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ON HECKE AND ASYMPTOTIC CATEGORIES FOR COMPLEX REFLECTION GROUPS

ABEL LACABANNE, DANIEL TUBBENHAUER AND PEDRO VAZ

ABSTRACT. Generalizing the dihedral picture for $G(M, M, 2)$, we construct Hecke algebras (and categories) and asymptotic counterparts. We think of these as associated with the complex reflection group $G(M, M, N)$.

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1. INTRODUCTION

Weyl groups are among the most important objects in algebra, as they govern the representation theory of their associated reductive group. Weyl groups are real reflection groups and special cases of *complex reflection groups*, and it is an interesting question what kind of ‘reductive group’ is associated with a complex reflection group. These ‘reductive groups’ are believed to exist in a certain sense, and they were famously named by Broué–Malle–Michel [BMM99] after the Greek island Spetses: such a ‘reductive group’ is called a *spets*.

We cannot provide a definitive answer to what spetses are, but recent developments indicate that Soergel bimodules, also known as *Hecke categories*, can often serve as a replacement whenever an associated geometric or Lie theoretic picture is missing.

In this paper, we focus on the complex reflection groups of type $G(M, M, N)$ for $M \geq N$ and suggest that they have an associated Hecke algebra and category. These arise from Chebyshev polynomials associated with root systems, have Kazhdan–Lusztig (KL) type combinatorics, include asymptotic categories, are related to Calogero–Moser (CM) families, and encode Fourier matrices for $G(M, M, N)$.

1A. From dihedral group to the main results. The dihedral group $G(M, M, 2)$ of order $2M$ is one of the simplest examples of a complex reflection group that is not a Weyl group (unless M is small), yet it still exhibits behavior typically associated with objects from Lie theory.

Let us list a few of these, all of which are for the middle cell:

- (A) The KL basis is determined by the coefficients of the Chebyshev polynomials [dC06, Eli16a, Tub22], and the Chebyshev polynomials determine the characters of the simple representations of SL_2 .
- (B) The Drinfeld centers of the asymptotic categories associated to the dihedral Hecke algebras are modular categories whose S -matrices coincide with the so-called Fourier matrices [RT23], the base change between ‘unipotent characters’ and ‘unipotent character sheaves’, constructed in the 90s [Lus94] in an ad hoc fashion. Hence, one can think of these centers as ‘unipotent character sheaves’, matching [Lus15] which proposed ‘unipotent character sheaves’ associated to Coxeter groups that encode the Fourier matrices.
- (C) The simple representations of the (complexified) asymptotic Hecke algebra are given by the KL family associated to the cell [Lus87].
- (D) The $\mathbb{Z}_{\geq 0}$ -representations of the dihedral group, or the 2-representations of the dihedral Hecke category, are indexed by ADE Dynkin diagrams [KMMZ19, MT19]. This is closely related to (but does not quite match) with three different objects: irreducible conformal field theories (CFT for short) for SU_2 [Pas87], subgroups of quantum SU_2 [KO02, Ost03] and module categories of the SL_2 Verlinde category [Ost03].

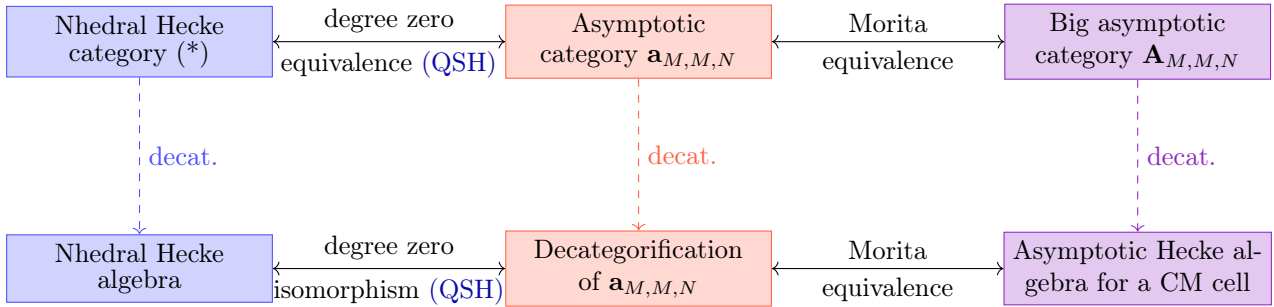
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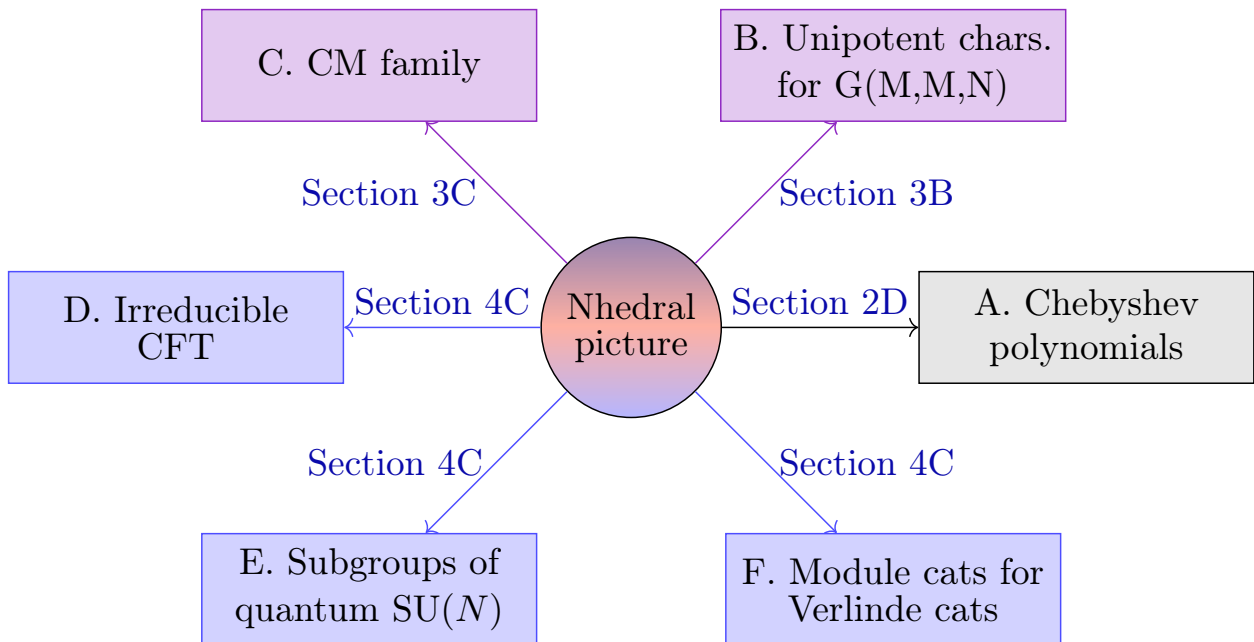
In this paper, we generalize all the above to the case $N \geq 2$ (for $N = 1$ the group $G(M, M, N)$ is trivial and we from now on assume that $N > 1$). The outline is as follows.

- (a) *Generalizing (A)*. We define a certain subalgebra T_∞ of the Hecke algebra of affine type A_{N-1} , and then a finite dimensional quotient T_e , the **Nhedral Hecke algebra** of level e , obtained from T_∞ by annihilating certain KL basis elements. *This algebra is immediately truncated to the analog of the aforementioned middle cell*. As we will show, the algebra T_∞ has its own KL theory and representation theory akin to the dihedral Hecke algebra. Its KL basis is determined by the coefficients of the Chebyshev polynomials associated with simple representations of SL_N .
- (b) *Generalizing (B)*. There is a categorification of T_e , called **Nhedral Hecke category** (or Nhedral Soergel bimodules) of level e . This category is positively graded and its degree zero part is what we call the **asymptotic category** $\mathfrak{a}_{M,M,N}$. We show that the Drinfeld center of $\mathfrak{a}_{M,M,N}$ is a modular category and compute its S and T matrices. We show that the S matrix coincides with Malle’s Fourier matrix for $G(M, M, N)$ [Mal95].
- (c) *Generalizing (C)*. We then define a matrix category $\mathbf{A}_{M,M,N}$ over $\mathfrak{a}_{M,M,N}$, the **big asymptotic category**, which, by construction, is Morita equivalent to $\mathfrak{a}_{M,M,N}$. We explain how this category is related to a CM family for $G(M, M, N)$.
- (d) *Generalizing (D)*. Returning to T_e , we show that most of Zuber’s generalized ADE Dynkin diagrams [Zub98] give rise to $\mathbb{Z}_{\geq 0}$ -representations of T_e . This is closely related (but does not quite match) with three different objects: irreducible CFT for SU_N as e.g. in [PZ96], subgroups of quantum SU_N as e.g. in [Ocn02] and module categories of the SL_N Verlinde category as, for example, in [Gan94].

Let us summarize the main points of the paper with two overview diagrams. The first diagram illustrates how the main algebras and categories are related:



The “(QSH)” is explain in Section 1C below, while (*) means that we believe this is definable and interesting to define, though we do not do this here since the relevant technology is missing while writing this paper.



In this picture we indicate where the reader can find more details about the connection.

1B. Related works and speculations. There are two infinite families of spetsial complex reflection groups, which represent two different extremes, yet both might permit some Hecke category combinatorics. Additionally, there is a twisted variant that appears to fit into the Hecke category framework, and we comment on all three here.

The first infinite family is $G(M, M, N)$, as discussed in this paper, and we think of the Nhedral picture as the associated Hecke category combinatorics. Furthermore, for $N = 3$, (A) and (D) above were originally generalized in [MMMT20]. The paper [MMMT20] also provides a classification of $\mathbb{Z}_{\geq 0}$ -representations of T_e for small e , a task we expect to be achievable for at least small N as well. Moreover, exciting so-called exotic nilCoxeter and deformed affine NilHecke algebras are obtained in [EJY24b, EJY24a] (with some results also for $N > 3$), but we do not know how to relate this work to ours.

The other infinite family of spetsial complex reflection groups consists of the groups $G(M, 1, N)$, where the corresponding Hecke algebra is the Ariki–Koike algebra [AK94, BM93]. Fourier matrices and families of unipotent characters for these groups have been studied quite extensively; see, for example, [Mal95, Cun07, BR20, Lac20b, Lac21]. However, we are not aware of any general Hecke category combinatorics for $G(M, 1, N)$. The paper [LTV23] identifies the Ariki–Koike algebra as a subalgebra of webs on an annulus, suggesting the possibility of using annular foams, as in [RW20], to describe a Hecke category. For $N = 1$, a Hecke category has been proposed in [GT20], but it seems quite different from the annular web picture.

There is also the story of Fourier matrices associated with groups having an automorphism, such as the Ree groups; see, for example, [GM03, Lac20a]. The corresponding Fourier matrices are not symmetric and often not integral, so they may only arise via equivariant module categories and not via modular categories. The automorphism ‘twists’ the setting in a certain sense and one might hope that the ‘twisted’ 2-representations of the Hecke category as e.g. [MMM⁺23] can be used to define degree zero module categories with crossed S -matrices in the sense of [Des17] might play a role.

1C. Additional remarks. For some proofs in this paper we need the following.

The *quantum Satake hypothesis (QSH)* is:
 (QSH) [Eli17, Theorem 5.35] is true.
 Whenever we assume (QSH) holds, we state this explicitly.

Remark 1C.1. The paper [Eli17] proves this for $N = 2, 3$, and in these cases we have further assumptions. \diamond

Remark 1C.2. We postponed several proofs to Section 5. If the reader is missing a proof, then they should be able to find it there. Moreover, in Section 2 and Section 4, we will generalize some of the main results of [MMMT20], following their exposition. Some results will have proofs that work mutatis mutandis and we will be brief with these, and we will point out when the arguments are sufficiently different. \diamond

Remark 1C.3. The paper is readable in black-and-white but we recommend reading it in color. \diamond

Remark 1C.4. Code for some of the calculations in this paper is available on GitHub: [LTV24]. \diamond

1D. Table of notation and general conventions. The following is the list of the most important concepts.

Symbol	Name	Description
$-$	Placeholder	Used as a placeholder symbol
$[-]_{\oplus}$	Grothendieck group	The additive Grothendieck group of $-$ (for semisimple categories this equals the abelian one)
$- \rightarrow$	A cyclic action	See Section 2B.
$\overline{-}$	The dual	If $\mathbf{m} = (m_1, \dots, m_{N-1})$, then $\overline{\mathbf{m}} = (m_{N-1}, \dots, m_1)$
$(-)_i$	Degree i part	If the category $-$ is graded, then this denotes the degree i part
$A(-)$	Adjacency matrix	The adjacency matrix of a graph $-$
$\mathbf{A}_{M,M,N}$	The big asymptotic category	The big asymptotic category for $G(M, M, N)$ defined in Definition 3C.2
$\mathbf{a}_{M,M,N}$	The asymptotic category	The asymptotic category for $G(M, M, N)$ defined in Definition 3A.1
$c_{y,y}^x$	Nhedral KL elements	See Section 4A

Symbol	Name	Description
$d_{\mathbf{m}}^{\mathbf{k}}$	Change-of-basis coefficients	See (2B.2)
$\dim_c _$	Categorical dimension	The categorical dimension of $_$
$\dim_{\eta} _$	Quantum dimension	The quantum dimension of $_$
$\dim_{\mathbb{K}} _$	Usual dimension	The dimension of $_$ over \mathbb{K}
e	Level	The level, a number $e \in \mathbb{Z}_{\geq 0}$ that we fix
η	Root of unity	The primitive $2M$ th root of unity $\exp(i\pi/M)$
$\eta^{1/N}$	Root of unity	The N th root of η given by $\exp(i\pi/NM)$
\mathcal{F}	Family of unipotent characters	A family of unipotent characters for $G(M, M, N)$, see Section 3B
\mathcal{F}_0	Principal series	The principal series of \mathcal{F}
$G(M, M, N)$	Complex reflection group	The imprimitive complex reflection group of order $M^{N-1}N!$
Γ	CM cell	The two-sided CM cell associated with \mathcal{F} , see Section 3C
$h_{y, y}^x$	Nhedral Bott-Samelson elements	See Section 4A
i	Imaginary unit	The usual square root of -1
I	Vertices	$I = \{0, \dots, N-1\}$, the vertices of the affine type A_{N-1} Dynkin diagram
J_e	Vanishing ideal	Defined in Definition 2E.1
M	Level (plus Coxeter number)	This is $e + N$, the ‘level’ of $G(M, M, N)$ such that $G(M, M, 2)$ is the dihedral group of order $2M$
M	Nhedral representation	Representation of the Nhedral Hecke algebra; potentially decorated with symbols
N	Rank (potentially plus one)	The rank, a number $N \in \mathbb{Z}_{\geq 2}$ that we fix
\varkappa	Quantum $N-1$ factorial	$[N-1]_{\mathbf{v}}!$
ζ	Root of unity	The primitive N th root of unity $\exp(2i\pi/N)$
p_N^e	Simplicial polytopic numbers	$p_N^e = \binom{e+N-1}{N-1} = \binom{M-1}{N-1}$
q	Quantum parameter	The quantum generic parameter
$[k]_{\mathbf{v}}$	Quantum numbers	The k th quantum number, $[k]_{\mathbf{v}} = \frac{q^k - q^{-1}}{q - q^{-1}}$
$[k]_{\mathbf{v}}!$	Quantum factorials	The k th quantum factorial, $[k]_{\mathbf{v}}! = [1]_{\mathbf{v}} \dots [k]_{\mathbf{v}}$
$\text{rk} _$	Rank	The rank of the category $_$ (the number of indecomposable objects)
$\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)$	Representation category	The fusion category of $U_{\eta}(\mathfrak{sl}_N)$ -representations
$\mathbf{Rep}_q(\mathfrak{sl}_N)$	Representation category	The category of (type 1) $U_q(\mathfrak{sl}_N)$ -representations
S	An S -matrix	The matrix S of a modular category involved in the action of the modular group
\bar{S}	Complex conjugate	Entry-wise complex conjugate of S
\mathbb{S}	Fourier matrix	The Fourier matrix of the family \mathcal{F}
$\text{Si}(_)$	Simple objects	The set of isomorphism classes of simple objects of $_$
$\Sigma \mathbf{m}$	Sum of the entries	For $\mathbf{m} = (m_1, \dots, m_r)$ we let $\Sigma \mathbf{m} = m_1 + \dots + m_r$
$\text{stab} _$	Stabilizer	Cardinal of a stabilizer
T	A T -matrix	The matrix T of a modular category involved in the action of the modular group
$T_{_}$	Nhedral Hecke algebra	The Nhedral Hecke algebra of level $_$, see e.g. Definition 4A.6
θ	Ribbon structure	The ribbon structure on $\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)$ defined via $\eta^{1/N}$
θ_i	Nhedral generators	The generators of $T_{_}$
$U_{\mathbf{m}}$	Chebyshev polynomials for \mathfrak{sl}_N	The higher versions of Chebyshev polynomials, see Definition 2D.1, due to Koornwinder and Eier-Lidl

Symbol	Name	Description
$U_{\mathfrak{q}}(\mathfrak{sl}_N)$	Quantum \mathfrak{sl}_N	The generic quantum group for quantum parameter \mathfrak{q}
$U_{\eta}(\mathfrak{sl}_N)$	Another quantum \mathfrak{sl}_N	The quantum group specialized quantum parameter η
V_e	Koornwinder variety	Defined in Definition 2E.1
V'_e	Reparametrization of V_e	Defined in Definition 2E.7
$\mathbf{V}(N)^e$	Verlinde category	The Verlinde category for \mathfrak{sl}_N defined using η , with ribbon structure θ
W	Affine Weyl group	The Weyl group for affine type A_{N-1}
W_C	Parabolic subgroup	The parabolic subgroup for the vertices C ; here $C \subset I$ which we often also write indicating the missing values
χ_c	Central character	The “colors” associated with simple $U_{\mathfrak{q}}(\mathfrak{sl}_N)$ -representations
X_i	Fundamental variables	The variables used for the fundamental $U_{\mathfrak{q}}(\mathfrak{sl}_N)$ -representations
$\mathbf{Z}(-)$	Drinfeld center	The Drinfeld center of a category $-$
Z_i	Koornwinder’s Z-functions	Defined in Definition 2E.5

As a few general conventions that hold throughout unless otherwise stated:

- (a) Our conventions for rings and fields, generic symbol \mathbb{K} , are: subscripts mean we adjoin a certain element to the ring/field, e.g. $\mathbb{C}^{\mathfrak{q}} = \mathbb{C}(\mathfrak{q})$. We write square brackets if we adjoin these elements as Laurent polynomials, e.g. $\mathbb{Z}^{[v]} = \mathbb{Z}[v, v^{-1}]$.
- (b) All \mathbb{K} -vector spaces are finite dimensional in this paper, or free of finite rank if \mathbb{K} is a ring.
- (c) All modules that we consider are left modules, and representations of quantum enveloping algebras are always of type 1 (in the sense of e.g. [\[APW91, Section 1.4\]](#)).
- (d) We often say “is a XYZ” instead of “can be equipped with the structure of an XYZ” to not overload statements. For example, we say a category is modular.
- (e) We also often say e.g. “there are only finitely many simple objects” instead of “there are only finitely many isomorphism classes of simple objects”.

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2. SOME \mathfrak{sl}_N COMBINATORICS

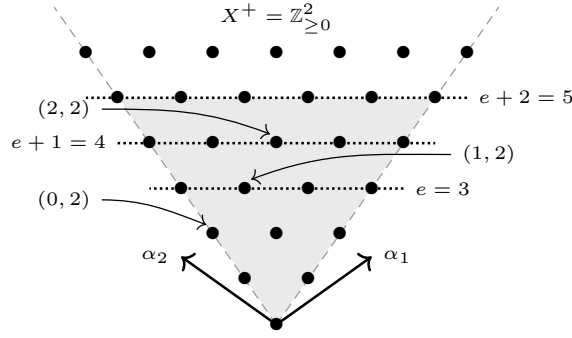
We start by fixing some notation regarding \mathfrak{sl}_N . Most of the material is known, but our exposition for some parts is new.

2A. Root combinatorics. Denote by $(\varepsilon_1, \dots, \varepsilon_N)$ the standard basis of \mathbb{R}^N , which we equip with the standard symmetric bilinear form $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$. Denote by \mathfrak{S}_N the symmetric group on N letters, which acts naturally on \mathbb{R}^N by permutation. We let $E = \{(x_1, \dots, x_N) \in \mathbb{R}^N \mid x_1 + \dots + x_N = 0\}$ and let $\alpha_i = \varepsilon_{i+1} - \varepsilon_i$ for $1 \leq i < n$. The vectors $\alpha_1, \dots, \alpha_{N-1}$ are the simple roots and we fix the coroots $\alpha_i^{\vee} \in E^*$ such that $\langle \alpha_i, \alpha_j^{\vee} \rangle = a_{ij}$, where (a_{ij}) is the usual Cartan matrix of type A_{N-1} and $\langle -, - \rangle$ is the duality pairing. The weight lattice is $X = \{\lambda \in E \mid \langle \lambda, \alpha_i^{\vee} \rangle \in \mathbb{Z} \text{ for all } 1 \leq i < N\}$ and the dominant weights are $X^+ = \{\lambda \in E \mid \langle \lambda, \alpha_i^{\vee} \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } 1 \leq i < N\}$. We also denote by $\omega_1, \dots, \omega_{N-1} \in E$ the fundamental weights which are defined through the equalities $\langle \omega_i, \alpha_j^{\vee} \rangle = \delta_{i,j}$, and by $\rho = \omega_1 + \dots + \omega_{N-1}$ the sum of the fundamental weights, or equivalently the half-sum of positive roots. Using the basis of fundamental weights, we identify X with \mathbb{Z}^{N-1} and X^+ with $\mathbb{Z}_{\geq 0}^{N-1}$.

We will also work with cut-offs $X^+(e)$ of the weight lattice, which will depend on the level $e \in \mathbb{Z}_{\geq 0}$, which are defined by $X^+(e) = \{\lambda \in X^+ \mid \langle \lambda, \alpha_1^{\vee} + \dots + \alpha_{N-1}^{\vee} \rangle \leq e\}$. Therefore, a weight $\lambda = \lambda_1 \omega_1 + \dots + \lambda_{N-1} \omega_{N-1}$ is in $X^+(e)$ if and only if each λ_i is nonnegative and $\Sigma \lambda \leq e$.

Notation 2A.1. As above, we repeatedly will sum over entries of tuples, and we use the following shorthand notation: $\Sigma \mathbf{m} = m_1 + \dots + m_k$ for $\mathbf{m} = (m_1, \dots, m_k)$. \diamond

The following picture is stolen from [MMMMT20]. We stole it because it summarizes our conventions for $N = 3$ and $e = 3$:



2B. Quantum group generically and semisimplified. Our conventions follow [Jan96, Chapters 4-7].

For q a formal variable, let $U_q(\mathfrak{sl}_N)$ denote the *quantum enveloping (\mathbb{C}^q -)algebra associated to \mathfrak{sl}_N* , with the Hopf algebra structure as chosen in [Jan96, Section 4.8]. Let us denote its category of finite dimensional representations by $\mathbf{Rep}_q(\mathfrak{sl}_N)$. This category is semisimple and has the same combinatorics as the corresponding category for \mathfrak{sl}_N itself, so all the below follows from classical theory. In particular, the simple $U_q(\mathfrak{sl}_N)$ -representations (we also write \mathfrak{sl}_N -representations for short) are parameterized by the integral positive Weyl chamber:

$$\{\mathbf{L}_m \mid \mathbf{m} = (m_1, \dots, m_{N-1}) \in X^+\}.$$

The highest weight here is $m_1\omega_1 + \dots + m_{N-1}\omega_{N-1}$. Let $[-]_{\oplus}$ denote the (additive) Grothendieck group, which is a \mathbb{Z} -algebra. We have a \mathbb{Z} -basis given by (the set of the elements)

$$[\mathbf{L}_m] = [\mathbf{L}_m]_{\oplus} \in [\mathbf{Rep}_q(\mathfrak{sl}_N)]_{\oplus}.$$

Scalar extension gives a \mathbb{C} -algebra:

$$[\mathbf{Rep}_q(\mathfrak{sl}_N)]_{\oplus}^{\mathbb{C}} = [\mathbf{Rep}_q(\mathfrak{sl}_N)]_{\oplus} \otimes_{\mathbb{Z}} \mathbb{C}.$$

The fundamental \mathfrak{sl}_N -representations are our generating variables:

$$X_i = L_{\omega_i}.$$

Let $\bar{\mathbf{m}} = (m_{N-1}, \dots, m_1)$. We have $(\mathbf{L}_m)^* \cong \mathbf{L}_{\bar{\mathbf{m}}}$ for all $\mathbf{m} \in X^+$.

We will see them as variables in a polynomial ring, i.e. let $\mathbb{Z}[X_i] = \mathbb{Z}[X_i \mid i \in \{1, \dots, N-1\}]$ where we use the X_i as variables. Write $X_i^k = X_i^{\otimes k}$, for short, and also use $X_i X_j = X_j X_i$. For $\mathbf{k} \in \mathbb{Z}_{\geq 0}^{N-1}$, write $X^{\mathbf{k}} = X_1^{k_1} \cdots X_{N-1}^{k_{N-1}}$. This is justified by:

Lemma 2B.1. *We have two bases of $[\mathbf{Rep}_q(\mathfrak{sl}_N)]_{\oplus}$ given by $\{\mathbf{L}_m \mid \mathbf{m} \in X^+\}$ and $\{[X^{\mathbf{k}}] \mid \mathbf{k} \in \mathbb{Z}_{\geq 0}^{N-1}\}$. Moreover, as \mathbb{Z} -algebras $[\mathbf{Rep}_q(\mathfrak{sl}_N)]_{\oplus} \cong \mathbb{Z}[X_i]$.*

Proof. By classical theory. □

Lemma 2B.1 motivates the definition of the *change-of-basis coefficients*:

$$(2B.2) \quad [\mathbf{L}_m] = \sum_{\mathbf{k}} d_{\mathbf{m}}^{\mathbf{k}} \cdot [X^{\mathbf{k}}], \quad d_{\mathbf{m}}^{\mathbf{k}} \in \mathbb{Z}.$$

Note that this sum is finite since $d_{\mathbf{m}}^{\mathbf{k}} = 0$ unless $\Sigma \mathbf{k} \leq \Sigma \mathbf{m}$. The numbers $d_{\mathbf{m}}^{\mathbf{k}}$ can be computed inductively, as explained in Section 2D below, and we have $d_{\mathbf{m}}^{\bar{\mathbf{k}}} = d_{\mathbf{m}}^{\mathbf{k}}$ and $d_{\mathbf{m}}^{\mathbf{m}} = 1$.

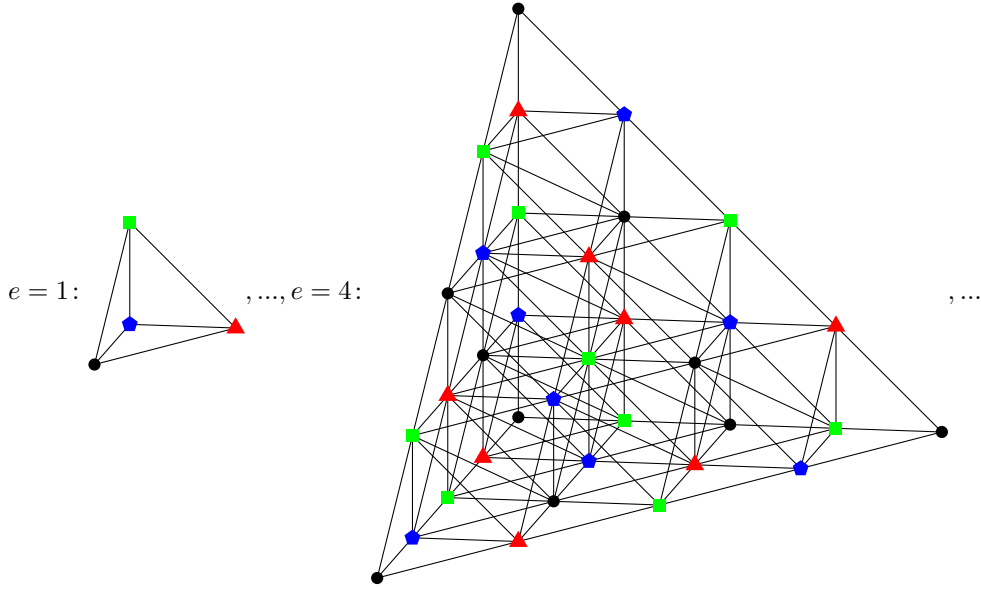
The center of SU_N is $\mathbb{Z}/N\mathbb{Z}$, which, following [MMMMT20], motivates:

Definition 2B.3. Let us define “*colors*” associated to the simple $U_q(\mathfrak{sl}_N)$ -representations:

$$(2B.4) \quad \chi_c(\mathbf{L}_m) = m_1 + 2m_2 + \dots + (N-1)m_{N-1} \in \mathbb{Z}/N\mathbb{Z}.$$

We call $\chi_c(\mathbf{L}_m)$ the *central character* of \mathbf{L}_m . ◇

Example 2B.5. For $N = 2$, this is the parity coloring of $\mathbb{Z}_{\geq 0}$, for $N = 3$ see [MMMT20, Example 3.16], and for $N = 4$, we have the following pictures:



The color 0 corresponds to black circles, 1 to red triangles, 2 to green squares and 3 to blue pentagons. In these pictures the edges correspond to the action of the direct sum of the fundamental \mathfrak{sl}_4 -representations. For example, for $e = 1$ the black circle represents the trivial \mathfrak{sl}_4 -representation, the red triangle the defining \mathfrak{sl}_4 -representation, the blue pentagon its dual and the green square the 6 dimensional simple \mathfrak{sl}_4 -representation. Moreover, tensoring with either of them jumps from the black circle to the other vertices. \diamond

These colors define a grading by $\mathbb{Z}/N\mathbb{Z}$ on $\mathbf{Rep}_{\mathfrak{q}}(\mathfrak{sl}_N)$ since χ_c is a character, by classical theory again.

Lemma 2B.6. All simple summands of $\mathbf{X}^{\mathbf{k}}$ have central character $\chi_c(\mathbf{L}_{\mathbf{k}})$.

Proof. All summands of $\mathbf{X}^{\mathbf{k}}$ have the same color and $\mathbf{L}_{\mathbf{k}}$ is a summand of $\mathbf{X}^{\mathbf{k}}$. \square

Specializing \mathfrak{q} to a primitive $2M$ th root of unity η (using the integral form), where $M = e + N$, we obtain $U_{\eta}(\mathfrak{sl}_N)$, as defined and used in e.g. [Lus90], [APW91]. There is an associated category of representations, which we can semisimplify as usual, see e.g. [AP95] for details. This category is denoted by $\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)$. The Grothendieck group of this category is a \mathbb{Z} -algebra by the fusion product, and has \mathbb{Z} -bases given by:

Lemma 2B.7. We have two bases of $[\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)]_{\oplus}$ given by $\{[\mathbf{L}_{\mathbf{m}}] \mid \mathbf{m} \in X^+(e)\}$ and $\{[\mathbf{X}^{\mathbf{k}}] \mid \mathbf{k} \in X^+(e)\}$.

Proof. Well-known by [AP95], see for example [EGNO15, Example 8.18.5]. \square

The following are known as *simplicial polytopical numbers*, since they count points in simplices.

Definition 2B.8. We define $p_N^e = \binom{e+N-1}{N-1} = \binom{M-1}{N-1}$. \diamond

We have shifted them when compared to the usual definition in the sense that our p_N^e is what is often denoted $P_{N-1}(e-1)$.

Example 2B.9. For $N = 2$ we have the linear numbers $p_2^e = e + 1$, for $N = 3$ we have the triangular numbers $p_3^e = \frac{(e+1)(e+2)}{2}$, see e.g. [OEI23, A000217]. \diamond

Lemma 2B.10. We have $[\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)]_{\oplus} \cong \mathbb{Z}^{\oplus p_N^e}$ as free \mathbb{Z} -modules.

Proof. By Lemma 2B.7, this is just a count. \square

The object $\mathbf{L}_{e\omega_1}$ is invertible in $\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)$ and for any $\mathbf{m} \in X^+(e)$ we have $\mathbf{L}_{e\omega_1} \otimes \mathbf{L}_{\mathbf{m}} \simeq \mathbf{L}_{\mathbf{m}^{\rightarrow}}$ where $\mathbf{m}^{\rightarrow} = (e - \sum \mathbf{m}, m_1, \dots, m_{N-2})$. This defines an action of $\mathbb{Z}/N\mathbb{Z}$ on $X^+(e)$ and we will denote by $\text{stab}_{\mathbf{m}}$ the size of the stabilizer of \mathbf{m} .

As for $\mathbf{Rep}_{\mathfrak{q}}(\mathfrak{sl}_N)$, we have a grading by $\mathbb{Z}/N\mathbb{Z}$ on $\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)$ by colors. For later use we recall:

Lemma 2B.11. The grading on $\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)$ by colors induces a grading on its Drinfeld center, also by colors.

Proof. Easy and omitted. \square

We denote the full subcategory of $\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)$ of objects of color i by $\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_i$, and similarly for the Drinfeld center.

2C. Braiding and ribbon structure. The fusion category $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$ is equipped with a braided structure $\beta_{-, -}$. This structure is obtained through the use of the universal R -matrix of $U_\eta(\mathfrak{sl}_N)$, as one can adapt from e.g. [Jan96, Chapter 7], and a choice of an N th root of η , which we denote by $\eta^{1/N}$. We also endow $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$ with a spherical structure, the corresponding quantum trace Tr and the corresponding ribbon θ , we refer to [EGNO15, Definition 8.10.1] for the definition of a ribbon structure. We will denote by $S_{\mathbf{m}, \mathbf{m}'}$ the values of the S -matrix as defined in [EGNO15, Definition 8.13.2].

Lemma 2C.1. *For any $\mathbf{m}, \mathbf{m}' \in X^+(e)$, we have the following formula for the S -matrix of $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$:*

$$S_{\mathbf{m}, \mathbf{m}'}^{\mathfrak{sl}} = \frac{\sum_{w \in \mathfrak{S}_N} (-1)^{l(w)} \eta^{2(\mathbf{m} + \rho, w(\mathbf{m}' + \rho))}}{\sum_{w \in \mathfrak{S}_N} (-1)^{l(w)} \eta^{2(\rho, w(\rho))}}.$$

For any $\mathbf{m} \in X^+(e)$, the ribbon $\theta_{\mathbf{m}}$ on $\mathbf{L}_{\mathbf{m}}$ is given by multiplication by the scalar $\eta^{(\mathbf{m}, \mathbf{m} + 2\rho)}$.

Proof. A proof of the lemma can be found in e.g. [BK01, Theorem 3.3.20]. \square

Using Bruguières's criterion [Bru00, Section 5], the fusion category $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$ is modular if and only if $\eta^{1/N}$ is a primitive $2MN$ th root of unity.

Lemma 2C.2. *Let $\mathbf{m} \in X^+(e)$. Then $\beta_{\mathbf{L}_{e\omega_1}, \mathbf{L}_{\mathbf{m}}} \circ \beta_{\mathbf{L}_{\mathbf{m}}, \mathbf{L}_{e\omega_1}} = \zeta^{-\chi_c(\mathbf{L}_{\mathbf{m}})} \cdot \mathrm{id}$.*

Proof. Since $\mathbf{L}_{\mathbf{m}^\rightarrow} = \mathbf{L}_{\mathbf{m}} \otimes \mathbf{L}_{e\omega_1}$ and since θ is a ribbon element, we have $\theta_{\mathbf{m}^\rightarrow} = \theta_{\mathbf{m}} \otimes \theta_{e\omega_1} \circ \beta_{\mathbf{L}_{e\omega_1}, \mathbf{L}_{\mathbf{m}}} \circ \beta_{\mathbf{L}_{\mathbf{m}}, \mathbf{L}_{e\omega_1}}$. Using Lemma 2C.1, we therefore obtain $\beta_{\mathbf{L}_{e\omega_1}, \mathbf{L}_{\mathbf{m}}} \circ \beta_{\mathbf{L}_{\mathbf{m}}, \mathbf{L}_{e\omega_1}} = \eta^{(\mathbf{m}^\rightarrow, \mathbf{m}^\rightarrow + 2\rho) - (\mathbf{m}, \mathbf{m} + 2\rho) - (e\omega_1, e\omega_1 + 2\rho)} \cdot \mathrm{id}$. One may finally show that $(\mathbf{m}^\rightarrow, \mathbf{m}^\rightarrow + 2\rho) - (\mathbf{m}, \mathbf{m} + 2\rho) - (e\omega_1, e\omega_1 + 2\rho) = -2e\chi_c(\mathbf{L}_{\mathbf{m}})/N$. \square

The grading by color is then retrieved using the braiding as in [EGNO15, Lemma 8.22.1].

2D. Chebyshev-like polynomials. The following polynomials are due to Koornwinder [Koo74], and Eierlidl [EL82]. We mildly change the conventions for convenience.

Definition 2D.1. For each \mathbf{m} we define *Chebyshev polynomials (of the second kind) for \mathfrak{sl}_N* , denoted by, $U_{\mathbf{m}} \in \mathbb{Z}[X_i]$ as

$$U_{\mathbf{m}} = \sum_{\mathbf{k}} d_{\mathbf{m}}^{\mathbf{k}} \cdot X^{\mathbf{k}},$$

with $d_{\mathbf{m}}^{\mathbf{k}} \in \mathbb{Z}$ as in (2B.2).

By convention, $U_{\mathbf{m}}$ and $\mathbf{L}_{\mathbf{m}}$ with negative subscripts m_i are zero. \diamond

Lemma 2D.2. *Let w_j^i denote the weights of \mathbf{L}_{ω_i} . We have the following Chebyshev-like recursion relations*

$$U_{\mathbf{m}}(X_1, \dots, X_{N-1}) = U_{\overline{\mathbf{m}}}(X_{N-1}, \dots, X_1), \quad X_i U_{\mathbf{m}} = \sum_{w_j^i} U_{\mathbf{m} + w_j^i}.$$

Together with the starting conditions for $e = 0, 1$ (as in Example 2D.3 for $N = 4$), these recursion relations determine the polynomials $U_{\mathbf{m}}$ for all \mathbf{m} .

Proof. By construction and classical theory. \square

Example 2D.3. The examples for $N = 2$ are the Chebyshev polynomials of the second kind normalized using the variable $x/2$ instead of x , while [MMMT20, Example 2.6] lists examples for $N = 3$.

The next case is $N = 4$, we have three fundamental variables, X_1 , X_2 and X_3 , associated to the simple 4 (the vector representation), 6 and 4 dimensional \mathfrak{sl}_4 -representations. We have the following recursion. If $m_i < 0$, then $U_{m_1, m_2, m_3} = 0$, and $U_{0,0,0} = 1$ and:

$$\begin{aligned} U_{m_1, m_2, m_3} &= X_1 \cdot U_{m_1-1, m_2, m_3} - U_{m_1-2, m_2+1, m_3} - U_{m_1-1, m_2-1, m_3+1} - U_{m_1-1, m_2, m_3-1}, \\ U_{m_1, m_2, m_3} &= \begin{cases} X_2 \cdot U_{m_1, m_2-1, m_3} - U_{m_1+1, m_2-2, m_3+1} - U_{m_1-1, m_2-1, m_3+1} \\ - U_{m_1+1, m_2-1, m_3-1} - U_{m_1-1, m_2, m_3-1} - U_{m_1, m_2-2, m_3}, \end{cases} \\ U_{m_1, m_2, m_3} &= X_3 \cdot U_{m_1, m_2, m_3-1} - U_{m_1, m_2+1, m_3-2} - U_{m_1+1, m_2-1, m_3-1} - U_{m_1-1, m_2, m_3-1}. \end{aligned}$$

Now $U_{0,0,0} = 1$ and

$e = 0$	$U_{1,0,0} = X_1, U_{0,1,0} = X_2, U_{0,0,1} = X_3,$
$e = 1$	$U_{2,0,0} = X_1^2 - X_2, U_{1,1,0} = X_1 X_2 - X_3, U_{1,0,1} = X_1 X_3 - 1,$ $U_{0,2,0} = X_2^2 - X_1 X_3, U_{0,1,1} = X_2 X_3 - X_1, U_{0,0,2} = X_3^2 - X_2,$
$e = 2$	$U_{3,0,0} = X_1^3 - 2X_1 X_2 + X_3, U_{2,1,0} = X_1^2 X_2 - X_1 X_3 - X_2^2 + 1, U_{2,0,1} = X_1^2 X_3 - X_2 X_3 - X_1, U_{1,2,0} = X_1 X_2^2 - X_1^2 X_3 - X_2 X_3 + X_1,$ $U_{1,1,1} = X_1 X_2 X_3 - X_1^2 - X_3^2, U_{1,0,2} = X_1 X_3^2 - X_1 X_2 - X_3, U_{0,3,0} = X_3^3 - 2X_1 X_2 X_3 + X_1^2 + X_3^2 - X_2,$ $U_{0,2,1} = X_2^2 X_3 - X_1 X_3^2 - X_1 X_2 + X_3, U_{0,1,2} = X_2 X_3^2 - X_1 X_3 - X_2^2 + 1, U_{0,0,3} = X_3^3 - 2X_2 X_3 + X_1.$

For $e = 3$ there are already 15 polynomials and for $e = 4$ there are 21, so we omit to put them here. Instead let us list the polynomials for the symmetric powers $U^{(k)} = U_{(k,0,0)}$ varying k :

k	0	1	2	3	4	5		
$U^{(k)}$	1	X_1	$X_1^2 - X_2$	$X_1^3 - 2X_1X_2 + X_3$	$X_1^4 - 3X_1^2X_2 + 2X_1X_3 + X_2^2 - 1$	$X_1^5 - 4X_1^3X_2 + 3X_1^2X_3 + 3X_1X_2^2 - 2X_2X_3 - 2X_1$		
k	6		7	8	9	10	11	12
$U^{(k)}$	$X_1^6 \pm \dots + 2X_2$		$X_1^7 \pm \dots - 2X_3$	$X_1^8 \pm \dots + 1$	$X_1^9 \pm \dots + 3X_1$	$X_1^{10} \pm \dots - 3X_2$	$X_1^{11} \pm \dots + 3X_3$	$X_1^{12} \pm \dots - 1$

We will see in the proof of [Lemma 2D.4](#) below how one can compute these fairly efficiently. \diamond

Lemma 2D.4. *The polynomial $U_{\mathbf{m}}$ has a nonzero constant term only if $\chi_c(\mathbf{L}_{\mathbf{m}}) = 0$. Moreover, all Chebyshev polynomial $U_{\mathbf{m}}$ with $\Sigma \mathbf{m} = e + 1$ have a zero constant term if and only if $e \equiv 0 \pmod{N}$.*

Proof. This proof is much more involved than in [\[MMMMT20, Lemma 2.8\]](#).

The trivial representation is of color 0, which implies that if $\chi_c(\mathbf{L}_{\mathbf{m}}) \neq 0$ then $U_{\mathbf{m}}$ has a zero constant coefficient.

The vanishing of the constant term is understood using the recursion [\[Bee91, \(5.18\)\]](#), which is nicely expressed using partitions instead of highest weights. This recursion implies that the Chebyshev polynomial $U_{\mathbf{m}}$ has a non zero constant coefficient if and only if the residues modulo N of $(m_1 + \dots + m_{N-1} + N - 1, m_2 + \dots + m_{N-1} + N - 2, \dots, m_{N-1} + 1, 0)$ are all different. Therefore, if $\Sigma \mathbf{m} \equiv 1 \pmod{N}$, the residue 0 appears at least twice and $U_{\mathbf{m}}$ has zero constant term. Conversely, given $e \equiv k \pmod{N}$ with $0 < k < N$, one may check that for $\mathbf{m} = \omega_k + e\omega_{N-1}$, the Chebyshev polynomial $U_{\mathbf{m}}$ has a nonzero constant term.

To be self-contained, the alternative recursion that we mention above comes from the following observation: It is remarkably easy to find a recursion for the k th symmetric power of the vector \mathfrak{sl}_N -representation, which correspond to $U_{(k,0,\dots,0)}$ in our notation. For $N = 2$ this recursion is the standard recursion since all simple \mathfrak{sl}_2 -representations are symmetric powers. For $N > 2$ let $U^{(k)}(X_1, \dots, X_{N-1}) = U_{(k,0,\dots,0)}(X_1, \dots, X_{N-1})$ denote these polynomials. The recursion then takes the form

$$U^{(k+N)} - X_1 U^{(k+N-1)} + \dots (-1)^{N-1} X_{N-1} U^{(k+1)} + (-1)^N U^{(k)} = 0,$$

with some additional starting conditions. The main observation that makes [\[Bee91, \(5.18\)\]](#) is then that this recursion also takes a nice form for other entries than the first when done entry-wise in the notation of the highest weights using partitions. In this recursion setting all variable to zero (the constant term) gives

$$U^{(k+N)} + (-1)^N U^{(k)} = 0,$$

and the claim then follows easily. \square

A classical result of Kostant [\[Kos76\]](#) gives a formula for certain powers of the Dedekind η -function by summing over simple \mathfrak{sl}_N -representations $\mathbf{L}_{\mathbf{m}}$, and the appearing coefficients $\epsilon_{\mathbf{m}}$ are in $\{0, 1, -1\}$. The following is a fun side observation:

Proposition 2D.5. *The constant coefficient of the Chebyshev polynomial $U_{\mathbf{m}}$ is equal to the trace of $a(ny)$ Coxeter element acting on the zero weight space of $\mathbf{L}_{\mathbf{m}}$ which in turn is equal to $\epsilon_{\mathbf{m}}$.*

Proof. This follows by comparing [\[Bee91, \(5.18\)\]](#) and [\[AF04, Theorem 1.2\]](#). \square

2E. Koornwinder variety. All proofs in this section are much more intricate than their rank 2 or 3 counterparts in [\[MMMMT20\]](#). We define:

Definition 2E.1. Let J_e be the ideal generated by

$$\{U_{\mathbf{m}} \mid \Sigma \mathbf{m} = e + 1\} \subset \mathbb{Z}[X_i].$$

We call J_e the *vanishing ideal* of level e . Associated to it is the *Koornwinder variety* of level e

$$V_e = \{\gamma = (\gamma_1, \dots, \gamma_{N-1}) \in \mathbb{C}^{N-1} \mid p(\gamma) = 0 \text{ for all } p \in J_e\} \subset \mathbb{C}^{N-1}$$

which we consider as a complex variety. \diamond

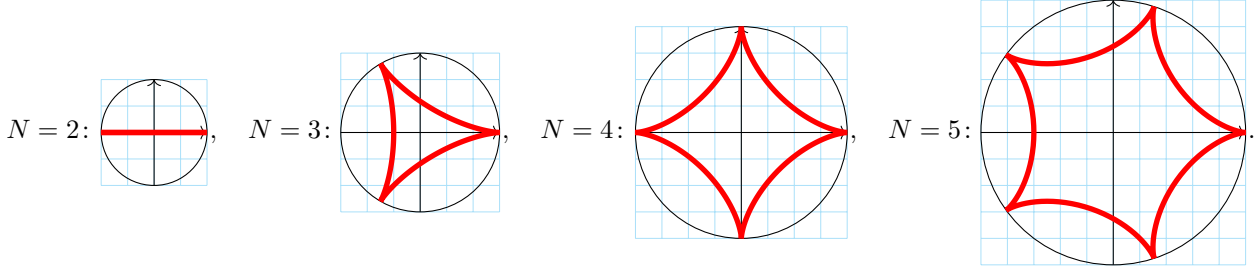
Remark 2E.2. Following history, one could also call V_e the *Chebyshev–Eier–Koornwinder–Lidl variety*, cf. [\[Koo74\]](#), which discusses the case $N = 3$ and [\[EL82\]](#), which discusses the general case, but that is a mouthful. \diamond

Example 2E.3. All roots of Chebyshev polynomials $U_{\mathbf{m}}$ have their first coordinate in the interior of the *N -cusped hypocycloid* of parametric equation

$$x(\theta) = (N - 1) \cos(\theta) + \cos((N - 1)\theta) \text{ and } y(\theta) = (N - 1) \sin(\theta) - \sin((N - 1)\theta).$$

This is a folk result, and can, for example, be explicitly found in [\[Kai06, Section 3\]](#). Recall that these are plane curves generated by following a point on a circle of radius 1 that rolls within a circle of radius N . Here

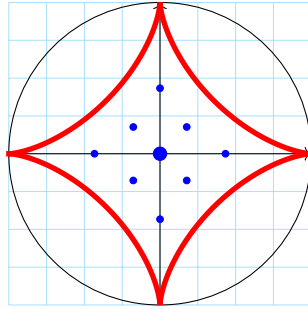
are the pictures:



To be completely explicit, let $N = 4$ and $e = 2$. The common roots of the Chebyshev-like polynomials are:

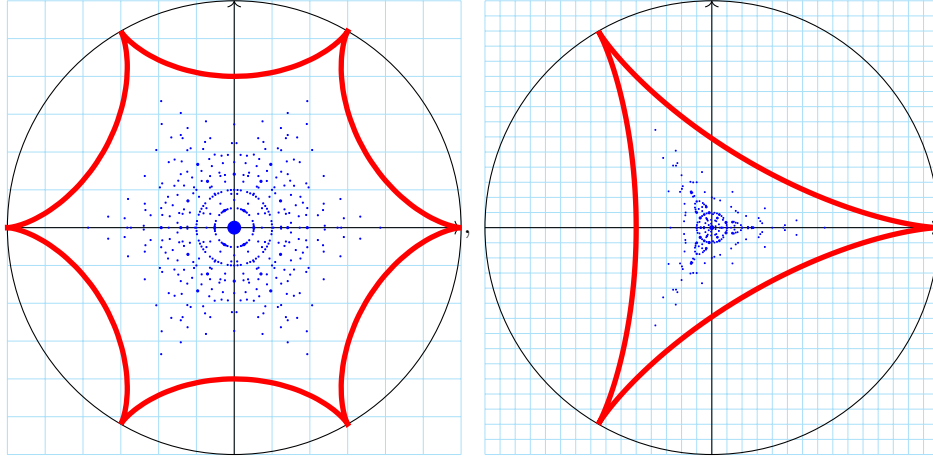
$$(0, -1, 0), (0, 1, 0), (\sqrt{3}i, -2, -\sqrt{3}i), (-\sqrt{3}i, -2, \sqrt{3}i), (\sqrt{3}, 2, \sqrt{3}), (-\sqrt{3}, 2, -\sqrt{3}), \\ (e^{2i\pi/8}, 0, e^{14i\pi/8}), (e^{6i\pi/8}, 0, e^{10i\pi/8}), (e^{10i\pi/8}, 0, e^{6i\pi/8}), (e^{14i\pi/8}, 0, e^{2i\pi/8}).$$

We have the following plot of the first coordinates:



The point at zero is illustrated thick since it correspond to two values. Similar conventions are used throughout, i.e. the thickness of points indicates their multiplicity. \diamond

Example 2E.4. For $N = 6$ and $e = 6$, the Koornwinder variety has $p_6^6 = 462$ points. We have the following plot of the first two coordinates:



We have not included the third coordinate since the points are on the real line. The fourth and fifth coordinates are the complex conjugates of the second and the first. Note that the radius of the circle of the i th coordinate equals the dimension of the \mathfrak{sl}_N -representation L_{ω_i} . \diamond

Definition 2E.5. For $1 \leq i \leq N - 1$, we introduce *Koornwinder's Z-functions* $Z_i: E \rightarrow \mathbb{C}$ defined by

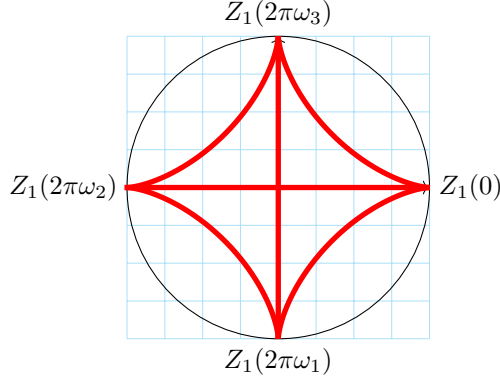
$$Z_i(\sigma) = \sum_j \exp(\mathfrak{i}(\sigma, w_j^i)),$$

where we recall that $(w_j^i)_j$ are the weights of the fundamental representation L_{ω_i} . \diamond

Remark 2E.6. The map Z_1 is $2\pi Y$ periodic, where Y is the root lattice, and is invariant under the action of \mathfrak{S}_N . The fundamental domain of $E/2\pi Y$ for this action is

$$D = \left\{ \sum_{i=1}^{N-1} \lambda_i \alpha_i \mid 2\lambda_i \geq \lambda_{i-1} + \lambda_{i+1} \text{ for all } 1 \leq i < N, \lambda_1 + \lambda_{N-1} \leq 2\pi \right\}.$$

Note that, in contrast to [Koo74] (this is $N = 3$), the map Z_1 is not injective on the fundamental domain D for $N > 3$. For example, for $N = 4$, D is a tetrahedron and the image of the six edges under Z_1 are the following:



The edge joining the vertices 0 and $2\pi\omega_2$ and the one joining the vertices $2\pi\omega_1$ and $2\pi\omega_3$ are mapped to the parts of the real and the imaginary axis inside the hypocycloid. \diamond

To describe V_e using the functions Z_i , we introduce a different parametrization.

Definition 2E.7. Define V'_e as the set $V'_e = \left\{ \frac{2\pi}{e+N}(\mathbf{k} + \rho) \mid \mathbf{k} \in X^+(e) \right\}$. For $\sigma = \frac{2\pi}{e+N}(\mathbf{k} + \rho) \in V'_e$, we denote by $\sigma \rightarrow$ the element $\frac{2\pi}{e+N}(\mathbf{k} \rightarrow + \rho) \in V'_e$. \diamond

Lemma 2E.8. Given $\sigma \in V'_e$, we have $(Z_1(\sigma), \dots, Z_{N-1}(\sigma)) \in V_e$.

Example 2E.9. For $N = 3$, the coordinates of the vectors of V'_e in the basis (α_1, α_2) coincide with the expression as in [MMMT20, (2-11)]. For $N = 4$, we get

$$V'_e = \left\{ \frac{2\pi}{4(e+4)}(3k_1 + 2k_2 + 1k_3 + 6, 2k_1 + 4k_2 + 2k_3 + 8, 1k_1 + 2k_2 + 3k_3 + 6) \mid 0 \leq k_1 + k_2 + k_3 \leq e \right\},$$

from Definition 2E.7, where the coordinates are given in the basis $(\alpha_1, \alpha_2, \alpha_3)$. \diamond

In the next theorem we use the Möbius function μ .

Theorem 2E.10. We have the following:

- (a) As \mathbb{Z} -algebras we have $[\mathbf{Rep}_\eta(\mathfrak{sl}_N)]_\oplus \cong \mathbb{Z}[\mathbf{X}_i]/J_e$.
- (b) $\#V_e = p_N^e$.
- (c) The number of points in V_e with stabilizer of size m under multiplication by ζ is

$$\frac{N}{M} \sum_{k|g} \mu(k) \binom{M/mk}{N/mk},$$

where $g = \gcd(N/m, M/m)$.

Proof. Part (a). That we get the claimed isomorphism of free \mathbb{Z} -modules follows from the definitions. Moreover, the quantum Racah formula [Saw06, Corollary 8] implies that this is actually an isomorphism of \mathbb{Z} -algebras.

Parts (b)+(c). These are proven in Section 5A. \square

3. ASYMPTOTIC $G(M, M, N)$

Below we will use some standard terminology that can be found e.g. in [EGNO15].

3A. The asymptotic category and its Drinfeld center. We define the asymptotic category $\mathbf{a}_{M,M,N}$ as a matrix category indexed by $\mathbb{Z}/N\mathbb{Z}$, with entries in suitable subcategories of $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$.

Definition 3A.1. The *asymptotic category* $\mathbf{a}_{M,M,N}$ is the monoidal additive \mathbb{C} -linear category whose:

- (a) indecomposable objects are matrices $(\mathbf{X}_{ij})_{i,j \in \mathbb{Z}/N\mathbb{Z}}$, with \mathbf{X}_{ij} a simple object in $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$ and $\chi_c(\mathbf{X}_{ij}) = i - j$,
- (b) morphisms between $(\mathbf{X}_{ij})_{i,j \in \mathbb{Z}/N\mathbb{Z}}$ and $(\mathbf{Y}_{ij})_{i,j \in \mathbb{Z}/N\mathbb{Z}}$ are matrices $(f_{ij})_{i,j \in \mathbb{Z}/N\mathbb{Z}}$ with f_{ij} being a morphism between \mathbf{X}_{ij} and \mathbf{Y}_{ij} in $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$,

and the tensor product is given by multiplication of matrices. We equip this category with the ribbon structure inherited from $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$. \diamond

Example 3A.2. When $e = 0$, then $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$ is equivalent to the category of \mathbb{C} -vector spaces, and $\mathbf{a}_{M,M,N}$ is then equivalent to the direct sum of N copies of the category of \mathbb{C} -vector spaces. \diamond

Lemma 3A.3. *The asymptotic category is a multifusion category of rank Np_N^e and is indecomposable if and only if $e \neq 0$.*

Proof. The first statement follows from [Lemma 2B.10](#), applied row-by-row. For $e = 0$ see [Example 3A.2](#). Finally, if $e \neq 0$, the grading of $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$ by colors is faithful. \square

We now describe the Drinfeld center of the asymptotic category when $e \geq 0$. The first case is easy:

Example 3A.4. If $e = 0$, then the Drinfeld center is simply N copies of the category of vector spaces, with a trivial braiding. \diamond

We now assume that $e > 0$. Let us denote by $J_3(k) = k^3 \prod_{p|k} (1 - \frac{1}{p^3})$ Jordan's totient function. See [\[OEI23, A059376\]](#) for explicit values. Moreover, recall the *modular closure* as in e.g. [\[Bru00\]](#) or [\[Müg00\]](#), which is a special case of de-equivariantization from [\[EGNO15, Section 8.23\]](#).

Theorem 3A.5. *We have the following.*

- (a) $\mathbf{Z}(\mathbf{a}_{M,M,N})$ is equivalent as a ribbon category to the modular closure of $\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))_0$. In particular, it is a modular category.
- (b) Write $\text{stab}_{\mathbf{k},\mathbf{m}} = \gcd(\text{stab}_{\mathbf{k}}, \text{stab}_{\mathbf{m}})$. Simple objects of $\mathbf{Z}(\mathbf{a}_{M,M,N})$ are indexed by the set

$$\{(\mathbf{m}, \mathbf{k}, i) \mid (\mathbf{m}, \mathbf{k}) \in X^+(e)^2 / (\mathbb{Z}/N\mathbb{Z}) \text{ with } \chi_c(\mathbf{L}_{\mathbf{m}}) = \chi_c(\mathbf{L}_{\mathbf{k}}), \text{ and } i \in \mathbb{Z}/\text{stab}_{\mathbf{m},\mathbf{k}}\mathbb{Z}\}.$$

Here $\mathbb{Z}/N\mathbb{Z}$ acts diagonally on $X^+(e)^2 = X^+(e) \times X^+(e)$.

- (c) The rank of $\mathbf{Z}(\mathbf{a}_{M,M,N})$ is

$$\text{rk } \mathbf{Z}(\mathbf{a}_{M,M,N}) = \frac{1}{M^2} \sum_{k|\gcd(N,M)} J_3(k) \left(\frac{M/k}{N/k}\right)^2.$$

- (d) For N fixed and $M \rightarrow \infty$ we have

$$\text{rk } \mathbf{Z}(\mathbf{a}_{M,M,N}) \sim \frac{1}{(N!)^2} \cdot M^{2N-2}.$$

- (e) The S -matrix $S^{\mathbf{a}}$ of $\mathbf{Z}(\mathbf{a}_{M,M,N})$ satisfies:

$$\sum_{i=1}^{\text{stab}_{\mathbf{k},\mathbf{m}}} S_{(\mathbf{m},\mathbf{k},i),(\mathbf{m}',\mathbf{k}',i')}^{\mathbf{a}} = \frac{1}{\text{stab}_{\mathbf{k},\mathbf{m}}} S_{\mathbf{m},\mathbf{m}'}^{\mathfrak{sl}} \overline{S}_{\mathbf{k},\mathbf{k}'}^{\mathfrak{sl}}.$$

Moreover, if N is prime, there exists a unique $(\mathbf{m}, \mathbf{k}) \in X^+(e)^2 / (\mathbb{Z}/N\mathbb{Z})$ with $\text{stab}_{\mathbf{k},\mathbf{m}} \neq 1$ if and only if $e \equiv 0 \pmod{N}$ and we have

$$S_{(\mathbf{m},\mathbf{k},i),(\mathbf{m},\mathbf{k},j)}^{\mathbf{a}} = \frac{1}{N^2} \begin{cases} S_{\mathbf{m},\mathbf{m}}^{\mathfrak{sl}} \overline{S}_{\mathbf{k},\mathbf{k}}^{\mathfrak{sl}} + (N-1)\theta_{\mathbf{m},\mathbf{k}}^3 \dim_c \mathbf{Rep}_\eta(\mathfrak{sl}_N) & \text{if } i = j, \\ S_{\mathbf{m},\mathbf{m}}^{\mathfrak{sl}} \overline{S}_{\mathbf{k},\mathbf{k}}^{\mathfrak{sl}} - \theta_{\mathbf{m},\mathbf{k}}^3 \dim_c \mathbf{Rep}_\eta(\mathfrak{sl}_N) & \text{if } i \neq j, \end{cases}$$

where $\dim_c \mathbf{Rep}_\eta(\mathfrak{sl}_N)$ is the categorical dimension. (See, for example, [\[BK01, \(3.3.9\)\]](#) for the value of $\dim_c \mathbf{Rep}_\eta(\mathfrak{sl}_N)$.) If N is not prime, see [Remark 3A.8](#).

- (f) The T -matrix of $\mathbf{Z}(\mathbf{a}_{M,M,N})$ is the matrix with entries $T_{(\mathbf{m},\mathbf{k},i),(\mathbf{m}',\mathbf{k}',i')}^{\mathbf{a}} = \delta_{\mathbf{m},\mathbf{m}'} \delta_{\mathbf{k},\mathbf{k}'} \delta_{i,i'} \theta_{\mathbf{m}}^{-1} \theta_{\mathbf{k}}$.

Example 3A.6. Let $N = 3$, $e = 3$ and $M = 6$. The category $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$ has 10 simple objects and $\mathbf{Z}(\mathbf{a}_{M,M,N})_0$ has then 34 simple objects. Among these 34 object, there are 11 orbits of size 3 and one of size 1 under the action of $\mathbb{Z}/N\mathbb{Z}$. Therefore, $\mathbf{Z}(\mathbf{a}_{M,M,N})$ has 14 simple objects, since each orbit of size 3 will give one simple object, and the orbit of size 1 will split into 3 simple objects.

The S -matrix of $\mathbf{Z}(\mathbf{a}_{M,M,N})$ is the 14-by-14 matrix

$$S^{\mathbf{a}} = \begin{pmatrix} \begin{array}{cccc|cccccccc|cccc} 1 & 1 & 3 & 1 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 \\ 1 & 1 & 3 & 1 & 4\Upsilon & 4\Upsilon & 4\Upsilon & 4\Upsilon^2 & 4\Upsilon^2 & 4\Upsilon^2 & 3 & 3 & 3 & 3 \\ 3 & 3 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & -3 & -3 & -3 \\ 1 & 1 & 3 & 1 & 4\Upsilon^2 & 4\Upsilon^2 & 4\Upsilon^2 & 4\Upsilon & 4\Upsilon & 4\Upsilon & 3 & 3 & 3 & 3 \\ 4 & 4\Upsilon & 0 & 4\Upsilon^2 & 4 & 4\Upsilon & 4\Upsilon^2 & 4 & 4\Upsilon^2 & 4\Upsilon & 0 & 0 & 0 & 0 \\ 4 & 4\Upsilon & 0 & 4\Upsilon^2 & 4\Upsilon & 4\Upsilon^2 & 4 & 4\Upsilon^2 & 4\Upsilon & 4 & 0 & 0 & 0 & 0 \\ 4 & 4\Upsilon & 0 & 4\Upsilon^2 & 4\Upsilon^2 & 4 & 4\Upsilon & 4\Upsilon & 4 & 4\Upsilon^2 & 0 & 0 & 0 & 0 \\ 4 & 4\Upsilon^2 & 0 & 4\Upsilon & 4 & 4\Upsilon^2 & 4\Upsilon & 4 & 4\Upsilon & 4\Upsilon^2 & 0 & 0 & 0 & 0 \\ 4 & 4\Upsilon^2 & 0 & 4\Upsilon & 4\Upsilon^2 & 4\Upsilon & 4 & 4\Upsilon & 4\Upsilon^2 & 4 & 0 & 0 & 0 & 0 \\ 4 & 4\Upsilon^2 & 0 & 4\Upsilon & 4\Upsilon & 4 & 4\Upsilon^2 & 4\Upsilon^2 & 4 & 4\Upsilon & 0 & 0 & 0 & 0 \\ 3 & 3 & 9 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -3 & -3 & -3 \\ 3 & 3 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 9 & -3 & -3 \\ 3 & 3 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -3 & 9 & -3 \\ 3 & 3 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -3 & -3 & 9 \end{array} \end{pmatrix},$$

where $\Upsilon = \eta^4$. The top colored block corresponds to the S -matrix of $\mathbf{Rep}_\eta(\mathfrak{sl}_N)_0$ and the bottom colored block corresponds to the object of $\mathbf{Z}(\mathbf{a}_{M,M,N})_0$ that splits in three. \diamond

Remark 3A.7. Let \mathfrak{A}_4 denote the alternating group of order twelve. The matrix in [Example 3A.6](#) is the same as the S -matrix of Drinfeld center of \mathfrak{A}_4 -graded vector spaces (the latter, by duality, is the same as the Drinfeld center of the category of \mathfrak{A}_4 -representations). However, the appearance of nonnegative integers in the first row and column of the matrix is a small number coincidence and we do not expect any nice description of the S -matrix in terms of a group in general. \diamond

Proof of Theorem 3A.5. Part (a). Since the asymptotic category $\mathbf{a}_{M,M,N}$ is indecomposable by [Lemma 3A.3](#), [\[KZ18, Theorem 2.5.1\]](#) shows that its Drinfeld center is equivalent to the Drinfeld center of any of its diagonal components $\mathbf{Rep}_\eta(\mathfrak{sl}_N)_0$. We then observe that $\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N)_0)$ is monoidally equivalent to the modular closure of $\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))_0$, which follows immediately from [\[EGNO15, Proposition 8.23.11\]](#).

Part (b). Let $(-)^{\text{rev}}$ denote the reverse category as in [\[EGNO15, Definition 8.1.4\]](#). Since $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$ is modular, its Drinfeld center is equivalent to $\mathbf{Rep}_\eta(\mathfrak{sl}_N) \boxtimes \mathbf{Rep}_\eta(\mathfrak{sl}_N)^{\text{rev}}$, see, for example, [\[EGNO15, Proposition 8.6.3\]](#). The degree zero part of $\mathbf{Rep}_\eta(\mathfrak{sl}_N) \boxtimes \mathbf{Rep}_\eta(\mathfrak{sl}_N)^{\text{rev}}$ is $\bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} \mathbf{Rep}_\eta(\mathfrak{sl}_N)_i \boxtimes (\mathbf{Rep}_\eta(\mathfrak{sl}_N)^{\text{rev}})_i$, and its symmetric center is generated by the object $L_{e\omega_1} \boxtimes L_{e\omega_1}$. Since $L_{e\omega_1} \otimes L_{\mathbf{m}} = L_{\mathbf{m}}$, the result follows from [\[Bru00, Remarques 4.5, 1\)\]](#) or [\[Müg00, Corollary 5.3\]](#).

Part (c). This is done in [Section 5B](#).

Part (d). For fixed N , $\text{rk } \mathbf{Z}(\mathbf{a}_{M,M,N})$ is polynomial in M and the leading term is obtained for $k = 1$. The result follows from $J_3(1) = 1$ and $\binom{M}{N} \sim \frac{M^N}{N!}$ when $M \rightarrow \infty$.

Part (e). This is done in [Section 5B](#).

Part (f). This follows immediately from [\[Müg00, Proposition 4.2\]](#). \square

Remark 3A.8. When N is not prime, the calculation of the S -matrix of $\mathbf{Z}(\mathbf{a}_{M,M,N})$ is more intricate. For $\text{gcd}(e, N) \neq 1$, [Lemma 5B.2](#) shows that several objects of $\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))_0$ can split in the modular closure. \diamond

3B. A family of unipotent characters. Malle [\[Mal95\]](#) has introduced the notion of unipotent characters for the complex reflection group $G(M, M, N)$. These characters are partitioned into *families*, similarly to the unipotent characters of a finite group of Lie type as e.g. in [\[Lus84, Chapter 4\]](#). We consider a certain family \mathcal{F} of unipotent characters which, in terms of the combinatorics developed by Malle, are indexed by so-called *M-symbols* with entries in the multiset $\{0^e, 1^N\}$, but we do not use this explicitly.

Remark 3B.1. For $N = 2$ the family \mathcal{F} corresponds to the subregular KL cell of \mathbf{a} -value 1. In general, the families we consider have \mathbf{a} -value $N(N-1)/2$. \diamond

The elements of \mathcal{F} are indexed by orbits under an action of $\mathbb{Z}/M\mathbb{Z}$, which we will define later in [Section 5B](#) (on M -symbols, this action is given by a cyclic shift of the rows), on the following set. Let $Y = \{1, \dots, M\}$ and $\pi: Y \rightarrow \mathbb{Z}_{\geq 0}$ defined by $\pi(y) = 1$ if $1 \leq y \leq N$ and $\pi(y) = 0$ otherwise. We consider the set

$$\Psi(Y, \pi) = \left\{ f: Y \rightarrow \{0, \dots, M-1\} \mid \sum_{y \in Y} f(y) \equiv \binom{M}{2} \pmod{M}, f_{|\pi^{-1}(i)} \text{ is strictly increasing for all } i \in \mathbb{Z}_{\geq 0} \right\}.$$

We denote equivalence classes by $[f]$. An equivalence class $[f]$ might index several unipotent characters of \mathcal{F} , which we will denote by $([f], i)$ for $i \in \mathbb{Z}_{\geq 0}$. We make this precise in [Section 5B](#).

To each family of unipotent characters, Malle [\[Mal95\]](#) has attached a *Fourier matrix* and eigenvalues of the *Frobenius*, generalizing the constructions from the groups of Lie type as, for example, in [\[Lus84\]](#). The

eigenvalue of the Frobenius of the unipotent character of \mathcal{F} indexed by $([f], i)$ is

$$(3B.2) \quad \text{Frob}([f], i) = \eta^{-\alpha(f)} \quad \text{with} \quad \alpha(f) = \frac{M(1-M^2)}{6} - \sum_{y \in Y} (f(y)^2 + Mf(y)).$$

For the Fourier matrix, we will adopt the conventions of [Las12, Section 5.4] which differ from [Mal95, Section 6C] by some signs. For $f \in \Psi(Y, \pi)$, we set $\varepsilon(f) = (-1)^{c(f)+\gamma(f)}$ where

$$c(f) = |\{(y, y') \in Y^2 \mid y < y', f(y) < f(y')\}| \quad \text{and} \quad \gamma(f) = \frac{M-1}{M} \left(\binom{M}{2} - \sum_{y \in Y} f(y) \right).$$

Let $\mathcal{S} = (\eta^{-2ij})_{0 \leq i, j < M}$ and define $\tau = (-1)^{\binom{M}{2}} \det(\mathcal{S})$. We view \mathcal{S} as the matrix of an endomorphism of $V = \bigoplus_{i=0}^{M-1} \mathbb{C}v_i$. Given $f: Y \rightarrow \{0, \dots, M-1\}$ strictly increasing on $\pi^{-1}(1) = \{1, \dots, N\}$ and on $\pi^{-1}(0) = \{N+1, \dots, M\}$, we set

$$v_f = (v_{f(1)} \wedge \dots \wedge v_{f(N)}) \otimes (v_{f(N+1)} \wedge \dots \wedge v_{f(M)}) \in \Lambda^N V \otimes \Lambda^e V.$$

Such vectors form a basis of $\Lambda^N V \otimes \Lambda^e V$ and we denote by $(\Lambda^N \mathcal{S} \otimes \Lambda^e \mathcal{S})_{f,g}$ the entries of the matrix of the endomorphism $\Lambda^N \mathcal{S} \otimes \Lambda^e \mathcal{S}$ in this basis.

We define the **pre-Fourier matrix** $\tilde{\mathbb{S}}$ of the family \mathcal{F} , which is indexed by orbits of $\Psi(Y, \pi)$ under the action of $\mathbb{Z}/M\mathbb{Z}$, by

$$\tilde{\mathbb{S}}_{[f],[g]} = \frac{(-1)^{M-1} M}{\tau} \varepsilon(f) \varepsilon(g) \overline{(\Lambda^N \mathcal{S} \otimes \Lambda^e \mathcal{S})_{f,g}},$$

for all $f, g \in \Psi(Y, \pi)$. In contrast to the pre-Fourier matrix, the Fourier matrix itself \mathbb{S} of the family \mathcal{F} is not entirely determined, see [Las12, Remark 5.4.35].

Let ι be the bijection defined in Section 5B. Recall the T -matrix $T^{\mathbf{a}}$ of the modular category $\mathbf{Z}(\mathbf{a}_{M,M,N})$. We denote the S -matrix of the degree zero part of the Drinfeld center of $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$ by S^0 .

Theorem 3B.3. *We have the following.*

- (a) *The cardinal of \mathcal{F} is equal to the rank of $\mathbf{Z}(\mathbf{a}_{M,M,N})$.*
- (b) *For all $f, g \in \Psi(Y, \pi)$,*

$$\tilde{\mathbb{S}}_{[f],[g]} = \text{factor}(f, g) \cdot S_{\iota(f), \iota(g)}^0, \quad \text{factor}(f, g) = \frac{(-1)^{\sum_{i=1}^N (f(i)+g(i))} \varepsilon(f) \varepsilon(g)}{\sqrt{\dim(\mathbf{Z}(\mathbf{a}_{M,M,N}))}}.$$

Thus, up to a diagonal change of basis, the pre-Fourier matrix and the S -matrix of $\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))_0$ coincide.

- (c) *For all $f \in \Psi(Y, \pi)$,*

$$\text{Frob}([f], i) = T_{\iota(f), \iota(f)}^{\mathbf{a}}.$$

Remark 3B.4. By Theorem 3A.5.(e), the S -matrix of $\mathbf{Z}(\mathbf{a}_{M,M,N})$ satisfies [Las12, Conjecture 5.4.34], and we think it hence deserves to be called the Fourier matrix of the family \mathcal{F} . \diamond

3C. The big asymptotic category related to a Calogero–Moser cell. We say that a unipotent character represented by $([f], i)$ is in the principal series if $|f^{-1}(i)| = 1$ for every $0 \leq i \leq e-1$. We denote by \mathcal{F}_0 the unipotent characters of \mathcal{F} lying in the principal series. They are in bijection with a subset of irreducible complex representations of $G(M, M, N)$, which are represented by M -partitions of N with each entry being (1) or empty, see [Mal95, Section 6A] for more details. By a (slight) abuse of notations, we will also denote by \mathcal{F}_0 the set of irreducible complex representations of $G(M, M, N)$ corresponding to the unipotent characters in the principal series. If $M > N$, a result of Bellamy [Bel12] states that \mathcal{F}_0 is a CM family of irreducible complex representations of $G(M, M, N)$ for a specific choice of parameters, the so-called **spetsial** ones.

Remark 3C.1. The spetsial parameters correspond to equal parameters for Iwahori–Hecke algebras. \diamond

In [BR17], Bonnafé–Rouquier construct a partition of a finite complex reflection group into left, right and two-sided CM cells, and conjecture that these cells coincide with the KL cells provided the complex reflection group is a Coxeter group. They also construct a bijection between the CM families and the two-sided CM cells. Let us denote by Γ the two-sided CM cell associated with the family \mathcal{F}_0 .

We now define what we call the **big asymptotic category**.

Definition 3C.2. Let $\mathbf{A}_{M,M,N}$ be the matrix category over $\mathbf{a}_{M,M,N}$ of size $(N-1)!$. \diamond

Since $\mathbf{a}_{M,M,N}$ is a ribbon category, so it the big asymptotic category $\mathbf{A}_{M,M,N}$.

Theorem 3C.3. *We have the following.*

- (a) *$\mathbf{a}_{M,M,N}$ and $\mathbf{A}_{M,M,N}$ are Morita equivalent (they have the same 2-representation theory).*

(b) The Drinfeld centers of $\mathfrak{a}_{M,M,N}$ and $\mathbf{A}_{M,M,N}$ are equivalent as modular categories.

(c) $\text{rk } \mathbf{A}_{M,M,N} = \frac{(N!)^2}{M} \binom{M}{N} = |\Gamma|$.

(d) $[\mathbf{A}_{M,M,N}]_{\oplus}^{\mathbb{C}} \simeq \bigoplus_{V \in \mathcal{F}_0} \text{Mat}_{\dim_{\mathbb{C}} V}(\mathbb{C})$.

Proof. Part (a). This is [MMM⁺21, Proposition 2.27].

Part (b). This follows from [KZ18, Theorem 2.5.1, (7)].

Part (c). The rank of $\mathbf{A}_{M,M,N}$ is $((N-1)!)^2 \text{rk } \mathfrak{a}_{M,M,N} = ((N-1)!)^2 N p_N^e$. The second equality follows from [BR17, Theorem 12.2.7, (c)] and taking dimensions in (d).

Part (d). See Section 5B. □

4. NHEDRAL HECKE ALGEBRAS AND CATEGORIES

While reading this section we recommend having Lemma 2B.1 and Theorem 2E.10.(a) in mind that we are going to “color”.

4A. Nhedral Hecke algebras. Let $I = \{0, \dots, N-1\}$, identified with the set of vertices of the affine type A_{N-1} Dynkin diagram. The associated Weyl group is $W = \langle s_i \mid i \in I \rangle / (\text{relations in (4A.2)})$.

Notation 4A.1. To easy notation, we write i instead of s_i for the standard generators of W . We will also often write e.g. 145 for $C = \{1, 4, 5\}$. ◇

Summarized in one picture:

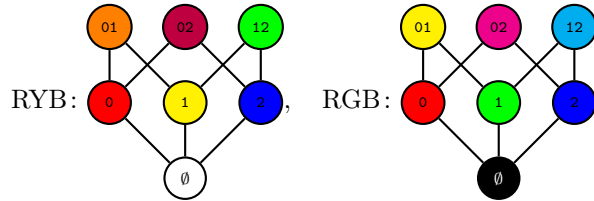
$$(4A.2) \quad \tilde{A}_{N-1} = A_{N-1}^{(1)}: \quad \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ 3 \quad 1 \\ \diagup \quad \diagdown \\ 4 \quad 0 \\ \vdots \\ N-1 \end{array}, \quad \text{relations: } ii = 1 \text{ and } \begin{array}{l} 010 = 101 \\ 121 = 212 \\ \dots \\ (N-1)0(N-1) = 0(N-1)0 \end{array} .$$

For $k \in \{0, \dots, N-1\}$ the k th colors are the subsets $C \subset I$ of size k . For such a C its *ingredient colors* are the $D \subset I$ of size $k-1$ such that $\#C \Delta D = 1$, i.e. they differ by precisely one element. The 0th color is \emptyset , also called *white*.

Remark 4A.3. The colors correspond to *finite parabolic subgroups* $W_C = \langle i \mid i \in C \rangle \subset W$. Note that there is no black color: I is not a color as the corresponding subgroup would be W itself. ◇

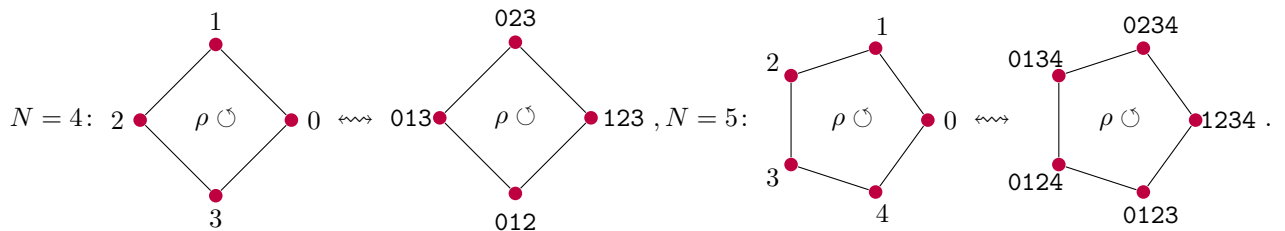
The $(N-1)$ th colors are the subsets $C \subset I$ of size $N-1$, and since we use them often we call these *top colors*. These correspond to maximal finite parabolic subgroups of W .

Example 4A.4. For $N = 3$, the color analogy can be used at its fullest, either in the RYB model (as in [MMMT20], the convention we follow) or the RGB model: Here 0 is red, 1 is either yellow or green, and 2 is blue. Moreover, 01 is orange or yellow, 02 is purple or magenta, and 12 is green or cyan.



(Note that white is black in RGB.) Beyond $N = 3$ the color analogy gets a bit shaky, but is still useful to keep in mind. ◇

Label the vertices of an N gon from 0 to $N-1$ counterclockwise. Let us put the top colors on the vertices of such a regular N gon, so that C corresponds to the vertex i with $C \cup i = I$. Let $\mathbb{Z}/N\mathbb{Z} \cong \langle \rho_0, \dots, \rho_{N-1} \mid \rho_i \rho_j = \rho_{i+j \pmod N} \rangle$ act on this configuration by rotation, i.e. $\rho = \rho_1 : i \rightarrow i+1 \pmod N$. For example, for $N = 4$ and $N = 5$:



Notation 4A.5. We will always have a set of size $N - 1$ inside a set of size N , and we will index the subsets by the missing value. Similarly, the parabolic subgroups of W corresponding to these subsets will be indexed by the missing value. For example, if $N = 4$, then 023 will be denoted 1 and the corresponding parabolic subgroup by W_1 . \diamond

Recall that \mathfrak{v} is a generic parameter. The quantum numbers in this variable will be denoted by $[a]_{\mathfrak{v}}$ etc.

Definition 4A.6. The *Nhedral Hecke algebra* T_{∞} of level ∞ is the associative unital ($\mathbb{C}^{\mathfrak{v}}$ -)algebra generated by N elements $\{\theta_i | i \in I\}$ subject to the following relations.

$$(4A.7) \quad \theta_i^2 = [N]_{\mathfrak{v}}! \cdot \theta_i, \quad \theta_{k+i+j}\theta_{k+i}\theta_k = \theta_{k+i+j}\theta_{k+j}\theta_k \text{ for all } i, j, k \in I,$$

where indices are taken modulo N . The second type of relation is called *fundamental commutativity*. \diamond

Remark 4A.8. Recall from [Remark 3B.1](#) that the \mathfrak{a} -value of the CM cell is $N(N - 1)/2$, which is half of the span of $[N]_{\mathfrak{v}}!$. So after shift, at $\mathfrak{v} = 0$ the relation $\theta_i^2 = [N]_{\mathfrak{v}}! \cdot \theta_i$ becomes $\theta_i^2 = \theta_i$. \diamond

We identify ρ_i with the fundamental variable X_i . The fundamental commutativity is then mimicking $X_i X_j = X_j X_i$. Moreover, given an expression $\theta_j \dots \theta_i$, we call i the *starting* and j the *ending color*.

Example 4A.9. Let $N = 4$. Then, up to changing the starting color, we have

$$\begin{aligned} \theta_3 \theta_1 \theta_0 &= \theta_3 \theta_2 \theta_0 \rightsquigarrow X_2 X_1 = X_1 X_2, \\ \theta_0 \theta_1 \theta_0 &= \theta_0 \theta_3 \theta_0 \rightsquigarrow X_3 X_1 = X_1 X_3, \\ \theta_1 \theta_2 \theta_0 &= \theta_1 \theta_3 \theta_0 \rightsquigarrow X_3 X_2 = X_2 X_3, \end{aligned}$$

as the fundamental commutativity relations. \diamond

Every i defines a word of length $\frac{1}{2}N(N-1)$ as follows. Say $i = N-1$, the general case being similar, then

$$w_i = 012\dots(N-2)\dots 012010 \in W_i.$$

Note that the w_i are representatives of the longest elements in their associated parabolic subgroups W_i .

Let $H(\tilde{A}_{N-1})$ denote the Hecke algebra of affine type A_{N-1} . Following the conventions of [\[Soe97\]](#), this algebra is generated by $\{\Theta_i | i \in I\}$ subject to (reading indices modulo N):

$$\Theta_i^2 = [2]_{\mathfrak{v}} \Theta_i, \quad \Theta_i \Theta_j \Theta_i - \Theta_i = \Theta_j \Theta_i \Theta_j - \Theta_j \text{ for } |i - j| = 1, \quad \Theta_i \Theta_j = \Theta_j \Theta_i \text{ for } |i - j| > 1.$$

Let Θ_w denote the KL basis element for $w \in W$, see e.g. [\[Soe97\]](#). We have:

Lemma 4A.10. (*This assumes (QSH).*) The algebra homomorphism given by

$$\theta_i \mapsto \Theta_{w_i},$$

defines an embedding $T_{\infty} \hookrightarrow H(\tilde{A}_{N-1})$ of algebras.

The proof of [Lemma 4A.10](#) is different from the one given in [\[MMMMT20, Lemma 3.2\]](#), and is postponed to [Section 5C](#).

The Nhedral KL combinatorics works as follows. For $\mathbf{k} = (k_1, \dots, k_{N-1}) \in X^+$ and starting color i , let

$$h_i^{\mathbf{k}} = \theta_{i_{\Sigma \mathbf{k}}} \dots \theta_{i_1} \theta_{i_0}$$

with $i_0 = i$ and $i_{r+1} = \rho^j(i_r) = i_r + j$ for any k_j values of r . We will call $\Sigma \mathbf{k}$ the ending color. Reversing the order, let

$${}^{\mathbf{k}}h = \theta_{i_0} \theta_{i_1} \dots \theta_{i_{\Sigma \mathbf{k}}}$$

be defined similarly as $h_i^{\mathbf{k}}$ but using ρ^{-j} instead of ρ^j .

Lemma 4A.11. For any starting color $i \in I$ and any $\mathbf{k} \in X^+$, the element $h_i^{\mathbf{k}}$ and ${}^{\mathbf{k}}h$ only depends on \mathbf{k} and not on the chosen sequence $i = i_0, i_1, \dots, i_{\Sigma \mathbf{k}}$.

Proof. Any word representing $h_i^{\mathbf{k}}$ is equivalent to the word $\theta_{i_{\Sigma \mathbf{k}}} \dots \theta_{i_0}$ with $i_0 = i$ and

$$i_{r+1} = \rho^j(i_r) \text{ for all } k_1 + \dots + k_{j-1} \leq r < k_1 + \dots + k_j.$$

Indeed, if there exists a subword of the form $\theta_{i_r+j+k} \theta_{i_r+j} \theta_{i_r}$ with $1 \leq k < j < N-1$ then using the fundamental commutativity relation [\(4A.7\)](#), we replace this subword by $\theta_{i_r+j+k} \theta_{i_r+k} \theta_{i_r}$. \square

Remark 4A.12. Via [\(QSH\)](#), the element $h_i^{\mathbf{k}}$ is associated to $X_1^{k_1} \dots X_{N-1}^{k_{N-1}}$ because its definition involves k_j times the application of ρ^j . \diamond

Example 4A.13. Let us choose 0 as the starting color. For $N = 4$ and $\Sigma \mathbf{k} = 2$, we have

$$\begin{aligned} h_0^{2,0,0} &= \theta_2 \theta_1 \theta_0 \rightsquigarrow \mathbf{X}_1^2, & h_0^{0,2,0} &= \theta_0 \theta_2 \theta_0 \rightsquigarrow \mathbf{X}_2^2, & h_0^{0,0,2} &= \theta_2 \theta_3 \theta_0 \rightsquigarrow \mathbf{X}_3^2, \\ h_0^{1,1,0} &= \theta_3 \theta_1 \theta_0 \rightsquigarrow \mathbf{X}_2 \mathbf{X}_1, & h_0^{1,0,1} &= \theta_0 \theta_1 \theta_0 \rightsquigarrow \mathbf{X}_3 \mathbf{X}_1, & h_0^{0,1,1} &= \theta_1 \theta_2 \theta_0 \rightsquigarrow \mathbf{X}_3 \mathbf{X}_2. \end{aligned}$$

Note that, for example, $h_0^{1,1,0} = \theta_3 \theta_2 \theta_0$ by (4A.7), and on the representation side by $\mathbf{X}_2 \mathbf{X}_1 = \mathbf{X}_1 \mathbf{X}_2$. \diamond

Recall that $\varkappa = [N-1]_{\mathbf{v}}!$ and $d_{\mathbf{m}}^{\mathbf{k}}$ denote the numbers from (2B.2). For each $\mathbf{m} \in X^+$, we define (*right*) **Nhedral KL basis elements**:

$$c_i^{\mathbf{m}} = \sum_{\mathbf{k}} \varkappa^{-\Sigma \mathbf{k}} d_{\mathbf{m}}^{\mathbf{k}} \cdot h_i^{\mathbf{k}}.$$

Note that the three sums are finite, because $d_{\mathbf{m}}^{\mathbf{k}} = 0$ unless $\Sigma \mathbf{k} \leq \Sigma \mathbf{m}$. Similarly, using ${}^{\mathbf{k}}h$ instead of $h_i^{\mathbf{k}}$, we define the (left) **Nhedral KL basis elements** ${}^{\mathbf{m}}c$. Note that the ending color of every term $h_i^{\mathbf{k}}$ appearing in the sum have the same ending color.

Remark 4A.14. Via (QSH), the element $c_i^{\mathbf{m}}$ is associated to the Chebyshev polynomial $U_{\mathbf{m}}(\mathbf{X}_1, \dots, \mathbf{X}_{N-1})$. \diamond

Example 4A.15. For $N = 4$ and $\Sigma \mathbf{k} = 2$, we have

$$\begin{aligned} c_0^{2,0,0} &= \varkappa^{-2} \theta_2 \theta_1 \theta_0 - \varkappa^{-1} \theta_2 \theta_0 \rightsquigarrow U_{2,0,0} = \mathbf{X}_1^2 - \mathbf{X}_2, & c_0^{0,2,0} &= \varkappa^{-2} \theta_0 \theta_2 \theta_0 - \varkappa^{-2} \theta_0 \theta_1 \theta_0 \rightsquigarrow U_{0,2,0} = \mathbf{X}_2^2 - \mathbf{X}_3 \mathbf{X}_1, \\ c_0^{0,0,2} &= \varkappa^{-2} \theta_2 \theta_3 \theta_0 - \varkappa^{-1} \theta_2 \theta_0 \rightsquigarrow U_{0,0,2} = \mathbf{X}_3^2 - \mathbf{X}_2, & c_0^{1,1,0} &= \varkappa^{-2} \theta_3 \theta_1 \theta_0 - \varkappa^{-1} \theta_3 \theta_0 \rightsquigarrow U_{1,1,0} = \mathbf{X}_2 \mathbf{X}_1 - \mathbf{X}_3, \\ c_0^{1,0,1} &= \varkappa^{-2} \theta_0 \theta_1 \theta_0 - \theta_0 \rightsquigarrow U_{1,0,1} = \mathbf{X}_3 \mathbf{X}_1 - 1, & c_0^{0,1,1} &= \varkappa^{-2} \theta_1 \theta_2 \theta_0 - \varkappa^{-1} \theta_1 \theta_0 \rightsquigarrow U_{0,1,1} = \mathbf{X}_3 \mathbf{X}_2 - \mathbf{X}_1, \end{aligned}$$

with 0 as the starting color. \diamond

Lemma 4A.16. For any $i \in I$ and $\mathbf{k} \in X^+$, let j be the ending color of $h_i^{\mathbf{k}}$. We have $h_i^{\mathbf{k}} = {}^{\mathbf{k}}h$ and similarly for the KL basis elements.

Proof. This follows immediately from Lemma 4A.11. \square

Lemma 4A.17. (This assumes (QSH).) For all $i \in I$ and $\mathbf{m} \in X^+$, let j be the ending color of $c_i^{\mathbf{m}}$. Let $0 \leq k < N$. With the notation in Lemma 2D.2, we have

$$\theta_{j+k} c_i^{\mathbf{m}} = \begin{cases} [N]_{\mathbf{v}}! c_i^{\mathbf{m}} & k = 0, \\ \varkappa \sum_l c_i^{\mathbf{m} + w_l^k} & \text{otherwise,} \end{cases}$$

where terms with negative entries are zero. Similarly for the left KL elements.

Remark 4A.18. Coming back to Remark 4A.8, at $\mathbf{v} = 0$, Lemma 4A.17 becomes

$$\theta_{j+k} c_i^{\mathbf{m}} = \begin{cases} c_i^{\mathbf{m}} & k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

since \varkappa has a smaller span than $N(N-1)$. This justifies the ‘‘degree 0 isomorphism’’ from T_e in Definition 4A.21 to the Grothendieck ring of the asymptotic category $\mathbf{a}_{M,M,N}$ in Definition 3A.1. \diamond

Proposition 4A.19. (This assumes (QSH).) Each of the four sets

$$\begin{aligned} \mathbf{H}^{\infty} &= \{1\} \cup \{h_i^{\mathbf{k}} | \mathbf{k} \in X^+, i \in I\}, & {}^{\infty}\mathbf{H} &= \{1\} \cup \{{}^{\mathbf{k}}h | \mathbf{k} \in X^+, i \in I\}, \\ \mathbf{C}^{\infty} &= \{1\} \cup \{c_i^{\mathbf{m}} | \mathbf{m} \in X^+, i \in I\} & {}^{\infty}\mathbf{C} &= \{1\} \cup \{{}^{\mathbf{m}}c | \mathbf{m} \in X^+, i \in I\} \end{aligned}$$

is a basis of T_{∞} . The first two are called **Nhedral Bott–Samelson bases**, the final two **Nhedral KL bases**.

Recall the definition of left, right and two-sided cells for T_{∞} using the **Nhedral KL basis**, see for example [MMMT20, Definition 3.10]. The unit forms its own cell, that we call the trivial cell.

Proposition 4A.20. The nontrivial cells for the algebra T_{∞} are

$$\begin{aligned} \mathbf{L}_i &= \{c_i^{\mathbf{m}} | \mathbf{m} \in X^+\}, & {}_i\mathbf{R} &= \{{}^{\mathbf{m}}c | \mathbf{m} \in X^+\}, & \text{for } i \in I, \\ \mathbf{J} &= \{c_i^{\mathbf{m}} | \mathbf{m} \in X^+, i \in I\} = \{{}^{\mathbf{m}}c | \mathbf{m} \in X^+, i \in I\}, \end{aligned}$$

where \mathbf{L}_i , ${}_i\mathbf{R}$ and \mathbf{J} are left, right and two-sided cells respectively.

Proof. As in [MMMT20, Proof of Proposition 3.11], this follows from Lemma 4A.17. \square

We now define finite dimensional quotients of T_{∞} , which are compatible with the cell structure in Proposition 4A.20.

Definition 4A.21. For our fixed e , let I_e be the two-sided ideal in T_∞ generated by

$$\{c_i^{\mathbf{m}} \mid i \in I, \mathbf{m} \in X^+, \Sigma \mathbf{m} = e + 1\}.$$

We define the *Nhedral Hecke algebra of level e* as

$$T_e = T_\infty / I_e$$

and we call I_e the *vanishing ideal* of level e . ◇

Proposition 4A.22. *The set*

$$\mathbb{C}^e = \{1\} \cup \{c_i^{\mathbf{m}} \mid i \in I, \mathbf{m} \in X^+(e)\} \stackrel{(*)}{=} {}^e\mathbb{C} = \{1\} \cup \{c_i^{\mathbf{m}} \mid i \in I, \mathbf{m} \in X^+(e)\}$$

(the equality $(*)$ follows from [Lemma 4A.16](#)) is a basis of T_e . Thus, we have $\dim_{\mathbb{C}^v} T_e = 1 + Np_N^e$.

Proof. Similarly as [\[MMMT20, Proof of Proposition 3.14\]](#). □

Proposition 4A.23. *The nontrivial cells for the algebra T_e are*

$$\begin{aligned} L_i &= \{c_i^{\mathbf{m}} \mid \mathbf{m} \in X^+(e)\}, & {}_iR &= \{c_i^{\mathbf{m}} \mid \mathbf{m} \in X^+(e)\}, & \text{for } i \in I, \\ J &= \{c_i^{\mathbf{m}} \mid \mathbf{m} \in X^+(e), i \in I\} = \{c_i^{\mathbf{m}} \mid \mathbf{m} \in X^+(e), i \in I\}, \end{aligned}$$

where L_i , ${}_iR$ and J are left, right and two-sided cells respectively. In particular, each left and right cell is of size p_N^e , and J is of size Np_N^e .

Proof. This follows from the previous results. □

Example 4A.24. Left and right cells correspond to the cut-off of the positive Weyl chamber of type A_{N-1} , similarly as in [\[MMMT20, Example 3.16\]](#). ◇

4B. Nhedral complex representations. We now classify all simple representations of T_e on \mathbb{C}^v -vector spaces, and this classification implies that T_e is semisimple.

To this end, for $\lambda = (\lambda_i)_{i \in I} \in (\mathbb{C}^v)^I$ define

$$M_\lambda: \mathbb{C}^v \langle \theta_i \mid i \in I \rangle \rightarrow \mathbb{C}^v, \theta_i \mapsto \lambda_i,$$

which determines a one dimensional T_e -representation in the following cases. We let $(x, i) \in (\mathbb{C}^v)^I$ the element with the i th entry x and zero otherwise.

Proposition 4B.1. *The following table*

$$(4B.2) \quad \begin{array}{c|c} e \equiv 0 \pmod{N} & e \not\equiv 0 \pmod{N} \\ \hline M_{0, \dots, 0}, M_{[N]_v!, i} \text{ for } i \in I & M_{0, \dots, 0} \\ \hline N + 1 \text{ in total} & \text{only one} \end{array}$$

gives a complete and irredundant list of one dimensional T_e -representations.

Proof. That M_0 is well-defined and simple is immediate. The remaining parts follow from [Theorem 4B.4](#) below. □

We now define some representations of dimension N , which are parametrized by the points in V'_e . Given $\sigma \in E$, we define

$$\begin{aligned} M_0(\sigma) &= \varkappa \begin{pmatrix} [N]_v & Z_1(\sigma) & Z_2(\sigma) & \dots & Z_{N-1}(\sigma) \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, & M_1(\sigma) &= \varkappa \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ Z_{N-1}(\sigma) & [N]_v & Z_1(\sigma) & \dots & Z_{N-2}(\sigma) \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \dots, \\ M_{N-1}(\sigma) &= \varkappa \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ Z_1(\sigma) & Z_2(\sigma) & \dots & Z_{N-1}(\sigma) & [N]_v \end{pmatrix}. \end{aligned}$$

Lemma 4B.3. *The assignment $\theta_i \mapsto M_i(\sigma)$ is a well-defined representation of T_e if and only if $\sigma \in V'_e$.*

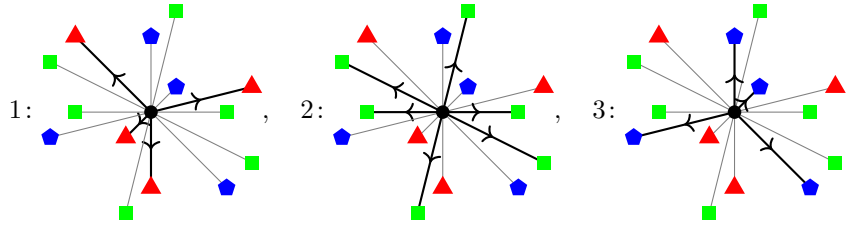
Proof. The defining relations [\(4A.7\)](#) of T_∞ are easy to check, since the matrices have only one nonzero row. It factors through J_e if and only if $(Z_1(\sigma), \dots, Z_N(\sigma)) \in V_e$ as in [\[MMMT20, Lemma 3.18\]](#). □

For $\sigma \in V'_e$, denote by $M(\sigma)$ the above defined T_e -representation.

Theorem 4B.4. *We have the following.*

- (a) *For $\sigma \in V'_e$, the representation $M(\sigma)$ decomposes as the direct sum of m equidimensional nonisomorphic simple representations, where m is the order of the stabilizer of σ for the $\mathbb{Z}/N\mathbb{Z}$ -action.*

If Γ is the graph of the type as in [Example 2B.5](#), then the Γ_i are obtained by putting an orientation on some edges, and removing other edges. Generically, the picture is:



These three graphs correspond to tensoring by L_{ω_i} . Note that the generic picture for 1 and 3 have four edges, while 2 has six edges, matching the dimensions of the L_{ω_i} . \diamond

Let us now assume that Γ is such that the condition in [Lemma 4C.3](#) holds.

Lemma 4C.5. *The T_∞ -representation $M(\Gamma)$ descends to a T_e -representation if and only if*

$$U_{\mathbf{m}}(A(\Gamma_1), \dots, A(\Gamma_{N-1})) = 0 \text{ for all } \mathbf{m} \text{ with } \Sigma \mathbf{m} = e + 1.$$

Proof. By construction. \square

Some examples of Ncolored graphs Γ satisfying [Lemma 4C.5](#) are obtained through the fusion rules of $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$. Here is a (non-exhaustive) list:

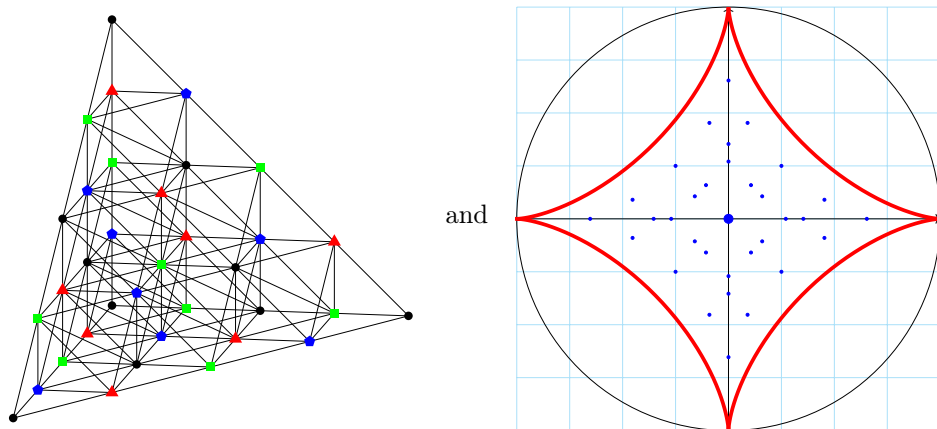
Definition 4C.6. We define the *graph of type A* of rank N and level e as the graph Γ with set of vertices $X^+(e)$, colored using the color χ_c , and the vertices \mathbf{m} and \mathbf{k} are adjacent if and only if $L_{\mathbf{m}}$ is a summand of $\bigoplus_{i=1}^{N-1} L_{\omega_i} \otimes L_{\mathbf{k}}$. The graph Γ_i is then the fusion graph of the object L_{ω_i} .

Moreover, we define *graphs of type D* in [Section 5C](#).

For $N = 4$, we have additional graphs that we will not describe here but rather refer to [[Ocn02](#), Figures 3 and 4, denoted $2A_e^c$, $2(A_e^c/2)$ and E]. These are called *graphs of conjugate type A* (these two are infinite family), and *graphs of type E* (there are six of these). \diamond

Remark 4C.7. There appears to be a small typo in [[Ocn02](#), Figures 3 and 4] for the graphs labeled $2A_e^c$ and $2(A_e^c/2)$: all double edges should be colored blue instead of red (in 2024, we got a colored version of Ocneanu's paper from <https://cel.hal.science/cel-00374414/document>). \diamond

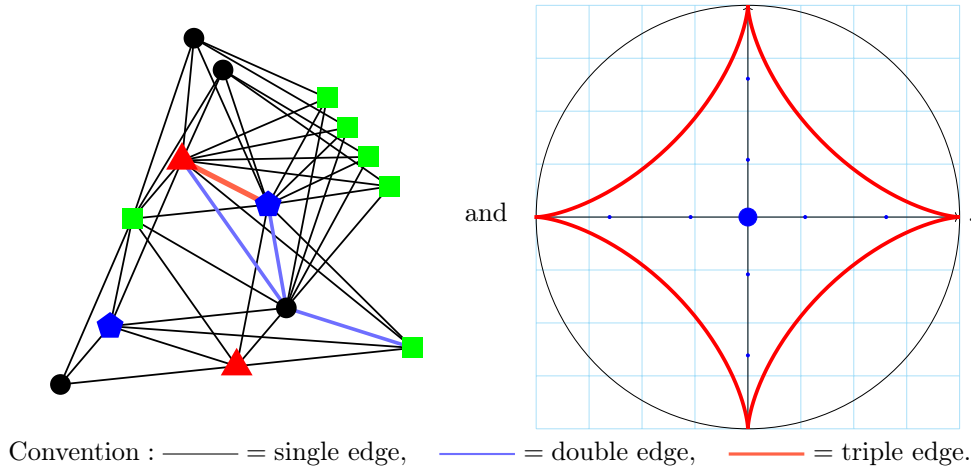
Example 4C.8 (Type A). For $N = 4$ and $e = 4$, the graph of type A is the tetrahedron of [Example 2B.5](#) and the plot of the eigenvalues of $A(\Gamma_1)$ are:



The joint spectrum of $(A(\Gamma_1), A(\Gamma_2), A(\Gamma_3))$ is the Koornwinder variety V_e . \diamond

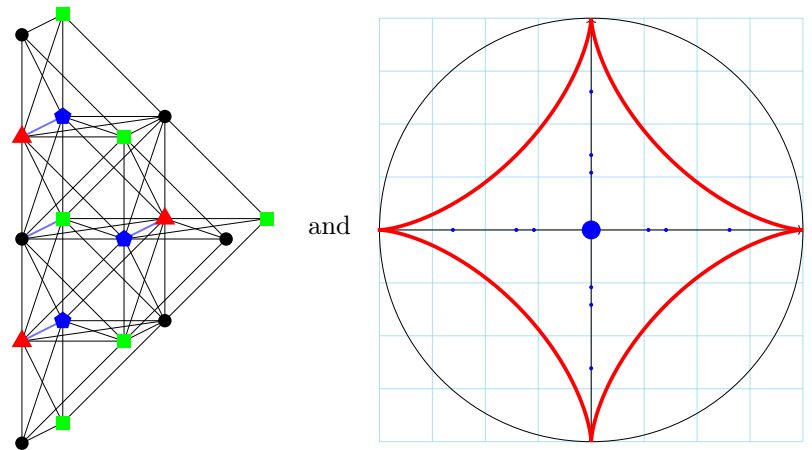
Example 4C.9 (Type D). For $N = 2$ the type D graphs are type D Dynkin diagrams, for $N = 3$ see [[MMMT20](#), Appendix 1], and for $N = 4$ see [[Ocn02](#), Figures 3 and 4]. Here are the graph of type D for

$e = 4$ and $N = 4$ and the plot of the eigenvalues of $A(\Gamma_1)$:



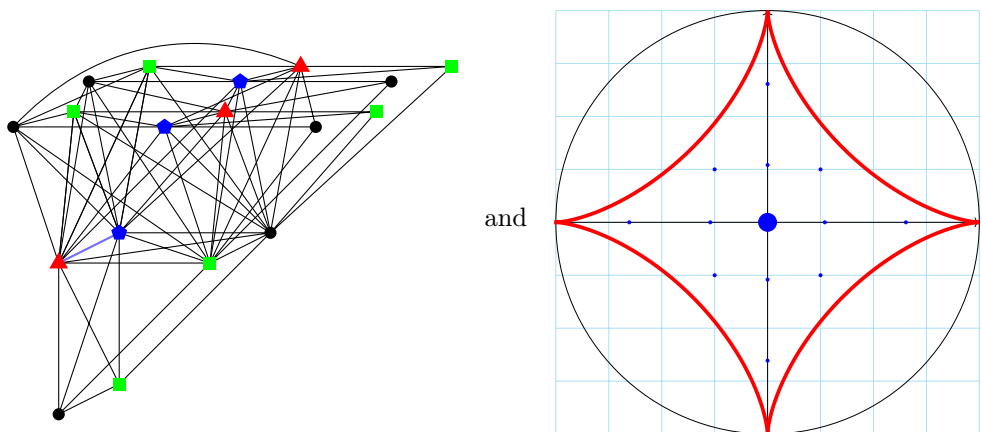
Here, similarly as below, the blue slightly thicker edges are double edges and the red thicker edge is a triple edge. This graph is not included in [Ocn02] and the joint spectrum of $(A(\Gamma_1), A(\Gamma_2), A(\Gamma_3))$ is a subset (with extra multiplicities) of the Koornwinder variety: there are 8 points with multiplicity 1 and 3 points with multiplicity 2. \diamond

Example 4C.10 (Type A conjugate). For $e = 4$, the graph of type $2A_e^c$ and the plot eigenvalues of $A(\Gamma_1)$ are



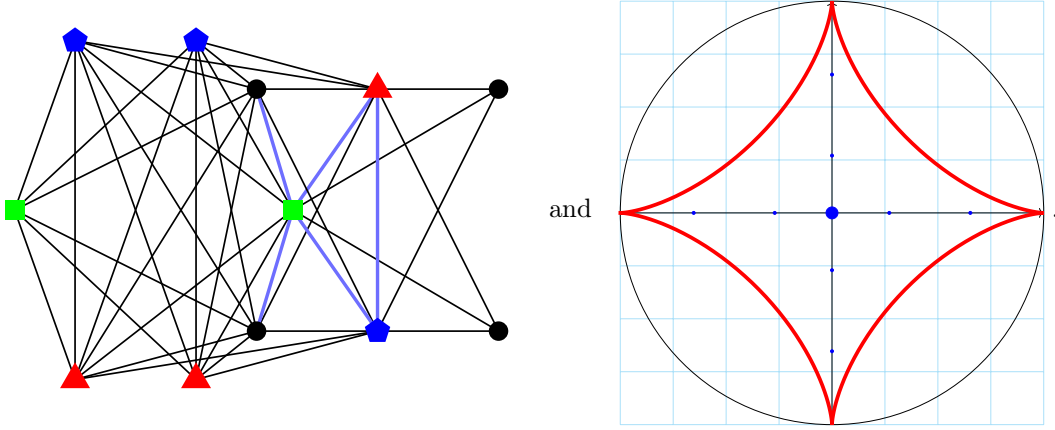
The joint spectrum of $(A(\Gamma_1), A(\Gamma_2), A(\Gamma_3))$ is a subset (with extra multiplicities) of the Koornwinder: there are 12 points with multiplicity 1 and 3 points with multiplicity 2.

For $e = 4$, the graph of type $2(A_e^c/2)$ and the plot eigenvalues of $A(\Gamma_1)$ are



The joint spectrum of $(A(\Gamma_1), A(\Gamma_2), A(\Gamma_3))$ is a subset (with extra multiplicities) of the Koornwinder: there are 12 points with multiplicity 1 and 3 points with multiplicity 2. \diamond

Example 4C.11 (Type E). The exceptional graph E_4 and the plot eigenvalues of $A(\Gamma_1)$ are



The joint spectrum of $(A(\Gamma_1), A(\Gamma_2), A(\Gamma_3))$ is a subset (with extra multiplicities) of the Koornwinder: there are 8 points with multiplicity 1 and 1 point with multiplicity 4. \diamond

Recall, from e.g. [MMMT20, Definition 5.6], the notion of a transitive $\mathbb{Z}^{[v]}$ -representation. These representations are the simple objects in the category of $\mathbb{Z}^{[v]}$ -representations, and we call them simple $\mathbb{Z}^{[v]}$ -representations.

Theorem 4C.12. *We have the following (potentially conjectural as indicated below). All of the following give well-defined simple $\mathbb{Z}^{[v]}$ -representations of T_e .*

- (a) $M(\Gamma)$ for Γ of type A of rank N and level e . (Proven.)
- (b) $M(\Gamma)$ for Γ of type D of rank N and level e . (Conjectural; verified in small cases.)
- (c) $M(\Gamma)$ for Γ of conjugate type A of rank N and level e . (Conjectural; verified in small cases.)
- (d) $M(\Gamma)$ for Γ of type E. (Proven via computer, see [LTV24].)

Proof. This is proven in Section 5C \square

Remark 4C.13. One could ask the following problem: could one classify the graphs such that the condition in Lemma 4C.5 is satisfied? This question has been addressed in the literature for special cases, see e.g. [Zub98], [Ocn02], [MMMT20]. We do not know the answer to this in general, not even for $N = 3$. In particular, there will be more simple $\mathbb{Z}^{[v]}$ -representations in Theorem 4C.12 in general. \diamond

4D. Categorification. There is a categorification of the above story, but we decided not to include it here for brevity and to not overload the paper with more combinatorics. (Another problem is that several notions we would need are not in the literature while writing this paper.) Instead, we list here what one needs to change when compared to [MMMT20].

- (i) The diagrammatic 2-category in [MMMT20, Section 4.1] should be replaced by its affine type A_{N-1} analog. While writing this paper, at least to the best of our knowledge, there is no diagrammatic presentation of the relevant 2-category but one rather has to work with algebraic singular Soergel bimodules as in [Wil11]. For the quantization of this category one needs to use the quantum Cartan matrix as in [Eli17].
- (ii) *Nhedral Soergel bimodules* of level ∞ can then be defined similarly as in [MMMT20, Section 4.2], replacing ‘secondary color’ therein by a subset of I of size $N - 1$. The main statement is then the analog of [MMMT20, Proposition 4.31], which holds verbatim and provides a categorification of T_∞ .
- (iii) Under (QSH), one can then copy [MMMT20, Section 4.3] using N -colored webs (using e.g. the webs from [CKM14]) and gets the analog of [MMMT20, Proposition 4.48], categorifying T_e for a fixed e . Hereby, the Nhedral version of the clasps can be defined using [Eli15] (which has been proven by now, see [MS22], and which might even give bases cf. [AST18]).
- (iv) Finally, [MMMT20, Section 5.2] can be, mutatis mutandis, used to categorify Theorem 4C.12 in the expected way. One exploits here the algebra objects that we give in Section 5C and the algebra object technology for Soergel bimodules developed in [MMMT19], [MMM⁺23], [MMM⁺21]. Alternatively, this could be done using a quiver similarly to [AT17], but the quiver is already complicated for $N = 3$, see [MMMT20, Section 5.3].

5. APPENDIX: SOME MORE TECHNICAL THINGS

5A. Missing proofs for Section 2.

Proof of Lemma 2E.8. Given $\mathbf{m} \in X^+$, we define $E_{\mathbf{m}}^-: E \rightarrow \mathbb{C}$ by

$$E_{\mathbf{m}}^-(\boldsymbol{\sigma}) = \sum_{w \in \mathfrak{S}_N} (-1)^{l(w)} \exp(\mathbf{i}(w(\mathbf{m}), \boldsymbol{\sigma})).$$

The functions Z_i and $E_{\mathbf{m}}^-$ are invariant by $\boldsymbol{\sigma} \mapsto \boldsymbol{\sigma} + 2\pi\alpha_i$ and therefore define functions on the torus $T^{N-1} = E/2\pi Y$, where Y is the root lattice. Moreover, the functions Z_i are invariant and the functions $E_{\mathbf{m}}^-$ are antiinvariant under the action of the symmetric group. The fundamental domain of the quotient T^{N-1}/\mathfrak{S}_N is equal to

$$D = \left\{ \sum_{i=1}^{N-1} \lambda_i \alpha_i \mid 2\lambda_i \geq \lambda_{i-1} + \lambda_{i+1} \text{ for all } 1 \leq i < N, \lambda_1 + \lambda_{N-1} \leq 2\pi \right\}.$$

Weyl's denominator formula implies that the zeroes of E_{ρ}^- are on the boundary of D . Therefore the functions Z_i and $E_{\mathbf{m}+\rho}^-/E_{\rho}^-$ are defined on the interior of D .

Using Weyl's character formula we have, for $\boldsymbol{\sigma}$ in the interior of D , we have

$$U_{\mathbf{m}}(Z_1(\boldsymbol{\sigma}), \dots, Z_{N-1}(\boldsymbol{\sigma})) = E_{\mathbf{m}+\rho}^-/E_{\rho}^-.$$

If moreover $\mathbf{m} \in X^+$ is such that $\sum \mathbf{m} = e + 1$, then [BK01, Equation (3.3.11)] implies that, for $\boldsymbol{\sigma} \in V'_e$, we have $U_{\mathbf{m}}(Z_1(\boldsymbol{\sigma}), \dots, Z_{N-1}(\boldsymbol{\sigma})) = 0$. \square

Proof of Theorem 2E.10. Part (b). Firstly, (a) and Hilbert's Nullstellensatz (via e.g. [Ful89, Corollary I.7.4]) imply that $\#V_e \leq p_N^e$. For the converse, we use the following important lemma:

Lemma 5A.1. *The map $\boldsymbol{\sigma} \mapsto (Z_1(\boldsymbol{\sigma}), \dots, Z_{N-1}(\boldsymbol{\sigma}))$ is an injection on the interior of the fundamental domain D .*

Proof. In [Bee91, Equation 5.9], Beerends calculated the value of the Jacobian $\left| \frac{\partial(Z_1, \dots, Z_{N-1})}{\partial(\sigma_1, \dots, \sigma_{N-1})} \right|$:

$$\left| \frac{\partial(Z_1, \dots, Z_{N-1})}{\partial(\sigma_1, \dots, \sigma_{N-1})} \right| = \prod_{1 \leq i < j \leq N} |e^{\mathbf{i}(\sigma_i - \sigma_{i-1})} - e^{\mathbf{i}(\sigma_j - \sigma_{j-1})}| = 2^{\binom{N}{2}} \prod_{1 \leq i < j \leq N} \left| \sin \left(\frac{\sigma_i - \sigma_{i+1} - \sigma_j + \sigma_{j+1}}{2} \right) \right|.$$

Therefore, the Jacobian vanishes only on the reflecting hyperplanes of the representation E of the symmetric group, and we are done. \square

Since V'_e lies in the interior of D , Lemma 2E.8 implies that we have found p_N^e points in V_e .

Part (c). We need the following two lemmas.

Lemma 5A.2. *Given $\boldsymbol{\sigma} = \sum_{j=1}^{N-1} \sigma_j \alpha_j \in E$, the function $Z_i(\boldsymbol{\sigma})$ is the evaluation of the i th elementary symmetric function at $(e^{\mathbf{i}\sigma_1}, e^{\mathbf{i}(\sigma_2 - \sigma_1)}, \dots, e^{\mathbf{i}(\sigma_{N-1} - \sigma_{N-2})}, e^{-\mathbf{i}\sigma_{N-1}})$.*

Proof. The function Z_i is the evaluation of the character of the fundamental \mathfrak{sl}_N -representation L_{ω_i} at the diagonal matrix $\text{diag}(\mathbf{i}\sigma_1, \mathbf{i}(\sigma_2 - \sigma_1), \dots, \mathbf{i}(\sigma_{N-1} - \sigma_{N-2}), -\mathbf{i}\sigma_{N-1})$. The claim follows from the usual correspondence between the character of L_{ω_i} and the i th elementary symmetric function. \square

Using Newton's identities, we can hence express $Z_i(\boldsymbol{\sigma})$ in terms of the power sums $Z_1(k\boldsymbol{\sigma})$.

Lemma 5A.3. *For $\boldsymbol{\sigma} \in V'_e$, we have $Z_i(\boldsymbol{\sigma}^{\rightarrow}) = \zeta^{-i} Z_i(\boldsymbol{\sigma})$.*

Proof. For $\mathbf{k} = \sum_{i=1}^{N-1} k_i \omega_i$, we write $\lambda_i(\mathbf{k})$ its coordinates in the basis $(\alpha_1, \dots, \alpha_{N-1})$. Using [Bou02, Plate I], we explicitly have

$$\lambda_i(\mathbf{k}) = \frac{1}{N} \left((N-i) \sum_{j=1}^{i-1} j k_j + (N-i) i k_i + i \sum_{j=i+1}^{N-1} (N-j) k_j \right).$$

Then, one easily show that $N(\lambda_i(\mathbf{k}) - \lambda_{i-1}(\mathbf{k})) = -\sum_{j=1}^{N-1} j k_j + N \sum_{j=i}^{N-1} k_j$. A straightforward calculation also shows that, for $\mathbf{k} \in X^+(e)$ and $1 \leq i \leq N-1$, we have

$$\begin{aligned} \lambda_{i+1}(\mathbf{k}^{\rightarrow} + \rho) - \lambda_i(\mathbf{k}^{\rightarrow} + \rho) &= \lambda_i(\mathbf{k} + \rho) - \lambda_{i-1}(\mathbf{k} + \rho) - \frac{N+e}{N} \quad \text{and} \\ \lambda_1(\mathbf{k}^{\rightarrow} + \rho) &= -\lambda_{N-1}(\mathbf{k} + \rho) - \frac{N+e}{N} + e + N. \end{aligned}$$

The result then follows from Lemma 5A.2. \square

Lemma 5A.3 and **Theorem 2E.10.(a)** that the map $X^+(e) \rightarrow V_e$, $\mathbf{k} \mapsto (Z_i(2i\pi(\mathbf{k} + \rho)/(e + N)))_{1 \leq i \leq N-1}$ is a bijection which is $\mathbb{Z}/N\mathbb{Z}$ -equivariant. We then do the counting in $X^+(e)$ instead of in the Koornwinder variety. Denote by $c(N, e, m)$ the number of weights in $X^+(e)$ with stabilizer of order m . We first notice that necessarily $m \mid N$ and $m \mid e$ and that $c(N, e, m) = c(N/m, e/m, 1)$. It then suffices to prove the formula for $m = 1$.

We proceed by induction on $\gcd(N, e)$. If $\gcd(N, e) = 1$ then all elements of $X^+(e)$ have a stabilizer of order 1. Therefore, $c(N, e, 1) = p_N^e = \binom{N-1}{M-1} = \frac{N}{M} \binom{N}{M}$.

Now suppose that $\gcd(N, e) > 1$. We have

$$c(N, e, 1) = \frac{N}{M} \binom{M}{N} - \sum_{\substack{k \mid \gcd(M, N) \\ k \neq 1}} c(N, e, k) = \frac{N}{M} \binom{M}{N} - \sum_{\substack{k \mid \gcd(M, N) \\ k \neq 1}} c(N/k, M/k, 1).$$

Since $k \neq 1$ in the last sum, we can apply the induction hypothesis and we find that

$$c(N, e, 1) = \frac{N}{M} \binom{M}{N} - \sum_{\substack{k \mid \gcd(M, N) \\ k \neq 1}} \frac{N}{M} \sum_{k' \mid \gcd(N/k, M/k)} \mu(k') \binom{M/kk'}{N/kk'}.$$

But

$$\sum_{k \mid \gcd(M, N)} \frac{N}{M} \sum_{k' \mid \gcd(N/k, M/k)} \mu(k') \binom{M/kk'}{N/kk'} = \frac{N}{M} \sum_{k \mid \gcd(N, M)} \sum_{k' \mid k} \mu\left(\frac{k}{k'}\right) \binom{M/k}{N/k}$$

and we conclude using the fact the Möbius function is the convolution inverse of the identity function. \square

5B. Missing proofs for Section 3.

Proof of Theorem 3A.5. Part (c). We will first compute the rank of $\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))_0$ and then the number of simple objects with a stabilizer of a given order. We will denote by $\mathbf{L}_{\mathbf{m}, \mathbf{k}}$ the simple object $\mathbf{L}_{\mathbf{m}} \boxtimes \mathbf{L}_{\mathbf{k}}$. In these lemmas we use Euler's totient function φ and Möbius function μ .

Lemma 5B.1. *The rank of $\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))_0$ is*

$$\frac{N}{M^2} \sum_{k \mid \gcd(M, N)} \varphi(k) \binom{M/k}{N/k}^2.$$

Proof. The simple objects are indexed by pairs $(\mathbf{m}, \mathbf{k}) \in X^+(e)$ such that $\chi_c(\mathbf{L}_{\mathbf{m}}) = \chi_c(\mathbf{L}_{\mathbf{k}})$. Let V be a complex vector space with basis (v_0, \dots, v_{N-1}) equipped with the endomorphism Ω defined by $\Omega(v_i) = \omega^i v_i$, where ω is a primitive N th root of unity. This endows V with an action of $\mathbb{Z}/N\mathbb{Z}$ and then also the symmetric algebra $S(V \times V^*) = S(V) \otimes S(V^*)$. We notice that the rank of $\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))_0$ is equal to the dimension of $(S^e(V) \otimes S^e(V^*))^{\mathbb{Z}/N\mathbb{Z}}$. Using a multigraded version of Molien's formula, one finds that the rank of $\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))_0$ is the coefficient of $u^e v^e$ in

$$\frac{1}{N} \sum_{i \in \mathbb{Z}/N\mathbb{Z}} \frac{1}{\det_V(1 - \Omega^i u) \det_{V^*}(1 - \Omega^i v)}.$$

Since Ω is diagonal in the basis (v_0, \dots, v_{N-1}) , we have $\det_V(1 - \Omega^i u) = \prod_{j=0}^{N-1} (1 - \omega^{ij} u) = (1 - u^{k_i})^{n/k_i}$, where k_i is the order of the root ω^i . Similarly, $\det_{V^*}(1 - \Omega^i v) = (1 - v^{k_i})^{n/k_i}$. Since there are exactly $\varphi(k)$ powers of ω of order k , one has

$$\begin{aligned} \frac{1}{N} \sum_{i \in \mathbb{Z}/N\mathbb{Z}} \frac{1}{\det_V(1 - \Omega^i u) \det_{V^*}(1 - \Omega^i v)} &= \frac{1}{N} \sum_{k \mid N} \frac{\varphi(k)}{(1 - u^k)^{N/k} (1 - v^k)^{N/k}} \\ &= \frac{1}{N} \sum_{k \mid N} \varphi(k) \sum_{i, j \geq 0} \binom{N/k + i - 1}{N/k - 1} \binom{N/k + j - 1}{N/k - 1} u^{ki} v^{kj}. \end{aligned}$$

Only the terms with $k \mid e$ will contribute to the coefficient of $u^e v^e$ and we finally obtain that the rank of $\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))_0$ is

$$\frac{1}{N} \sum_{k \mid \gcd(e, N)} \varphi(k) \binom{N/k + e/k - 1}{N/k - 1}^2 = \frac{N}{M^2} \sum_{k \mid \gcd(M, N)} \varphi(k) \binom{M/k}{N/k}^2,$$

the last equality is obtained using $e + N = M$. \square

Lemma 5B.2. *Let $m \in \mathbb{Z}$. If $m \mid \gcd(N, e)$, then the number of simple objects of $\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))_0$ with stabilizer of order m under $L_{e\omega_1, e\omega_1} \otimes -$ is*

$$c(N, M, m) = \frac{mN}{M^2} \sum_{k \mid \gcd(N/m, M/m)} \mu(k) \left(\frac{M/mk}{N/mk} \right)^2,$$

and is 0 otherwise.

Proof. The count is similar to the count of [Theorem 2E.10.\(c\)](#). The base case is dealt by [Lemma 5B.1](#), and, at the end of the inductive step, we use the fact Euler's totient function is the convolution of the Möbius function and of the identity function. \square

Each orbit of $X^+(e)^2$ of stabilizer of order m will contribute to m simple objects in the modular closure of $\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))_0$, and there are $\frac{m}{N}c(N, M, m)$ such orbits. Hence, the rank of $\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))_0$ is

$$\begin{aligned} \sum_{m \mid \gcd(M, N)} \frac{m^2}{N} c(N, M, m) &= \frac{1}{M^2} \sum_{m \mid \gcd(M, N)} \sum_{k \mid \gcd(M/m, N/m)} m^3 \mu(k) \left(\frac{M/mk}{N/mk} \right)^2 \\ &= \frac{1}{M^2} \sum_{m \mid \gcd(M, N)} \sum_{k \mid m} k^3 \mu\left(\frac{m}{k}\right) \left(\frac{M/m}{N/m} \right)^2. \end{aligned}$$

We obtain the formula in (c) since J_3 is the convolution of the Möbius function μ and the cube function.

Part (e). We will use the description of the modular closure of $\mathbf{R} = (\mathbf{Rep}_\eta(\mathfrak{sl}_N) \boxtimes \mathbf{Rep}_\eta(\mathfrak{sl}_N)^{\text{rev}})_0$ as in [\[Müg00, Definition 3.12\]](#). Recall that it is obtained as the idempotent completion of the category where the objects are the same as in \mathbf{R} , but morphisms between X and Y are $\bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} \text{Hom}_{\mathbf{R}}(X, L_{e\omega_i, e\omega_i} \otimes Y)$. If we denote by γ an isomorphism $L_{\mathbf{m}, \mathbf{k}} \simeq L_{e\omega_1, e\omega_1} \otimes L_{\mathbf{m}, \mathbf{k}}$ different from the identity and of order $\text{stab}_{\mathbf{m}, \mathbf{k}}$, then the primitive idempotents of $\text{End}(L_{\mathbf{m}, \mathbf{k}})$ in the modular closure are $p_j = \frac{1}{\text{stab}_{\mathbf{m}, \mathbf{k}}} \sum_{k=1}^{\text{stab}_{\mathbf{m}, \mathbf{k}}} \xi^{jk} \gamma^k$ for $j \in \mathbb{Z}/N\mathbb{Z}$, where ξ is a primitive $\text{stab}_{\mathbf{m}, \mathbf{k}}$ th root of unity. Using similar notations for the object $L_{\mathbf{m}', \mathbf{k}'}$, we have

$$\begin{aligned} S_{(\mathbf{m}, \mathbf{k}, i), (\mathbf{m}', \mathbf{k}', j)} &= \text{Tr}(p_i \otimes p'_j \circ \beta_{L_{\mathbf{m}, \mathbf{k}}, L_{\mathbf{m}, \mathbf{k}}} \circ \beta_{L_{\mathbf{m}, \mathbf{k}}, L_{\mathbf{m}, \mathbf{k}}}) \\ &= \frac{1}{\text{stab}_{\mathbf{m}, \mathbf{k}}} \sum_{k=1}^{\text{stab}_{\mathbf{m}, \mathbf{k}}} \xi^{ik} \text{Tr}(\gamma^k \otimes p'_{j'} \circ \beta_{L_{\mathbf{m}', \mathbf{k}'}, L_{\mathbf{m}, \mathbf{k}}} \circ \beta_{L_{\mathbf{m}, \mathbf{k}}, L_{\mathbf{m}', \mathbf{k}'}}). \end{aligned}$$

Therefore, summing over i , we obtain

$$\sum_{i=1}^{\text{stab}_{\mathbf{m}, \mathbf{k}}} S_{(\mathbf{m}, \mathbf{k}, i), (\mathbf{m}', \mathbf{k}', i')} = \text{Tr}(\text{id} \otimes p'_{j'} \circ \beta_{L_{\mathbf{m}', \mathbf{k}'}, L_{\mathbf{m}, \mathbf{k}}} \circ \beta_{L_{\mathbf{m}, \mathbf{k}}, L_{\mathbf{m}', \mathbf{k}'}}) = \frac{1}{\text{stab}_{\mathbf{m}, \mathbf{k}}} \text{Tr}(\beta_{L_{\mathbf{m}', \mathbf{k}'}, L_{\mathbf{m}, \mathbf{k}}} \circ \beta_{L_{\mathbf{m}, \mathbf{k}}, L_{\mathbf{m}', \mathbf{k}'}}),$$

since the other terms in the sum defining $p'_{j'}$ are of trace 0, as they are traces of morphisms between different objects in \mathbf{R} .

Now suppose that N is prime. Then [Lemma 5B.2](#) shows that there exists a unique pair $(\mathbf{m}, \mathbf{k}) \in X^+(e)^2$ with $\chi_c(L_{\mathbf{m}}) = \chi_c(L_{\mathbf{k}})$ and $\text{stab}_{\mathbf{m}, \mathbf{k}} \neq 1$ if and only if $e \equiv 0 \pmod{N}$, and one easily checks that $\mathbf{m} = \mathbf{k} = (e/N, \dots, e/N)$ provides such a pair. A similar argument as above shows that $S_{(\mathbf{m}, \mathbf{k}, i), (\mathbf{m}, \mathbf{k}, j)} = S_{(\mathbf{m}, \mathbf{k}, i'), (\mathbf{m}, \mathbf{k}, j')}$ if $i - j \equiv i' - j' \pmod{N}$.

We now use the relation $\tau^{-1} T^{-1} S T^{-1} = S T S$, where τ^{-1} is the Gauss sum as in [\[EGNO15, Definition 8.15.1\]](#). This relation is satisfied since the modular closure is a modular category. The entry $((\mathbf{m}, \mathbf{k}, 0), (\mathbf{m}, \mathbf{k}, i))$ of this relation gives

$$\begin{aligned} \tau^{-1} \theta_{\mathbf{m}, \mathbf{k}}^2 S_{(\mathbf{m}, \mathbf{k}, 0), (\mathbf{m}, \mathbf{k}, i)} &= \sum_{(\mathbf{m}', \mathbf{k}') \neq (\mathbf{m}, \mathbf{k})} \theta_{\mathbf{m}', \mathbf{k}'}^{-1} S_{(\mathbf{m}, \mathbf{k}, 0), (\mathbf{m}', \mathbf{k}')} S_{(\mathbf{m}', \mathbf{k}'), (\mathbf{m}, \mathbf{k}, 0)} \\ &\quad + \theta_{\mathbf{m}, \mathbf{k}}^{-1} \sum_{j=1}^N S_{(\mathbf{m}, \mathbf{k}, 0), (\mathbf{m}, \mathbf{k}, j)} S_{(\mathbf{m}, \mathbf{k}, j), (\mathbf{m}, \mathbf{k}, i)}. \end{aligned}$$

Let $k \not\equiv 0 \pmod{N}$. Multiplying by ξ^{ik} and summing then gives

$$\begin{aligned} \tau^{-1} \theta_{\mathbf{m}, \mathbf{k}}^3 \sum_{i=1}^N \xi^{ik} S_{(\mathbf{m}, \mathbf{k}, 0), (\mathbf{m}, \mathbf{k}, i)} &= \sum_{i, j=1}^N \xi^{ik} S_{(\mathbf{m}, \mathbf{k}, 0), (\mathbf{m}, \mathbf{k}, j)} S_{(\mathbf{m}, \mathbf{k}, j), (\mathbf{m}, \mathbf{k}, i)} \\ &= \sum_{i, j=1}^N \xi^{ik} S_{(\mathbf{m}, \mathbf{k}, 0), (\mathbf{m}, \mathbf{k}, j)} S_{(\mathbf{m}, \mathbf{k}, 0), (\mathbf{m}, \mathbf{k}, j-i)} = \left(\sum_{i=1}^N \xi^{ik} S_{(\mathbf{m}, \mathbf{k}, 0), (\mathbf{m}, \mathbf{k}, i)} \right)^2. \end{aligned}$$

But $\sum_{i=1}^N \xi^{ik} S_{(\mathbf{m}, \mathbf{k}, 0), (\mathbf{m}, \mathbf{k}, i)} \neq 0$ since a similar calculation shows that

$$\sum_{i=1}^N \xi^{ik} S_{(\mathbf{m}, \mathbf{k}, 0), (\mathbf{m}, \mathbf{k}, i)} = \sum_{i=1}^N (S^2)_{(\mathbf{m}, \mathbf{k}, 0), (\mathbf{m}, \mathbf{k}, i)}$$

and that S^2 is, up to a nonzero constant, the permutation matrix given by the duality on simple objects.

We finally obtain that

$$S_{(\mathbf{m}, \mathbf{k}, i), (\mathbf{m}, \mathbf{k}, j)} = \frac{1}{N^2} \begin{cases} S_{(\mathbf{m}, \mathbf{k}), (\mathbf{m}, \mathbf{k})} + N(N-1)\theta_{\mathbf{m}, \mathbf{k}}^3 \tau^- & \text{if } i = j, \\ S_{(\mathbf{m}, \mathbf{k}), (\mathbf{m}, \mathbf{k})} - N\theta_{\mathbf{m}, \mathbf{k}}^3 \tau^- & \text{if } i \neq j. \end{cases}$$

It remains to notice that the Gauss sum τ^- is equal to $\dim \mathbf{Rep}_\eta(\mathfrak{sl}_N)/N$. \square

Now we come to [Section 3B](#), so the reader might want to recall the various terminology. We define an action of $\mathbb{Z}/M\mathbb{Z}$ on $\Psi(Y, \pi)$ as follows. Given $f \in \Psi(Y, \pi)$ and $k \in \mathbb{Z}/M\mathbb{Z}$, we define $f + k$ as the unique function $Y \rightarrow \{0, \dots, M-1\}$ such that $(f + k)(y) \equiv f(y) + k \pmod{M}$ for all $y \in Y$, and f^{+k} as the unique function in $\Psi(Y, \pi)$ such that $f^{+k}(\pi^{-1}(i)) = (f + k)(\pi^{-1}(i))$ for all $i \in \mathbb{Z}_{\geq 0}$. Given $f \in \Psi(Y, \pi)$, we denote by $s(f)$ the cardinal of the stabilizer of f for this action of $\mathbb{Z}/M\mathbb{Z}$. The unipotent characters lying in the family \mathcal{F} are then indexed by pairs $([f], i)$ where $[f]$ is an equivalence class of $\Psi(Y, \pi)$ under the action of $\mathbb{Z}/M\mathbb{Z}$ and $1 \leq i \leq s(f)$.

We define a map $\iota: \Psi(Y, \pi) \rightarrow (X^+(e))^2$, $f \mapsto (\mathbf{m}_f, \mathbf{k}_f)$ where

$$\mathbf{m}_f = (f(2) - f(1) - 1, \dots, f(N) - f(N-1) - 1)^{\rightarrow r_f} \text{ and } \mathbf{k}_f = (\bar{f}(2) - \bar{f}(1) - 1, \dots, \bar{f}(N) - \bar{f}(N-1) - 1),$$

with $r_f \in \mathbb{Z}$ being the unique integer such that $\sum_{i=1}^N (f(i) - \bar{f}(i)) = r_f e$.

Proof of [Theorem 3B.3](#). Part (a). The computation of the cardinal of \mathcal{F} is similar to [Theorem 3A.5\(c\)](#). We replace the N dimensional space of the proof of [Lemma 5B.1](#) by a M dimensional space, and the symmetric powers by exterior powers, and we obtain

$$|\Psi(Y, \pi)| = \frac{1}{M} \sum_{k | \gcd(N, M)} \varphi(k) \binom{M/k}{N/k}^2.$$

A similar argument as in [Lemma 5B.2](#) shows that the number of $f \in \Psi(Y, \pi)$ with $s(f) = m$ is

$$\frac{m}{M} \sum_{k | \gcd(N/m, M/m)} \mu(k) \binom{M/mk}{N/mk}^2.$$

The end of the proof is then the same as [Theorem 3A.5\(c\)](#).

Before proving part (b) and (c) of [Theorem 3B.3](#), we show that ι induces an isomorphism between \mathcal{F} and $\text{Si}(\mathbf{Z}(\mathbf{a}_{M, M, N}))$.

Lemma 5B.3. *For any $f \in \Psi(Y, \pi)$, we have $\chi_c(\mathbf{L}_{\mathbf{m}_f}) = \chi_c(\mathbf{L}_{\mathbf{k}_f})$.*

Proof. Easy and omitted. \square

Lemma 5B.4. *The map ι induces a bijection between the orbits of $\Psi(Y, \pi)$ under the action of $\mathbb{Z}/M\mathbb{Z}$ and the orbits of $\{(\mathbf{m}, \mathbf{k}) \in (X^+(e))^2 \mid \chi_c(\mathbf{L}_{\mathbf{m}}) = \chi_c(\mathbf{L}_{\mathbf{k}})\}$ under the action of $\mathbb{Z}/N\mathbb{Z}$.*

Proof. We first show that ι is constant on the orbits of $\Psi(Y, \pi)$ under the action of $\mathbb{Z}/M\mathbb{Z}$. Let $f \in \Psi(Y, \pi)$ and we discuss whether $f(N)$ and $\bar{f}(N)$ are equal to $M-1$ or not. If $f(N) = M-1$ and $\bar{f}(N) \neq M-1$ then $f^+(1) = 0$, $f^+(i) = f(i-1) + 1$ for all $1 < i \leq N$ and $\bar{f}^+(i) = \bar{f}(i) + 1$ for all $1 \leq i \leq N$. Therefore $r_{f^+} = r_f - 1$ and $\iota(f^+) = \iota(f)$. The other cases are similar.

We now prove that if $f, g \in \Psi(Y, \pi)$ are such that $\iota(f)$ and $\iota(g)$ are in the same orbit of $(X^+(e))^2$ under the action of $\mathbb{Z}/N\mathbb{Z}$, then f and g are in the same orbit of $\Psi(Y, \pi)$ under the action of $\mathbb{Z}/M\mathbb{Z}$. Let $1 \leq l \leq N$ such that $\iota(f) = \iota(g)^{\rightarrow l}$, and $0 \leq k < M$ such that $\bar{g}(N-l+1) = M-k$. Then $\iota(g^{+k}) = \iota(g)^{\rightarrow l}$ and we may and will suppose that $\iota(f) = \iota(g)$. Since ι is constant on the orbits of $\Psi(Y, \pi)$ under \rightarrow , we also may and will suppose that $\bar{f}(N) = \bar{g}(N)$.

With these extra assumptions, it remains to show that $f = g$. Since $\bar{f}(N) = \bar{g}(N)$ and $\mathbf{k}_f = \mathbf{k}_g$, we easily deduce that $\bar{f}(i) = \bar{g}(i)$ for all $1 \leq i \leq N$. Let $0 \leq r'_f, r'_g < N$ such that $r'_f = r_f \pmod{N}$ and $r'_g = r_g \pmod{N}$. Since $\mathbf{m}_f = \mathbf{m}_g$ we have:

$$\begin{aligned} 0 &= \sum_{i=1}^{N-1} i(\mathbf{m}_{f,i} - \mathbf{m}_{g,i}) = \sum_{i=1}^N (g(i) - f(i)) + N(f(N - r'_f) - g(N - r'_g)) + M(r'_f - r'_g) \\ &= M(r'_f - r_f + r_g - r'_g) + N(f(N - r'_f) - g(N - r'_g)). \end{aligned}$$

We hence deduce that M divides $f(N - r'_f) - g(N - r'_g)$, which is between $-M + 1$ and $M - 1$ so that $f(N - r'_f) = g(N - r'_g)$. The equality $\mathbf{m}_f = \mathbf{m}_g$ then implies that $f(i - r'_f) = g(i - r'_g)$ for all i and that $r_f = r_g$ since $M(r_f - r_g) = \sum_{i=1}^N (f(i) - g(i))$. We obtain that $f = g$ as expected.

Therefore ι induces an injection between the $\mathbb{Z}/M\mathbb{Z}$ orbits of $\Psi(Y, \pi)$ and the $\mathbb{Z}/N\mathbb{Z}$ orbits of $\{(\mathbf{m}, \mathbf{k}) \in (X^+(e))^2 \mid \chi_c(\mathbf{L}_\mathbf{m}) = \chi_c(\mathbf{L}_\mathbf{k})\}$. Since the number of such orbits is the same we deduce that this injection is a bijection. \square

Recall that the symmetric group \mathfrak{S}_N acts on the weights. We will denote by \bullet the action shifted by ρ . We need the following two technical lemmas.

Lemma 5B.5. *Let $f \in \Psi(Y, \pi)$ and $w \in \mathfrak{S}_N$. Then*

$$(w \bullet \mathbf{m}_f)_i = f(w^{-1}(i+1) - r_f) - f(w^{-1}(i) - r_f) - 1 \pmod{M} \text{ and} \\ (w \bullet \mathbf{k}_f)_i = \bar{f}(w^{-1}(i+1)) - \bar{f}(w^{-1}(i)) - 1 \pmod{M}.$$

Proof. It suffices to treat the case of a simple reflection s_j through the hyperplane orthogonal to α_j . By definition, we have $s_j \bullet \mathbf{k}_f = \mathbf{k}_f - \langle \mathbf{k}_f + \rho, \alpha_j^\vee \rangle \alpha_j$. Since $\langle \mathbf{k}_f + \rho, \alpha_j^\vee \rangle = \bar{f}(j+1) - \bar{f}(j) \pmod{M}$, we conclude using $\alpha_j = -\omega_{j-1} + 2\omega_j - \omega_{j+1}$. The proof for $s_j \bullet \mathbf{m}_f$ is similar. \square

Lemma 5B.6. *Let $f \in \Psi(Y, \pi)$ and $w, w' \in \mathfrak{S}_N$. Then $w \bullet \mathbf{m}_f - w' \bullet \mathbf{k}_f = \sum_{i=1}^{N-1} \nu_i \alpha_i$ with*

$$\nu_i = \sum_{j=1}^i (\bar{f}(w'^{-1}(j)) - f(w^{-1}(j) - r_f)) \pmod{M}.$$

Proof. By definition of the fundamental roots, we have $\nu_i = \langle w \bullet \mathbf{m}_f - w' \bullet \mathbf{k}_f, \omega_i \rangle$. As in Lemma 5B.5, we have

$$\begin{aligned} \nu_i &= M \langle \omega_{r_f}, \omega_i \rangle + \sum_{j=1}^{N-1} (f(w^{-1}(j+1) - r_f) - f(w^{-1}(j) - r_f) - \bar{f}(w'^{-1}(j+1)) + \bar{f}(w'^{-1}(j))) \langle \omega_j, \omega_i \rangle \\ &= M \langle \omega_{r_f}, \omega_i \rangle + \sum_{j=1}^N (f(w^{-1}(j) - r_f) - \bar{f}(w'^{-1}(j))) \langle \omega_{j-1} - \omega_j, \omega_i \rangle. \end{aligned}$$

Since $\langle \omega_i, \omega_j \rangle = \min(i, j) - \frac{ij}{N}$, we deduce that

$$\begin{aligned} \nu_i &= M \langle \omega_{r_f}, \omega_i \rangle + \frac{i}{N} \sum_{j=1}^N (f(w^{-1}(j) - r_f) - \bar{f}(w'^{-1}(j))) - \sum_{j=1}^j (f(w^{-1}(j) - r_f) - \bar{f}(w'^{-1}(j))) \\ &= M \min(r_f, i) + \sum_{j=1}^j (\bar{f}(w'^{-1}(j)) - f(w^{-1}(j) - r_f)), \end{aligned}$$

the last equality following from the definition of r_f . \square

Part (b). We first give another expression for the pre-Fourier matrix of the family \mathcal{F} , which will be more suitable for the comparison with the modular data arising from the asymptotic category.

Lemma 5B.7. *Let $0 \leq i_1 < \dots < i_e \leq M - 1$ and $0 \leq j_1 < \dots < j_e \leq M - 1$. Let \mathbf{i} be the e -tuple (i_1, \dots, i_e) , \mathbf{j} be the e -tuple (j_1, \dots, j_e) and ${}^c\mathbf{i}$ be the strictly increasing n -tuple obtained from the complement of \mathbf{i} in $\{0, \dots, M - 1\}$ and similarly for ${}^c\mathbf{j}$. Then*

$$(\Lambda^e \mathcal{S})_{\mathbf{i}, \mathbf{j}} = (-1)^{\sum_{k=1}^e ({}^c i_k + {}^c j_k)} \frac{\det(\mathcal{S})}{M^N} (\Lambda^N \bar{\mathcal{S}})_{{}^c\mathbf{i}, {}^c\mathbf{j}}.$$

Proof. Since $\frac{1}{M} \bar{\mathcal{S}}$ is the inverse of \mathcal{S} , the inverse of $\Lambda^N \mathcal{S}$ is $\frac{1}{M} \Lambda^N \bar{\mathcal{S}}$. But the inverse of $\Lambda^N \mathcal{S}$ can also be expressed in terms of the N th adjugate matrix of \mathcal{S} . The lemma follows then from the explicit form of the adjugate matrix. \square

As $\tau = (-1)^{\binom{M}{2}} \det(\mathcal{S})$, we then deduce that, given $f, g \in \Psi(Y, \pi)$, the pre-Fourier matrix of \mathcal{F} have the following expression:

$$\tilde{\mathcal{S}}_{[f], [g]} = \frac{(-1)^{\sum_{i=1}^N (\bar{f}(i) + \bar{g}(i))} \varepsilon(f) \varepsilon(g)}{M^{N-1}} \sum_{w, w' \in \mathfrak{S}_N} (-1)^{l(w) + l(w')} \prod_{i=1}^N \eta^{2(f(w(i))g(i) - \bar{f}(w'(i))\bar{g}(i))}.$$

Given $f, g \in \Psi(Y, \pi)$, using the explicit formula for S -matrix of $\mathbf{Z}(\mathbf{a}_{M,M,e})_0$, we have

$$S_{\iota(f), \iota(g)} = \frac{\sum_{w, w' \in \mathfrak{S}_N} (-1)^{l(w)+l(w')} \eta^{2\langle \mathbf{m}_f + \rho, w(\mathbf{m}_g + \rho) \rangle - 2\langle \mathbf{k}_f + \rho, w'(\mathbf{k}_g + \rho) \rangle}}{\sum_{w, w' \in \mathfrak{S}_N} (-1)^{l(w)+l(w')} \eta^{2\langle \rho, w(\rho) \rangle - 2\langle \rho, w'(\rho) \rangle}}.$$

We are then concerned with the value, modulo M , of $\langle \mathbf{m}_f + \rho, w(\mathbf{m}_g + \rho) \rangle - \langle \mathbf{k}_f + \rho, w'(\mathbf{k}_g + \rho) \rangle$. Using the shifted action of the symmetric group, we have

$$\langle \mathbf{m}_f + \rho, w(\mathbf{m}_g + \rho) \rangle - \langle \mathbf{k}_f + \rho, w'(\mathbf{k}_g + \rho) \rangle = \langle \mathbf{m}_f - \mathbf{k}_f, w' \bullet \mathbf{k}_g + \rho \rangle + \langle w \bullet \mathbf{m}_g - w' \bullet \mathbf{k}_g, w' \bullet \mathbf{k}_g + \rho \rangle$$

Using [Lemma 5B.5](#) and [Lemma 5B.6](#), we find

$$\langle \mathbf{m}_f - \mathbf{k}_f, w' \bullet \mathbf{k}_g + \rho \rangle = \sum_{j=1}^{N-1} (\bar{g}(w'^{-1}(N)) - \bar{g}(w'^{-1}(j))) (\bar{f}(j) - f(j - r_f)) \pmod{M}$$

and

$$\langle w \bullet \mathbf{m}_g - w' \bullet \mathbf{k}_g, w' \bullet \mathbf{k}_g + \rho \rangle = \sum_{j=1}^{N-1} (f(n - r_f) - f(j - r_f)) (\bar{g}(w'^{-1}(j)) - g(w^{-1}(j) - k_g)) \pmod{M}.$$

Since $f \in \Psi(Y, \pi)$ we have $\sum_{i=1}^N (f(i) - \bar{f}(i)) = 0 \pmod{M}$ and similarly for g . Therefore, we obtain

$$\langle \mathbf{m}_f + \rho, w(\mathbf{m}_g + \rho) \rangle - \langle \mathbf{k}_f + \rho, w'(\mathbf{k}_g + \rho) \rangle = \sum_{j=1}^N (f(j - r_f) - g(w^{-1}(j) - r_g) - \bar{f}(j) \bar{g}(w'^{-1}(j))) \pmod{M}$$

and since $w \mapsto (j \mapsto w(j + r_g) - r_g)$ is a signature preserving bijection of \mathfrak{S}_N we have

$$S_{\iota(f), \iota(g)} = \frac{\sum_{w, w' \in \mathfrak{S}_N} (-1)^{l(w)+l(w')} \prod_{i=1}^N \eta^{f(w(j))g(j) - \bar{f}(w'(j))\bar{g}(j)}}{\sum_{w, w' \in \mathfrak{S}_N} (-1)^{l(w)+l(w')} \eta^{2\langle \rho, w(\rho) \rangle - 2\langle \rho, w'(\rho) \rangle}}.$$

It remains to renormalize the S -matrix by the positive square root of the categorical dimension of $\mathbf{Z}(\mathbf{a}_{M,M,N})$. By [\[EGNO15, Corollary 8.23.12\]](#), we have $\dim(\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))) = N^2 \dim(\mathbf{Z}(\mathbf{a}_{M,M,N}))$. Since $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$ is modular, we deduce that $\dim(\mathbf{Z}(\mathbf{Rep}_\eta(\mathfrak{sl}_N))) = \dim(\mathbf{Rep}_\eta(\mathfrak{sl}_N))^2$. Using [\[BK01, Theorem 3.3.20\]](#), we find that the positive square root of $\mathbf{Z}(\mathbf{a}_{M,M,N})$ is then $\frac{\dim(\mathbf{Rep}_\eta(\mathfrak{sl}_N))}{N} = M^{N-1} |\sum_{w \in \mathfrak{S}_N} (-1)^{l(w)} \eta^{2\langle \rho, w(\rho) \rangle}|^{-2}$.

Part (c). We first rewrite slightly the eigenvalue of the Frobenius.

Lemma 5B.8. *Given $f \in \Psi(Y, \pi)$ and $1 \leq i \leq s(f)$, we have $\text{Frob}([f], i) = \eta^{-\alpha(f)}$ with*

$$\alpha(f) \equiv -2 \sum_{i=1}^N (\bar{f}(i) - f(i - r_f)) \left(\sum_{j=1}^{i-1} \bar{f}(j) - \sum_{j=1}^i f(j - r_f) \right) \pmod{2M}.$$

Proof. The value of $\alpha(f)$ is given in [\(3B.2\)](#). First, using the definition of \bar{f} , we have

$$\sum_{y \in Y} (f(y)^2 + Mf(y)) = \sum_{i=1}^N (f(i)^2 + Mf(i) - \bar{f}(i)^2 - M\bar{f}(i)) + \sum_{j=1}^M (j^2 + Mj)$$

so that $\alpha(f) \equiv \sum_{i=1}^N (\bar{f}(i)^2 + M\bar{f}(i) - f(i)^2 - Mf(i)) \pmod{2M}$. Now, we rewrite $\bar{f}(N)$ using the fact that $f \in \Psi(Y, \pi)$ and we obtain

$$\begin{aligned} \bar{f}(N)^2 + M\bar{f}(N) &= \left(f(N - r_f) + \sum_{i=1}^{N-1} (f(i - r_f) - f(i)) - er_f \right)^2 \\ &\quad + M \left(f(N - r_f) + \sum_{i=1}^{N-1} (f(i - r_f) - f(i)) - er_f \right). \end{aligned}$$

Expanding the square, we obtain

$$\begin{aligned} \bar{f}(N)^2 + M\bar{f}(N) &\equiv f(N - r_f)^2 + \left(\sum_{i=1}^{N-1} (f(i - r_f) - f(i)) \right)^2 + 2f(N - r_f) \sum_{i=1}^{N-1} (f(i - r_f) - f(i)) \\ &\quad + M \left(f(N - r_f) + \sum_{i=1}^{N-1} (f(i - r_f) - f(i)) \right) \pmod{2M}. \end{aligned}$$

We then check that this implies that

$$\alpha(f) \equiv 2 \sum_{i=1}^{N-1} (f(i-r_f) - \bar{f}(i)) \left(\sum_{j=i+1}^N f(j-r_f) - \sum_{j=i}^{N-1} \bar{f}(j) \right) \pmod{2M},$$

and we conclude the proof using once again that $\sum_{i=1}^N (\bar{f}(i) - f(i)) \equiv 0 \pmod{M}$. \square

Given $g \in \Psi(Y, \pi)$, the value for the ribbon in $\mathbf{Z}(\mathbf{a}_{M,M,N})$ is $\theta_{\iota(f)} = \eta^{\langle \mathbf{m}_f, \mathbf{m}_f + 2\rho \rangle - \langle \mathbf{k}_f, \mathbf{k}_f + 2\rho \rangle}$. We notice the equality $\langle \mathbf{m}_f, \mathbf{m}_f + 2\rho \rangle - \langle \mathbf{k}_f, \mathbf{k}_f + 2\rho \rangle = \langle \mathbf{m}_f - \mathbf{k}_f, \mathbf{m}_f - \mathbf{k}_f \rangle + 2\langle \mathbf{m}_f - \mathbf{k}_f, \mathbf{k}_f + \rho \rangle$. On the one hand, using [Lemma 5B.6](#), $\langle \alpha_i, \alpha_j \rangle = a_{i,j}$ and $\sum_{i=1}^N (f(i) - \bar{f}(i)) \equiv 0 \pmod{M}$ we have

$$\langle \mathbf{m}_f - \mathbf{k}_f, \mathbf{m}_f - \mathbf{k}_f \rangle \equiv 2 \sum_{i=1}^N \left(\sum_{j=1}^i (\bar{f}(j) - f(j-r_f)) \right) (\bar{f}(i) - f(i-r_f)) \pmod{2M}.$$

On the other hand, using once again [Lemma 5B.6](#), we have

$$\langle \mathbf{m}_f - \mathbf{k}_f, \mathbf{k}_f + \rho \rangle \equiv \sum_{i=1}^{N-1} \left(\sum_{j=1}^i (\bar{f}(j) - f(j-r_f)) \right) (\bar{f}(i+1) - \bar{f}(i)) \equiv - \sum_{i=1}^N \bar{f}(i) (\bar{f}(i) - f(i-r_f)) \pmod{M}$$

Therefore $\theta_{\iota(f)} = \eta^{2\beta(f)}$, where

$$\beta(f) = \sum_{i=1}^N (\bar{f}(i) - f(i-r_f)) \left(\sum_{j=1}^{i-1} \bar{f}(j) - \sum_{j=1}^i f(j-r_f) \right).$$

Then [Lemma 5B.8](#) shows that the ribbon and the eigenvalue of the Frobenius coincide, and we conclude since the T -matrix is the diagonal matrix with entries the inverse of the ribbon. \square

Proof of [Theorem 3C.3\(d\)](#). Since $\mathbf{A}_{M,M,N}$ is a matrix category over $\mathbf{a}_{M,M,N}$, we first compute the Artin–Wedderburn decomposition of $[\mathbf{a}_{M,M,N}]_{\oplus}^{\mathbb{C}}$, which is a subring of $\text{Mat}_N([\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)]_{\oplus}^{\mathbb{C}})$, by determining all the primitive central idempotents.

It is easy to see that the center of $[\mathbf{a}_{M,M,N}]_{\oplus}^{\mathbb{C}}$ consists of diagonal matrices with entries in the center of $[\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_0]_{\oplus}^{\mathbb{C}}$. A central idempotent of $[\mathbf{a}_{M,M,N}]_{\oplus}^{\mathbb{C}}$ is then a diagonal matrix with coefficients central idempotents of $[\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_0]_{\oplus}^{\mathbb{C}}$.

Lemma 5B.9. *The primitive central idempotents of $[\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_0]_{\oplus}^{\mathbb{C}}$ are in bijection with the $L_{e\omega_1} \otimes -$ -orbits on $\text{Si}(\mathbf{Rep}_{\eta}(\mathfrak{sl}_N))$.*

Proof. Since $\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_0$ is a braided fusion category, its complexified Grothendieck ring is commutative and semisimple. The primitive central idempotents of $[\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_0]_{\oplus}^{\mathbb{C}}$ are then in bijection with the characters $[\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_0]_{\oplus}^{\mathbb{C}} \rightarrow \mathbb{C}$. Given $L_{\mathbf{m}} \in \text{Si}(\mathbf{Rep}_{\eta}(\mathfrak{sl}_N))$, the linear extension $\chi_{L_{\mathbf{m}}}$ of $[L_{\mathbf{k}}] \mapsto S_{\mathbf{m},\mathbf{k}}^{\mathfrak{sl}} / \dim(L_{\mathbf{m}})$ is a character of $[\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_0]_{\oplus}^{\mathbb{C}}$, see [\[EGNO15, Proposition 8.13.11\]](#). Since $\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)$ is a modular category, it follows from [\[EGNO15, Theorem 8.20.7 and Corollary 8.20.11\]](#) that all this exhaust all characters $[\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_0]_{\oplus}^{\mathbb{C}} \rightarrow \mathbb{C}$, and that $\chi_{L_{\mathbf{m}}} = \chi_{L_{\mathbf{m}'}}$ if and only if $L_{\mathbf{m}}$ and $L_{\mathbf{m}'}$ are in the same $(\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_0)'$ -module component. Here, $(\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_0)'$ denotes the centralizer of $\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_0$ in $\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)$ as in [\[EGNO15, Section 8.20\]](#). One may easily show that $(\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_0)'$ is the fusion subcategory generated by $L_{e\omega_1}$, which implies that $L_{\mathbf{m}}$ and $L_{\mathbf{m}'}$ are in the same $(\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_0)'$ -module component if and only if $L_{\mathbf{m}}$ and $L_{\mathbf{m}'}$ are in the same orbit under $L_{e\omega_1} \otimes -$. \square

The idempotent of $[\mathbf{Rep}_{\mathfrak{q}}(\mathfrak{sl}_N)]_{\oplus}^{\mathbb{C}}$ corresponding to the object $L_{\mathbf{m}}$ is then a scalar multiple of $R_{\mathbf{m}} = \sum_{X \in \text{Si}(\mathbf{Rep}_{\eta}(\mathfrak{sl}_N))} \chi_{L_{\mathbf{m}}}(X)[X^*]$. Let us denote by $R_{\mathbf{m},i} = \sum_{X \in \text{Si}(\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)_i)} \chi_{L_{\mathbf{m}}}(X)[X^*]$ so that the idempotent $e_{\mathbf{m}}$ of $[\mathbf{Rep}_{\mathfrak{q}}(\mathfrak{sl}_N)_0]_{\oplus}^{\mathbb{C}}$ corresponding to the object $L_{\mathbf{m}}$ is a scalar multiple of $R_{\mathbf{m},0}$. One can also check that $R_{\mathbf{m},0}[L_{\mathbf{k}}] = \chi_{L_{\mathbf{m}}}(L_{\mathbf{k}})R_{\mathbf{m},-i}$ for $L_{\mathbf{k}}$ a simple object of color i .

Lemma 5B.10. *Let $\mathbf{m} \in X^+(e)$. Define $A_{\mathbf{m}} = \{i \in \mathbb{Z}/N\mathbb{Z} \mid \exists L_{\mathbf{k}}, \chi_{L_{\mathbf{m}}}(L_{\mathbf{k}}) \neq 0 \text{ and } \chi_c(L_{\mathbf{k}}) = i\}$. Then $A_{\mathbf{m}}$ is a subgroup of $\mathbb{Z}/N\mathbb{Z}$ and the diagonal matrix $E_{\mathbf{m},i+A_{\mathbf{m}}}$ supported by the right coset $i + A_{\mathbf{m}}$ with nonzero entries equal to $e_{\mathbf{m}}$ is a primitive central idempotent of $[\mathbf{a}_{M,M,N}]_{\oplus}^{\mathbb{C}}$.*

Proof. First, it is clear that $A_{\mathbf{m}}$ is an additive subgroup of $\mathbb{Z}/N\mathbb{Z}$. One may easily show that each diagonal matrix supported by a right coset of $A_{\mathbf{m}}$ with nonzero entries equal to $e_{\mathbf{m}}$ is a central idempotent of $[\mathbf{a}_{M,M,N}]_{\oplus}^{\mathbb{C}}$.

Let f be a primitive central idempotent of $[\mathbf{a}_{M,M,N}]_{\oplus}^{\mathbb{C}}$ and choose i such that the diagonal entry $f_{i,i}$ is nonzero. Since e is primitive, there exists \mathbf{m} such that $f_{i,i} = e_{\mathbf{m}}$. Suppose that there exists a simple object $L_{\mathbf{k}}$ of color j such that $\chi_{L_{\mathbf{m}}}(L_{\mathbf{k}}) \neq 0$ and consider $x \in [\mathbf{a}_{M,M,N}]_{\oplus}^{\mathbb{C}}$ whose only non zero entry is at the position $(i, i+j)$ and is equal to $[L_{\mathbf{k}}]$. Then, since f is central, we obtain that $x_{i,i+j} f_{i+j,i+j} = f_{i,i} x_{i,i+j} = e_{\mathbf{m}}[L_{\mathbf{k}}]$

which is a scalar multiple of $R_{\mathbf{m},-j}$. Since $\chi_{\mathbf{m}}(\mathbf{L}_{\mathbf{k}}) \neq 0$, the element $R_{\mathbf{m},j}$ is nonzero as well as $x_{i,i+j}f_{i+j,i+j}$. In particular, the coefficient $f_{i+j,i+j}$ of f is nonzero and a scalar multiple of $R_{\mathbf{m},0}$. Therefore $f_{i+j,i+j} = e_{\mathbf{m}}$. We deduce that $f = E_{m,i+A_{\mathbf{m}}}$. \square

In the Artin–Wedderburn decomposition of $[\mathbf{a}_{M,M,N}]_{\oplus}^{\mathbb{C}}$, the idempotent $E_{m,i+A_{\mathbf{m}}}$ will contribute to a matrix algebra of size $|A_{\mathbf{m}}|$. Let $\text{Stab}_{\mathbf{m}}$ be the stabilizer subgroup of $\mathbf{L}_{\mathbf{m}}$ under the action $\mathbf{L}_{e\omega_1} \otimes -$.

Lemma 5B.11. *Let $\mathbf{m} \in X^+(e)$ and $r \in \mathbb{Z}/N\mathbb{Z}$. Then $r \in \text{Stab}_{\mathbf{m}}$ if and only if $ri \equiv 0 \pmod{N}$ for all $i \in A_{\mathbf{m}}$.*

Proof. Using Lemma 2C.2, we have, for any $\mathbf{m}, \mathbf{k} \in X^+(e)$ and $r \in \mathbb{Z}_{\geq 0}$:

$$\chi_{\mathbf{m} \rightarrow r}(\mathbf{L}_{\mathbf{k}}) = \frac{\dim(\mathbf{L}_{\mathbf{k}})}{\dim(\mathbf{L}_{\mathbf{m}})} \chi_{\mathbf{k}}(\mathbf{L}_{\mathbf{m} \rightarrow r}) = \frac{\dim(\mathbf{L}_{\mathbf{k}})}{\dim(\mathbf{L}_{\mathbf{m}})} \chi_{\mathbf{k}}(\mathbf{L}_{\mathbf{m}}) \chi_{\mathbf{k}}(\mathbf{L}_{e\omega_r}) = \eta^{-2re\chi_c(\mathbf{L}_{\mathbf{k}})/N} \chi_{\mathbf{m}}(\mathbf{L}_{\mathbf{k}}).$$

Suppose that $r \in \text{Stab}_{\mathbf{m}}$ and let $i \in A_{\mathbf{m}}$. We then choose $\mathbf{k} \in X^+(e)$ such that $\chi_c(\mathbf{L}_{\mathbf{k}}) = i$ and $\chi_{\mathbf{m}}(\mathbf{L}_{\mathbf{k}}) \neq 0$. By assumption, $\mathbf{m} \rightarrow r = \mathbf{m}$ and therefore we have $\chi_{\mathbf{m}}(\mathbf{L}_{\mathbf{k}}) = \chi_{\mathbf{m} \rightarrow r}(\mathbf{L}_{\mathbf{k}}) = \eta^{-2re\chi_c(\mathbf{L}_{\mathbf{k}})/N} \chi_{\mathbf{m}}(\mathbf{L}_{\mathbf{k}})$. As $\chi_{\mathbf{m}}(\mathbf{L}_{\mathbf{k}}) \neq 0$, we obtain that $r\chi_{\mathbf{m}}(\mathbf{L}_{\mathbf{k}}) \equiv 0 \pmod{N}$.

Conversely, suppose that $ri \equiv 0 \pmod{N}$ for all $i \in A_{\mathbf{m}}$. Since $\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)$ is modular, $\chi_{\mathbf{m}} = \chi_{\mathbf{m} \rightarrow r}$ implies that $\mathbf{L}_{\mathbf{m}} \simeq \mathbf{L}_{\mathbf{m} \rightarrow r}$. Let $\mathbf{k} \in X^+(e)$. If $\chi_c(\mathbf{L}_{\mathbf{k}}) \notin A_{\mathbf{m}}$ then $\chi_{\mathbf{m}}(\mathbf{L}_{\mathbf{k}}) = 0$ and $\chi_{\mathbf{m} \rightarrow r}(\mathbf{L}_{\mathbf{k}}) = \eta^{-2re\chi_c(\mathbf{L}_{\mathbf{k}})/N} \chi_{\mathbf{m}}(\mathbf{L}_{\mathbf{k}}) = 0$. Otherwise, $r\chi_c(\mathbf{L}_{\mathbf{k}}) \equiv 0 \pmod{N}$ and we have $\chi_{\mathbf{m} \rightarrow r}(\mathbf{L}_{\mathbf{k}}) = \eta^{-2re\chi_c(\mathbf{L}_{\mathbf{k}})/N} \chi_{\mathbf{m}}(\mathbf{L}_{\mathbf{k}}) = \chi_{\mathbf{m}}(\mathbf{L}_{\mathbf{k}})$. Therefore the characters $\chi_{\mathbf{m}}$ and $\chi_{\mathbf{m} \rightarrow r}$ coincide. \square

We hence deduce that $|A_{\mathbf{m}}|$ is equal to the cardinal of the orbit of $\mathbf{L}_{\mathbf{m}}$ under $\mathbf{L}_{e\omega_1} \otimes -$.

Lemma 5B.12. *We have*

$$[\mathbf{a}_{M,M,N}]_{\oplus}^{\mathbb{C}} \simeq \bigoplus_{m|\text{gcd}(M,N)} \text{Mat}_{N/m}(\mathbb{C})^{\oplus n_m}, \quad [\mathbf{A}_{M,M,N}]_{\oplus}^{\mathbb{C}} \simeq \bigoplus_{m|\text{gcd}(M,N)} \text{Mat}_{N!/m}(\mathbb{C})^{\oplus n_m},$$

where $n_m = \frac{m^2}{M} \sum_{k|\text{gcd}(N/m, M/m)} \mu(k) \binom{M/mk}{N/mk}$.

Proof. To obtain the Artin–Wedderburn decomposition of $\mathbf{a}_{M,M,N}$, it remains to compute the number of orbits of simple objects of $\mathbf{Rep}_{\eta}(\mathfrak{sl}_N)$ under $\mathbf{L}_{e\omega_1} \otimes -$ of a given cardinality m , which is done similarly to Lemma 5B.2. Indeed, each such orbit gives rise to m idempotents with a matrix algebra of size N/m . The Artin–Wedderburn decomposition of $\mathbf{A}_{M,M,N}$ follows from the definition of the big asymptotic category. \square

To finish the proof, it remains to understand the dimensions of the representations of the Calogero–Moser family \mathcal{F}_0 .

Lemma 5B.13. *The Calogero–Moser family \mathcal{F}_0 contains only representations of dimension $N!/m$ for every $m|\text{gcd}(M,N)$. The number of representations of dimension $N!/m$ is n_m .*

Proof. Recall that in term of M -partitions of N , the representations of \mathcal{F}_0 are indexed by orbits of multi-partitions with entries equal to (1) or \emptyset under the action of $\mathbb{Z}/M\mathbb{Z}$ by cyclic shifting. Each orbit of size m parametrizes M/m representations of dimension $N!/m$.

To count the number of representations of a given dimension, one may proceed similarly to the proof of Theorem 3B.3.(a). \square

The proof is complete. \square

5C. Missing proofs for Section 4 and type D graphs.

Proof of Lemma 4A.10. We have two things to check: that the map is well-defined and that it is injective. The first is significantly more difficult than in [MMMT20].

Well-defined. To see that the Θ_{w_i} satisfy the defining relations of T_{∞} in (4A.7) we first observe that $\Theta_{w_i} \Theta_{w_i} = [N]_v! \cdot \Theta_{w_i}$ as follows from, for example, [Eli16b, (2.8)] or [Bon17, Proposition 10.5.2.(d)] since $\#W_i = N!$. The second relation in (4A.7) is much more difficult to prove and we use (QSH) to do so: under (QSH) going from θ_k to θ_{k+i} corresponds to tensoring with \mathbf{L}_{ω_i} and from θ_{k+i} to θ_{k+i+j} corresponds to tensoring with \mathbf{L}_{ω_j} , and similarly, but reversed, for the other side of the equation. In particular, this equation in $H(\hat{A}_{N-1})$ holds since we have $\mathbf{L}_{\omega_j} \otimes \mathbf{L}_{\omega_i} \cong \mathbf{L}_{\omega_i} \otimes \mathbf{L}_{\omega_j}$.

Injective. The same argument as in [MMMT20, Proof of Lemma 3.2] works. \square

Proof of Theorem 4B.4. We start with a lemma.

Lemma 5C.1. *Let $\sigma \in V'_e$. The stabilizer of σ is of order $\text{gcd}(\{j|Z_j(\sigma) \neq 0\} \cup \{N\})$.*

Proof. Let m be such that $Z_j(\sigma) = 0$ if $m \nmid j$. By Lemma 5A.3, we have $Z_i(\sigma \rightarrow) = \zeta^{-i} Z_i(\sigma)$ for all $1 \leq i \leq N$. This implies that $Z_i(\sigma \rightarrow^{N/m}) = Z_i(\sigma)$, either because $Z_i(\sigma) = 0$, if $m \nmid j$, or because $\zeta^{-iN/m} = 1$, otherwise. By Lemma 5A.1, we have $\sigma \rightarrow^{N/m} = \sigma$. \square

(a). Since $Z_j(\sigma) = 0$ if $j \not\equiv 0 \pmod{m}$, the representation $M(\sigma)$ decomposes as $L(\sigma)_0 \oplus \dots \oplus L(\sigma)_{m-1}$ where $\theta_i, \theta_{i+m}, \dots, \theta_{i+(N/m-1)m}$ act on $L(\sigma)_i$ by the matrices

$$\chi \begin{pmatrix} [N]_v & Z_m(\sigma) & Z_{2m}(\sigma) & \dots & Z_{(N/m-1)m}(\sigma) \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad \chi \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ Z_{(N/m-1)m}(\sigma) & [N]_v & Z_m(\sigma) & \dots & Z_{(N/m-2)m}(\sigma) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \dots,$$

$$\chi \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ Z_m(\sigma) & Z_{2m}(\sigma) & \dots & Z_{(N/m-1)m}(\sigma) & [N]_v \end{pmatrix}$$

and by zero otherwise.

Lemma 5C.2. *The endomorphism ring of $L(\sigma)_i$ is one dimensional.*

Proof. Let f be a matrix intertwiner for $L(\sigma)_i$. Since $[N]_v \neq 0$, we immediately obtain that f must be a diagonal matrix $\text{diag}(f_0, \dots, f_{N/m-1})$. The relation $f\theta_{i+km} = \theta_{i+km}f$ implies that $f_k Z_{rm}(\sigma) = Z_{rm}(\sigma) f_{k+r}$ so that $f_k = f_{k+r}$ for all $0 \leq k < N/m$ and $0 < r < N/m$ such that $Z_{rm}(\sigma) \neq 0$, where the indices are taken modulo N/m . By Lemma 5C.1, the set $\{1 \leq r < N/m \mid Z_{rm}(\sigma) \neq 0\}$ generates $\mathbb{Z}/(N/m)\mathbb{Z}$. Therefore, we obtain that f is diagonal. \square

Since $[N]_v \neq 0$, the explicit form of the matrices above implies that the representations $L(\sigma)_i$ are pairwise nonisomorphic. They are moreover simple because their endomorphism rings are one dimensional by Lemma 5C.2.

(b). This follows from (a) and the following lemma.

Lemma 5C.3. *The representations $M(\sigma)$ and $M(\sigma')$ are isomorphic if and only if σ and σ' are in the same orbit.*

Proof. We have two directions.

Case \Leftarrow . Let j be such that $\sigma' = \sigma^{-j}$. By Lemma 5A.3, this implies that $Z_i(\sigma') = \zeta^{-ij} Z_i(\sigma)$. An easy calculation shows that, for all $i \in I$, we have

$$M_i(\sigma') = \text{diag}(1, \zeta^j, \dots, \zeta^{(N-1)j}) M_i(\sigma) \text{diag}(1, \zeta^j, \dots, \zeta^{(N-1)j})^{-1}.$$

Therefore the representations $M(\sigma)$ and $M(\sigma')$ are isomorphic.

Case \Rightarrow . As in the proof of Lemma 5C.2, an invertible matrix intertwiner $M(\sigma) \rightarrow M(\sigma')$ is necessarily a diagonal matrix $\text{diag}(f_0, \dots, f_{N-1})$. The relation $M_i(\sigma)f = fM_i(\sigma')$ implies that $f_i Z_k(\sigma) = Z_k(\sigma') f_{i+k}$ for all $0 \leq i < N$ and $1 \leq k < N$. Since f is invertible, we obtain that $Z_k(\sigma') = 0$ if and only if $Z_k(\sigma) = 0$. By Lemma 5C.1, the stabilizers of σ and σ' have the same order m .

If for all $1 \leq k < N$ we have $Z_k(\sigma) = Z_k(\sigma') = 0$, then, by Lemma 5A.1, we have $\sigma = \sigma'$. We then suppose that all Z -functions do not vanish at σ (and consequently at σ'). By Lemma 5C.1, m is a generator of the subgroup of $\mathbb{Z}/N\mathbb{Z}$ generated by the $1 \leq j < N$ with $Z_j(\sigma) \neq 0$. Therefore, using the relations $f_i Z_k(\sigma) = Z_k(\sigma') f_{i+k}$ with k such that $Z_k(\sigma) \neq 0 \neq Z_k(\sigma')$, one may show that the ratios f_i/f_{i+m} are all equal. As

$$1 = \frac{f_0}{f_m} \cdot \frac{f_m}{f_{2m}} \dots \frac{f_{(N/m-1)m}}{f_0} = \left(\frac{f_0}{f_m} \right)^{N/m},$$

the ratio f_0/f_m is equal to ζ^{-mj} for some j .

The relation $f_0 Z_k(\sigma) = Z_k(\sigma') f_k$ gives then $Z_k(\sigma') = \zeta^{-jk} Z_k(\sigma)$. Indeed, either k is a multiple of m and this follows from $f_0 = \zeta^{-mj} f_{rm}$, either k is not a multiple of m and $Z_k(\sigma) = Z_k(\sigma') = 0$. We finally obtain that $Z_k(\sigma') = Z_k(\sigma^{-j})$ for all $1 \leq k < N$ and then $\sigma' = \sigma^{-j}$ by Lemma 5A.1. \square

(c).+(d). The simple representations we found in (a) satisfy

$$\dim_{\mathbb{C}^v}(M_0)^2 + \sum_{\sigma \in \text{orbits}} \sum_{i=0}^{m-1} \dim_{\mathbb{C}^v}(L(\sigma)_i)^2 = \dim_{\mathbb{C}^v}(T_e).$$

This follows from Theorem 2E.10.(c). Indeed, each orbit in V_e with stabilizer of order m defines m representations of dimension N/m and there are $\frac{m}{N} \frac{N}{M} \sum_{k \mid \gcd(N/m, M/m)} \mu(k) \binom{M/mk}{N/mk}$ such orbits.. Hence

$$\begin{aligned} \sum_{\sigma \in \text{orbits}} \sum_{i=0}^{m-1} \dim_{\mathbb{C}^v}(L(\sigma)_i)^2 &= \sum_{m \mid \gcd(N, e)} m \frac{m}{N} \frac{N^2}{m^2} \frac{N}{M} \sum_{k \mid \gcd(N/m, M/m)} \mu(k) \binom{M/mk}{N/mk} \\ &= \frac{N^2}{M} \sum_{m \mid \gcd(N, e)} \sum_{k \mid \gcd(N/m, M/m)} \mu(k) \binom{M/mk}{N/mk} \stackrel{(*)}{=} \frac{N^2}{M} \binom{M}{N} = N p_e^N, \end{aligned}$$

where the equality (\star) follows from the convolution property of the Möbius function. The result now follows from classical theory. \square

We also construct other examples of Ncolored graph through the fusion rules of module categories over $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$, constructed as categories of modules over an algebra in $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$. This is similar to the orbifold procedure of [Kos88].

Given a rank N and e , the pointed subcategory generated by $L_{e\omega_i}$ has the fusion rules of $\mathbb{Z}/N\mathbb{Z}$. However, this subcategory is not a symmetric subcategory in general. Recall, for a finite group G and a central element $z \in G$ with $z \neq 1$ and $z^2 = 1$, the braided fusion category $\mathbf{Rep}(G, z)$ as in [EGNO15, Example 9.9.1.(3)]. Let $g = \gcd(N, e)$ and $p = N/g$.

Lemma 5C.4. *The pointed subcategory \mathbf{P} generated by $L_{e\omega_p}$ is a symmetric subcategory of $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$. Moreover \mathbf{P} is braided equivalent to $\mathbf{Rep}(\mathbb{Z}/g\mathbb{Z}, z)$ if $(e/g$ and p are odd and g is even) and to $\mathbf{Rep}(\mathbb{Z}/g\mathbb{Z})$ otherwise.*

Proof. The twist of the simple object $L_{e\omega_i}$ is given by $\eta^{(e\omega_i, e\omega_i + 2\rho)} = \exp(i\pi i(N - i)/N)$. Therefore, since $L_{e\omega_{pi}} \otimes L_{e\omega_{pj}} \simeq L_{e\omega_{p(i+j)}}$, we have $\beta_{L_{e\omega_{pi}}, L_{e\omega_{pj}}} = \theta_{e\omega_{p(i+j)}} \theta_{e\omega_{pi}}^{-1} \theta_{e\omega_{pj}}^{-1} = \text{id}$, and the subcategory \mathbf{P} is symmetric.

By [EGNO15, Corollary 9.9.25], it remains to compute the braiding of the generating object $L_{e\omega_p}$ with itself. Since the object $L_{e\omega_p} \otimes L_{e\omega_p}$ is simple, the braiding $\beta_{L_{e\omega_p}, L_{e\omega_p}}$ is a scalar multiple of the identity. By [EGNO15, Exercise 8.10.15], since the invertible object $L_{e\omega_p}$ is of quantum dimension 1, we have $\beta_{L_{e\omega_p}, L_{e\omega_p}} = \theta_{e\omega_p}$. One may check that $\theta_{e\omega_p} = (-1)^{ep(g-1)/g} \text{id}$, which is equal to -1 if and only if e/g , p and $g - 1$ are all odd. \square

Let A^D be the algebra in \mathbf{P} corresponding to the algebra of functions on $\mathbb{Z}/g\mathbb{Z}$.

Lemma 5C.5. *In $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$, we have $A^D \simeq L_0 \oplus L_{e\omega_p} \oplus \dots \oplus L_{e\omega_{(g-1)p}}$.*

Proof. The categories $\mathbf{Rep}(\mathbb{Z}/g\mathbb{Z})$ and $\mathbf{Rep}(\mathbb{Z}/g\mathbb{Z}, z)$ are equivalent to the category of $\mathbb{Z}/g\mathbb{Z}$ -graded vector spaces. We now apply [EGNO15, Example 7.8.3.(4)]. \square

Remark 5C.6. For $N = 3$, in [MMMT20, Proposition 5.4.3] the calculation of the algebra object A^D was done using symmetric webs [RT16], [RW20], [LT21]. We instead use a corollary of Deligne theorem on Tannakian categories, see [EGNO15, Theorem 9.9.22]. \diamond

We consider the category \mathbf{M} of (right) modules over A^D in $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$. In the case case \mathbf{P} is equivalent to $\mathbf{Rep}(\mathbb{Z}/g\mathbb{Z})$, the category \mathbf{M} is the de-equivariantization of $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$, it is a braided $\mathbb{Z}/g\mathbb{Z}$ -crossed category.

Lemma 5C.7. *The category \mathbf{M} is a finite semisimple module category over $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$.*

Proof. The only nontrivial statement is the semisimplicity, which follows from the separability of A^D , by [EGNO15, Proposition 7.8.30]. By Remark 5C.6, separability is equivalent to the digon scalar to be invertible, which is immediate from the definition of symmetric webs. \square

There is $\mathbb{Z}/N\mathbb{Z}$ -grading on \mathbf{M} . The simple objects of \mathbf{M} are all obtained as summands of free objects $A^D \otimes L_{\mathbf{m}}$. Since all summands of A^D are of color 0, $A^D \otimes L_{\mathbf{m}}$ is of color $\chi_c(L_{\mathbf{m}})$. This grading is moreover compatible with the grading of $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$.

We define the graph *graph of type D* of rank N and level e as the graph Γ with set of vertices $\text{Si}(\mathbf{M})$, colored with the above $\mathbb{Z}/N\mathbb{Z}$ -grading, and the vertices X and Y are adjacent if and only if Y is a summand of $\bigoplus_{i=1}^{N-1} L_{\omega_i} \otimes X$. The graph Γ_i is then the fusion graph of the object L_{ω_i} .

Proof of Theorem 4C.12. For type A, this is explained in [Zub98, Section 1.3, Example 2]. We give a self-contained proof.

Lemma 5C.8. *For Γ of type A of rank N and level e , the matrices $(A(\Gamma_1), \dots, A(\Gamma_{N-1}))$ can be simultaneously diagonalized and their joint spectrum is the Koornwinder variety V_e .*

Proof. The matrix $A(\Gamma_i)$ is the fusion matrix of L_{ω_i} in the category $\mathbf{Rep}_\eta(\mathfrak{sl}_N)$. The S -matrix diagonalizes the fusion rules [EGNO15, Corollary 8.14.5] and the joint spectrum of $(A(\Gamma_1), \dots, A(\Gamma_{N-1}))$ is:

$$\left\{ (S_{\mathbf{k}, \omega_1}^{\mathfrak{sl}} / S_{\mathbf{k}, 0}^{\mathfrak{sl}}, \dots, S_{\mathbf{k}, \omega_{N-1}}^{\mathfrak{sl}} / S_{\mathbf{k}, 0}^{\mathfrak{sl}}) \mid \mathbf{k} \in X^+(e) \right\}.$$

A direct application of Weyl character formula shows that

$$\frac{S_{\mathbf{k}, \omega_i}^{\mathfrak{sl}}}{S_{\mathbf{k}, 0}^{\mathfrak{sl}}} = \sum_j \eta^{2(\mathbf{k} + \rho, w_j^i)} = Z_i \left(\frac{2\pi}{e + N} (\mathbf{k} + \rho) \right).$$

We conclude using Lemma 2E.8 and Theorem 2E.10. \square

By definition of the Koornwinder variety and Lemma 4C.5, $\text{M}(\Gamma)$ is a T_e -representation for Γ of type A. \square

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