



HAL
open science

Ants on the highway

Anahí Gajardo, Victor Lutfalla, Michaël Rao

► **To cite this version:**

| Anahí Gajardo, Victor Lutfalla, Michaël Rao. Ants on the highway. 2024. hal-04696322

HAL Id: hal-04696322

<https://hal.science/hal-04696322>

Preprint submitted on 13 Sep 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Ants on the highway

Anahí Gajardo, Victor H. Lutfalla, Michaël Rao

Departamento de Ingeniería Matemática & Centro de Investigación
en Ingeniería Matemática, Universidad de Concepción, Casilla
160-C, Concepción, Chile
`anahi@ing-mat.udec.cl`
Aix-Marseille Univ, CNRS, I2M, Marseille, France
`victor.lutfalla@math.cnrs.fr`
CNRS& LIP, Ecole Normale Supérieure de Lyon, France
`michael.rao@ens-lyon.fr`

Abstract

We perform intensive computations of Generalised Langton's Ants, discovering rules with a big number of highways. We depict the structure of some of them, formally proving that the number of highways which are possible for a given rule does not need to be bounded, moreover it can be infinite. The frequency of appearing of these highways is very unequal within a given generalised ant rule, in some cases these frequencies where found in a ratio of $1/10^7$ in simulations, suggesting that those highways that appears as the only possible asymptotic behaviour of some rules, might be accompanied by a big family of very infrequent ones.

Keywords: Langton's ant, Turing machines

1 Introduction

Introduced in the 80's independently by different researchers [3,9,11], the automaton mostly known as *Langton's ant* remains intriguing. It is cited in vulgarisation media as a paradigm of 'emergent phenomena', 'unpredictability', 'abnormal diffusion' and as evidence of the impossibility of a 'theory of everything'.

Its *emerging phenomenon* consists in a pattern that makes the ant to move in such a way that after 104 steps, it is shifted by (-2,2) and the same pattern appears again in the configuration, also shifted. Given this, the ant movements will repeat again and again, producing a periodic behaviour known as the *highway* (see figure 2(a)). In the terminology of [10], this behaviour is due to the presence of a periodic point in the *Moving Tape Model*, with the particularity that in the propagating direction, the background is homogeneous.

The intriguing fact is that this pattern spontaneously seems to appear on every simulation over initially finite configurations, the transient phase vary mostly from 1000 to 100,000 iterations for initial configurations with only one black cell in a radius of 3 cells around the ant position; and until now, no simulation suggest the conjecture might be false.

Several generalisations have been proposed: with more movements [2], with more states [7], on other grids [5], and on other dimensions [4]. The generalization that seems to better preserve the Langton’s ant properties is the one we call *generalised ants* [7]. In this class, emergent behaviours are also observed, with different periods, or different shapes. In addition, there are rules where no highway or any similar asymptotic behaviour arrives for some initial finite configurations.

Motivated by the fact that computational simulations can show only a very restricted part of the whole (and infinite) space of possibilities, and that on such complex systems strange phenomena may appear only on extremely big initial configurations, we started a series of high performance simulations, looking for new behaviours that could put in evidence this limitation of the computational tools.

Section 2 introduces the model and makes a journey over the main theoretical founding about generalised ants since their discovery; also it formally defines the concept of “*highway*”. Section 3 resumes what simulations gave us. In Section 4, we establish our first theoretical result: *there is a family with a growing set of highways*; while our second result is demonstrated in Section 5: *there is a generalised ant with an infinite set of highways*. We leave some reflections and overtures in Section 6.

2 Generalised ants

Langton’s ant is an agent, provided with an orientation in the plane $\{\rightarrow, \uparrow, \leftarrow, \downarrow\}$ that moves and turns either to the right or to the left depending on the colour of its underlying cell, which it flips after leaving the cell. Cells can have only one of two colours, *white* or *black* (0 or 1). Over a white cell, the ant turns to the left and over a black cell the ant turns to the right. Generalised ants have more colours, its movement rule is given by the meaning of each colour, and this one changes following a simple increasing cycle.

Definition 1. [7] *A generalised ant with rule word $w \in \{L, R\}^+$ is an automaton with state set $Q = \{\rightarrow, \uparrow, \leftarrow, \downarrow\}$, representing the four unitary vectors of \mathbb{Z}^2 , for example $\leftarrow = (-1, 0)$, and alphabet $A = \{0, 1, \dots, |w| - 1\} = \mathbb{Z}/\mathbb{Z}_n$, that moves over the grid \mathbb{Z}^2 . A configuration is an element from $A^{\mathbb{Z}^2} \times \mathbb{Z}^2 \times Q$, that represents:*

- *an assignment of symbols from A to each cell in \mathbb{Z}^2 called picture,*
- *a marked cell representing the position of the ant, and*
- *the current state of the ant.*

For a given configuration $(C, (i, j), d) \in A^{\mathbb{Z}^2} \times \mathbb{Z}^2 \times Q$, the global transition function T_w is defined by $T_w(C, (i, j), d) = (C', (i', j'), d')$, where:

- $C'(k, l) = \begin{cases} C(k, l) + 1 \pmod{|w|} & \text{if } (k, l) = (i, j) \\ C(k, l) & \text{otherwise} \end{cases}$,
- d' *is a rotation of d by 90° clockwise if $w_{C(i, j)} = R$, or counterclockwise if $w_{C(i, j)} = L$,*

- $(i', j') = (i, j) + d'$.

We call *trace* of ant w from configuration $(C, (i, j), d)$ the sequence of symbols that the ant encounters when evolving from configuration $(C, (i, j), d)$. A trace contains all the necessary information to recover the initial picture, up to rotations, at least on the set of visited cells [6].

We assume that the rule word has at least one L and one R , otherwise we call it *trivial*, because in that case its behaviour will simply consist in constantly turning in a fixed sense.

An ant simulator is provided at lutfalla.fr/ant/, most of the results we present in this paper are illustrated there.

Very few theoretical results exist about this system. The first remark that should be done is that the dynamics is reversible, that is, the transition function T_w is bijective, thus no information is lost through the time. After this, the most important result about ants establishes that: *independently from the initial configuration, the ant orbit will always be unbounded*. This result was formally proved by Bunimovitch and Troubetzkoy in [2] for the rule LR , but it is easy to see that it is also true for the whole family of non-trivial ants. The proof is very simple but clever, and it is based on a very strong property of this system: if the ant starts with an horizontal direction over a cell (i, j) , then it will always be with horizontal direction over every cell (i', j') such that $i + j = i' + j' \pmod 2$, and with vertical direction otherwise (Figure 1). This is because generalised ants always alternate between vertical and horizontal directions, and also the parity of the cell where they lands is always alternating between odd and even, which couples these two situations.

The most important consequence of this fact is that each cell has only two fixed entering sides: the ant can only enter a horizontal cell from east or west, and it can only enter a vertical cell from north or south. Then if an ant trajectory was finite, the set of visited cells should contain corners, that is cells with only two adjacent cells inside the set, and these forming an angle of $\pi/2$. Independently on the orientation of this cell, horizontal or vertical, it has only one entering cell inside the set and also only one output, which is either to the left or to the right of the first, this implies that the cell can be only visited finitely many times, since the rule word contains at least one L and one R . But this is impossible, because the system is reversible, thus any orbit on a finite set is periodic, and the corner must be visited infinitely many times.

As a corollary, if in a given trajectory we consider the set of cells which are visited infinitely many times, we know that this set (if not empty) cannot have any corner.

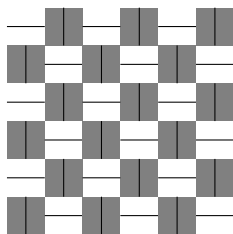


Figure 1: Each cell has two fixed entering sides: either vertical or horizontal.

Up to our knowledge, the class of generalised ants was introduced by the

first time in [7], where different behaviours are presented. Since then, other researchers started making simulations, finding quite surprising behaviours [1]. Figure 2 shows nice rules taken from mathtician.weebly.com/langtons-ant.html.

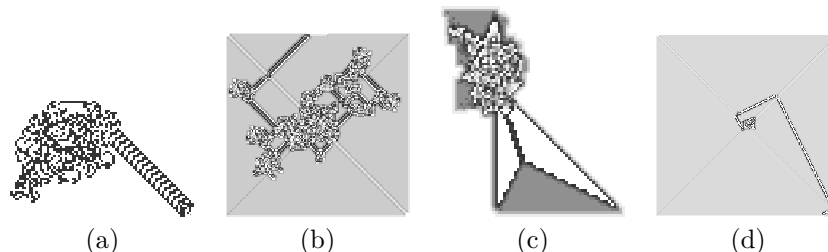


Figure 2: Zoology of emergent behaviours of generalised ants: (a) rule LR , the original rule; (b) rule $LRRRRLLR$ builds a textured square over which highways repetitively born and crash; (c) rule $LLRRLLRRLLL$ builds a growing triangle that travels on the plane; (d) rule $LRRRLLLRRL$ builds a square with a logarithmic spiral inside.

Another beautiful result is proved in [8]: *All non-trivial rules whose code word w belongs to $\{LL, RR\}^+$ will produce an evolution composed by an infinite sequence of closed trajectories that always comes back to the initial cell if the initial configuration satisfy the appropriate conditions [8].* This result has two consequences: on the initially white configuration the ant evolution produces patterns with bilateral symmetry, as butterflies (figure 3); and also, over this set of initial configurations, no *highway* will ever appear.

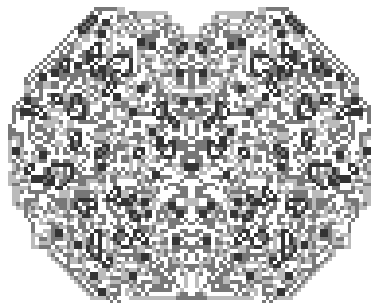


Figure 3: Configuration at iteration 817888 of rule $LLRR$ starting with the initial configuration where all the cells are white (0).

The last known result about this class shows that predicting the trajectory of a non-trivial generalised ant is P-complete and some predictions are undecidable. More precisely, any non-trivial ant rule can compute logical circuits if the appropriate initial configuration is given. Moreover, there exists a periodical initial configuration such that, the dynamics of the ant can simulate a universal Turing machine over any finite input, if the appropriate finite perturbing pattern is applied [12]. This result somehow explains why the ant analysis is so hard.

In this paper we are interested in highways, that is, emergent patterns that makes the ant to follow a periodical movement that will propagate over an homogeneous background. In order to formalise these phenomena, we need to

introduce some notations and concepts.

We say that a configuration $(C, (i, j), d) \in A^{\mathbb{Z}^2} \times \mathbb{Z}^2 \times Q$ is *finite* if the set of cells with a symbol different than 0 is finite: $|\{(a, b) : C(a, b) \neq 0\}| < \infty$, in this case we also say that the picture C is finite. The picture full of 0s is called *white*.

A *pattern* is an assignment P of symbols to a finite set of cells S , that is, a function $P : S \rightarrow A$ with $S \subseteq \mathbb{Z}^2$. The set S is the *support* of P , and it is denoted by $\text{supp}(P)$. A restriction of a picture C to a finite set S is *the pattern of C on S* .

The transition function T_w of a generalised ant can be extended to patterns if also a position and direction inside the pattern support is given, that is: T_w may be applied to a tuple $(P, (i, j), d)$ if and only if $(i, j) \in \text{supp}(P)$.

Finally, patterns, pictures and configurations can be shifted by a vector $(a, b) \in \mathbb{Z}^2$. The shift function $\sigma^{(a,b)}$ is defined on pictures, on configurations, on sets, and patterns by:

- $\sigma^{(a,b)}(C)_{(i,j)} = C_{(i-a, j-b)}$,
- $\sigma^{(a,b)}(C, (i, j), d) = (\sigma^{(a,b)}(C), (i + a, i + j), d)$,
- $\sigma^{(a,b)}(S) = S + \{(a, b)\}$,
- $\sigma^{(a,b)}(P) : \sigma^{(a,b)}(S) \rightarrow A, \sigma^{(a,b)}(P)_{(i,j)} = P_{(i-a, j-b)}$.

Definition 2. A highway of period N and drift (a, b) of an ant rule w is represented by a pattern P , an ant position (i, j) inside its support and an ant state d such that the transition function T_w can be applied N times over it, after which, the new pattern $T_w^N(P)$ has the ant at position $(i, j) + (a, b)$ in state d and the pattern P appears inside $T_w^N(P)$ but shifted by (a, b) , more precisely:

$$\forall (x, y) \in \text{supp}(P), P(x, y) = \begin{cases} T_w^N(P)_{(x+a, y+b)} & \text{if } (x + a, y + b) \in \text{supp}(P) \\ 0 & \text{otherwise} \end{cases}$$

This definition ensures that if the pattern appears in a configuration where all the cells in direction (a, b) from $\text{supp}(P)$ are in state 0, then the same pattern will appear every N steps and both the ant trace and ant movements will be N periodic. Behind, the cells in $\text{supp}(P) \setminus \text{supp}(\sigma^{(a,b)}(P))$ will not be visited any more; repeated, they will configure the *prints* leaved by the ant on the highway.

Of course, any of the patterns appearing during the evolution of a highway can be used to represent the same highway; we can understand a highway as a global phenomenon, or as a cyclic sequence of patterns. Also, any rotation of a highway is the same highway, but with a rotated drift.

A highway can be characterized in terms of the trace of the ant, indeed if the trace of an ant that started on a finite configuration is eventually periodic, it is clear that the ant has entered a highway, because every periodic trace comes from a semiinfinite periodic pattern [6], and the only semiinfinite periodic pattern contained in a finite configuration is the white one. We use this characterisation to automatically detect highways in computer simulations.

Example 1. The ant LR has a highway of period 104 and drift $(-2, 2)$. Figure 4 shows one of the patterns that represent it. In gray are marked the non null cells, this smaller pattern is the one that we find shifted after 104 steps. In light gray

are drawn the cells with symbol 0. If the background in direction $(-2, 2)$ is full of 0s, the ant finds the whole pattern after 104 iterations, shifted by $(-2, 2)$ and the history will repeat.

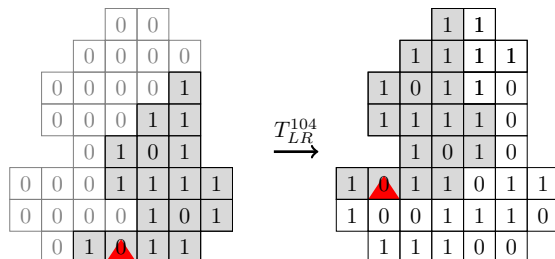


Figure 4: After 104 iterations the ant has moved by the vector $(-2, 2)$, and the configuration which produced that movement is also shifted by $(-2, 2)$. If the cells in direction $(-2, 2)$ from this configuration have all the symbol 0, then this movement will repeat itself forever.

3 Simulations and classification

Our goal is to tackle the highway conjecture. In order to better understand the classical Langton's ant, we decided to study the highways and emerging behaviour of generalised ants. The first step in this study was a large number of simulations of generalised ants over random perturbations of the 0-uniform configuration.

For each computation, an initial pattern P_0 of size 11×11 is taken uniformly at random, we then compute the trace t of the ant for 10^5 steps from the picture C_0 which is 0-uniform with the pattern P_0 around the origin (and the ant at the centre of P_0). The trace t is then analysed to determine if a highway was reached. For each ant, we ran the simulation at least 10^8 times. Table 1 shows the statistics of these simulations for some generalised ants which have remarkable properties.

For these simulation we are only interested in the length or period of the highways and not in the exact configuration, pattern or trajectory of the highway. This means that multiple highways with different trace, drift or trajectory are counted as the same if they have the same period.

Note that each highway that was found is an actual highway of the corresponding ant, however we make no claim that we found all the possible highways for any of these ants, nor that the proportions we observed are close to the actual probability of reaching the corresponding highway from an initial perturbation of size $n \times n$ taken uniformly at random.

We also tried the same process with another shape of the initial random pattern (for example cross shaped support instead of square support). In our experience, this did not change significantly the behaviours or frequencies.

From these computations, we draw a few observations.

The L^+R ants behave nicely. These ants are the simplest generalised ants and they seem to behave very nicely, though the original LR ant stands out in

Ant word	Highways	Period and frequency of the highways
LR	100%	104:100%
LLR	100%	18:100%
$LLLR$	100%	52:99.88%, 156:0.12%
$LLLLR$	100%	34:30.1%, 68:69.9%
L^5R	100%	84:100%
L^6R	100%	50:18.63%, 100:81.37%
L^7R	100%	116:100%
L^8R	100%	66:13.4%, 132:86.6%
L^9R	100%	148:100%
$L^{10}R$	100%	82:10.45%, 164:89.55%
$L^{11}R$	100%	180:100%
$L^{12}R$	100%	98:8.58%, 196:91.42%
$LLRL$	22.95%	308: $3 \cdot 10^{-4}\%$, 380: $4 \cdot 10^{-7}\%$, 384:99.9997%, 388: $4 \cdot 10^{-6}\%$, 928: $2 \cdot 10^{-7}\%$
$LLRLRL$	1.52%	220:29.8%, 244:10.6%, 268:5.3%, 284:0.7%, 292:2.5%, 300:9.4%, 308:0.9%, 316:1.5%, 324:8%, 332:0.6%, 340:1%, 348:6.7%, 356:0.9%, 364:1%, 372:4.1%, 380:0.8%, 388:4.2%, 396:2.9%, 404:0.3%, 412:1%, 420:2.2%, 428:0.3%, 436:0.8%, 444:1.5%, 452:0.4%, 460:0.6%, 468:0.7%, 1196:0.6%, 1268:0.3%, 1388:0.03%, 1412:0.1%

Table 1: Highway behaviours from simulations.

This table is read as follows: for ant $LLRL$, in our computations from initial random configurations, we reached a highway behaviour in 22.95% of the computations. Amongst those highway behaviour there is a dominant highway of length 384 which was reached in 99.9997% of cases and other rare highways of length 308 (freq. $3 \cdot 10^{-4}\%$), 380 (freq. $4 \cdot 10^{-7}\%$) and 928 (freq. $2 \cdot 10^{-7}\%$).

the behaviour we observe. Our first observation is that a highway behaviour is reached in every computation we ran. We then observe two clear subfamilies : the $L^{2k}R$ ants seem to each have a highway of period $16k + 2$ and for $k \geq 2$ a second highway of period $32k + 4$; while the $L^{2k+1}R$ ants seem to each have only a highway of period $32k + 20$, but the rule L^3R has two highways, one of them quite infrequent, 3 times longer and with a much bigger support. One can ask whether other $L^{2k+1}R$ ants have another big and very infrequent highway as L^3R does?

Additionally, we observe that for ant $L^{2k}R$, as k grows, the short (call *fundamental*) highway gets rare and the long (call *harmonic*) highway gets more and more dominant. This is partly explained by the fact that the harmonic highway is actually $k - 1$ distinct highways with the same period as detailed in Section 4.

The $LLRL$ ant has an overwhelmingly dominant highway, and very rare highways. From most initial configurations, no highway behaviour seems to be reached quickly (that is, before $\sim 10^5$ steps). Amongst the configuration where a highway is reached there is a dominant highway of length 384 that is

reached in 99.9997% of cases and a few very rare other highways of length 308, 380, 388 and 928 which all combined are reached in about 0.0003% of cases. In particular, from the initial 0-uniform grid, the ant reaches the dominant 384 period highway after 256100 steps. These highways are illustrated on our online simulator¹.

This behaviour that we observe, with a highway which exists but is extremely rare, comforts us in the idea that it just might be possible that the LR ant has other possible asymptotic behaviours but they are so extremely rare that they were never encountered in computations.

The $LLRLRL$ ant has infinitely many highways. Among the few initial configurations that reached a highway, it is quite remarkable that no highway is too infrequent, as was the case for rule $LLRL$. More remarkably still, we observe arithmetic progressions in the period of the highways, correlated with some tipicity in the frequencies of these. In particular we find a family² of highways of length $220 + 24n$, though we also find lengths of the form $300 + 24n$ and $308 + 24n$. When looking at the highways them selves, we see that the period does not determine the highway, we find different trajectories with the same period. Note that, from the initial 0-uniform grid, and even after 10^{10} steps, the $LLRLRL$ ant does not present any highway behaviour (nor any emerging behaviour).

This ant, though remarkable, does not seem to be the only ant with infinitely many highways. Our simulations suggest that the ants with rule words $LLRLRLRL$, $LLRLRRLRL$, $LLRLRRRL$, $LLRLRRRLR$ and $LLRRRLRL$ all have infinitely many highways.

4 Multiple highways for simple family of ants

The simplest generalised ants are the L^+R ants, that is ants whose rule word has n L states and then a R state. The classical Langton's ant being the special case for $n = 1$.

This family of generalised ants already contains many asymptotic highway behaviours and in this section we are interested in the subfamily $(LL)^+R$, or $L^{2k}R$ ants.

These ants and the highways we present in this section are illustrated on lutfalla.fr/ant/highway.html?antword=L2KR&k=4&i=2. Values for k (number of (LL) s in the ant word) and i (variant of the highway) can be modified in the interface and url.

Lemma 1 ($L^{2k}R$ cycles). *Let $a, b, c, d \in \mathbb{N}$ such that $2k \geq a \geq b, c, d \geq 0$. Let $b' = b + (2k - a)$, $c' = c + (2k - a)$, $d' = d + (2k - a)$. Let P and P' be the patterns of Fig. 5.*

Then $T_{L^{2k}R}^{A(2k-a)}(P) = P'$.

Proof. We first remark that the lemma is trivially true when $a = 2k$, because in that case $P = P'$ and $4(2k - a) = 0$, we can thus assume $a < 2k$. As we have $a, b, c, d < 2k$, the ant $L^{2k}R$ turns left and stays in the domain $\text{supp}(P)$ until

¹lutfalla.fr/ant/highway.html?antword=LLRL.

²lutfalla.fr/ant/highway.html?antword=LLRLRL.

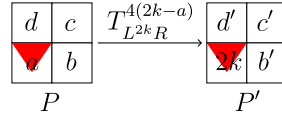


Figure 5: An elementary cycle of the $L^{2k}R$ ant.

it reaches a state $2k$. As $a \geq b, c, d$, the first state to become $2k$ and the first $2k$ state to be reached will be in the lower left position (where the state a and the ant initially were). Thus, the ant turns left and does $2k - a$ complete left 4-cycles and reaches configuration P' . \square

Denote $\widehat{a.b.c.d} = a.b.c.d.a+1 \dots 2k-1.b+2k-a-1.c+2k-a-1.d+2k-a-1$ the trace of the $L^{2k}R$ ant from P to P' , .

Lemma 2 ($L^{2k}R$ almost highways). *Let $0 < i < 2k$.*

Let P_i and P'_{2k-i} be the patterns of Fig. 6.

We have $T_{L^{2k}R}^{24k-8i+2}(P_i) = P'_{2k-i}$.

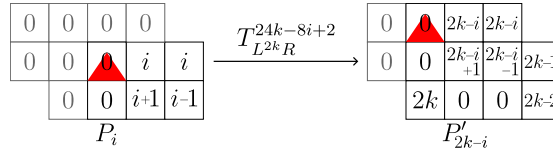


Figure 6: The almost highway of the $L^{2k}R$ ant.

Proof. We decompose the trace of the $L^{2k}R$ ant starting from P_i in three elementary cycles of lengths $8k$, $8k - 4i$, $8k - 4(i + 1)$ and in six individual moves as depicted in Fig. 7.

This means that the trace from P_i to P'_{2k-i} is $t_{k,i}$, where

$$t_{k,i} := \widehat{0.0.0.0.2k.i.0.0.0.2k.i+1.i-1.i.0.2k.2k.2k-i.2k}.$$

\square

Additionally, we remark that for $i \neq j$ the trace $t_{k,i} \cdot t_{k,i}$ does not contain the factor subword $j + 1.j - 1.j.0$ contained in $t_{k,j}$ so none of these almost-highways is a sub-dynamic of another.

Now we explain how the “almost highways” can be used to construct complete highways.

Theorem 1 (Fundamental and harmonic highway for $L^{2k}R$). *For any integer $k > 1$, the $L^{2k}R$ ant admits at least two different highways :*

- *a fundamental highway of period $16k + 2$ and drift $(\pm 1, \pm 1)$,*
- *a harmonic highway of period $32k + 4$ and drift $(\pm 2, \pm 2)$.*

Proof. Let $k > 1$ and consider the and $L^{2k}R$.

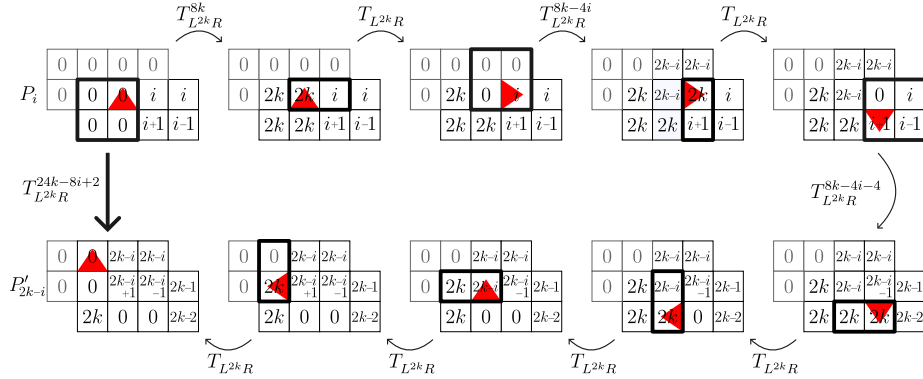


Figure 7: Decomposition of the almost highway for the $L^{2k}R$ ant. At each step, the domain of the next move or elementary cycle is emphasised with bold boundary.

For $n \geq 0$, let $C_{k,i}^n$ be the \mathbb{Z}^2 picture defined as

$$C_{k,i}^n(x, y) = \begin{cases} i & \text{if } y = n \wedge x \in \{1 - n, 2 - n\} \\ i + 1 & \text{if } (x, y) = (1 - n, n - 1) \\ i - 1 & \text{if } (x, y) = (2 - n, n - 1) \\ 2k & \text{if } y = -x - 2 \wedge -n \leq x \leq -1 \\ 2k - 1 & \text{if } y = -x + 2 \wedge 2 \leq x \leq n + 1 \\ 2k - 2 & \text{if } y = -x + 1 \wedge 2 \leq x \leq n + 1 \\ 0 & \text{otherwise} \end{cases}$$

In particular, $C_{k,k}^0$ is the 0-uniform picture with the pattern P_k of Lemma 2 at the origin, $C_{k,k}^1$ is the same with pattern P'_k , and in general $C_{k,i}^n$ has pattern P_i at position $(-n, n)$. We remark that P'_k also has pattern P_k but at position $(-1, 1)$.

We first show the fundamental highway. Applying Lemma 2 with $i = k$ we obtain that for any n ,

$$T_{L^{2k}R}^{16k+2}(C_{k,k}^n, (-n, n), \uparrow) = (C_{k,k}^{n+1}, (-n - 1, n + 1), \uparrow)$$

and the trace of this transition is $t_{k,k}$. In other words, $(C_{k,k}^0, (0, 0), \uparrow)$ starts a highway of period $16k + 2$, drift $(-1, 1)$ and trace $t_{k,k}$.

Now we prove the harmonic highway. Remark that, by applying Lemma 2 with $i = 1$ and $i = 2k - 1$ we have for any n both

$$T_{L^{2k}R}^{24k-6}(C_{k,1}^n, (-n, n), \uparrow) = (C_{k,2k-1}^{n+1}, (-n - 1, n + 1), \uparrow)$$

with trace $t_{k,1}$ and

$$T_{L^{2k}R}^{8k+10}(C_{k,2k-1}^{n+1}, (-n - 1, n + 1), \uparrow) = (C_{k,1}^{n+2}, (-n - 2, n + 2), \uparrow),$$

with trace $t_{k,2k-1}$.

Combining the two we obtain

$$T_{L^{2k}R}^{32k+4}(C_{k,1}^n, (-n, n), \uparrow) = (C_{k,1}^{n+2}, (-n - 2, n + 2), \uparrow).$$

In other words, $(C_{k,1}^0, (0,0), \uparrow)$ starts a highway of period $32k + 4$, drift $(-2, 2)$ and trace $t_{k,1} \cdot t_{k,2k-1}$.

The periodic repetition $(t_{k,1} \cdot t_{k,2k-1})^\omega$ has $32k + 4$ as a smallest period because the factor subword 2.0.1.0 only appears once in $t_{k,1}$ and does not appear in $t_{k,2k-1}$; so $32k + 4$ is indeed the highway period of this dynamics. \square

But what is done for $i = 1$ can be done for every $0 < i < k$, thus we have a family of harmonic highways.

Theorem 2 (Variants of the harmonic highway for $L^{2k}R$). *For any integer $k > 1$, there are at least $k - 1$ distinct harmonic highways of the $L^{2k}R$ ant.*

Proof. Let $k > 1$ and define the \mathbb{Z}^2 pictures $C_{k,i}^n$ as in the proof of Theorem 1. For any $0 < i < k$, we have for any n , we apply Lemma 2 for i and $2k - i$ we obtain:

$$T_{L^{2k}R}^{24k-8i+2}(C_{k,i}^n, (-n, n), \uparrow) = (C_{k,2k-i}^{n+1}, (-n-1, n+1), \uparrow)$$

with trace $t_{k,i}$ and

$$T_{L^{2k}R}^{8k+8i+2}(C_{k,2k-i}^{n+1}, (-n-1, n+1), \uparrow) = (C_{k,i}^{n+2}, (-n-2, n+2), \uparrow),$$

with trace $t_{k,2k-i}$. Combining the two we obtain:

$$T_{L^{2k}R}^{32k+4}(C_{k,i}^n, (-n, n), \uparrow) = (C_{k,i}^{n+2}, (-n-2, n+2), \uparrow)$$

with trace $t_{k,i} \cdot t_{k,2k-i}$. In other words, $(C_{k,i}^0, (0,0), \uparrow)$ starts a highway of period $32k + 4$, drift $(-2, 2)$ and trace $t_{k,i} \cdot t_{k,2k-i}$.

These highways are indeed distinct, as mentioned above, since the trace factor subword $i + 1.i - 1.i.0$ does not appear in $t_{k,j} \cdot t_{k,2k-j}$ for $i \neq j$. \square

Remark 1. *It is quite remarkable that, for a given $k > 1$, all of these k highways (the fundamental and the $k - 1$ variants of the harmonic) leave the same print behind consisting of a diagonal of $2k$, a diagonal of $2k - 1$ and a diagonal of $2k - 2$. See Fig. 8.*

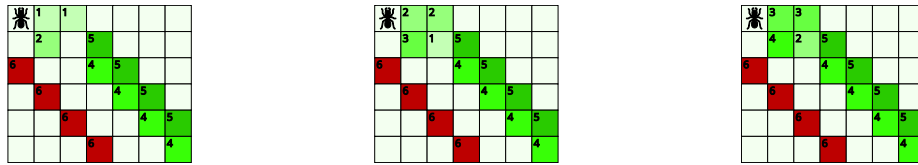


Figure 8: The three highways for the L^6R ant, the rightmost is the fundamental highway.

Remark 2 (LLR). *The LLR ant does not have these harmonic highways of length 36 as there is no i such that $0 < i < 1$. However the fundamental highway of length 18 described above exists.*

Remark 3 ($L^{2k+1}R$ ants). *Analogously, it can be proved that for any $k > 1$, the ant with rule word $L^{2k+1}R$ has a highway of length $32k + 20$. Its dynamics, for any k , can be seen in lutfalla.fr/ant/highway.html?antword=L2K1R&k=2, and its initial pattern is shown in Figure 9.*

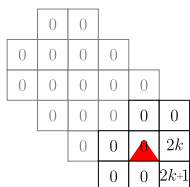


Figure 9: The configuration that starts a highway of period $32k + 20$ for the ant $L^{2k+1}R$.

5 Infinitely many highways for a single ant

Theorem 3. *The ant LLRLRL has infinitely many highways.*

In particular for any n , LLRLRL admits a highway of period $220 + 24n$.

Proof. To prove this result we introduce three widgets, two of them in two versions: a main widget M_1 and M_2 and a bounce back widget B_1 and B_2 ; also a link widget in four versions: L_1 , L_2 , L_3 , and L_4 .

We will prove that, for any $n \in \mathbb{N}$, the pattern $c_n = M_1 \cdot L_1^n \cdot B_1$ starts a highway of period $220 + 24n$, and drift $(-2, -2)$. Figure ?? shows a picture of this highway where the widgets can be identified, together with its print leaved after several periods. The proof is performed by describing the different stages of the ant movement, which can be verified by the reader by hand or by means of any ant simulator, for example lutfalla.fr/ant/highway.html?antword=LLRLRL.

Start. The ant start over $c_n = M_1 \cdot L_1^n \cdot B_1$ at the red arrow, as Figure 10 shows. The ant will spend 84 steps inside M_1 , to finally exit and enter the first of the series of n L_1 s at the position and direction marked by the yellow triangle. It will take 12 steps to transverse each L_1 to finally enter pattern B_1 at the position and direction marked by the cyan arrow, where it will spend only 4 steps. At this point, M_1 has been transformed into M_2 concatenated with a half L_2 and each of the L_1 s has been transformed into an L_2 , except for the last one, which slightly differs from the others and will be called L'_2 .

Rebound 1. Now the ant faces L'_2 and a series of $n - 1$ L_2 s, it takes 4 steps to transverse each of them (see Figure 11). After the ant visit, this series of n widgets is modified. The column of the first L_2 , that was adjacent to B_1 in the previous stage, will fuse with B_1 to become B_2 . Starting from the other column of this widget, a new series of n link widgets is formed, that we call L_3 .

New start. Now the ant is on $M_2 \cdot L_3^n \cdot B_2$ at the position marked by the red arrow (see Figure 12). The ant will spend 100 steps inside M_2 , to finally exit it to enter the series of n L_3 s at the position and direction marked by the yellow triangle. It will take only 4 steps to transverse each L_3 to finally enter pattern B_2 at the position and direction marked by the cyan arrow, where it will spend 30 steps. At this point, M_2 has been transformed into M_1 concatenated with a half of L_4 and each of the L_3 s has been transformed into an L_4 , except for the last one, which slightly differs from the others and will be called L'_4 .

Rebound 2. Now the ant faces L'_4 and a series of $n - 1$ L_4 s, again it takes 4 steps to transverse each of them (see Figure 13). After the ant visit, this series of n widgets is modified. The column of the first L_4 , that was adjacent to B_2 in the previous stage, together with two cells of B_2 become B_1 . Starting from the other column of this widget, the series of n L_1 appears.

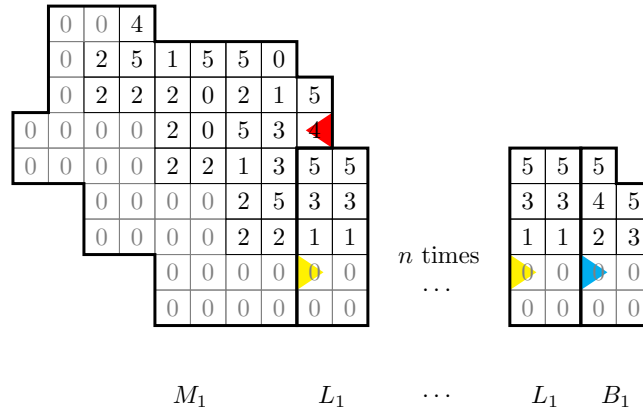


Figure 10: Patterns M_1 , B_1 , and L_1 . The red triangle marks the starting ant's position and direction. The other 3 triangles mark the entering positions of the ant to the respective widget.

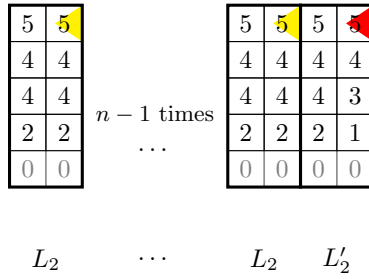


Figure 11: Pattern L_2 . The ant will transverse each block in 4 steps, transforming them into a series of L_3 s.

After these four stages, c_n appears again, but shifted by $(-2,-2)$, covering the gray 0s, and if enough 0s are placed in the background, a new cycle of $220 + 24n$ steps starts. \square

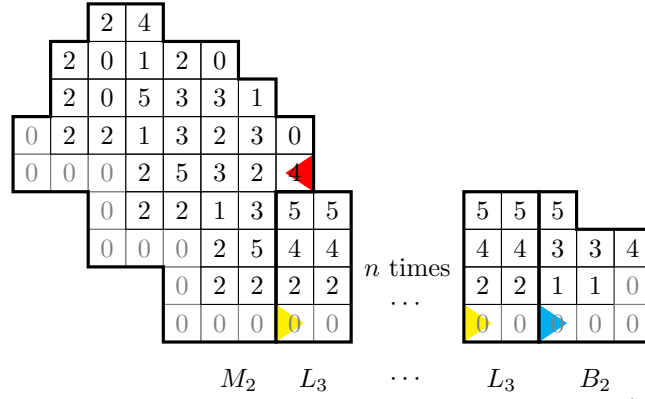


Figure 12: Patterns M_2 , B_2 , and L_3 . The red triangle marks the starting ant's position and direction of this stage. The other 3 triangles mark the entering positions of the ant to the respective widget.

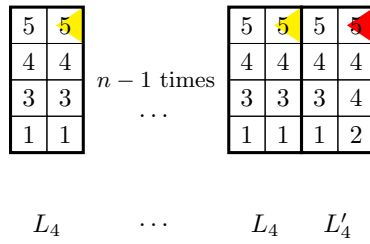


Figure 13: Pattern L_4 . The ant will transverse each block in 4 steps, transforming it into L_1 .

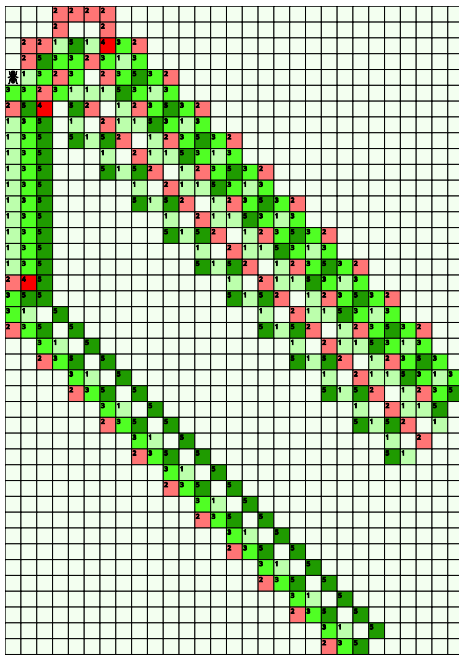


Figure 14: The configuration produced by a highway of period $220 + 5 \cdot 24$ for the *LLRLRL* ant after $11 \cdot (220 + 5 \cdot 24)$ iterations. We can identify the *L* widgets acting as a corridor between the two “walls” *M*, on top, and *B*, in the opposite extreme of the series of *Ls*. The two diagonals on the picture are the prints left by widgets *M* and *B*. Note that left-turning (resp. right-turning) states are emphasized in green (resp. red).

References

- [1] Olivier Beuret and Marco Tomassini. Behaviour of multiple generalized Langton's ants. In *Artificial Life V*, page 45–50. MIT Press, Cambridge, Mass., 1997.
- [2] Leonid Bunimovich and Serge Troubetzkoy. Recurrence properties of Lorentz lattice gas cellular automata. *J. of Stat. Phys.*, 67(1):289–302, 1992.
- [3] Alexander Keewatin Dewdney. Computer recreations: Two-dimensional Turing machines and turmites make tracks on a plane. *Scientific American*, 261:180–183, sep 1989.
- [4] Paul Dorbec and Anahí Gajardo. Langton's flies. *Journal of Physics A: Mathematical and Theoretical*, 41(40):405101, sep 2008.
- [5] Anahí Gajardo, Andrés Moreira, and Eric Goles. Complexity of Langton's ant. *Discrete Appl. Math.*, 117(1-3):41–50, 2002.
- [6] Anahí Gajardo. Sofic one head machines. In B. Durand, editor, *Journées Automates Cellulaires*, pages 54–64, 2008.
- [7] David Gale and Jim Propp. Further Ant-ics. *Math. Intelligencer*, 16:37–42, 1994.
- [8] David Gale, Jim Propp, Scott Sutherland, and Serge Troubetzkoy. Further Travels with My Ant. *Math. Intelligencer*, 17:48–56, 1995.
- [9] Xiang-Peng Kong and Ezechiele Godert David Cohen. Diffusion and propagation in triangular Lorentz lattice gas cellular automata. *Journal of Statistical Physics*, 62:737–757, 1991.
- [10] Petr Kůrka. On topological dynamics of Turing machines. *Theoretical Computer Science*, 174(1-2):203–216, 1997.
- [11] Christopher Langton. Studying artificial life with cellular automata. *Physica D*, 22:120–149, 86.
- [12] Diego Maldonado, Anahí Gajardo, Benjamin Hellouin de Menibus, and Andrés Moreira. Nontrivial turmites are Turing-universal. *J. of Cell. Automata*, 13(5/6):373–392, 2018.