j}} (\check{G}_i(x,\nu))_{j\ell} \alpha_i^\ell(t+\nu, 1) d\nu. \quad (39)$$

$$\alpha_i^j(t,x) = \alpha_i^j(t - \frac{x}{\lambda_i^j}, 0) + \sum_{\ell=j+1}^{n_i} \int_0^{\frac{x}{\lambda_i^j}} (\tilde{G}_i(x,\nu))_{j\ell} \alpha_i^\ell(t-\nu, 0) d\nu. \quad (40)$$

Proof 7: We will only prove that equation (39). The proof of equation (40) is analogous. The proof will be done recursively. Applying the method of characteristics to equation (17), we obtain for all $1 \leq j \leq n_i$, for all $x \in [0,1]$, and for all $t > t_0 + \frac{1}{\lambda_i^1}$,

$$\alpha_i^j(t,x) = \alpha_i^j(t + \frac{1-x}{\lambda_i^j}, 1) - \sum_{k=j+1}^{n_i} \int_0^{\frac{1-x}{\lambda_i^j}} (G_i(x + \lambda_i^j \nu))_{jk} \alpha_i^k(t+\nu, x + \lambda_i^j \nu) d\nu. \quad (41)$$

In particular, since the matrix G_i is strictly upper-triangular, we obtain

$$\alpha_i^{n_i}(t,x) = \alpha_{n_i}^j(t + \frac{1-x}{\lambda_i^{n_i}}, 1).$$

Let us now consider $1 < j \leq n_i$ and assume that there exist matrix functions $\check{G}_i$ such that equation (39) holds for any $k \geq j$. Applying equation (39) to equation (41), we obtain

$$\alpha_i^{j-1}(t,x) = \alpha_i^{j-1}(t + \frac{1-x}{\lambda_i^{j-1}}, 1) - \sum_{k=j}^{n_i} \int_0^{\frac{1-x}{\lambda_i^{j-1}}} (G_i(x + \lambda_i^{j-1}\nu))_{j-1,k} [\alpha_i^k(t + \nu + \frac{1-x-\lambda_i^{j-1}\nu}{\lambda_i^k}, 1) + \sum_{\ell=k+1}^{n_i} \int_0^{\frac{1-x-\lambda_i^{j-1}\nu}{\lambda_i^k}} (\check{G}_i(x + \lambda_i^{j-1}\nu, \eta))_{k\ell} \alpha_i^\ell(t+\nu+\eta, 1) d\eta] d\nu.$$

We obtain the desired expression by performing changes of variables and applying Fubini's theorem. ■

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