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Finite dimensional Galerkin approximations for mildly-coupled bilinear quantum systems

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Abstract—Several infinite dimensional bilinear quantum systems encountered in the physics literature can be described, with good precision, by appropriate finite dimensional approximations. We present a regularity condition sufficient for the existence of these approximations. We also show a counterexample of a system that is approximately controllable while its infinite dimensional dynamics cannot be precisely described by its finite dimensional Galerkin approximations.

I. INTRODUCTION

A. Quantum control

The state of a quantum system evolving in a finite dimensional Riemannian manifold Ω is described by the wave function, namely a point ψ in the unit sphere of the space $L^2(\Omega, \mathbf{C})$. The square of the modulus of the wave function $|\psi|^2 : \Omega \rightarrow [0, +\infty)$ is the density of probability for the state of the system. Classical examples of choices for Ω could be the set $\{0, 1\}$ to model the spin of an electron, or $(0, +\infty)$ endowed with the usual distance to model the length of a chemical bond, or $SO(3)$ endowed with its bi-invariant Riemannian structure to model the orientation of a rigid molecule in space.

When the system is suitably isolated from external influence and submitted to a sufficiently smooth external excitation (e.g., an electric field), the wave function evolves following the Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = -\Delta \psi + V(x)\psi + u(t)W(x)\psi, \quad x \in \Omega \quad (1)$$

where Δ is the Laplace Beltrami operator on Ω , $V : \Omega \rightarrow \mathbf{R}$ is a physical potential (accounting for the properties of the system) and $W : \Omega \rightarrow \mathbf{R}$ a potential accounting for the properties of the external excitation. The dynamics is usually called *bilinear* in order to emphasize the bilinear dependence of the term $u(t)W(x)\psi$ in (u, ψ) .

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The aim of quantum control is to design a (open-loop) control $u : [0, T] \rightarrow \mathbf{R}$ such that the solution of the Schrödinger equation (1) satisfies desirable properties, for instance: the final point $\psi(T)$ is close, in some sense, to a given target.

B. Finite dimensional approximations

As soon as Ω is not a finite set, the state space (i.e., the unit sphere of the Hilbert space $L^2(\Omega, \mathbf{C})$) is infinite dimensional. This leads to considerable difficulties, and especially - in the field of quantum control- for the definition or the computation of the solutions of (1), or the study of the controllability of (1).

A widely used and remarkably efficient techniques consists in replacing (1) with a finite dimensional problem, for which the aforementioned problems of definitions, existence or computation of solutions are easily tackled with standard calculus and geometric techniques.

An important issue, both from a practical and a theoretical viewpoint, is to ensure that the results obtained on the finite dimensional approximations are true (or almost true, in a sense to be precised) for the original infinite dimensional system (1).

C. Content

In this note, we study the existence of the so-called *Good Galerkin Approximations* (GGA for short), i.e. a finite dimensional Galerkin approximation of the infinite dimensional system whose evolution is “close”, in some suitable topology, to the one of the original system. The term “Good” is chosen in order to emphasize the facts that these approximations are not only valid for a specific control u , but for every control, and every times, provided the energy in the control is bounded. Our aim is to present the possible difficulties arising when approaching infinite dimensional bilinear conservative dynamics with their finite dimensional approximations. The two main original results of this note are a positive one (Theorem 3) with sufficient condition for the existence of a GGA and a negative one (given in Section V), namely an example for which a GGA cannot exist.

The paper is organized as follows. In Section II, we reformulate the problem (1) in the abstract framework of conservative dynamics in Hilbert spaces and recall well-known well-posedness and controllability results, along with their link with the regularity of the systems under consideration. In particular, we recall the notion of mildly-coupled systems (Section II-C). Section III is devoted to various concepts of finite dimensional approximation and to the introduction

to the central notion of Good Galerkin Approximations. Section IV states an original result on the existence of GGA for mildly-coupled systems. Finally, we provide in Section V an original counter-example showing that a GGA does not exist in general.

II. FRAMEWORK AND ASSUMPTIONS

Let \mathcal{H} be a complex Hilbert space with Hilbert product $\langle \cdot, \cdot \rangle$, and associated norm $\| \cdot \|$. We consider a conservative bilinear control system on \mathcal{H}

$$\psi'(t) = (A + u(t)B)\psi(t), \quad (2)$$

where $\psi(t)$ is the state at time t and belongs to \mathcal{H} . The linear operators A and B are skew-symmetric operators and u is a scalar function called control taking value in a $U \subset \mathbf{R}$. The operator A is sometimes called *drift* and B is called *control* operator.

A. Solutions and Propagators

The definition of solution for system (2) when \mathcal{H} is infinite dimensional is, in general, not trivial, indeed, even the definition of the sum of unbounded operators is not obvious. In order to define the solutions of system (2), we make the following assumptions.

Assumption 1: Let (A, B, U) be such that

- 1) A is a linear skew-adjoint operator with domain $D(A) \subset \mathcal{H}$ such that $A : D(A) \rightarrow \mathcal{H}$ is invertible;
- 2) B is a linear skew-symmetric operator with domain $D(B) \subset \mathcal{H}$ such that $D(A) \subset D(B)$;
- 3) The *control set* U is a subset of \mathbf{R} containing 0;
- 4) For every u in U , the operator $A + uB$ with domain $D(A)$ is essentially skew-adjoint.

Since, by Assumption 1, $A + uB$ is essentially skew-adjoint, it generates a group of unitary operators. We can therefore define the solution and the propagator of system (2) associated with a piecewise constant control. Namely, for any $T > 0$, given any initial state ψ_0 , and any piecewise constant control

$$u = \sum_{i=1}^k u_i \mathbf{1}_{[t_i, t_{i+1})}, \quad (3)$$

for a partition $t_1 = 0 < t_2 < \dots < t_{k+1} = T$ of $[0, T]$ and a finite sequence $(u_i)_{1 \leq i \leq k}$ in U , Equation (2), with value ψ_0 at $t = 0$, can be integrated in \mathcal{H} providing a unique continuous solution given by $t \in [0, T] \mapsto \Upsilon_t^u \psi_0$ where for $t \in [t_i, t_{i+1}]$

$$\Upsilon_t^u := e^{(t-t_i)(A+u_i B)} \circ \dots \circ e^{(t_2-t_1)(A+u_1 B)}, \quad (4)$$

is called the *propagator* associated with $A + u(t)B$.

In order to define the propagator of (2) associated with a bounded variation controls we denote by $BV(I, E)$ the space of bounded variation functions from $I \subset \mathbf{R}$ interval to $E \subset \mathbf{R}$ endowed with the norm $\| \cdot \|_{BV} := \| \cdot \|_{L^1} + \text{TV}(\cdot)$, where $\text{TV}(\cdot)$ denotes the *total variation*. On $BV(I, E)$, we consider the following notion of convergence : $(u_k)_{k \in \mathbf{N}} \in BV(I, E)$ converges to $u \in BV(I, E)$ if $(u_k)_{k \in \mathbf{N}}$ is a bounded sequence in $BV(I, E)$ pointwise convergent to

$u \in BV(I, E)$. The definition of propagators extends by continuity to BV controls, see for instance [1, Corollary 9]

Definition 1: The operator B is said *relatively bounded* with respect to A , or A -bounded, if $D(A) \subset D(B)$ and there exist $a, b > 0$ such that for every ψ in $D(A)$,

$$\|B\psi\| \leq a\|A\psi\| + b\|\psi\|. \quad (5)$$

In this case we call $\|B\|_A$ the infimum among all the constants a satisfying (5) (and, in principle, it can be 0).

As a consequence of Kato–Rellich Theorem (see, for instance, [2, Theorem X.12]) the A -boundedness of B is a sufficient condition for Assumption 1.4.

B. Controllability

We say that system (2) is controllable if, given an initial state $\psi_0 \in \mathcal{H}$ and a final state $\psi_1 \in \mathcal{H}$, with $\|\psi_0\| = 1 = \|\psi_1\|$, there exists a (open-loop) control $u(t)$ in $BV((0, \infty), U)$ and $T > 0$, such that $\Upsilon_T^u \psi_0 = \psi_1$. The controllability of system (2) is a well-established topic when the state space \mathcal{H} is finite-dimensional, i.e. when the manifold Ω is finite, see, for instance [3] or [4] and references therein. Many controllability results in the finite-dimensional framework rely on general controllability methods for left-invariant control systems on compact Lie groups ([5], [6], [7], [8]).

When the state space \mathcal{H} is infinite-dimensional the situation is complicated by the subtleties of the evolution in Banach spaces and the consequent fragmental nature of controllability theory for PDEs.

One of the first known results is a negative one: indeed when B is a bounded operator the bilinear Schrödinger equation is not exactly controllable, namely the attainable set has empty interior as a meagre set [9], [10]. The obstruction to exact controllability holds even when considering very large class of controls [1], as for instance L^1 controls [11] or Radon measures [12]. In certain cases, it is possible to prove exact controllability for the potential well in suitable functional spaces on a real interval (see [13], [14], [15]). The results extend to a system describing a particle confined on a radially symmetric 2D domains [16]. However in higher dimension and for more general systems the exact description of the reachable set is a difficult task. The literature hence focuses on weaker controllability properties. Approximate controllability results have been obtained with different techniques: adiabatic control ([17], [18]), Lyapunov methods ([19], [20], [21]), and Lie-algebraic methods ([22], [23], [24], [25], [26], [27], [28]).

C. Mildly coupled systems

In this paper we use the notion of *mild-coupling*, firstly introduced in [1]. This is a regularity assumption on system (2) and it is a generalization of the notion of weak-coupling introduced in [29]. Given a skew-adjoint operator A and $k \in \mathbf{R}$, $k \geq 0$, we define

$$\|\psi\|_{k/2} = \sqrt{\langle |A|^k \psi, \psi \rangle}.$$

Definition 2 (Mild coupling): Let $k \in \mathbf{R}$, $k \geq 0$. A pair of skew-adjoint operators (A, B) is *k-mildly coupled* if

- (i) A is invertible with bounded inverse from $D(A)$ to \mathcal{H} ,
- (ii) for any $t \in \mathbf{R}$, $e^{tB}D(|A|^{k/2}) \subset D(|A|^{k/2})$,
- (iii) there exists $c \geq 0$ and $c' \geq 0$ such that $B - c$ and $-B - c'$ generate contraction semigroups on $D(|A|^{k/2})$ for the norm $\|\cdot\|_{k/2}$.

Remark 1: In Definition 2.(iii) the restriction to $D(|A|^{k/2})$ of the group generated by $B - c$, in principle acting on \mathcal{H} , defines a contraction semigroup. We thus have two generators and, thanks to Definition 2.(ii), the domain of the second one is included in the first and the smallest domain is dense in \mathcal{H} . The same comment can be made for $-B - c'$. Neglecting the constant c and c' and the minus sign, we identify these three operators (which are closed in \mathcal{H} and $D(|A|^{k/2})$ respectively) and with an abuse of notation we denote them by the same symbol B as they are restrictions of B . The domains of $B - c$ and $-B - c'$ acting on $D(|A|^{k/2})$ are, in fact, both equal to $\{\phi \in D(|A|^{k/2}) \cap D(B) \mid B\phi \in D(|A|^{k/2})\}$. They both contain this set since whenever ϕ is in this set, then

$$e^{\pm tB}\phi - \phi \mp tB\phi = \int_0^t (e^{\pm tB} - 1)B\phi ds = o(|t|),$$

in \mathcal{H} or $D(|A|^{k/2})$ and, $t \mapsto e^{\pm tB}\phi$ is differentiable in $D(|A|^{k/2})$, then it is also differentiable in \mathcal{H} .

The notion of mild coupling is related to the notion of *weak coupling*, introduced in [29, Definition 1] for Schrödinger operators with pure point spectrum.

Assumption 2: Let (A, B, U, Φ) be such that

- 1) (A, B, U) satisfies Assumption 1
- 2) The skew-adjoint operator A has pure point spectrum with an associated complete orthonormal basis $\Phi = (\phi_k)_{k \in \mathbf{N}}$ of eigenvectors ;

Definition 3: Let $k \in \mathbf{R}$, $k \geq 0$. A pair of skew-adjoint operators (A, B) is *k-weakly coupled* if:

- (i) (A, B, \mathbf{R}) satisfies Assumption 2
- (ii) for every $u \in \mathbf{R}$,

$$D(|A + uB|^{k/2}) = D(|A|^{k/2})$$

- (iii) there exists a constant C such that for every ψ in $D(|A|^k)$,

$$|\Re \langle |A|^k \psi, B\psi \rangle| \leq C |\langle |A|^k \psi, \psi \rangle|.$$

The relation between these two definitions is given by the following lemma.

Lemma 1: Let (A, B) be *k-weakly coupled* for some $k \geq 0$ and A be invertible. Then (A, B) is *k-mildly coupled*.

Proof: This is [1, Lemma 21]. ■

Remark 2: The assumption on the invertibility of A is needed to ensure that $\|\cdot\|_{k/2}$ is a norm equivalent to the graph norm of $D(|A|^{k/2})$ and, for many examples in the physics literature is not restrictive. Indeed whenever the operator A is skew-adjoint with a spectral gap up to replacing A by $A - \lambda i$ for a suitable λ in \mathbf{R} , one can assume that A is invertible. This translation on A is physically irrelevant since it corresponds to a phase shift on the propagator.

Remark 3: In general the converse of Lemma 1 is not true. There are examples of systems with operators (A, B) that are *k-mildly coupled* but not *k-weakly coupled*. Indeed, Assumption (ii) in Definition 3 is quite a strong one. For instance, consider the pair of operators $A = \frac{d}{dx}$ and $B = i \frac{d^2}{dx^2}$ in $\mathcal{H} = L^2([0, 1])$ with respective domain $D(A) = \{\phi \in AC[0, 1] \mid \phi(0) = \phi(1)\}$ and $D(B) = \{\phi \in L^2([0, 1]) \mid \phi' \in AC[0, 1], \phi(0) = \phi(1), \phi'(0) = \phi'(1)\}$ corresponding, up to a time-reparametrization, to the conservative bilinear system

$$i \frac{d}{dt} \varphi(t, x) = -\Delta_x \varphi(t, x) + iu \frac{d}{dx} \varphi(t, x)$$

with $\varphi(t, 0) = \varphi(t, 1)$ and $d_x \varphi(t, 0) = d_x \varphi(t, 1)$. This system is 2-mildly coupled (see for instance [30, Section VIII.2] and [2, Example 1, Section X.1]) while it is not weakly coupled since $D(A + uB) = D(B) \subsetneq D(A)$.

More in general, given every invertible skew-adjoint unbounded operator A , the pair (A, iA^2) is 2-mildly coupled and *not* 2-weakly coupled since $D(A + iA^2) = D(A^2) \neq D(A)$.

Remark 4: Many systems encountered in the physics literature are sufficiently regular to be weakly-coupled, hence mildly-coupled. See, for instance, [29, Section III and Section IV] or the recent paper [31]. The main advantage of Definition 2 is that no assumptions are made on the spectrum of A : the results in this paper may also apply in presence of continuous spectrum. This is the case, for instance, of the Morse potential controlled by a (bounded) external field [32].

III. GALERKIN APPROXIMATIONS

A. Compressions and dynamics

For every N in \mathbf{N} , we define

$$\mathcal{L}_N^\Phi = \text{span}\{\phi_1, \dots, \phi_N\},$$

and by π_N^Φ the projection of \mathcal{H} on \mathcal{L}_N^Φ , namely

$$\begin{aligned} \pi_N^\Phi : \mathcal{H} &\rightarrow \mathcal{H} \\ \psi &\mapsto \sum_{j=1}^N \langle \phi_j, \psi \rangle \phi_j. \end{aligned} \quad (6)$$

Remark 5: We stress the fact that the projections in (6) and, as a consequence, the compressions of operators strongly depend on the basis Φ . However, for the sake of readability, from now on we drop the mention to Φ .

Remark 6: When it does not create ambiguities we identify $\text{Im}(\pi_N) = \mathcal{L}_N$ with \mathbf{C}^N .

We define

$$A^{(N)} = \pi_N A \pi_N \quad \text{and} \quad B^{(N)} = \pi_N B \pi_N.$$

as the *compressions* of A and B (respectively) associated with \mathcal{L}_N .

Definition 4: The Galerkin approximation of order n of system (2) is the system in \mathbf{C}^N described by the equation

$$\dot{x} = \left(A^{(N)} + uB^{(N)} \right) x. \quad (7)$$

Since \mathcal{L}_N is a finite dimensional $(A^{(N)}, B^{(N)}, \mathbf{R})$ satisfies Assumptions 2. We can therefore define the unitary propagator $Y_{t,0}^{(N)}(u)$ of (7) associated with a BV control u .

B. A general (yet deceptive) result

Proposition 2: Let (A, B, Φ, U) satisfy Assumption 2. For every n in \mathbf{N} , for every $T > 0$, for every $u : [0, T] \rightarrow U$ piecewise constant, for every $\varepsilon > 0$, there exists N in \mathbf{N} , such that for every $k \geq N$, for every t in $[0, T]$,

$$\|Y_{t,0}^{(N)}(u)\pi_N - \Upsilon_t^u \pi_N\| \leq \varepsilon.$$

Proof: The proof of Proposition 2 is inspired by the recent [33, Proposition 2.11], stated in the framework of ultra-weak solutions in the sense of Lions. We present here an elementary proof that does not rely on advanced notions of solutions of PDEs.

Without loss of generality we consider u constant, and conclude in the general case by concatenation.

The subspace $\text{span}(\Phi)$ is dense in \mathcal{H} and, hence, it is a core of $A^{(k)} + uB^{(k)}$, $k \in \mathbf{N}$ and of $A + uB$ as well (this is a reformulation of Assumption 2). For any ϕ in $\text{span}(\Phi)$, $((A^{(k)} + uB^{(k)})\phi)_{k \in \mathbf{N}}$ converges to $(A + uB)\phi$. We deduce from [34, Theorem VIII.25.a], that $(A^{(k)} + uB^{(k)})_{k \in \mathbf{N}}$ converges to $A + uB$ in the strong resolvent sense (for a precise definition of convergence in the strong resolvent sense, see [34, Section VIII.7]).

From Trotter's theorem, [34, Theorem VIII.21], the sequence $(e^{t(A^{(k)} + uB^{(k)})})_{k \in \mathbf{N}}$ converges strongly to $e^{t(A + uB)}$ uniformly in t in any bounded interval. ■

Proposition 2 guarantees the possibility to approach any conservative bilinear dynamics with a finite dimensional one as soon as the drift A has pure point spectrum and the eigenvectors of the drift are in the domain of the control operator B . These conditions are fulfilled, to the best of our knowledge, for all the bilinear quantum systems with discrete spectrum encountered in the physics literature.

While the result of Proposition 2 may seem appealing, its importance should not be overestimated. The main drawback is that the finite dimensional approximation strongly depends on the time and on the control u . It is not clear at all (and false in general, see Section V) that the same approximation will provide the same precision when using controls with the same size (say L^1 norm, which is the most natural to consider on the set of control laws, see [35]). In particular, it is not clear a priori how Proposition 2 can be used to obtain approximate controllability results.

To overcome this difficulty, we introduce here the notion of Good Galerkin Approximations.

IV. GOOD GALERKYN APPROXIMATIONS AND MILD-COUPLING

In [22] and [23], precise estimates of the distance between the infinite dimensional systems and some of its Galerkin approximations are used to prove that systems of type (1) are approximately controllable under physical conditions of non-degeneracy of the discrete spectrum of $-\Delta + V$. These estimates are derived for a sequence of *ad hoc* controls designed to steer the system from a given source to a given target. Besides the discreteness of the spectrum of $-\Delta + V$, very few regularity assumptions are made on (1). Since the potential W is not assumed to be bounded or regular (say, not

even continuous), the estimates obtained for a control u can possibly fail to hold for controls close to u , for instance, in a small neighborhood of u for some H^k norm. In this paper we give a sufficient condition for mildly coupled system to admit a sequence of Good Galerkin Approximations (see Theorem 3 below). In other words the propagator of a mildly-coupled system, can be precisely approached by propagators of finite dimensional bilinear Galerkin approximations.

Theorem 3: Let (A, B) be k -mildly coupled for some $k \geq 0$. Assume that B is A -bounded and that $B(1 - A)^{-1}$ is compact. Let $U = [-1/(2\|B\|_A), 1/(2\|B\|_A)]$. Then for every $\varepsilon > 0$, $L \geq 0$, $p \in \mathbf{N}$, $s \in [0, 1 + k/2]$, and $(\psi_j)_{1 \leq j \leq p}$ in $D(|A|^{1+k/2})^p$ there exists $N \in \mathbf{N}$ such that for any $u \in BV([0, T], U)$,

$$\|u\|_{BV} < L \Rightarrow \|\Upsilon_t^u(\psi_j) - Y_{t,0}^{(N)}(u)\pi_N \psi_j\|_s < \varepsilon,$$

for every $t \geq 0$ and $j = 1, \dots, p$.

The proof of Theorem 3 relies on energy estimates for the propagator of (2) and is presented in Section IV-B below.

A. Energy bounds for mildly coupled systems

If (A, B) is k -mildly coupled the A -boundedness of B extends naturally to $D(|A|^{k/2})$. Hence it is not restricting to consider $\mathcal{H} = D(|A|^{k/2})$ as stated in the following result.

Lemma 4: Let k be a nonnegative real. Let (A, B) be k -mildly coupled and such that B is A -bounded. Then

$$\inf_{\lambda > 0} \|B(A - \lambda)^{-1}\|_{L(D(|A|^{\frac{k}{2}}), D(|A|^{\frac{k}{2}}))} \leq \|B\|_A$$

Proof: This is [1, Lemma 23] ■

As mentioned in Remark 3 the condition $D(|A + uB|^{k/2}) = D(|A|^{k/2})$ in Definition 3 is quite strong and, in principle is hard to check in practice. The following result provides a easy verifiable condition. Roughly speaking, the idea is that if the control u is “sufficiently small” then $A + uB$ is a small perturbation of A .

Lemma 5: Let k be a positive real, (A, B) be k -mildly coupled, and $u \in \mathbf{R}$ such that $|u| < 1/\|B\|_A$. Then $D(|A|^s) = D(|A + uB|^s)$ for every $s \in [0, 1 + k/2]$.

Proof: This is [1, Lemma 24] ■

Proposition 6: Let $k \in \mathbf{R}$, $k \geq 0$ and let (A, B) be k -mildly coupled with B be A -bounded. Let $U = [-1/(2\|B\|_A), 1/(2\|B\|_A)]$. Then

(i) for any $t \in [0, T]$ and for any $\psi_0 \in D(|A|^{k/2})$ there exists C (depending only on A , B , and k) such that

$$\|\Upsilon_t^u(\psi_0)\|_{k/2} \leq C e^{\|u\|_{BV}} \|\psi_0\|_{k/2}.$$

(ii) for any $t \in [0, T]$ and for any $\psi_0 \in D(|A|^{1+k/2})$ there exists M (depending only on A , B , k , and $\|u\|_{L^\infty([0, T])}$) such that

$$\|\Upsilon_t^u(\psi_0)\|_{1+k/2} \leq M e^{M\|u\|_{BV}} \|\psi_0\|_{1+k/2}.$$

Proof: This result is a consequence of [1, Proposition 25]. ■

Lemma 7: Let k be a positive real. Let (A, B) be k -mildly coupled and let Φ be an Hilbert basis made of eigenvectors of A . Then, for every N in \mathbf{N} , $(A^{(N)}, B^{(N)})$ is k -mildly coupled.

Proof: From [1, Proposition 20] there exists $C = C(A, B, k)$ such that

$$|\Re\langle |A|^k \psi, B\psi \rangle| \leq C \|\psi\|_{D(|A|^{k/2})},$$

for every $\psi \in D(|A|^{k/2})$. Hence restricted to \mathcal{L}_N , we obtain $(A^{(N)}, B^{(N)})$ that is k -mildly coupled. ■

B. Proof of the main result

Since the propagators are contractions in \mathcal{H} , it is sufficient to prove the statement for $(\psi_j)_{1 \leq j \leq p}$ in the dense subset $D(A)$. Fix an arbitrary $\psi \in D(A)$, let $N \in \mathbf{N}$ and u in BV with values in $[-1/(2\|B\|_A), 1/(2\|B\|_A)]$.

Consider $y^u : t \mapsto \pi_N \Upsilon_t^u(\psi)$, that is, an absolutely continuous function satisfying

$$\dot{y}^u(t) = (A^{(N)} + u(t)B^{(N)})y^u(t) + u(t)\pi_N B(I - \pi_N)\Upsilon_t^u(\psi),$$

for almost every t . Then, by variation of constants formula

$$\begin{aligned} y^u(t) &= Y_{t,0}^{(N)}(u)\psi \\ &+ \int_0^t Y_{s,t}^{(N)}(u)u(s)\pi_N B(I - \pi_N)\Upsilon_s^u(\psi)ds, \end{aligned}$$

for every t .

Now

$$\begin{aligned} \|\Upsilon_t^u(\psi) - Y_{t,0}^{(N)}(u)\pi_N \psi\| &\leq \|(I - \pi_N)\Upsilon_t^u(\psi)\| \\ &+ \|y^u(t) - Y_{t,0}^{(N)}(u)\psi\|. \end{aligned}$$

Since $Y_{t,0}^{(N)}(u)$ is unitary

$$\begin{aligned} \|y^u(t) - Y_{t,0}^{(N)}(u)\psi\| &\leq \|u\|_{L^1} \|\pi_N B(I - \pi_N)\Upsilon_t^u(\psi)\| \\ &\leq \|u\|_{L^1} \|\pi_N B(I - A)^{-1}(I - \pi_N)\Upsilon_t^u(\psi)\| \\ &+ \|u\|_{L^1} \|\pi_N B(I - A)^{-1}(I - \pi_N)A\Upsilon_t^u(\psi)\|. \end{aligned}$$

Finally, notice that $I - \pi_N$ tends strongly to 0, as N goes to infinity. Moreover since $B(I - A)^{-1}$ is compact, then $B(I - A)^{-1}(I - \pi_N)$ tends to 0 in norm. The conclusion for $s = 0$ then follows from Proposition 6 point (i) with $k = 0$.

Consider now the general case $s \in (0, 1)$. The sequence $(\pi_N \Upsilon_t^u(\psi) - Y_{t,0}^{(N)}(u)(\psi))_{N \in \mathbf{N}}$ converges to zero in \mathcal{H} as N tends to infinity, uniformly with respect to $t \geq 0$ and u with total variation less than L (this is the case treated previously). The same sequence $(\pi_N \Upsilon_t^u(\psi) - Y_{t,0}^{(N)}(u)(\psi))_{N \in \mathbf{N}}$ is bounded in $D(A)$ uniformly with respect to $t \geq 0$ and u with total variation less than L , indeed

$$\begin{aligned} \|\pi_N \Upsilon_t^u(\psi) - Y_{t,0}^{(N)}(u)(\psi)\|_1 &\leq \\ \|\pi_N \Upsilon_t^u(\psi)\|_1 + \|Y_{t,0}^{(N)}(u)(\psi)\|_1 &\end{aligned}$$

where both terms on the right hand side are bounded, uniformly with respect to N , u and t since the projections π_N are bounded in $D(A)$ as they commute with A . The conclusion follows from standard interpolation arguments (see, for instance, [1, Lemma 38]).

The only remaining point to check is the boundedness of $(\pi_N \Upsilon_t^u(\psi) - Y_{t,0}^{(N)}(u)(\psi))_{N \in \mathbf{N}}$ in $D(|A|^{1+k/2})$ which follows from Proposition 6.

Remark 7: The compactness of the operator $B(1 - A)^{-1}$ is crucial to estimate the norm

$$\|(I - \pi_N)B\pi_N \psi(t)\|,$$

uniformly in $t \in [0, T]$ and $\|u\|_{BV([0, T])} \leq L$. This compactness is used only for the uniform bound of the above norm.

V. EXAMPLE: A CONTROLLABLE SYSTEM WITHOUT GOOD GALERKIN APPROXIMATION

We present here a (non-physical) example which is simultaneously approximately controllable and cannot be suitably represented by its finite dimensional Galerkin approximations. It consists in system (2), in which the Hilbert space \mathcal{H} is equal to $L^2(\mathbf{R}, \mathbf{C})$ endowed with its standard scalar product $\langle f, g \rangle = \int_{\mathbf{R}} \bar{f}g$, A is equal to $i((\Delta + x^2) + (\Delta + x^2)^{-1})$ and B is the multiplication by x^4 . The skew symmetric operator is skew adjoint, with spectrum $Sp(A) = i\{(2k - 1)/2 + 2/(2k - 1), k \in \mathbf{N}\}$. The k^{th} Hermite function ϕ_k is an eigenstate of A associated with the eigenvalue $i(2k + 1)/2 + i2/(2k - 1)$. The family $\Phi = (\phi_k)_{k \in \mathbf{N}}$ is an Hilbert basis of \mathcal{H} .

Notice that B is not relatively bounded with respect to A , while, for every (constant) u in $[0, +\infty)$, $A + uB$ is the generator of group of unitary transformations. We define the solutions of (2) only for controls u that are piecewise constant.

For parity reasons, B leaves invariant the sets $\Phi^e = (\phi_{2k})_{k \in \mathbf{N}}$ and $\Phi^o = (\phi_{2k+1})_{k \in \mathbf{N}}$. Hence, no global controllability in \mathcal{H} is to be expected.

Defining $\phi_k^o = \phi_{2k+1}$ for every k in \mathbf{N} and $\Phi^o = (\phi_k^o)_{k \in \mathbf{N}}$, the closure \mathcal{H}^o of $\text{span}\Phi^o$ is an Hilbert space stable by A and B . The restriction of A and B to \mathcal{H}^o are denoted by A^o and B^o respectively.

The set $\{(j, k) \in \mathbf{N}^2, |j - k| = 1\}$ is a non resonant chain of connectedness of (A^o, B^o, Φ^o) (in the sense of [23, Definition 2.5]). Hence the control system (A^o, B^o) is approximately controllable in \mathcal{H}^o (see, for instance [23], [25], [28]). Moreover, for every j in \mathbf{N} , for every $\varepsilon > 0$, there exists a piecewise constant positive control $u_{j,\varepsilon}$ with L^1 norm less than $\sum_{l=1}^{j-1} \frac{\pi}{2n^2} < \frac{\pi^3}{12}$ that steers this system from the first eigenstate ϕ_1^o to an ε -neighborhood of ϕ_j^o . The $(A^o)^s$ norm of ϕ_j^o tends to $+\infty$ as j tends to infinity. Hence (A^o, B^o) does not satisfy the conclusions of Theorem 3.

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