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# Finite-dimensional homogeneous boundary control for a 1D reaction-diffusion equation

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**Abstract**—In this paper, we address the problem of finite-dimensional boundary stabilization of the 1D reaction-diffusion equation. Using the modal decomposition approach, we propose a finite-dimensional *homogeneous* controller, stabilizing the unstable dynamics while ensuring the stability of the residual part. The closed-loop system with homogeneous feedback is well-posed and stable. The proposed controller is proven superior to a finite-dimensional linear feedback controller in terms of closed-loop performance. The numerical simulations are presented to support the analytical results.

## I. INTRODUCTION

Parabolic partial differential equations (PDEs) play a crucial role in the mathematical description of complex systems involving diffusion phenomena (arising in biology, chemistry, spatial ecology, etc). Control design for complex systems modeled by PDEs has constituted, for many years, a central research area [1], [2]. A traditional way to act on those systems is through *in-domain* and *boundary control*. Traditionally, two main approaches have been used for boundary control and observer design: 1) *the backstepping method* [3] and 2) *modal decomposition approach*. The latter approach relies on separating a finite-dimensional unstable part from a stable infinite-dimensional part of the PDE. It allows for the design of finite-dimensional controllers and observers, which are easier to implement and offer greater flexibility, including the possibility of designing nonlinear boundary controllers (e.g., sliding mode output feedback control as described by [4]) and sampled-data controllers [5]. The roots of modal decomposition can be traced back to seminal works like [6], [7], [8]. In [7], for instance, a Galerkin projection on modal subspace is used to design a finite-dimensional control for the reduced order model of the reaction-diffusion equation so that the control law stabilizes the unstable part while maintaining the stability of the residual part. This method has gained renewed interest in recent years, with several contributions utilizing it to propose constructive ways to design finite-dimensional observers and controllers (see, e.g., [9], [10], [11], [12], [13] through Lyapunov and LMI-based techniques). In [9], a finite-dimensional linear control has been constructed for 1-d reaction-diffusion equation with input delay at boundary. Based on some feasible LMIs conditions and using a direct Lyapunov method, the authors of

[10] constructed a finite-dimensional observer-based control for stabilization of a parabolic PDE with unbounded control operator and either bounded and unbounded observation operator and extend this approach to the case of Dirichlet boundary control and measurement for heat equation in [11]. In [13], they constructed a finite-dimensional control for semi-linear parabolic PDE. The authors of [12] extended the work [10] to the case of Dirichlet boundary control and either a Dirichlet or Neumann boundary observation.

In most cases, the resulting finite-dimensional controllers for PDEs are linear, which are suitable for application and implementation because of their simplicity and for enabling more accessible a well-posedness analysis of closed-loop systems. However, similar to finite-dimensional settings, alternative approaches like sliding-mode or homogeneous-based controllers might offer better performance with features like accelerated or even finite-time convergence compared to linear controllers.

Homogeneity is a sort of symmetry of an object (function or set) with respect to group transformation (dilation), which has been discovered for many physical models. As dilation, we can cite a uniform dilation introduced by Leonhard Euler in the 18th century, a weighted dilation [14], a geometric dilation generated by unstable  $C^1$  vector field [15] and a linear dilation generated by an anti-Hurwitz matrix [16] (as used in this paper). Homogeneous-based controllers (known to be an intermediary of linear control and sliding mode control) have been widely used in control and estimation design in finite-dimensional systems [16], [17] as better convergence rate, and lesser overshooting. The closed-loop system with homogeneous feedback are known to be highly stable (finite/fixed time stability).

While homogeneous-based in-domain control for parabolic PDEs exists [18], applying this approach to boundary actuation remains underdeveloped. Hence, in this paper, we propose a finite-dimensional homogeneous-based controller for the boundary stabilization of a reaction-diffusion PDE. This controller capitalizes on the advantages of homogeneity applied through boundary actuation and enables improved performance in such PDE systems.

Exploiting the modal-decomposition approach, we design stabilizing feedback (for the unstable modes) using the concept of generalized homogeneity. We state and give a sketch of the proof that the closed-loop system with the proposed nonlinear control is well-posed and performs better than a finite-dimensional linear control. In fact, we can achieve what is called *partial finite-time stability* (see e.g., [19] for more details on partial finite time stability).

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To our knowledge, this work constitutes a first attempt to design a finite-dimensional homogeneous-based *boundary* controller for a reaction-diffusion PDE in the literature.

The paper is organized as follows: In Section 2, we present some preliminaries on homogeneity notions. Section 3 deals with preliminaries on modal decomposition, linear finite-dimensional control design for reaction-diffusion equation, and homogeneous control design for the linear finite-dimensional system. The main result is presented in Section 4. Numerical results are presented in Section 5 showing the effectiveness of control strategy.

### A. Notation

$\mathbb{R}_+ = \{a \in \mathbb{R} : a > 0\}$ ;  $|z| = \sqrt{z^\top z}$  is Euclidean norm of  $z = (z_i)_{i=1}^n = (z_1, \dots, z_n)^\top \in \mathbb{R}^n$ ; we write  $P \succ 0$  if the symmetric matrix  $P = P^\top \in \mathbb{R}^{n \times n}$  is positive definite;  $\|x\| = \sqrt{x^\top P x}$  is the weighted Euclidean norm with the matrix  $P \succ 0$  defined below; given  $L > 0$ ,  $\langle \cdot, \cdot \rangle$  is a scalar product on  $L^2(0, L)$  such that  $\forall f, g \in L^2(0, L)$ ,  $\langle f, g \rangle = \int_0^L f(x)g(x)dx$ ,  $H^k(0, L)$  is Sobolev space having  $k$  square integrable weak derivatives;  $\mathbf{0}_N = (0, \dots, 0)^\top \in \mathbb{R}^N$ ,  $\mathbf{1}_N = (1, 0, \dots, 0)^\top \in \mathbb{R}^N$ ;  $C(\mathbb{R}_+)$  is a set of continuous function on  $\mathbb{R}_+$ . A function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be a class- $\mathcal{K}$  function if it is continuous, zero at zero, and strictly increasing. A class- $\mathcal{K}$  function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be a class- $\mathcal{K}_\infty$  function if it is unbounded with its argument.

## II. PRELIMINARIES ON HOMOGENEITY AND PROBLEM STATEMENT

### A. Preliminaries on Homogeneity

Homogeneity is a symmetry property with respect to a group of transformations and in this paper we consider a particular case of homogeneity with respect to linear dilation.

A family of linear operators  $\mathbf{d}(s) : \mathbb{R}^n \mapsto \mathbb{R}^n$  with  $s \in \mathbb{R}$  is called a dilation if it satisfies:

- *group property* :  $\mathbf{d}(0)x = x$ ,  $\mathbf{d}(s) \circ \mathbf{d}(t)x = \mathbf{d}(s+t)x$ ,  $\forall x \in \mathbb{R}^n, \forall s, t \in \mathbb{R}$ ;
- *continuous* if the mapping  $s \mapsto \mathbf{d}(s)x$  is continuous,  $\forall x \in \mathbb{R}^n$ ;
- *limit property*:  $\liminf_{s \rightarrow +\infty} \|\mathbf{d}(s)x\| = +\infty$  and  $\limsup_{s \rightarrow -\infty} \|\mathbf{d}(s)x\| = 0, \forall x \neq \mathbf{0}$ .

Any linear dilation  $\mathbf{d}$  in  $\mathbb{R}^n$  admits the representation [20]:

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} = \sum_{j=0}^{\infty} \frac{s^j G_{\mathbf{d}}^j}{j!}, \quad s \in \mathbb{R}. \quad (1)$$

where  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is an anti-Hurwitz matrix called a generator of  $\mathbf{d}$  with  $G_{\mathbf{d}}$ .

*Definition 1:* A dilation  $\mathbf{d}$  is monotone if  $s \mapsto \|\mathbf{d}(s)x\|$  is a monotone increasing function for any  $x \neq \mathbf{0}$ .

*Corollary 1:* A linear continuous dilation in  $\mathbb{R}^n$  is monotone with respect to the weighted Euclidean norm  $\|x\| = \sqrt{x^\top P x}$ ,  $0 \prec P = P^\top \in \mathbb{R}^{n \times n}$  if and only if

$$P G_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \quad P \succ 0. \quad (2)$$

Any linear continuous and monotone dilation in a normed vector space also introduces an alternative norm topology.

A function  $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto [0, +\infty)$  defined as follows:  $\|\mathbf{0}\|_{\mathbf{d}} = 0$  and

$$\|x\|_{\mathbf{d}} = e^{s_x}, \quad \text{where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0}, \quad (3)$$

is called canonical  $\mathbf{d}$ -homogeneous norm in  $\mathbb{R}^n$  where  $\mathbf{d}$  is a dilation. Note that  $\forall s \in \mathbb{R}$  and  $\forall x \in \mathbb{R}^n$ , one has:

$$\|\mathbf{d}(-\ln(\|x\|_{\mathbf{d}}))x\| = 1. \quad (4)$$

*Definition 2:* [15] A function  $h : \mathbb{R}^n \mapsto \mathbb{R}$  (resp., a vector field  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ ) is  $\mathbf{d}$ -homogeneous of degree  $\mu \in \mathbb{R}$  if

$$h(\mathbf{d}(s)x) = e^{\mu s} h(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}. \quad (5)$$

(resp., if  $f(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s)f(x)$ ,  $\forall x \in \mathbb{R}^n \quad \forall s \in \mathbb{R}$ .)

### B. Problem Statement

Let us consider the 1D reaction-diffusion equation :

$$\partial_t y(t, x) = \nu \partial_{xx} y(t, x) + qy(t, x), \quad (6)$$

$$y(t, 0) = 0, \quad (7)$$

$$y(t, L) = U(t), \quad (8)$$

$$y(0, x) = y_0(x), \quad (9)$$

$(t, x) \in \mathbb{R}_+ \times [0, L]$ ,  $L > 0$ , where  $y(t, \cdot)$  is the reaction-diffusion PDE state at time  $t$ ,  $\nu$  is the diffusion coefficient and  $q > 0$  is the reaction coefficient.  $U(t) \in \mathbb{R}$  is the control input. The aim of this paper is

- to design the homogeneous finite-time stabilizer for the system (6)-(9) using modal decomposition approach;
- to investigate if the use of the homogeneous feedback results in a smaller peaking than the linear feedback. More precisely, consider the following control problem: given initial state  $y_0 \in H^2(0, L)$  with  $y_0(0) = 0$ , a stabilization precision  $\epsilon > 0$ , a prescribed time  $T > 0$  and a maximal control amplitude  $\bar{U}$ , the closed-loop system (6)-(9) has the desired stabilization precision:

$$\|y(t, \cdot)\|_{H^1(0, L)} \leq \epsilon, \quad \forall t > T, \quad (10)$$

with the restricted control magnitude

$$\sup_{t > 0} |U(t)| \leq \bar{U}. \quad (11)$$

We aim to investigate if (for some given  $y_0 \in H^2(0, L)$  with  $y_0(0) = 0$ ) the homogeneous feedback may solve this problem, while linear feedback always violates one of the above conditions (see [21], [22]).

## III. MODAL DECOMPOSITION

In this section, we recall the main ideas for boundary control design using modal decomposition. We start by using the following change of variable (see [2, Definition 3.3.2]):

$$w(t, x) = y(t, x) - \frac{x}{L} U(t), \quad (12)$$

where, as in [11], we set  $U(0) = 0$ , to obtain the following equivalent PDE:

$$\partial_t w(t, x) = \nu \partial_{xx} w(t, x) + qw(t, x) - \frac{x}{L} \dot{U}(t) + \frac{qx}{L} U(t), \quad (13)$$

$$w(t, 0) = 0, \quad (14)$$

$$w(t, L) = 0, \quad (15)$$

$$w(0, x) = y_0(x), \quad (16)$$

provided that

$$w(0, x) = y_0(x) - \frac{x}{L} U(0) = y_0(x). \quad (17)$$

By introducing a new control input  $\xi(t) = \dot{U}(t)$ , one obtains a coupled ODE-PDE

$$\dot{U}(t) = \xi(t), \quad (18)$$

$$\partial_t w(t, x) = \nu \partial_{xx} w(t, x) + qw(t, x) - \frac{x}{L} \xi(t) + \frac{qx}{L} U(t), \quad (19)$$

$$w(t, 0) = 0, \quad (20)$$

$$w(t, L) = 0, \quad (21)$$

$$w(0, x) = y_0(x), \quad (22)$$

The solution to (19) can be represented as follows:

$$w(t, x) = \sum_{n=1}^{\infty} w_n(t) \phi_n(x), \quad w_n(t) = \langle w(t, \cdot), \phi_n \rangle, \quad (23)$$

with  $\phi_n(x) = \sqrt{\frac{2}{L}} \sin(\frac{n\pi x}{L})$ ,  $n = 1, 2, \dots$  and projecting onto the basis  $\{\phi_n\}_{n=1}^{\infty}$ , one has

$$\dot{U}(t) = \xi(t) \quad (24)$$

$$\dot{w}_n(t) = (\nu \lambda_n + q) w_n(t) - b_n \xi(t) + q b_n U(t), \quad (25)$$

$$w_n(0) = y_{0,n}, \quad (26)$$

where  $n \geq 1$ ,  $\lambda_n = -\frac{n^2 \pi^2}{L^2}$  is the eigenvalue of the operator  $\partial_{xx}$  corresponding to the eigenvector  $\phi_n$ ,  $y_{n,0} = \langle y_0, \phi_n \rangle$ , and

$$b_n = \frac{1}{L} \int_0^L x \phi_n(x) dx. \quad (27)$$

Since  $\lambda_n \rightarrow -\infty$  when  $n \rightarrow +\infty$ , then there exists an integer  $N \geq 1$  such that

$$\nu \lambda_n + q < -\delta, \quad \forall n \geq N + 1, \quad (28)$$

with  $\delta$  some positive constant.

By setting,  $w^N(t) = (U(t), w_1(t), w_2(t), \dots, w_N(t))^T$ ,  $\tilde{B}_N = (-\frac{1}{B_N})$ ,  $B_N = (b_n)_{n=1}^N$ ,  $\tilde{A}_N = \begin{pmatrix} 0 & 0_{1 \times N} \\ q B_N & A_N \end{pmatrix}$  and  $A_N = \text{diag}\{\nu \lambda_n + q\}_{n=1}^N$ , we obtain from (24)-(26), the following augmented system:

$$\dot{w}^N(t) = \tilde{A}_N w^N(t) + \tilde{B}_N \xi(t), \quad (29)$$

$$\dot{w}_n(t) = (\nu \lambda_n + q) w_n(t) - b_n \xi(t) + q b_n U(t), \quad (30)$$

and initial data  $(w^N(0), w_n(0)) = (y_0^N, y_{0,n})$ .

#### IV. HOMOGENEOUS-BASED CONTROLLER AND MAIN RESULT

We propose a nonlinear control following the concept of generalized homogeneity based on a linear dilation  $\mathbf{d}$  generated by an anti-Hurwitz matrix  $G_{\mathbf{d}}$  as defined in (1).

Since the pair  $(\tilde{A}_N, \tilde{B}_N)$  is controllable, then according to [23], the linear algebraic equation

$$\tilde{A}_N G_0 - G_0 \tilde{A}_N + \tilde{B}_N Y_0 = \tilde{A}_N, \quad G_0 \tilde{B}_N = \mathbf{0}, \quad (31)$$

has a solution  $Y_0 \in \mathbb{R}^{1 \times (N+1)}$ ,  $G_0 \in \mathbb{R}^{(N+1) \times (N+1)}$  such that the matrix  $G_0 - I_{N+1}$  is invertible. Moreover, the matrix

$$G_{\mathbf{d}} := I_{N+1} + \mu G_0, \quad (32)$$

is anti-Hurwitz for any  $\mu \in (-1, 0)$  and the matrix  $\tilde{A}_0 = \tilde{A}_N + \tilde{B}_N K_0$  with  $K_0 = Y_0 (G_0 - I_{N+1})^{-1}$  satisfies the identity

$$\tilde{A}_0 G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I_{N+1}) \tilde{A}_0, \quad G_{\mathbf{d}} \tilde{B}_N = \tilde{B}_N. \quad (33)$$

Furthermore, the Linear Matrix Inequalities

$$(\tilde{A}_0 + \delta G_{\mathbf{d}}) X_h + X_h (\tilde{A}_0 + \delta G_{\mathbf{d}})^T + \tilde{B}_N Y_h + Y_h^T \tilde{B}_N^T \preceq 0, \quad (34)$$

$$G_{\mathbf{d}} X_h + X_h G_{\mathbf{d}}^T \succ 0, \quad X_h = X_h^T \succ 0, \quad (35)$$

have solutions  $X_h \in \mathbb{R}^{(N+1) \times (N+1)}$ , and  $Y_h \in \mathbb{R}^{1 \times (N+1)}$ . The following result proposes a homogeneous control using a dilation  $\mathbf{d}$  that is generated by  $G_{\mathbf{d}}$  defined in (32). This controller can achieve finite-time stabilization of the finite-dimensional system (29).

*Lemma 1:* ([23]) Given any initial condition  $y_0^N \in \mathbb{R}^{N+1}$ , the system (29) with the feedback law

$$\xi(w^N(t)) = K_0 w^N(t) + \mathcal{N}(w^N(t)), \quad (36)$$

with

$$\mathcal{N}(w^N(t)) = \|w^N(t)\|_{\mathbf{d}}^{1+\mu} \tilde{K}_h \mathbf{d} (-\ln(\|w^N(t)\|_{\mathbf{d}})) w^N(t), \quad (37)$$

where  $\tilde{K}_h = Y_h X_h^{-1}$ ,  $Y_h, X_h$  solution of (34)-(35) and  $\mathbf{d}$  the dilation generated by  $G_{\mathbf{d}}$  defined in (32) is globally Lyapunov stable and satisfies

$$w^N(t) = \mathbf{0}_{N+1}, \quad \forall t \geq T(y_0^N), \quad (38)$$

with  $T(y_0^N) := \frac{\|y_0^N\|_{\mathbf{d}}^{-\mu}}{-\delta\mu}$ .

*Proof:* The proof follows the same lines as in the proof of [23, Lemma 5] showing that the functional  $V(t) = \|w^N(t)\|_{\mathbf{d}}$  satisfies  $\dot{V}(t) \leq -\delta V(t)^{1+\mu}$  along the solution of closed-loop system (29), (36), (37). ■

Based on this lemma, we propose a finite-dimensional homogeneous control stabilizing globally the infinite-dimensional system (6)-(9).

*Theorem 1:* For any initial condition  $y_0 \in H^2(0, L)$  such that  $y_0(0) = y_0(L) = 0$ , if  $\mu \in (-1, 0)$  then the system (6)-(9) with feedback law (36) has a unique solution and the origin  $y \equiv \mathbf{0}$  is globally asymptotically stable in  $H^1(0, L)$ . Moreover, for all  $t \geq T(y_0^N) := \frac{\|y_0^N\|_{\mathbf{d}}^{-\mu}}{-\delta\mu}$ , the control input  $U$  and  $y$  satisfy

$$U(t) = 0, \quad \langle y(t, \cdot), \phi_n \rangle = 0, \quad n = 1, \dots, N, \quad (39)$$

$$\|y(t, \cdot)\|_{H^1(0,L)}^2 \leq M e^{-2\delta(t-T(y_0^N))} \sum_{n>N} |\lambda_n| |y_n(T(y_0^N))|^2$$

with  $y_0^N = (0, \langle y_0, \phi_1 \rangle, \dots, \langle y_0, \phi_N \rangle)^\top$  and some positive  $M > 0$ . (40)

*Proof:* We provide a sketch of the proof, which is divided into two parts.

*Part 1:* (Existence and uniqueness)

The closed-loop system (18)-(22), (36)-(37) is rewritten in the following compact form:

$$\dot{Z}(t) = (\mathcal{A} + BK_0\mathcal{P})Z(t) + \mathcal{B}\mathcal{N}(\mathcal{P}Z(t)), \quad (41)$$

$$Z(0) = (0, y_0) \in \mathcal{H}, \quad (42)$$

where  $\mathcal{H} = \mathbb{R} \times L^2(0, L)$  is a Hilbert space with inner product given by  $\langle Z_1, Z_2 \rangle_{\mathcal{H}} = U_1 U_2 + \langle w_1, w_2 \rangle$ , with  $Z_1 = (U_1, w_1) \in \mathcal{H}$ ,  $Z_2 = (U_2, w_2) \in \mathcal{H}$ ,  $Z(t) = (U(t), w(t, \cdot)) \in \mathcal{H}$  is the system state, the operator  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  is given by

$$\mathcal{A}Z(t) = \begin{bmatrix} 0 & \mathbf{0} \\ qr & \nu \partial_{xx} + q\mathbf{1} \end{bmatrix} Z(t),$$

$$\mathcal{D}(\mathcal{A}) = \{(U, w) \in \mathbb{R} \times H^2(0, L) | w(0) = w(L) = 0\}, \quad (43)$$

$r(x) = \frac{x}{L}$ , using the identity operator  $\mathbf{1}$  on  $\mathcal{H}$  and null operator  $\mathbf{0}$  on  $\mathcal{H}$ , the operator  $\mathcal{P} : \mathcal{H} \rightarrow \mathbb{R}^{N+1}$  defined as  $\mathcal{P}Z(t) = w^N(t)$  is linear and bounded and the operator  $\mathcal{B} : \mathbb{R} \rightarrow \mathcal{H}$  defined as  $\mathcal{B}\xi(t) = [-r(x)] \xi(t)$  is linear and bounded. Since the Laplacian operator  $\partial_{xx}$  generates a  $\mathcal{C}_0$ -semigroup on  $L^2(0, L)$  and the operator  $Z(t) \in \mathcal{H} \mapsto q[r(\cdot), \mathbf{1}]Z(t) \in \mathcal{H}$  is linear bounded in  $\mathcal{H}$  then, the operator  $\mathcal{A}$  generates a strongly continuous semigroup on  $\mathcal{H}$  ([24, Theorem 3.2]), and in parallel, since  $\mathcal{B}, \mathcal{P}$  are linear and bounded, then  $\mathcal{A} + BK_0\mathcal{P}$  generates a strongly continuous semigroup  $(\mathcal{S}(t))_{t \geq 0}$  on  $\mathcal{H}$ . Moreover the control operator  $\mathcal{B}$  is admissible (see [25, Section 4.2]) for the semigroup generated by  $\mathcal{A} + BK_0\mathcal{P}$ .

Following same lines as in [26, Theorem 1] and using [25, Proposition 4.2.5 and 4.2.10], the closed loop system (41)-(42) admits a mild solution  $Z(t) \in C([0, +\infty), \mathcal{D}(\mathcal{A})) \cap H_{loc}^1([0, +\infty), \mathcal{H})$  for  $\mu \in (-1, \frac{-1}{2})$  and a classical solution  $Z(t) \in C([0, +\infty), \mathcal{D}(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H})$  for  $\mu \in [-\frac{1}{2}, 0)$  defined by

$$Z(t) = \mathcal{S}(t)Z(0) + \int_0^t \mathcal{S}(t-s)\mathcal{B}\mathcal{N}(\alpha(s))ds, \quad (44)$$

where  $\alpha(t)$  is solution of

$$\dot{\alpha}(t) = \tilde{A}_0\alpha(t) + \tilde{B}_N(\mathcal{N}(\alpha(t))), \quad (45)$$

$$\alpha(0) = \mathcal{P}(0, y_0) \in \mathbb{R}^{N+1}, \quad (46)$$

for any initial condition  $y_0 \in H^2(0, L)$  such that  $y_0(0) = y_0(L) = 0$ ,  $\tilde{A}_0$  and  $\tilde{B}_N$  are such that (33) is verified. Notice that the solution  $\alpha(t)$  of (45) is unique and continuously differentiable on  $[0, +\infty)$  (see [16, Theorem 9.1]); the map

$$t \mapsto \mathcal{N}(\alpha(t)) \in \begin{cases} L_{loc}^2([0, +\infty), \mathbb{R}), & \text{if } \mu \in (-1, -\frac{1}{2}), \\ H_{loc}^1([0, +\infty), \mathbb{R}), & \text{if } \mu \in [-\frac{1}{2}, 0). \end{cases} \quad (47)$$

The proof of existence of solution of closed-loop system (41)-(42) is complete.

The uniqueness of the solution of closed-loop system (41)-(42) can be shown by contradiction. In fact if  $Z_1$  and  $Z_2$  are two solutions of (41)-(42) and using the fact that (45)-(46) admits unique solution, one derives  $Z_1 = Z_2$ . The proof is complete.

*Part 2:* (Asymptotic stability)

Let us consider the system

$$\dot{w}^N(t) = \tilde{A}_0 w^N(t) + \tilde{B}_N \mathcal{N}(w^N(t)), \quad (48)$$

$$\dot{w}_n(t) = (\nu\lambda_n + q)w_n(t) - b_n f(w^N(t)), \quad (49)$$

where  $n > N$ ,  $w^N(t) = (U(t), w_1(t), w_2(t), \dots, w_N(t))^\top$ ,  $f(w^N(t)) = \tilde{K}_0 w^N(t) + \mathcal{N}(w^N(t))$ ,  $\tilde{K}_0 = K_0 - q\mathbf{1}_{N+1}^\top$  and  $\tilde{A}_0 = \tilde{A}_N + \tilde{B}_N K_0$ .

From (12) and using Wirtinger's inequality along with [11, Lemma 1], one has

$$\|y(t, \cdot)\|_{H^1(0,L)} \leq M_1 \left( \left( \sum_{n=1}^{\infty} |\lambda_n| |w_n|^2 \right)^{\frac{1}{2}} + |U(t)| \right), \quad (50)$$

for some positive  $M_1$ .

Since  $\mathbf{0}_{N+1}$  is Lyapunov stable for (48) and using [16, Lemma 7.2] there exists  $\bar{\sigma}_1 \in \mathcal{K}_\infty$  such that

$$\forall t \geq 0, \quad |w^N(t)| \leq \bar{\sigma}_1(|y_0^N|). \quad (51)$$

This implies that there exists  $\sigma_1 \in \mathcal{K}_\infty$  such that

$$\left( \sum_{n \leq N} |\lambda_n| |w_n|^2 \right)^{\frac{1}{2}} + |U(t)| \leq \sigma_1(|y_0^N|). \quad (52)$$

Since  $w^N(t) = \mathbf{0}_{N+1}, \forall t \geq T(y_0^N)$ , then  $\forall n \geq N+1$  and  $\forall t \geq 0$ ,

$$w_n(t) = e^{\bar{\lambda}_n t} y_{n,0} - b_n \int_0^{T(y_0^N)} e^{\bar{\lambda}_n(t-s)} f(w^N(s)) ds, \quad (53)$$

with  $\bar{\lambda}_n = \nu\lambda_n + q$ .

Then using [16, Lemma 7.2] and (51), there exists  $\sigma_2 \in \mathcal{K}_\infty$  such that  $\forall t \geq 0$ ,

$$|w_n(t)| \leq |y_{n,0}| + \frac{|b_n|}{|\nu\lambda_n + q|} \sigma_2(|y_0^N|), \quad \forall n > N. \quad (54)$$

Thus,

$$\left( \sum_{n>N} |\lambda_n| |w_n|^2 \right)^{\frac{1}{2}} \leq M_2 \left[ \left( \sum_{n>N} |\lambda_n| |y_{n,0}|^2 \right)^{\frac{1}{2}} + \sigma_2(|y_0^N|) \right], \quad (55)$$

with  $M_2 = \sqrt{2} \max\{1, \left( \sum_{n>N} \frac{|\lambda_n| |b_n|^2}{|\nu\lambda_n + q|^2} \right)^{\frac{1}{2}}\}$ .

Using the inequalities (50) and (52),(55), there exists  $\sigma_3 \in \mathcal{K}_\infty$  such that

$$\forall t \geq 0, \quad \|y(t, \cdot)\|_{H^1(0,L)} \leq \sigma_3(\|y_0\|_{H^1(0,L)}). \quad (56)$$

This means that  $y \equiv \mathbf{0}$  is Lyapunov stable in  $H^1$ .

From (49),  $\forall t \geq T(y_0^N)$ , one has

$$\begin{aligned} \dot{w}^N(t) &= \mathbf{0}_{N+1}, \\ \dot{w}_n(t) &= (\lambda_n + q)w_n(t), \quad n \geq N. \end{aligned} \quad (57)$$

Then using (50), one derives  $\forall t \geq T(y_0^N)$ ,

$$\|y(t, \cdot)\|_{H^1} \leq M_1 e^{-\delta(t-T(y_0^N))} \left[ \sum_{n>N} |\lambda_n| |w_n(T(y_0^N))|^2 \right]^{\frac{1}{2}}, \quad (58)$$

which implies

$$\lim_{t \rightarrow +\infty} \|y(t, \cdot)\|_{H^1(0,L)} = 0. \quad (59)$$

This means  $y \equiv \mathbf{0}$  is asymptotically attractive in  $H^1(0, L)$ . Since  $\forall t \geq T(y_0^N)$ ,  $U(t) = 0$  then (12) implies that

$$\forall t \geq T(y_0^N), \quad y(t, \cdot) = w(t, \cdot). \quad (60)$$

Using (38) one derives

$$\forall t \geq T(y_0^N), \langle y(t, \cdot), \phi_n \rangle = 0, \quad n = 1, \dots, N. \quad (61)$$

The proof is complete.  $\blacksquare$

*Remark 1:* The homogeneous control (36) is a nonlinear static feedback, which is Non-Lipschitz at  $w^N = \mathbf{0}_{N+1}$ . In this case, Theorem 1.6 from [27], commonly used to justify the well-posedness of infinite dimensional systems with locally Lipschitz non-linearities, is inapplicable to the entire space  $\mathcal{H}$ .

*Remark 2:* Theorem 1 proves that the closed-loop system with the homogeneous control is *partially finite-time stable*. The proposed homogeneous controller steers the first  $N$  modes of the distributed state  $y$  to zero in a finite time, while all other modes decay exponentially to zero.

## V. NUMERICAL SIMULATIONS

We perform numerical simulations on the system (6)-(9), by using system (18)-(22) and transformation (12). The initial condition and parameters are:  $y_0(x) = x(\pi - x)$ ,  $q = 1.4$ ,  $\nu = 1$ ,  $L = \pi$  and  $N = 1$ . We consider (23) with  $M = 10$  truncated basis. We use the control toolbox on Matlab to compute the following linear control gains:

$$\begin{aligned} K_{l_1} &= (-13.5600, -106.6818), \\ K_{l_2} &= (-17.7100, -147.8641) \end{aligned} \quad (62)$$

We use the Homogeneous Control Systems (HCS) Toolbox for MATLAB [28] to compute  $\mu, \|\cdot\|_{\mathbf{d}}, K_0, \tilde{K}_h, X_h, G_{\mathbf{d}}$ :

$$\begin{aligned} \mu &= -0.1, K_0 = (-0.5600, -2.0918), \\ \tilde{K}_h &= (-13.0000, -104.5900), \\ X_h &= \begin{pmatrix} 0.0269 & 0.3096 \\ 0.3096 & 3.8570 \end{pmatrix}, G_{\mathbf{d}} = \begin{pmatrix} 0.9600 & -0.5229 \\ 0.0107 & 1.1400 \end{pmatrix}. \end{aligned} \quad (63)$$

Figure 1 shows the simulations in logarithmic scale of the norm  $\|y(t, \cdot)\|_{H^1(0,\pi)}$  of the closed-loop system with linear control (blue line), with high-gain linear control (black dashed line), and with homogeneous control (red line). We can observe that the closed-loop system with homogeneous control converges faster to zero than a closed-loop system with linear control. One can also observe that only a closed-loop system with high-gain linear control and homogeneous control achieves

$$\|y(t, \cdot)\|_{H^1(0,\pi)} \leq \epsilon := 10^{-4}, \quad \forall t \geq T := 6, \quad (64)$$

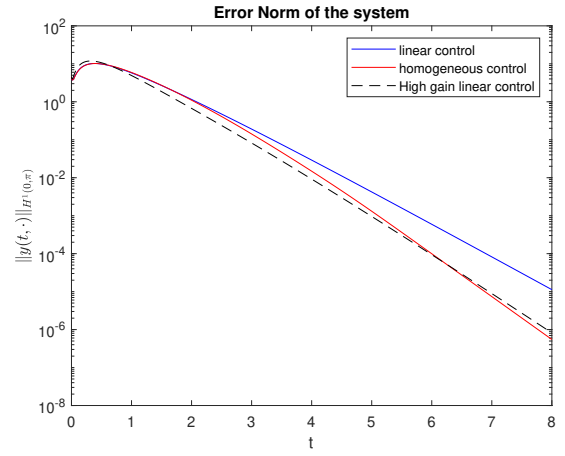


Fig. 1: Evolution of  $\|y(t, \cdot)\|_{H^1(0,\pi)}$  in a logarithmic scale of the closed-loop system with linear control (blue and black dashed lines) and homogeneous control (red line)

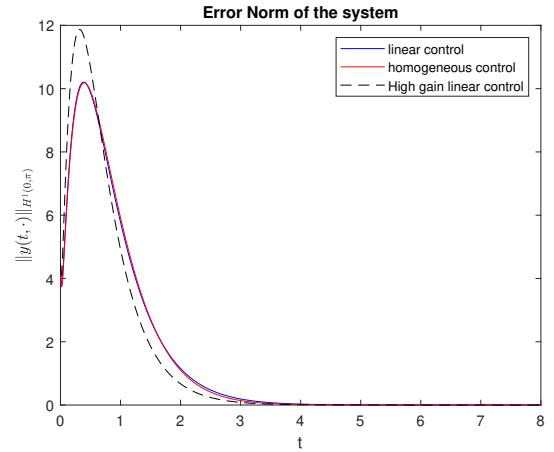


Fig. 2: Evolution of  $\|y(t, \cdot)\|_{H^1(0,\pi)}$  of the closed-loop system with linear control (blue and black) and homogeneous control (red) during the initial phase of stabilization

with the price to pay, the linear controller violates the conditions stated in (11). Indeed, from Figure 3 only the homogeneous control achieves (64) with the control restriction

$$\sup_{t>0} |U(t)| \leq \bar{U} := 9.5. \quad (65)$$

One can also observe in Figure 2 and Figure 3 that using high-gain linear control to achieve (64) implies a large deviation of solution closed-loop system (a peaking) and an overshoot of the control signal during the initial phase of stabilization. Note that using high-gain linear control may result in an unbounded peak and a very large overshoot when  $N$  is very large. Hence, we can observe the advantage of using a homogeneous control while making it possible to obtain faster convergence without a peaking effect and with a small overshoot.

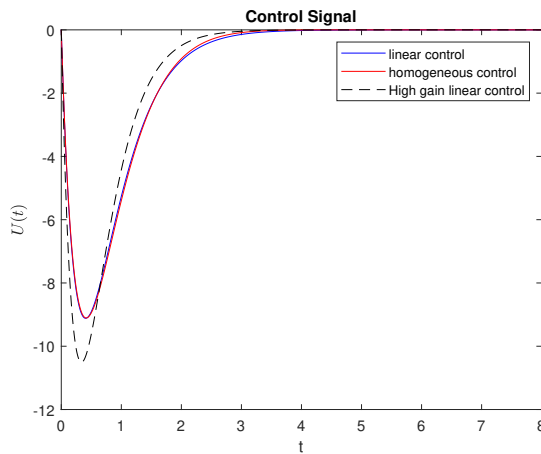


Fig. 3: Time-evolution of the control signal  $U(t)$  of the linear control (black and blue line) and homogeneous control (red line) during the initial phase of stabilization

## VI. CONCLUSIONS

In this paper, we designed a finite-dimensional nonlinear homogeneous controller for stabilizing a 1D reaction-diffusion equation. We provided a statement and sketch of the proof for the existence of the closed-loop solution and demonstrated the system's stability. Simulations revealed that the homogeneous controller outperformed the linear feedback controller in terms of closed-loop performance, eliminating peaking effects and reducing overshoot. Future work will focus on developing homogeneous observer-based controllers for reaction-diffusion PDEs and extending these results to nonlinear PDEs, such as the 1D viscous Burgers equation.

## REFERENCES

- [1] P. Christofides and J. Chow, "Nonlinear and robust control of pde systems: Methods and applications to transport-reaction processes," *Appl. Mech. Rev.*, vol. 55, no. 2, pp. B29–B30, 2002.
- [2] R. Curtain and H. Zwart, *An introduction to infinite-dimensional linear systems theory*, vol. 21. Springer Science & Business Media, 2012.
- [3] M. Krstic and A. Smyshlyaev, *Boundary control of PDEs: A course on backstepping designs*. SIAM, 2008.
- [4] Y. Orlov, Y. Lou, and P. Christofides, "Robust stabilization of infinite-dimensional systems using sliding-mode output feedback control," *International Journal of Control*, vol. 77, no. 12, pp. 1115–1136, 2004.
- [5] R. Katz, E. Fridman, and A. Selivanov, "Network-based boundary observer-controller design for 1D heat equation," in *2019 IEEE 58th Conference on Decision and Control (CDC)*, pp. 2151–2156, IEEE, 2019.
- [6] D. L. Russell, "Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions," *Siam Review*, vol. 20, no. 4, pp. 639–739, 1978.
- [7] M. Balas, "Feedback control of flexible systems," *IEEE Transactions on Automatic Control*, vol. 23, no. 4, pp. 673–679, 1978.
- [8] M. Balas, "Finite-dimensional controllers for linear distributed parameter systems: exponential stability using residual mode filters," *Journal of Mathematical Analysis and Applications*, vol. 133, no. 2, pp. 283–296, 1988.
- [9] C. Prieur and E. Trélat, "Feedback stabilization of a 1-d linear reaction–diffusion equation with delay boundary control," *IEEE Transactions on Automatic Control*, vol. 64, no. 4, pp. 1415–1425, 2019.
- [10] R. Katz and E. Fridman, "Constructive method for finite-dimensional observer-based control of 1-d parabolic pdes," *Automatica*, vol. 122, p. 109285, 2020.
- [11] R. Katz and E. Fridman, "Finite-dimensional control of the heat equation: Dirichlet actuation and point measurement," *European Journal of Control*, vol. 62, pp. 158–164, 2021.
- [12] H. Lhachemi and C. Prieur, "Finite-dimensional observer-based boundary stabilization of reaction-diffusion equations with either a Dirichlet or Neumann boundary measurement," *Automatica*, vol. 135, p. 109955, 2022.
- [13] R. Katz and E. Fridman, "Global stabilization of a 1d semilinear heat equation via modal decomposition and direct lyapunov approach," *Automatica*, vol. 149, p. 110809, 2023.
- [14] V. Zubov, "On systems of ordinary differential equations with generalized homogeneous right-hand sides," *Izvestia vuzov. Mathematica (in Russian)*, vol. 1, pp. 80–88, 1958.
- [15] M. Kawski, "Families of dilations and asymptotic stability," in *Analysis of Controlled Dynamical Systems: Proceedings of a Conference held in Lyon, France, July 1990*, pp. 285–294, Springer, 1991.
- [16] A. Polyakov, *Generalized homogeneity in systems and control*. Springer, 2020.
- [17] A. Polyakov and M. Krstic, "Finite- and fixed-time nonovershooting stabilizers and safety filters by homogeneous feedback," *IEEE Transactions on Automatic Control*, 2023.
- [18] A. Polyakov, J.-M. Coron, and L. Rosier, "On Homogeneous Finite-Time Control For Evolution Equation in Hilbert Space," *IEEE Transactions on Automatic Control*, vol. 63, no. 9, pp. 3143–3150, 2017.
- [19] W. Haddad and A. L' Afflito, "Finite-time partial stability and stabilization, and optimal feedback control," *Journal of the Franklin Institute*, vol. 352, no. 6, pp. 2329–2357, 2015.
- [20] A. Polyakov, "Sliding mode control design using canonical homogeneous norm," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 3, pp. 682–701, 2019.
- [21] B. T. Polyak and G. Smirnov, "Large deviations for non-zero initial conditions in linear systems," *Automatica*, vol. 74, pp. 297–307, 2016.
- [22] R. Izmailov, "Peak effect in stationary linear-systems in scalar inputs and outputs," *Automation and Remote Control*, vol. 48, no. 8, pp. 1018–1024, 1987.
- [23] K. Zimenko, A. Polyakov, D. Efimov, and W. Perruquetti, "Robust feedback stabilization of linear mimo systems using generalized homogenization," *IEEE Transactions on Automatic Control*, vol. 65, no. 12, pp. 5429–5436, 2020.
- [24] R. S. Phillips, "Perturbation theory for semi-groups of linear operators," *Transactions of the American Mathematical Society*, vol. 74, no. 2, pp. 199–221, 1953.
- [25] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*. Springer Science & Business Media, 2009.
- [26] I. Balogoun, S. Marx, and F. Plestan, "Sliding mode control for a class of linear infinite-dimensional systems," 2022.
- [27] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, 1983.
- [28] A. Polyakov, "Homogeneous systems control toolbox (HCS toolbox) for MATLAB: <https://gitlab.inria.fr/polyakov/hcs-toolbox-for-matlab>," tech. rep., 2023.