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1 An observability estimate for the wave equation and applications to
 2 the Neumann boundary controllability for semi-linear wave
 3 equations

4 Sue Claret*

5 September 10, 2024

6 **Abstract**

7 We give a boundary observability result for a 1d wave equation with a potential. We then deduce with
 8 a Schauder fixed-point argument the existence of a Neumann boundary control for a semi-linear wave
 9 equation $\partial_{tt}y - \partial_{xx}y + f(y) = 0$ under an optimal growth assumption at infinity on f of the type $s \ln^2 s$.
 10 Moreover, assuming additional assumption on f' , we construct a minimizing sequence which converges
 11 to a control. Numerical experiments illustrate the results. This work extends to the Neumann boundary
 12 control case the work of Zuazua in 1993 and the work of Münch and Trélat in 2022.

13 **AMS Classifications:** 35L71, 93B05.

14 **Keywords:** Observability inequality, semi-linear wave equation, exact boundary controllability, fixed-
 15 point method, least-squares approach.

16 **1 Introduction and main results**

Let $\Omega := (0, 1)$, $T > 0$ and $Q_T := \Omega \times (0, T)$. We define the Hilbert space $H_{(0)}^1(\Omega) := \{z \in H^1(\Omega); z(0) = 0\}$
 endowed with the norm $\|\cdot\|_{H_{(0)}^1(\Omega)} := \|\partial_x \cdot\|_{L^2(\Omega)}$. We denote by $H_{(0)}^{-1}(\Omega)$ the dual space of $H_{(0)}^1(\Omega)$ equipped
 with the dual norm

$$\|w\|_{H_{(0)}^{-1}(\Omega)} := \sup_{z \in H_{(0)}^1(\Omega) \setminus \{0\}} \frac{\langle w, z \rangle_{-1,1}}{\|z\|_{H_{(0)}^1(\Omega)}},$$

17 where $\langle \cdot, \cdot \rangle_{-1,1}$ denotes the dual product between $H_{(0)}^{-1}(\Omega)$ and $H_{(0)}^1(\Omega)$. For any $A \in L^\infty(Q_T)$ and $B \in$
 18 $L^2(Q_T)$, we consider the following linear wave equation

$$\begin{cases} \partial_{tt}\varphi - \partial_{xx}\varphi + A\varphi = B, & Q_T, \\ \varphi(0, t) = 0, \quad \partial_x\varphi(1, t) = 0, & (0, T), \\ (\varphi(\cdot, 0), \partial_t\varphi(\cdot, 0)) = (\varphi_0, \varphi_1), & \Omega, \end{cases} \quad (\star)$$

19 where $\varphi = \varphi(x, t)$ is a state and $(\varphi_0, \varphi_1) \in L^2(\Omega) \times H_{(0)}^{-1}(\Omega)$ is a given initial data. Then, (\star) admits a unique
 20 solution in the sense of transposition in $\mathcal{C}^0([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; H_{(0)}^{-1}(\Omega))$, we refer to Definition 2 and
 21 Theorem 5. This paper is devoted to the boundary observability problem corresponding to (\star) . Precisely,
 22 our main result is as follows

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Theorem 1. *Let $T > 2$, $A \in L^\infty(Q_T)$ and $B \in L^2(Q_T)$. Then, for any φ solution in the sense of transposition of (\star) , there exists $C_{obs} = C(\Omega, T, A) > 0$ such that*

$$\|(\varphi(\cdot, 0), \partial_t \varphi(\cdot, 0))\|_{L^2(\Omega) \times H_{(0)}^{-1}(\Omega)}^2 \leq C_{obs} \left(\|\varphi(1, \cdot)\|_{L^2(0, T)}^2 + \|\partial_{tt} \varphi - \partial_{xx} \varphi + A\varphi\|_{L^2(Q_T)}^2 \right). \quad (\text{Obs})$$

Moreover, there exists $C = C(\Omega, T) > 0$ such that

$$C_{obs} = Ce^{C\sqrt{\|A\|_{L^\infty(Q_T)}}}.$$

Estimate (Obs) has been known for many years, see [LT89a, Theorem 3.1 (b) p. 271 or Theorem 3.4 (b) p. 274], [ILTZ05, Chapter 19, Lemma 19.A.1 p. 324] or [Cor07, Proposition 2.60, 2.61 p. 74]. In the above results, the constant C_{obs} depends on the potential A . However, the explicit estimate of C_{obs} as a function of the potential A as in [Zua93, Theorem 4 p. 120] or [Zha00, Theorem 3.1, 3.2] is a part of the problem and it is exactly the main novelty of this paper. The proof is based on the same method as in [Zua93, Section 3]. In particular, the one-dimensional character is used in a fundamental way.

By duality arguments, boundary observability estimates are equivalent to boundary controllability properties. In particular here, one motivation for Theorem 1, is the exact boundary controllability of the following semi-linear wave equation

$$\begin{cases} \partial_{tt} y - \partial_{xx} y + f(y) = 0, & Q_T, \\ y(0, \cdot) = 0, \quad \partial_x y(1, \cdot) = v, & (0, T), \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1), & \Omega, \end{cases} \quad (\star\star)$$

where $(u_0, u_1) \in H_{(0)}^1(\Omega) \times L^2(\Omega)$ is a given initial data, v is a control function and $f \in C^1(\mathbb{R})$ is a non-linear function. The problem of exact controllability associated with $(\star\star)$ is the following one:

Given a controllability time $T > 0$ large enough, for any initial data (u_0, u_1) and any final data (z_0, z_1) in $H_{(0)}^1(\Omega) \times L^2(\Omega)$, find a control function $v \in L^2(0, T)$ and $y \in C^0([0, T]; H_{(0)}^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ solution of $(\star\star)$ such that

$$(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1) \quad \text{in } \Omega. \quad (1)$$

The controllability problem of the non-linear wave equation has essentially been studied in the case of distributed control with mainly Dirichlet boundary conditions. The first work on the exact controllability of a finite-dimensional non-linear wave equation is due to Markus in [Mar65] by using an implicit function theorem. The above method was then applied to obtain local controllability results on non-linear wave equation in [Fat75] or [Che76]. After that, the global controllability problem for semi-linear wave equation is studied under the following growth assumption on the non-linearity :

$$\exists \beta > 0 \text{ small enough such that } \limsup_{|r| \rightarrow +\infty} \frac{|f(r)|}{|r| \ln^p |r|} \leq \beta \text{ for some } p \geq 0. \quad (\text{H})$$

In particular, by a Schauder fixed-point argument, Zuazua in [Zua93] proves the first distributed controllability result for the 1d semi-linear wave equation under the assumption (H) with $p = 2$. Subsequently, the above result is generalized by the same approach to the multidimensional case with $p = 1/2$ in [LZ00] and then with $p = 3/2$ in [FLZ19]. In the specific case of mixed boundary conditions with a Neumann control, to our knowledge, there are very few results. A global exact boundary controllability result for $(\star\star)$, under the assumption that the non-linearity f is continuous with first derivative uniformly bounded, was given in [ILTZ05, Chapter 19] in the space $H_{(0)}^1(\Omega) \times L^2(\Omega)$ for the multidimensional case. Another controllability result is proved in [Cor07, Theorem 4.22 p. 178] with a Schauder fixed-point argument adapted from [Zua93] assuming that f is at most linear.

A first consequence of Theorem 1 is the following

Theorem 2. *Let $T > 2$. Assume that $f \in C^1(\mathbb{R})$ such that*

$$\limsup_{|r| \rightarrow +\infty} \frac{|f(r)|}{|r| \ln^2 |r|} \leq \beta. \quad (\text{H1})$$

If β is small enough then the system $(\star\star)$ is exactly controllable in time T .

The proof is based on a Schauder fixed-point argument. In particular, the stability property of the operator results from the observability inequality (Obs) of Theorem 1 and the assumption (H1) on f ; which can be proved to be optimal in the power of the logarithm by the same argument as in [Zua93, Section 4].

Remark that Theorem 2 and all the previous cited results only give the existence of a control function for $(\star\star)$ and is not based on a constructive method. It is only very recently that the question about the construction of convergent control approximation has emerged: we refer to [MT22, BLM23b] using a least-squares approach or [CCR22, BLM23a, CLM24] where a Picard iterative scheme is proposed. Under the assumption that the non-linearity is continuous in time and Lipschitz in space with a Lipschitz constant independent of time, we mention the back-and-forth iterations method of [NZWF19] which is illustrated in the case of the boundary controllability of the Sine-Gordon equation with a mixed boundary conditions. Eventually, in [CT06], Coron and Trélat construct a control in a feedback form to move from any steady-state to any other one provided that they are in the same connected component of the set of steady-states.

Assuming an additional growth assumption on f' , a second consequence of Theorem 1 is the following

Theorem 3. *Let $T > 2$. Assume that f' is α -Hölder continuous, for some $\alpha \in [0, 1]$, and satisfies*

$$\limsup_{|r| \rightarrow +\infty} \frac{|f'(r)|}{\ln^2 |r|} < \beta^*, \quad (\text{H2})$$

with $\beta^ > 0$ small enough. In the case $\alpha = 0$, we assume moreover that $\|f'\|_{L^\infty(\mathbb{R})}$ is small enough. There exists a sequence $(y_k, v_k)_{k \in \mathbb{N}}$ which converges strongly, at least with order $1 + \alpha$ after a finite number of iterations, to a state-control pair (y, v) of $(\star\star)$.*

As in [MT22, BLM23b, LMGM21], Theorem 3 is obtained by a least-squares approach, which consists to minimize the functional

$$E(y, v) := \|\partial_{tt}y - \partial_{xx}y + f(y)\|_{L^2(Q_T)}^2$$

over all the pair $(y, v) \in L^2(Q_T) \times L^2(0, T)$ satisfying $(y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1)$ and $(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1)$. In particular, the observability inequality (Obs) is used in a fundamental way and the assumption (H2) on f' is again a consequence of the expression on the observability constant.

Outline. The paper is organized as follows. Section 2 is devoted to giving some theoretical results for linear wave equation with potential and source term. In particular, we give the existence, uniqueness and *a priori* estimates for weak solution in the regularity space $H_{(0)}^1(\Omega) \times L^2(\Omega)$ (see Section 2.1) and for solution in the sense of transposition in the space $L^2(\Omega) \times H_{(0)}^{-1}(\Omega)$ (see Section 2.2). In Section 3 we give the proof of the observability inequality (Obs) given by Theorem 1. We give then controllability results as an application of (Obs) in Section 4. Section 4.1 is devoted to controllability of a linear wave equation, while Section 4.2 and Section 4.3 are devoted to Theorem 2 and Theorem 3 respectively. Finally, we present some numerical simulations in Section 5 to illustrate our results and, we conclude and give some perspectives in Section 6.

2 Linear wave equation with mixed boundary condition

2.1 Existence and uniqueness of solutions in the space $H_{(0)}^1(\Omega) \times L^2(\Omega)$

Let $T > 2$, $B \in L^1(0, T; L^2(\Omega))$, $v \in L^2(0, T)$ and $A \in L^\infty(Q_T)$. We consider the following linear wave equation

$$\begin{cases} \partial_{tt}y - \partial_{xx}y + Ay = B, & Q_T, \\ y(0, \cdot) = 0, \quad \partial_x y(1, \cdot) = v, & (0, T), \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1), & \Omega, \end{cases} \quad (2)$$

where $y = y(x, t)$ is the state and $(u_0, u_1) \in H_{(0)}^1(\Omega) \times L^2(\Omega)$ is a given initial data.

We define the solutions of (2) in a weak sense.

Definition 1. We say that $y \in L^2(0, T; H_{(0)}^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \cap H^2(0, T; H_{(0)}^{-1}(\Omega))$ is a **weak solution** of (2) if and only if y is solution of the variational formulation

$$\langle \partial_{tt}y(\cdot, t), z \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)} - v(t)z(1) + \int_{\Omega} \partial_x y \partial_x z \, dx + \int_{\Omega} Ayz \, dx = \int_{\Omega} Bz \, dx, \quad \forall z \in H_{(0)}^1(\Omega), \quad (3)$$

for a.e. $t \in [0, T]$ and $(y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1)$ in Ω .

The well-posedness of system (2) is proved in the following theorem

Theorem 4. Let $A \in L^\infty(Q_T)$, $B \in L^1(0, T; L^2(\Omega))$, $v \in L^2(0, T)$ and $(u_0, u_1) \in H_{(0)}^1(\Omega) \times L^2(\Omega)$. There exists a unique weak solution y of (2) satisfying

$$y \in C^0([0, T]; H_{(0)}^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)).$$

Moreover, there exists a constant $C = C(\Omega, T) > 0$ such that, for any $t \in [0, T]$,

$$\|y(\cdot, t)\|_{H_{(0)}^1(\Omega)} + \|\partial_t y(\cdot, t)\|_{L^2(\Omega)} \leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \left(\|(u_0, u_1)\|_{H_{(0)}^1(\Omega) \times L^2(\Omega)} + \|B\|_{L^1(0, t; L^2(\Omega))} + \|v\|_{L^2(0, t)} \right). \quad (4)$$

Finally, $y(1, \cdot)$ belongs to $H^1(0, T)$ and there exists $C = C(\Omega, T) > 0$ such that, for any $t \in [0, T]$,

$$\|\partial_t y(1, \cdot)\|_{L^2(0, t)} \leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \left(\|(u_0, u_1)\|_{H_{(0)}^1(\Omega) \times L^2(\Omega)} + \|B\|_{L^1(0, t; L^2(\Omega))} + \|v\|_{L^2(0, t)} \right). \quad (5)$$

Proof. The proof is based on well-known arguments. For the existence, as in [LM72, Proof of Theorem 8.1 p. 265], a candidate solution is built using the Faedo-Galerkin method by considering a Hilbert basis of $L^2(\Omega)$ composed of eigenvalues of the Laplacian which is also an orthogonal basis of $H_{(0)}^1(\Omega)$ (see [Bre83, Théorème IX.31 and Remarque 29 p.192-193] for the existence of such a basis). First, we prove that this solution satisfies inequalities (4) and (5) and then we show that the above candidate solution is a solution of our problem. Uniqueness is treated separately in a similar way to [Eva10, Theorem 4 p.385]. \square

2.2 Existence and uniqueness of solutions in the space $L^2(\Omega) \times H_{(0)}^{-1}(\Omega)$

This section is dedicated to the existence and uniqueness of solution of the linear wave equation in the space $L^2(\Omega) \times H_{(0)}^{-1}(\Omega)$. In particular, we define the solution of (\star) by the transposition method (see [LM72, Chapter I, Section 4.2 p.47]). For any $g \in L^2(Q_T)$, let $w \in C^0([0, T]; H_{(0)}^{-1}(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ be the weak solution of the backward adjoint equation

$$\begin{cases} \partial_{tt}w - \partial_{xx}w + Aw = g, & Q_T, \\ w(0, \cdot) = 0, \quad \partial_x w(1, \cdot) = 0, & (0, T), \\ (w(\cdot, T), \partial_t w(\cdot, T)) = (0, 0), & \Omega. \end{cases} \quad (6)$$

Definition 2. We say that $\varphi \in L^2(0, T; L^2(\Omega))$ is a solution in the sense of transposition of (\star) if and only if φ is solution of the identity

$$\int_{Q_T} \varphi(x, t) g(x, t) \, dx dt = \int_{Q_T} B(x, t) w(x, t) \, dx dt + \langle \varphi_1, w(\cdot, 0) \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)} - \int_{\Omega} \varphi_0(x) \partial_t w(x, 0) \, dx, \quad \forall g \in L^2(Q_T), \quad (7)$$

where w is the weak solution of (6).

Theorem 5. Let $B = 0$, $A \in L^\infty(Q_T)$ and $(\varphi_0, \varphi_1) \in L^2(\Omega) \times H_{(0)}^{-1}(\Omega)$. Then, there exists a unique solution in the sense of transposition of (\star) satisfying

$$\varphi \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H_{(0)}^{-1}(\Omega)).$$

Moreover, there exists a constant $C = C(T, \Omega) > 0$ such that

$$\|\varphi(\cdot, t)\|_{L^2(\Omega)} + \|\partial_t \varphi(\cdot, t)\|_{H_{(0)}^{-1}(\Omega)} \leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \|(\varphi_0, \varphi_1)\|_{L^2(\Omega) \times H_{(0)}^{-1}(\Omega)}, \quad (8)$$

for any $t \in [0, T]$.

Proof. Let $(\varphi_0, \varphi_1) \in L^2(\Omega) \times H_{(0)}^{-1}(\Omega)$ and $A \in L^\infty(Q_T)$. We easily check that problem

$$b(\varphi, g) = \ell(g), \quad \forall g \in L^2(Q_T), \quad (9)$$

where

$$b(\varphi, g) = \int_{Q_T} \varphi g \, dx dt \quad \text{and} \quad \ell(g) = \langle \varphi_1, w_g(\cdot, 0) \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)} - \int_{\Omega} \varphi_0(x) \partial_t w_g(x, 0) \, dx,$$

admits a unique solution $\varphi \in L^2(Q_T)$ by the Lax-Milgram theorem (see [Bre83, Corollaire V.8 p.84]). Now, let us write

$$\varphi = \Psi_1 + \Psi_2$$

where $\Psi_1 \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H_{(0)}^{-1}(\Omega))$ and $\Psi_2 \in C^0([0, T]; H_{(0)}^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ are respectively the solution in the sense of transposition and the weak solution of

$$\begin{cases} \partial_{tt} \Psi_1 - \partial_{xx} \Psi_1 = 0, & Q_T, \\ \Psi_1(0, \cdot) = 0, \quad \partial_x \Psi_1(1, \cdot) = 0, & (0, T), \\ (\Psi_1(\cdot, T), \partial_t \Psi_1(\cdot, T)) = (\varphi_0, \varphi_1), & \Omega, \end{cases} \quad \begin{cases} \partial_{tt} \Psi_2 - \partial_{xx} \Psi_2 + A \Psi_2 = -A \Psi_1, & Q_T, \\ \Psi_2(0, \cdot) = 0, \quad \partial_x \Psi_2(1, \cdot) = 0, & (0, T), \\ (\Psi_2(\cdot, T), \partial_t \Psi_2(\cdot, T)) = (0, 0), & \Omega. \end{cases}$$

Remark that, $\varphi \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H_{(0)}^{-1}(\Omega))$. Since the energy associated with system in Ψ_1 is conserved, we have

$$\|(\Psi_1(\cdot, t), \partial_t \Psi_1(\cdot, t))\|_{L^2(\Omega) \times H_{(0)}^{-1}(\Omega)}^2 = \|(\varphi_0, \varphi_1)\|_{L^2(\Omega) \times H_{(0)}^{-1}(\Omega)}^2, \quad \forall t \in [0, T]. \quad (10)$$

Moreover, using estimates (4) and (10), we obtain

$$\begin{aligned} \|(\Psi_2(\cdot, t), \partial_t \Psi_2(\cdot, t))\|_{H_{(0)}^1(\Omega) \times L^2(\Omega)}^2 &\leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \|A \Psi_1\|_{L^2(Q_T)}^2 \\ &\leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} T \|A\|_{L^\infty(Q_T)}^2 \|(\varphi_0, \varphi_1)\|_{L^2(\Omega) \times H_{(0)}^{-1}(\Omega)}^2. \end{aligned} \quad (11)$$

Therefore, since the embeddings $H_{(0)}^1(\Omega) \hookrightarrow L^2(\Omega)$ and $L^2(\Omega) \hookrightarrow H_{(0)}^{-1}(\Omega)$ are continuous, we deduce estimate (8) using (10) and (11). \square

Remark 1. Let $\varphi \in \mathcal{C}^0([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; H_{(0)}^{-1}(\Omega))$ the solution in the sense of transposition of (\star) associated with $B = 0$. As a consequence of the time reversibility of the wave equation, φ satisfies

$$\|(\varphi_0, \varphi_1)\|_{L^2(\Omega) \times H_{(0)}^{-1}(\Omega)} \leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \left(\|\varphi(\cdot, t)\|_{L^2(\Omega)} + \|\partial_t \varphi(\cdot, t)\|_{H_{(0)}^{-1}(\Omega)} \right), \quad (12)$$

for all $t \in [0, T]$.

3 Observability estimate

This section is devoted to the proof of Theorem 1. It is an adaptation of [Zua93, Section 3 p.120] to the boundary case. Here and in the following, the Dirichlet-Neumann Laplacian inverse operator, noted $(-\partial_{xx})^{-1}$, is defined by

$$\begin{aligned} (-\partial_{xx})^{-1} : H_{(0)}^{-1}(\Omega) &\rightarrow H_{(0)}^1(\Omega) \\ f &\mapsto u \end{aligned}$$

where u is the unique solution of

$$\int_{\Omega} \partial_x u \partial_x v \, dx = \langle f, v \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)}, \quad \forall v \in H_{(0)}^1(\Omega).$$

In particular, $(-\partial_{xx})^{-1}$ is a continuous isomorphism (see [Lio88, p.201]) such that

$$\|f\|_{H_{(0)}^{-1}(\Omega)} = \left\| (-\partial_{xx})^{-1} f \right\|_{H_{(0)}^1(\Omega)}, \quad \forall f \in H_{(0)}^{-1}(\Omega).$$

Proposition 1. Hidden regularity. Let $\varphi \in \mathcal{C}^0([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; H_{(0)}^{-1}(\Omega))$ be the solution in the sense of transposition of (\star) associated with $A \in L^\infty(Q_T)$, $B = 0$ and $(\varphi_0, \varphi_1) \in L^2(\Omega) \times H_{(0)}^{-1}(\Omega)$. Then, $\varphi(1, \cdot)$ belongs to $L^2(0, T)$ and there exists a constant $C = C(T, \Omega) > 0$ such that

$$\|\varphi(1, \cdot)\|_{L^2(0, T)} \leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \|(\varphi_0, \varphi_1)\|_{L^2(\Omega) \times H_{(0)}^{-1}(\Omega)}. \quad (13)$$

Proof. Let $(\varphi_0, \varphi_1) \in H_{(0)}^1(\Omega) \times L^2(\Omega)$ and let $\varphi \in \mathcal{C}^0([0, T]; H_{(0)}^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega)) \cap \mathcal{C}^2([0, T]; H_{(0)}^{-1}(\Omega))$ be the weak solution of (\star) associated with (φ_0, φ_1) . We consider $\chi \in H_{(0)}^1(\Omega)$ the unique weak solution of

$$\begin{cases} -\partial_{xx}\chi = -\varphi_1, & \Omega, \\ \chi(0) = 0, \quad \partial_x \chi(1) = 0, \end{cases} \quad (14)$$

that is χ is solution of

$$\int_{\Omega} \partial_x \chi \partial_x v \, dx = -\langle \varphi_1, v \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)}, \quad \forall v \in H_{(0)}^1(\Omega).$$

We define

$$h(x, t) = \int_0^t \varphi(x, s) \, ds + \chi(x), \quad \forall (x, t) \in Q_T.$$

In particular, $h \in \mathcal{C}^1([0, T]; H_{(0)}^1(\Omega)) \cap \mathcal{C}^2([0, T]; L^2(\Omega)) \cap \mathcal{C}^3([0, T]; H_{(0)}^{-1}(\Omega))$. Since φ is a weak solution of system (\star) , by Definition 1, we have

$$\langle \partial_{tt} \varphi(\cdot, t), z \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)} + \int_{\Omega} \partial_x \varphi(x, t) z(x) \, dx + \int_{\Omega} A(x, t) \varphi(x, t) z(x) \, dx = 0, \quad \forall z \in H_{(0)}^1(\Omega)$$

for any $t \in [0, T]$. We integrate the previous equation on $[0, t], t \in (0, T]$ and we obtain

$$\begin{aligned}
& \int_0^t \langle \partial_{tt}\varphi(\cdot, s), z \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)} ds + \int_{\Omega} \int_0^t \partial_x \varphi(x, s) z(x) ds dx + \int_0^t \int_{\Omega} A(x, s) \varphi(x, s) z(x) ds dx = 0 \\
\Leftrightarrow & \int_0^t \partial_t \left(\langle \partial_t \varphi(\cdot, s), z \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)} \right) ds + \int_{\Omega} \int_0^t \partial_x \varphi(x, s) \partial_x z(x) ds dx = \int_{\Omega} \left(- \int_0^t A(x, s) \varphi(x, s) ds \right) z(x) dx \\
\Leftrightarrow & \langle \partial_t \varphi(\cdot, t), z \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)} - \langle \partial_t \varphi(\cdot, 0), z \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)} + \int_{\Omega} \int_0^t \partial_x \varphi(x, s) \partial_x z(x) ds dx \\
& = \int_{\Omega} \left(- \int_0^t A(x, s) \varphi(x, s) ds \right) z(x) dx \\
\Leftrightarrow & \langle \partial_t \varphi(\cdot, t), z \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)} + \int_{\Omega} \partial_x \chi(x) \partial_x z(x) dx + \int_{\Omega} \left(\int_0^t \partial_x \varphi(x, s) ds \right) \partial_x z(x) dx \\
& = \int_{\Omega} \left(- \int_0^t A(x, s) \varphi(x, s) ds \right) z(x) dx \\
\Leftrightarrow & \langle \partial_t \varphi(\cdot, t), z \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)} + \int_{\Omega} \left(\int_0^t \partial_x \varphi(x, s) ds + \partial_x \chi(x) \right) \partial_x z(x) dx = \int_{\Omega} \left(- \int_0^t A(x, s) \varphi(x, s) ds \right) z(x) dx.
\end{aligned}$$

Since $\partial_{tt}h(\cdot, t) = \partial_t \varphi(\cdot, t)$ in $H_{(0)}^{-1}(\Omega)$ and $\partial_x h(\cdot, t) = \int_0^t \partial_x \varphi(\cdot, s) ds + \partial_x \chi$ in $L^2(\Omega)$, we deduce

$$\langle \partial_{tt}h(\cdot, t), z \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)} + \int_{\Omega} \partial_x h(\cdot, t) \partial_x z(x) dx = \int_{\Omega} \left(- \int_0^t A(x, s) \varphi(x, s) ds \right) z(x) dx$$

1 i.e. h is a weak solution of

$$\begin{cases} \partial_{tt}h - \partial_{xx}h = - \int_0^t A \varphi d\sigma, & Q_T, \\ h(0, \cdot) = 0, \partial_x h(1, \cdot) = 0, & (0, T) \\ (h(\cdot, 0), \partial_t h(\cdot, 0)) = (\chi, \varphi_0), & \Omega, \end{cases} \quad (15)$$

with $(\chi, \varphi_0) \in H_{(0)}^1(\Omega) \times H_{(0)}^1(\Omega) \subset H_{(0)}^1(\Omega) \times L^2(\Omega)$. Thus, using (5), we get

$$\|\partial_t h(1, \cdot)\|_{L^2(0, T)} \leq C \left(\|\chi\|_{H_{(0)}^1(\Omega)} + \|\varphi_0\|_{L^2(\Omega)} + \left\| \int_0^t A(\cdot, \sigma) \varphi(\cdot, \sigma) d\sigma \right\|_{L^2(Q_T)} \right).$$

We have (using Cauchy-Schwarz inequality and estimate (8))

$$\begin{aligned}
\left\| \int_0^t A(\cdot, \sigma) \varphi(\cdot, \sigma) d\sigma \right\|_{L^2(Q_T)}^2 & \leq T \|A\|_{L^\infty(Q_T)}^2 \int_0^1 \left| \int_0^t \varphi(x, \sigma) d\sigma \right|^2 dx \\
& \leq T \|A\|_{L^\infty(Q_T)}^2 \int_0^1 \left| \sqrt{t} \left(\int_0^t |\varphi(x, \sigma)|^2 d\sigma \right)^{1/2} \right|^2 dx \\
& \leq T \|A\|_{L^\infty(Q_T)}^2 \int_0^1 T \|\varphi(x, \cdot)\|_{L^2(0, T)}^2 dx \leq T^2 \|A\|_{L^\infty(Q_T)}^2 \|\varphi\|_{L^2(Q_T)}^2 \\
& \leq T^2 C \|A\|_{L^\infty(Q_T)}^2 e^{C \sqrt{\|A\|_{L^\infty(Q_T)}}} \|(\varphi_0, \varphi_1)\|_{L^2(\Omega) \times H_{(0)}^{-1}(\Omega)}^2
\end{aligned}$$

and since $(-\partial_{xx})^{-1}$ is a continuous isomorphism, $\|\chi\|_{H_{(0)}^1(\Omega)} \leq C \|\varphi_1\|_{H_{(0)}^{-1}(\Omega)}$. Therefore

$$\|\partial_t h(1, \cdot)\|_{L^2(0, T)} \leq C e^{C \sqrt{\|A\|_{L^\infty(Q_T)}}} \|(\varphi_0, \varphi_1)\|_{L^2(\Omega) \times H_{(0)}^{-1}(\Omega)},$$

2 for all $(\varphi_0, \varphi_1) \in H_{(0)}^1(\Omega) \times L^2(\Omega)$. By a density argument, since $\partial_t h = \varphi$ in Q_T , we deduce the result. \square

Lemma 1. Let $T > 2$ and let $\varphi \in \mathcal{C}^0([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; H_{(0)}^{-1}(\Omega))$ be the solution in the sense of transposition of (\star) associated with $A \in L^\infty(Q_T)$, $B = 0$ and $(\varphi_0, \varphi_1) \in L^2(\Omega) \times H_{(0)}^{-1}(\Omega)$. There exists a constant $C = C(\Omega, T) > 0$ such that

$$\int_1^{T-1} \|\varphi(\cdot, t)\|_{L^2(\Omega)}^2 dt \leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \|\varphi(1, \cdot)\|_{L^2(0, T)}^2. \quad (16)$$

Proof. Let $\psi = \psi(t, x) \in \mathcal{C}^0([0, 1]; L^2(0, T)) \cap \mathcal{C}^1([0, 1]; H_{(0)}^{-1}(0, T))$ be the solution in the sense of transposition of the following wave equation where the role of the time and space variables has been interchanged:

$$\begin{cases} \partial_{xx}\psi - \partial_{tt}\psi - A\psi = 0, & Q_T, \\ \psi(0, \cdot) = 0, \quad \partial_t\psi(T, \cdot) = 0, & \Omega, \\ (\psi(\cdot, 1), \partial_x\psi(\cdot, 1)) = (\varphi(1, \cdot), \partial_x\varphi(1, \cdot)), & (0, T). \end{cases}$$

The estimate (8) applied to ψ becomes

$$\|\psi(\cdot, x)\|_{L^2(0, T)}^2 + \|\partial_x\psi(\cdot, x)\|_{H_{(0)}^{-1}(0, T)}^2 \leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \|(\psi(\cdot, 1), \partial_x\psi(\cdot, 1))\|_{L^2(0, T) \times H_{(0)}^{-1}(0, T)}^2, \quad \forall x \in \Omega.$$

Since $\psi(t, 1) = \varphi(1, t)$ and $\partial_x\psi(t, 1) = \partial_x\varphi(1, t) = 0$ in $[0, T]$, we obtain

$$\|\psi(\cdot, x)\|_{L^2(0, T)}^2 + \|\partial_x\psi(\cdot, x)\|_{H_{(0)}^{-1}(0, T)}^2 \leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \|\varphi(1, \cdot)\|_{L^2(0, T)}^2, \quad \forall x \in \Omega,$$

and we deduce

$$\|\psi\|_{L^2(Q_T)}^2 \leq C |\Omega| e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \|\varphi(1, \cdot)\|_{L^2(0, T)}^2. \quad (17)$$

Moreover, since $\varphi = \psi$ in $\tau(1) := \{(x, t) \in Q_T; t \in (1-x, x+T-1)\} \subset Q_T$, we have

$$\begin{aligned} \|\psi\|_{L^2(Q_T)}^2 &\geq \int_{\tau(1)} |\psi(t, x)|^2 dx dt = \int_{\tau(1)} |\varphi(x, t)|^2 dx dt \\ &= \int_0^1 \int_{1-x}^{x+T-1} |\varphi(x, t)|^2 dx dt \geq \int_0^1 \int_1^{T-1} |\varphi(x, t)|^2 dx dt. \end{aligned} \quad (18)$$

Therefore, using (17) and (18), we deduce the result. \square

Lemma 2. Let $T > 2$. There exists $C = C(\Omega, T) > 0$ such that the solution $\varphi \in \mathcal{C}^0([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; H_{(0)}^{-1}(\Omega))$ in the sense of transposition of (\star) associated with $A \in L^\infty(Q_T)$, $B = 0$ and $(\varphi_0, \varphi_1) \in L^2(\Omega) \times H_{(0)}^{-1}(\Omega)$ satisfies

$$\int_{t_1}^{t_2} \|\partial_t\varphi(\cdot, t)\|_{H_{(0)}^{-1}(\Omega)}^2 dt \leq C (1 + \|A\|_{L^\infty(Q_T)}) \int_1^{T-1} \|\varphi(\cdot, t)\|_{L^2(\Omega)}^2 dt, \quad (19)$$

for any $t_1, t_2 \in (1, T-1)$, $t_1 < t_2$.

Proof. Let $(\varphi_0, \varphi_1) \in (H^2(\Omega) \cap H_{(0)}^1(\Omega)) \times H_{(0)}^1(\Omega)$ and let $\varphi \in \mathcal{C}^0([0, T]; H^2(\Omega) \cap H_{(0)}^1(\Omega)) \cap \mathcal{C}^1([0, T]; H_{(0)}^1(\Omega)) \cap \mathcal{C}^2([0, T]; H_{(0)}^{-1}(\Omega))$ be the solution of (\star) associated with (φ_0, φ_1) . Let $t_1, t_2 \in (1, T-1)$ with $t_1 < t_2$. We take $r \in \mathcal{C}_c^1([1, T-1])$ such that

$$\begin{cases} r(t) = 1 \text{ for all } t \in [t_1, t_2] \\ \left| \frac{r'}{r} \right| \in L^\infty(1, T-1). \end{cases}$$

Multiplying the equation in (\star) by $r(t)(-\partial_{xx})^{-1}\varphi$ and integrating by part on $[1, T-1]$ the term in $\partial_{tt}\varphi$, we obtain

$$\begin{aligned} & - \int_1^{T-1} r'(t) \int_0^1 \partial_t\varphi(x, t) (-\partial_{xx})^{-1}\varphi(x, t) dx dt - \int_1^{T-1} r(t) \int_0^1 \partial_t\varphi(x, t) \partial_t(-\partial_{xx})^{-1}\varphi(x, t) dx dt \\ & - \int_1^{T-1} r(t) \int_0^1 \partial_{xx}\varphi(x, t) (-\partial_{xx})^{-1}\varphi(x, t) dx dt + \int_1^{T-1} \int_0^1 A(x, t)\varphi(x, t) r(t) (-\partial_{xx})^{-1}\varphi(x, t) dx dt = 0. \end{aligned}$$

We can check that, for any $t \in [0, T]$

$$\int_0^1 \partial_t \varphi(x, t) \partial_t (-\partial_{xx})^{-1} \varphi(x, t) dx = \|\partial_t \varphi(\cdot, t)\|_{H_{(0)}^{-1}(\Omega)}^2, \quad \int_0^1 \partial_{xx} \varphi(x, t) (-\partial_{xx})^{-1} \varphi(x, t) dx = \|\varphi(\cdot, t)\|_{L^2(\Omega)}^2.$$

Thus, we obtain

$$\begin{aligned} \int_1^{T-1} r(t) \|\partial_t \varphi(\cdot, t)\|_{H_{(0)}^{-1}(\Omega)}^2 dt &= - \int_1^{T-1} r'(t) \int_0^1 \partial_t \varphi(x, t) (-\partial_{xx})^{-1} \varphi(x, t) dx dt - \int_1^{T-1} r(t) \|\varphi(\cdot, t)\|_{L^2(\Omega)}^2 dt \\ &\quad + \int_1^{T-1} \int_0^1 A(x, t) \varphi(x, t) r(t) (-\partial_{xx})^{-1} \varphi(x, t) dx dt \\ &\leq \int_1^{T-1} \langle \sqrt{r(t)} \partial_t \varphi(\cdot, t), \frac{r'(t)}{\sqrt{r(t)}} (-\partial_{xx})^{-1} \varphi(\cdot, t) \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)} dt \\ &\quad + \|r\|_{L^\infty(1, T-1)} \int_1^{T-1} \|\varphi(\cdot, t)\|_{L^2(\Omega)}^2 dt \\ &\quad + C (1 + \|r\|_{L^\infty(1, T-1)} \|A\|_{L^\infty(Q_T)}) \int_1^{T-1} \|\varphi(\cdot, t)\|_{L^2(\Omega)}^2 dt \\ &\leq \frac{1}{2} \int_1^{T-1} r(t) \|\partial_t \varphi(\cdot, t)\|_{H_{(0)}^{-1}(\Omega)}^2 dt \\ &\quad + C \left(1 + \left\| \frac{r'(t)}{r(t)} \right\|_{L^\infty(1, T-1)} + (1 + \|A\|_{L^\infty(Q_T)}) \|r\|_{L^\infty(1, T-1)} \right) \\ &\quad \quad \quad \times \int_1^{T-1} \|\varphi(\cdot, t)\|_{L^2(\Omega)}^2 dt \end{aligned}$$

1 i.e. we have (19) for all φ regular, where C depends on r . We conclude the result by a density argument. \square

2 We are now able to establish Theorem 1.

Proof of Theorem 1. Let $\varphi \in \left\{ z \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H_{(0)}^{-1}(\Omega)); \partial_{tt} \varphi - \partial_{xx} \varphi + A\varphi \in L^2(Q_T) \right\}$. We decompose φ as $\varphi = \psi_1 + \psi_2$ where ψ_1 and ψ_2 are respectively solution in the sense of transposition and weak solution of

$$\begin{cases} \partial_{tt} \psi_1 - \partial_{xx} \psi_1 + A\psi_1 = 0, & Q_T, \\ \psi_1(0, \cdot) = 0, \quad \partial_x \psi_1(1, \cdot) = 0, & (0, T), \\ (\psi_1(\cdot, 0), \partial_t \psi_1(\cdot, 0)) = (\varphi_0, \varphi_1), & \Omega, \end{cases} \quad \begin{cases} \partial_{tt} \psi_2 - \partial_{xx} \psi_2 + A\psi_2 = \partial_{tt} \varphi - \partial_{xx} \varphi + A\varphi, & Q_T, \\ \psi_2(0, \cdot) = 0, \quad \partial_x \psi_2(1, \cdot) = 0, & (0, T), \\ (\psi_2(\cdot, 0), \partial_t \psi_2(\cdot, 0)) = (0, 0), & \Omega, \end{cases}$$

Using (12) then (19), we have for any $t_1, t_2 \in (1, T-1) \subset [0, T], t_1 < t_2$

$$\begin{aligned} (t_2 - t_1) \|(\varphi_0, \varphi_1)\|_{L^2(\Omega) \times H_{(0)}^{-1}(\Omega)} &\leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \int_{t_1}^{t_2} \|\psi_1(\cdot, t)\|_{L^2(\Omega)} + \|\partial_t \psi_1(\cdot, t)\|_{H_{(0)}^{-1}(\Omega)} dt \\ &\leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \int_1^{T-1} \|\psi_1(\cdot, t)\|_{L^2(\Omega)} dt, \end{aligned}$$

and using (16) and since $\psi_1 = \varphi - \psi_2$, we deduce

$$\begin{aligned} \|(\varphi_0, \varphi_1)\|_{L^2(\Omega) \times H_{(0)}^{-1}(\Omega)} &\leq \frac{1}{t_1 - t_2} C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \|\psi_1(1, \cdot)\|_{L^2(0, T)} \\ &\leq \frac{1}{t_1 - t_2} C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \left(\|\varphi(1, \cdot)\|_{L^2(0, T)} + \|\psi_2(1, \cdot)\|_{L^2(0, T)} \right). \end{aligned}$$

Moreover, since $\psi_2(1, 0) = 0$ and using (5), we have

$$\|\psi_2(1, \cdot)\|_{L^2(0, T)} \leq C \|\partial_t \psi_2(1, \cdot)\|_{L^2(0, T)} \leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \|\partial_{tt} \varphi - \partial_{xx} \varphi + A\varphi\|_{L^2(Q_T)},$$

3 and we deduce the result. \square

4 Application to the controllability of a semi-linear wave equation

This section is devoted to applications of Theorem 1. In particular, as mentioned in the Introduction, one motivation for inequality (Obs) is the exact boundary controllability for the semi-linear wave equation ($\star\star$).

4.1 Controllability of a linear wave equation

In this section, for any $T > 0$, the exact controllability problem associated with (2) is considered : given $(u_0, u_1), (z_0, z_1) \in H_{(0)}^1(\Omega) \times L^2(\Omega)$, we look for the existence of a control function $v \in L^2(0, T)$ such that the associated solution $y \in C^0([0, T]; H_{(0)}^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ of (2) satisfies $(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1)$ in Ω . Moreover, the aim is to get, thanks to estimate (Obs), precise estimates of a particular state-control pair in term of the data.

Let Φ be the space defined by

$$\Phi := \left\{ w \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H_{(0)}^{-1}(\Omega)); w \text{ is the solution of the transposition of } (\star) \right. \\ \left. \text{for some } B \in L^2(Q_T) \right\}.$$

From (Obs), Φ endowed with the scalar product given by

$$(p, q)_\Phi := \int_{Q_T} (\partial_{tt} p - \partial_{xx} p + Ap) (\partial_{tt} q - \partial_{xx} q + Aq) \, dx dt + \int_0^T p(1, t) q(1, t) dt, \quad \forall p, q \in \Phi,$$

is a Hilbert space. The main result of this section is a null-controllability result for the linear system (2).

Theorem 6. *Assume $T > 2$. For $A \in L^\infty(Q_T)$, $B \in L^2(Q_T)$ and $(u_0, u_1) \in H_{(0)}^1(\Omega) \times L^2(\Omega)$, there exists a unique function $p \in \Phi$ solution of*

$$(p, q)_\Phi = \int_{Q_T} Bq \, dx dt + \int_\Omega u_1(x) q(x, 0) \, dx - \langle \partial_t q(\cdot, 0), u_0 \rangle_{H_{(0)}^{-1}(\Omega), H_{(0)}^1(\Omega)}, \quad \forall q \in \Phi. \quad (20)$$

Moreover, $y = \partial_{tt} p - \partial_{xx} p + Ap$ is a controlled trajectory to zero for (2), $v = -p(1, \cdot)$ is the associated control function and there exists a constant $C = C(\Omega, T) > 0$ such that (y, v) satisfies

$$\|y\|_{L^2(Q_T)} + \|v\|_{L^2(0, T)} \leq C e^{C\sqrt{\|A\|_{L^\infty(Q_T)}}} \left(\|B\|_{L^2(Q_T)} + \|(u_0, u_1)\|_{H_{(0)}^1(\Omega) \times L^2(\Omega)} \right). \quad (21)$$

Proof. We refer to [CFCM13, Proposition 2.2 p.6] and [BLM23a, Theorem 6 p.7] where a similar result is obtained in the case of Dirichlet boundary control. \square

Remark 2. *The state-control pair given by Theorem 6 is the unique solution of the following extremal problem*

$$\begin{cases} \text{Minimize } \mathcal{J}(y, v) = \|y\|_{L^2(Q_T)}^2 + \|v\|_{L^2(0, T)}^2 \\ \text{Subject to } (y, v) \in C(u_0, u_1; T) \end{cases} \quad (22)$$

where $C(u_0, u_1; T) := \{(y, v) \in L^2(Q_T) \times L^2(0, T); (y, v) \text{ is solution of (2) with } y(\cdot, T) = \partial_t y(\cdot, T) = 0 \text{ in } \Omega\}$.

4.2 Proof of Theorem 2 by a Schauder fixed-point argument

Proof of Theorem 2. Let $R > 0$. We define the following class

$$\bar{B}_{\|\cdot\|_\infty}(0, R) := \{z \in L^\infty(Q_T); \|z\|_{L^\infty(Q_T)} \leq R\}.$$

1 It suffices to prove that the non-linear operator K defined by

$$K : \bar{B}_{\|\cdot\|_\infty}(0, R) \rightarrow \bar{B}_{\|\cdot\|_\infty}(0, R) \quad (23)$$

$$\xi \mapsto y$$

2 where y is the controlled solution of

$$\begin{cases} \partial_{tt}y - \partial_{xx}y + \widehat{f}(\xi)y = -f(0), & Q_T, \\ y(0, \cdot) = 0, \quad \partial_x y(1, \cdot) = v, & (0, T), \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1), & \Omega, \end{cases} \quad \text{with } \widehat{f}(r) = \begin{cases} \frac{f(r) - f(0)}{r} & \text{if } r \neq 0 \\ f'(0) & \text{if } r = 0 \end{cases} \quad (24)$$

given by Theorem 6, has a fixed-point for some R large enough. The stability of K is a consequence of the observability inequality (Obs). In particular, using (H1), there exists $\gamma > 0$ such that

$$\|\widehat{f}(\xi)\|_{L^\infty(Q_T)} \leq \gamma + \beta \ln^2(1 + \|\xi\|_{L^\infty(Q_T)}),$$

and we deduce, using (4) then (21),

$$\begin{aligned} \|y\|_{L^\infty(Q_T)} &\leq C e^{C\sqrt{\gamma + \beta \ln^2(1 + \|\xi\|_{L^\infty(Q_T)})}} \left(\|f(0)\|_{L^2(Q_T)} + \|(u_0, u_1)\|_{H^1_{(0)}(\Omega) \times L^2(\Omega)} \right) \\ &\leq C e^{C\sqrt{\gamma}} \left(1 + \|\xi\|_{L^\infty(Q_T)} \right)^{C\sqrt{\beta}} \left(\|f(0)\|_{L^2(Q_T)} + \|(u_0, u_1)\|_{H^1_{(0)}(\Omega) \times L^2(\Omega)} \right). \end{aligned}$$

From this estimate, we deduce that, if β is small enough, there exists $R > 0$ such that $\bar{B}_{\|\cdot\|_\infty}(0, R)$ is stable under the map K . The proof for the continuity of K in $\bar{B}_{\|\cdot\|_\infty}(0, R)$ and the fact that $K(\bar{B}_{\|\cdot\|_\infty}(0, R))$ is a relatively compact subset of $\bar{B}_{\|\cdot\|_\infty}(0, R)$ is very closed to [BLM23a, Proof of Proposition 2 p.92 and Proposition 3 p.93]. In particular, the proof is based on the compact embedding (see [Sim87, Corollary 8 p.90])

$$C^0([0, T]; H^1_{(0)}(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \hookrightarrow L^\infty(Q_T).$$

3 We conclude the existence of a fixed-point for K by the Schauder theorem. □

4 **Remark 3.** *Following the blow up argument of [Zua93, Section 4 p.124], we can prove that the exponent 2 in the logarithm in (H1) is optimal.*

5 **Remark 4.** *A priori, the operator K is not a contraction. In particular, we cannot explicitly construct a control using the Banach-Picard theorem. We refer to Section 5.3 where divergence of the sequence*

$$y_0 \in L^\infty(Q_T), \quad y_{k+1} = K(y_k), \quad k \geq 0. \quad (\text{PF1})$$

is observed numerically.

6 4.3 Proof of Theorem 3 by a least-squares approach

7 The motivation of this section is the approximation of exact controls for (**). Recently, a construction,
8 based on a least-squares approach, of convergent sequence have been initially proposed in [LMGM21] for the
9 heat equation and have been then adapting for the wave equation in [MT22] (and also in [BLM23b]) in the
10 internal control case. In particular, this section aims to show that the observability inequality (Obs) allows
11 to extend the result [MT22, Theorem 2 p.8] in our boundary control case.

For any $\alpha \in [0, 1]$, we define the space

$$W_\alpha := \{f \in C^1(\mathbb{R}); [f']_\alpha < +\infty\}, \quad [f']_\alpha := \sup_{a, b \in \mathbb{R}, a \neq b} \frac{|f'(a) - f'(b)|}{|a - b|^\alpha}.$$

12 The functional framework is as follows:

- 1 • We consider the Hilbert space \mathcal{H} defined by

$$\mathcal{H} := \left\{ (y, v) \in L^2(Q_T) \times L^2(0, T); y \in C^0([0, T]; H_{(0)}^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \text{ is the weak solution of (2) for some } B \in L^2(Q_T) \right\}$$

endowed with the scalar product

$$\begin{aligned} \left((y, v), (\bar{y}, \bar{v}) \right)_{\mathcal{H}} &:= \left(\partial_{tt}y - \partial_{xx}y, \partial_{tt}\bar{y} - \partial_{xx}\bar{y} \right)_{L^2(Q_T)} + (y, \bar{y})_{L^2(Q_T)} \\ &\quad + (v, \bar{v})_{L^2(0, T)} + \left((y(\cdot, 0), \partial_t y(\cdot, 0)), (\bar{y}(\cdot, 0), \partial_t \bar{y}(\cdot, 0)) \right)_{H_{(0)}^1(\Omega) \times L^2(\Omega)} \end{aligned}$$

- 2 and the norm $\| \cdot \|_{\mathcal{H}} := \sqrt{(\cdot, \cdot)_{\mathcal{H}}}$.

- We introduce \mathcal{A} and \mathcal{A}_0 the closed subspaces of \mathcal{H} defined by

$$\begin{aligned} \mathcal{A} &:= \left\{ (y, v) \in \mathcal{H}; (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1), (y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1) \right\}, \\ \mathcal{A}_0 &:= \left\{ (y, v) \in \mathcal{H}; (y(\cdot, 0), \partial_t y(\cdot, 0)) = (0, 0), (y(\cdot, T), \partial_t y(\cdot, T)) = (0, 0) \right\}. \end{aligned}$$

- 3 We assume (H2). In particular, using (4) and since $H^1(\Omega) \hookrightarrow C^0(\bar{\Omega})$, for any $(y, v) \in \mathcal{A}$, $y \in L^\infty(Q_T)$ and thus $f(y) \in L^2(Q_T)$. We then consider the following non convex (well-defined) extremal problem :

$$\boxed{\min_{(y, v) \in \mathcal{A}} E(y, v), \quad E(y, v) = \frac{1}{2} \|\partial_{tt}y - \partial_{xx}y + f(y)\|_{L^2(Q_T)}^2} \quad (25)$$

- 5 Remark that the infimum of E is reached and is equal to 0 since any controlled solution of $(\star\star)$, with its associated control, is a zero of E . Conversely, any zero of E is a state-control pair of $(\star\star)$. We have the following property for E :

Proposition 2. *Let $T > 2$. For any $(y, v) \in \mathcal{A}$, there exists a constant $C = C(T, \Omega) > 0$ such that*

$$\sqrt{E(y, v)} \leq C e^C \sqrt{\|f'(y)\|_{L^\infty(Q_T)}} \|E'(y, f)\|_{\mathcal{A}'_0}, \quad (26)$$

8 where \mathcal{A}'_0 is the topological dual^a of \mathcal{A}_0 .

^aendowed with the norm $\|E'(y, v)\|_{\mathcal{A}'_0} := \sup_{(Y, V) \in \mathcal{A}_0 \setminus \{0\}} \frac{E'(y, v) \cdot (Y, V)}{\|(Y, V)\|_{\mathcal{H}}}$.

- 9 *Proof.* For details, we refer to [MT22, Proposition 1, (iii) p.5]. □

We deduce that any critical point $(y, v) \in \mathcal{A}$ of E is a zero of E . In particular, any minimizing sequence $(y_k, v_k)_{k \in \mathbb{N}} \subset \mathcal{A}$ of E such that $\|f'(y_k)\|_{L^\infty(Q_T)}$ is uniformly bounded with respect to $k \in \mathbb{N}$ converges to a global minimum of E , and thus converges to a state-control pair for $(\star\star)$. Remark that, as in [MT22, Proposition 1, (ii) p.5], for any $(y, v) \in \mathcal{A}$

$$E'(y, v) \cdot (Y, V) = 2E(y, v),$$

- 10 where (Y, V) is the solution of

$$\begin{cases} \partial_{tt}Y_k - \partial_{xx}Y_k + f'(y_k)Y_k = \partial_{tt}y_k - \partial_{xx}y_k + f(y_k), & Q_T, \\ Y_k(0, \cdot) = 0, \quad \partial_x Y_k(1, \cdot) = V_k, & (0, T), \\ \left(Y_k(\cdot, 0), \partial_t Y_k(\cdot, 0) \right) = (0, 0), & \Omega, \end{cases} \quad (27)$$

1 associated with (y, v) . Thus, $-(Y, V)$ is a descent direction for E . This leads us to consider the sequence
 2 $(y_k, v_k)_{k \in \mathbb{N}}$ in \mathcal{A} defined by

$$\begin{cases} (y_0, v_0) \in \mathcal{A} \\ (y_{k+1}, v_{k+1}) = (y_k, v_k) - \lambda_k(Y_k, V_k) \\ \lambda_k = \operatorname{argmin}_{\lambda \in [0,1]} E((y_k, v_k) - \lambda(Y_k, V_k)) \end{cases} \quad (\text{LS})$$

3 where $(Y_k, V_k) \in \mathcal{A}_0$ is the solution of (27) satisfying the extremal problem (22).

4 The main result of this section is a convergence of the least-squares algorithm (LS):

Theorem 7. *Assume that $f \in W_\alpha$, for some $\alpha \in (0, 1]$, and that f' satisfies (H2) with β^* small enough. Let $(y_k, v_k)_{k \in \mathbb{N}}$ be the sequence defined by (LS). Then,*

- The sequence $(E(y_k, v_k))_{k \in \mathbb{N}}$ decays to zero as $k \rightarrow \infty$.
- The sequence $(\lambda_k)_{k \in \mathbb{N}}$ converges to 1 as $k \rightarrow \infty$.
- The sequence $(y_k, v_k)_{k \in \mathbb{N}}$ strongly converges to a state-control pair (\bar{y}, \bar{v}) of $(\star\star)$.

Moreover, the convergence of all these sequences is first at least linear and then at least with order $1 + \alpha$ after a finite number of iterations.

In the case $\alpha = 0$, the result is still true if we assume moreover that $\|f'\|_{L^\infty(\mathbb{R})}$ is small enough.

6 *Proof.* The calculations differ only slightly from [MT22, Section 3]. In the same way, the key point in the proof
 7 is the uniformly bounded character of the sequence $(\|y_k\|_{L^\infty(Q_T)})_{k \in \mathbb{N}}$. The above property can be proved by
 8 induction using (H2) (see [BLM23b, Proof of Theorem 2 p.13]) and allows us to keep a uniform bound of the
 9 sequence of observability constants $(Ce^{C\sqrt{\|f'(y_k)\|_{L^\infty(Q_T)}}})_{k \in \mathbb{N}}$ appearing in particular in Proposition 2. \square

Remark 5. *We can remove the assumption (H2), leading to a local controllability result: If $\|(u_0, u_1)\|_{H^1_{(0)}(\Omega) \times L^2(\Omega)}$ is small enough then the sequence $(y_k, v_k)_{k \in \mathbb{N}}$ defined by (LS) strongly converges to a state-control pair for $(\star\star)$. For details, we refer to [BLM23b, Proposition 5 p.15].*

11 5 Numerical simulations

12 5.1 Least-squares algorithm (LS)

13 In order to consider a control vanishing in time $t = 0$ and $t = T$, we introduce a cut-off function $\eta \in C_c^1([0, T])$
 14 and then we consider the state-control pairs in $L^2(Q_T) \times L^2_\eta(0, T)$ where $L^2_\eta(0, T) := \left\{ v; \|\eta^{-1}v\|_{L^2(0, T)} < +\infty \right\}$.
 15 The least-squares algorithm is therefore given in Algorithm 1.

Lemma 3. *Let (y_0, v_0) the unique controlled solution of $(\star\star)$ with $f = 0$ minimizing \mathcal{J} defined by (22). Then, $E(y_{k+1}, v_{k+1})$ is expressed explicitly in terms of $f(y_{k'})$ and $f'(y_{k'})$, $0 \leq k' \leq k$:*

$$E(y_{k+1}, v_{k+1}) = \frac{1}{2} \left\| \left(\prod_{i=0}^k (1 - \lambda_i) \right) f(y_0) + \left(\sum_{j=0}^{k-1} G_j(\lambda_j) \prod_{i=j+1}^k (1 - \lambda_i) \right) + G_k(\lambda_k) \right\|_{L^2(Q_T)}^2 \quad (30)$$

where

$$G_k(\lambda) = f(y_k - \lambda Y_k^1) - f(y_k) + \lambda f'(y_k) Y_k^1.$$

¹by the trichotomy method

Algorithm 1: Least-squares algorithm

Initialization

- Compute $(y_0, v_0) \in \mathcal{A}$ solution of

$$\begin{cases} \partial_{tt}y_0 - \partial_{xx}y_0 = 0, & Q_T, \\ y_0(0, \cdot) = 0, \quad \partial_x y_0(1, \cdot) = v_0, & (0, T), \\ (y_0(\cdot, 0), \partial_t y_0(\cdot, 0)) = (u_0, u_1), & \Omega, \\ (y_0(\cdot, T), \partial_t y_0(\cdot, T)) = (0, 0), & \Omega, \end{cases} \quad (28)$$

where v_0 minimizes the functional

$$\mathcal{J}(y, v) := \left(\|y\|_{L^2(Q_T)}^2 + \|v\|_{L^2_\eta(0, T)}^2 \right).$$

- Compute $E(y_0, v_0) = \frac{1}{2} \|\partial_{tt}y_0 - \partial_{xx}y_0 + f(y_0)\|_{L^2(Q_T)}^2$.

while $\sqrt{2E(y_k, v_k)} > 10^{-5}$ **do**

- Compute the optimal direction $(Y_k^1, V_k^1) \in \mathcal{A}_0$ solution of

$$\begin{cases} \partial_{tt}Y_k^1 - \partial_{xx}Y_k^1 + f'(y_k)Y_k^1 = \partial_{tt}y_k - \partial_{xx}y_k + f(y_k), & Q_T, \\ Y_k^1(0, \cdot) = 0, \quad \partial_x Y_k^1(1, \cdot) = V_k^1, & (0, T), \\ (Y_k^1(\cdot, 0), \partial_t Y_k^1(\cdot, 0)) = (0, 0), & \Omega, \\ (Y_k^1(\cdot, T), \partial_t Y_k^1(\cdot, T)) = (0, 0), & \Omega, \end{cases} \quad (29)$$

where V_k^1 minimizes the functional \mathcal{J} .

- Compute the optimal descent step¹

$$\begin{aligned} \lambda_k &= \operatorname{argmin}_{\lambda \in [0, 1]} E(y_k - \lambda Y_k^1, v_k - \lambda V_k^1) \\ &= \operatorname{argmin}_{\lambda \in [0, 1]} \|(1 - \lambda)(\partial_{tt}y_k - \partial_{xx}y_k + f(y_k)) + G_k(\lambda)\|_{L^2(Q_T)}^2 \end{aligned}$$

where $G_k(\lambda) = f(y_k - \lambda Y_k^1) - f(y_k) + \lambda f'(y_k)Y_k^1$.

- Update $(y_{k+1}, v_{k+1}) = (y_k - \lambda_k Y_k^1, v_k - \lambda_k V_k^1)$
- Compute $E(y_{k+1}, v_{k+1}) = \frac{1}{2} \|\partial_{tt}y_{k+1} - \partial_{xx}y_{k+1} + f(y_{k+1})\|_{L^2(Q_T)}^2$.
- Do $k = k + 1$

end

1 *Proof.* By induction, let us prove that

$$\partial_{tt}y_{k+1} - \partial_{xx}y_{k+1} + f(y_{k+1}) = \left(\prod_{i=0}^k (1 - \lambda_i) \right) f(y_0) + \left(\sum_{j=0}^{k-1} G_j(\lambda_j) \prod_{i=j+1}^k (1 - \lambda_i) \right) + G_k(\lambda_k). \quad (31)$$

For $k = 0$, we have

$$\begin{aligned} \partial_{tt}y_1 - \partial_{xx}y_1 + f(y_1) &= \partial_{tt}(y_0 - \lambda_0 Y_0) - \partial_{xx}(y_0 - \lambda_0 Y_0) + f(y_0 - \lambda_0 Y_0) \\ &= \partial_{tt}y_0 - \partial_{xx}y_0 + f(y_0) - \lambda_0 (\partial_{tt}Y_0 - \partial_{xx}Y_0 + f'(y_0)Y_0) + G_0(\lambda_0) \\ &= f(y_0) - \lambda_0 (\partial_{tt}y_0 - \partial_{xx}y_0 + f(y_0)) + G_0(\lambda_0) \\ &= (1 - \lambda_0)f(y_0) + G_0(\lambda_0). \end{aligned}$$

Assume (30) for some $k \in \mathbb{N}$. Then,

$$\begin{aligned} \partial_{tt}y_{k+1} - \partial_{xx}y_{k+1} + f(y_{k+1}) &= \partial_{tt}(y_k - \lambda_k Y_k) - \partial_{xx}(y_k - \lambda_k Y_k) + f(y_k - \lambda_k Y_k) \\ &= \partial_{tt}y_k - \partial_{xx}y_k + f(y_k) - \lambda_k (\partial_{tt}Y_k - \partial_{xx}Y_k + f'(y_k)Y_k) + G_k(\lambda_k) \\ &= \partial_{tt}y_k - \partial_{xx}y_k + f(y_k) - \lambda_k (\partial_{tt}y_k - \partial_{xx}y_k + f(y_k)) + G_k(\lambda_k) \\ &= (1 - \lambda_k) (\partial_{tt}y_k - \partial_{xx}y_k + f(y_k)) + G_k(\lambda_k) \\ &= (1 - \lambda_k) \left(\left(\prod_{i=0}^{k-1} (1 - \lambda_i) \right) f(y_0) + \left(\sum_{j=0}^{k-2} G_j(\lambda_j) \prod_{i=j+1}^{k-1} (1 - \lambda_i) \right) + G_{k-1}(\lambda_{k-1}) \right) + G_k(\lambda_k) \\ &= \left(\prod_{i=0}^k (1 - \lambda_i) \right) f(y_0) + \left(\sum_{j=0}^{k-2} G_j(\lambda_j) \prod_{i=j+1}^k (1 - \lambda_i) \right) + (1 - \lambda_k)G_{k-1}(\lambda_{k-1}) + G_k(\lambda_k) \\ &= \left(\prod_{i=0}^k (1 - \lambda_i) \right) f(y_0) + \left(\sum_{j=0}^{k-1} G_j(\lambda_j) \prod_{i=j+1}^k (1 - \lambda_i) \right) + G_k(\lambda_k). \end{aligned}$$

2 Therefore (31) is true for any $n \in \mathbb{N}$. Since $E(y_k, v_k) = \|\partial_{tt}y_k - \partial_{xx}y_k + f(y_k)\|_{L^2(Q_T)}^2$ the result follows. \square

3 **Remark 6.** Lemma 3 avoids second differentiation in time and space in the evaluation of $E(y_{k+1}, v_{k+1})$, for all $k \in \mathbb{N}$.

4 5.2 Discretization by a conformal space-time finite elements method

To compute (y_0, v_0) and (Y_k, V_k) for any $k \in \mathbb{N}$, we discretize the variational formulation (20) by using a conformal space-time finite elements method, we refer to [BLM23a, Section 5.1] and [CM15]. We introduce a triangulation \mathcal{T}_h of Q_T such that $\overline{Q_T} = \cup_{K \in \mathcal{T}_h} K$. We assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family. We approximate the variable p by the variable p_h in the finite-dimensional space

$$P_h := \left\{ p_h \in \mathcal{C}^1(\overline{Q_T}); p_h|_K \in \mathbb{P}(K) \text{ for all } k \in \mathcal{T}_h \right\} \subset \Phi$$

where $\mathbb{P}(K)$ denotes the reduced Hsieh-Clough-Tocher (HCT) \mathcal{C}^1 -finite element (see [Cia02, p. 340]) and the controlled solution y are approximated by y_h in the finite-dimensional space

$$Q_h := \left\{ q_h \in \mathcal{C}^0(\overline{Q_T}); q_h|_K \in \mathbb{Q}(K) \text{ for all } k \in \mathcal{T}_h \right\} \subset L^2(Q_T)$$

5 where $\mathbb{Q}(K)$ denotes the space of polynomials of degree one.

1 5.3 Experiments

We use a regular space-time mesh composed of 60000 triangles corresponding to the discretization parameter $h \approx 1.414 \times 10^{-2}$ and we denote by

$$k^* := \min_{k \geq 0} \left\{ \sqrt{2E(y_k, v_k)} > 10^{-5} \right\} + 1$$

2 the number of iterations obtained when the least-squares algorithm stops. We set the controllability time
3 equal to $T = 3$, the final data to $(z_0, z_1) = (0, 0)$ and we define the following non-linear function f

$$f(r) = c_f r \ln^2(1 + |r|), \quad \forall r \in \mathbb{R}, \quad (32)$$

for $c_f \in \mathbb{R}$. In particular, $f \in W_\alpha$ with $\alpha = 1$ and f' satisfies (H2). Note that the unfavorable case in which the norm of the corresponding uncontrolled solution of $(\star\star)$ grows corresponds to negative values of the parameter c_f . Finally, we define the following cut-off function

$$\eta(t) = \frac{e^{-\frac{1}{2(t+10^{-6})}} e^{-\frac{1}{2(T-t+10^{-6})}}}{e^{-\frac{1}{T+10^{-6}}} e^{-\frac{1}{T+10^{-6}}}}.$$

4 **I. Experiments with the initial state $u_0(x) = c_{u_0}(\cos(\pi x) - 1)$, $c_{u_0} \in \mathbb{R}$, and $u_1(x) = 0$ in Ω .**

5 **I-a. We fix (c_f, c_{u_0}) .** We compute the sequence $(y_k, v_k)_{k \in \mathbb{N}}$ given by the least-squares algorithm as-
6 sociated with the fixed parameters $c_f = -1$ and $c_{u_0} = 20$. The convergence of the algorithm is observed
7 after $k^* = 13$ iterations. Table 1 collects some numerical values with respect to the iterations k . Figure
8 1 represents the evolution of the error $\sqrt{E(y_k, v_k)}$ as well as the optimal steps λ_k with respect to k . As
9 expected, the sequence $(\lambda_k)_{k \in \mathbb{N}^*}$ converges to 1 and we observe the change in the convergence speed after a
10 finite number of iterations: first, the optimal step is close to zero and the error decreases linearly and then,
11 after 10 iterations, the optimal step reaches 1 while the error decreases quadratically. Finally, the controlled
12 solution y_{k^*} obtained is shown in Figure 2 as well as its associated control v_{k^*} .

k	$\sqrt{2E(y_k, v_k)}$	λ_k	$\ y_k\ _{L^2(Q_T)}$	$\ v_k\ _{L^2(0,T)}$	$\frac{\ y_k - y_{k-1}\ _{L^2(Q_T)}}{\ y_{k-1}\ _{L^2(Q_T)}}$	$\frac{\ v_k - v_{k-1}\ _{L^2(0,T)}}{\ v_{k-1}\ _{L^2(0,T)}}$
0	1.915×10^2	—	1.882×10^1	2.873×10^1	—	—
1	1.798×10^2	1.067×10^{-1}	2.125×10^1	2.471×10^1	4.926×10^{-1}	1.491
2	1.617×10^2	1.068×10^{-1}	2.714×10^1	6.360×10^1	5.618×10^{-1}	1.932
3	1.474×10^2	1.497×10^{-1}	3.476×10^1	1.219×10^2	4.566×10^{-1}	1.004
4	1.320×10^2	1.481×10^{-1}	4.296×10^1	1.755×10^2	3.525×10^{-1}	4.936×10^{-1}
5	1.195×10^2	1.480×10^{-1}	5.177×10^1	2.173×10^2	2.806×10^{-1}	3.298×10^{-1}
6	1.077×10^2	1.658×10^{-1}	6.125×10^1	2.573×10^2	2.357×10^{-1}	3.005×10^{-1}
7	9.502×10^1	2.039×10^{-1}	7.161×10^1	3.043×10^2	2.082×10^{-1}	3.048×10^{-1}
8	8.030×10^1	2.721×10^{-1}	8.300×10^1	3.659×10^2	1.900×10^{-1}	3.129×10^{-1}
9	6.191×10^1	4.013×10^{-1}	9.547×10^1	4.473×10^2	1.764×10^{-1}	3.136×10^{-1}
10	3.638×10^1	6.966×10^{-1}	1.087×10^2	5.489×10^2	1.614×10^{-1}	2.977×10^{-1}
11	4.818	1	1.145×10^2	6.001×10^2	6.512×10^{-2}	1.235×10^{-1}
12	3.146×10^{-3}	1	1.145×10^2	6.003×10^2	1.156×10^{-3}	2.428×10^{-3}
13	2.409×10^{-8}	1	1.145×10^2	6.003×10^2	2.826×10^{-7}	6.473×10^{-7}

Table 1: $c_{u_0} = 20$ and $c_f = -1$. Some norms with respect to the iterations k .

13 **I-b. We fix c_{u_0} and we consider several values for c_f .** We fix $c_{u_0} = 20$ and we compute $(y_k, v_k)_{k \in \mathbb{N}}$
14 for several values of c_f . Table 2 collects the results. Remark that the algorithm fails to converge when c_f
15 is large which is in agreement with our theoretical result that the constant c_f should be small enough. As

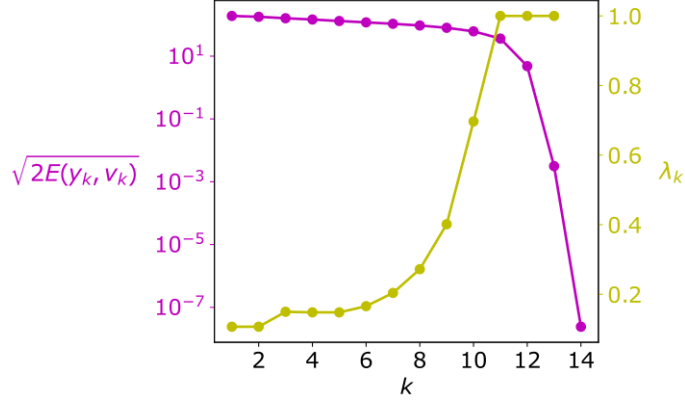


Figure 1: $c_{u_0} = 20$ and $c_f = -1$. Evolution of $\sqrt{2E(y_k, v_k)}$ and λ_k with respect to the iterations k .

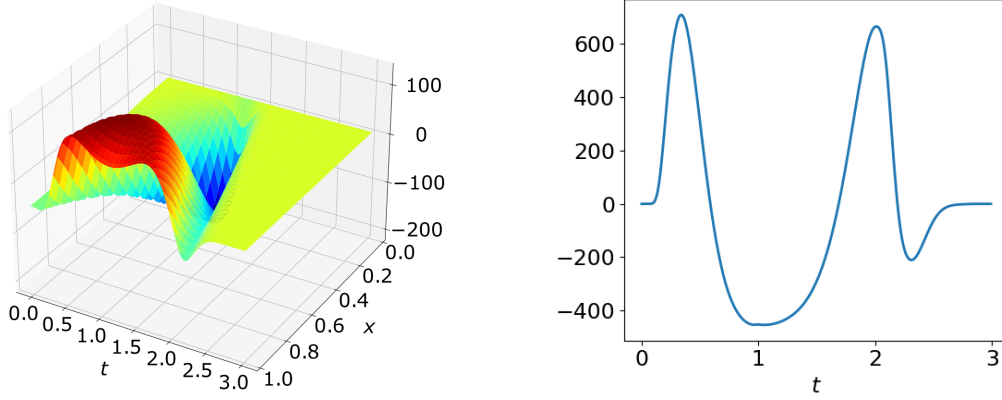


Figure 2: $c_{u_0} = 20$ and $c_f = -1$. **Left:** Representation of the semi-linear controlled solution y_{k^*} in the space-time domain Q_T . **Right:** Representation of the control v_{k^*} on $(0, T)$.

- 1 expected, the $\|\cdot\|_{L^2(Q_T)}$ -norm of y_{k^*} increases (and thus also the $\|\cdot\|_{L^2(0,T)}$ -norm of v_{k^*}) with the absolute
- 2 value of c_f . In particular, due to the non-linearity, for a given $|c_f|$, much more iterations are required in the
- 3 unfavorable case ($c_f < 0$) than in the favorable case ($c_f > 0$).

c_f	$\sqrt{2E(y_{k^*}, v_{k^*})}$	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\frac{\ y_{k^*} - y_{k^*-1}\ _{L^2(Q_T)}}{\ y_{k^*-1}\ _{L^2(Q_T)}}$	$\frac{\ v_{k^*} - v_{k^*-1}\ _{L^2(0,T)}}{\ v_{k^*-1}\ _{L^2(0,T)}}$	k^*
10	5.461×10^{-9}	3.635×10^1	1.547×10^3	4.805×10^{-7}	6.015×10^{-7}	27
5	1.726×10^{-9}	2.478×10^1	3.449×10^2	9.589×10^{-8}	1.654×10^{-8}	9
2	5.754×10^{-10}	2.025×10^1	8.554×10^1	4.392×10^{-8}	2.470×10^{-7}	7
1	2.512×10^{-8}	2.099×10^1	5.895×10^1	5.857×10^{-6}	2.399×10^{-5}	5
-0.5	3.656×10^{-6}	4.195×10^1	1.355×10^2	9.689×10^{-5}	2.928×10^{-4}	5
-1	2.409×10^{-8}	1.145×10^2	6.003×10^2	2.826×10^{-7}	6.473×10^{-7}	13
-1.5	3.314×10^{-8}	3.332×10^2	2.541×10^3	4.538×10^{-8}	1.635×10^{-7}	40
-2	1.217×10^{-9}	9.982×10^2	1.110×10^4	5.914×10^{-10}	1.408×10^{-9}	143

Table 2: $c_{u_0} = 20$. Some norms with respect to the parameter c_f .

1 **I-c. We fix c_f and we consider several values of c_{u_0} .** In this case, we fix $c_f = -1$ and we compute
2 $(y_k, v_k)_{k \in \mathbb{N}}$ for several values of c_{u_0} . Table 3 collects the results. We observe that the norm of the control
3 and the controlled solution increase with $|c_{u_0}|$. Moreover, as expected, the algorithm converges even for large
4 values of c_{u_0} .

c_{u_0}	$\sqrt{2E(y_{k^*}, v_{k^*})}$	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\frac{\ y_{k^*} - y_{k^*-1}\ _{L^2(Q_T)}}{\ y_{k^*-1}\ _{L^2(Q_T)}}$	$\frac{\ v_{k^*} - v_{k^*-1}\ _{L^2(0,T)}}{\ v_{k^*-1}\ _{L^2(0,T)}}$	k^*
1	7.081×10^{-6}	1.005	1.326	3.582×10^{-3}	1.150×10^{-2}	2
50	6.893×10^{-10}	8.640×10^2	6.700×10^3	4.838×10^{-10}	1.670×10^{-9}	31
100	9.124×10^{-11}	4.128×10^3	4.399×10^4	2.358×10^{-11}	4.319×10^{-11}	62
500	1.865×10^{-8}	1.891×10^5	2.974×10^6	8.448×10^{-11}	1.034×10^{-10}	444

Table 3: $c_f = -1$. Some norms with respect to the parameter c_{u_0} .

5 **I-d. Influence of the non-linearity f .** In this case, we fix $c_{u_0} = 1$. For $c_f \in \{-1, -2, -4\}$, Figure
6 3 represents, as a function of time, the $L^2(\Omega)$ -norm of the uncontrolled solution $y^*(\cdot, t)$, the $L^2(\Omega)$ -norm of
7 the linear controlled solution $y_0(\cdot, t)$ (used to initialize the algorithm) and the $L^2(\Omega)$ -norm of the controlled
8 solution $y_{k^*}(\cdot, t)$ obtained by the least-squares algorithm. The linear control v_0 associated to y_0 and the
9 control v_{k^*} associated with y_{k^*} are also depicted.

- 10 • For $c_f = -1$. The uncontrolled solution y^* oscillates and is bounded. The convergence of the least-
11 squares algorithm is quadratic and is observed after $k^* = 2$ iterations. The dynamic of the initial
12 state-control pair (y_0, v_0) and the final state-control pair (y_{k^*}, v_{k^*}) are similar: the non-linearity f
13 therefore has a reduced impact.
- 14 • For $c_f = -2$. The uncontrolled solution y^* oscillates more than the previous case and is still bounded.
15 The convergence of the least-squares algorithm is again quadratic and is observed after $k^* = 3$ iterations.
16 The initial and final dynamics are still similar.
- 17 • For $c_f = -4$. In this case, the uncontrolled solution grows exponentially with respect to the time
18 variable. The algorithm converges again with $k^* = 5$. The non-linearity f has a strong impact: the
19 controls v_0 and v_{k^*} no longer match. In particular, the final control v_{k^*} acts very strongly at the
20 beginning to balance the exponential growth of the uncontrolled solution.

21 As expected, there is a large gap between the initial control v_0 and the final control v_{k^*} as $|c_f|$ increases.

22 II. Comparison with other algorithms.

23 **II-a. Newton algorithm.** When $\lambda_k = 1$ for all $k \in \mathbb{N}$, the least-squares algorithm given by (LS)
24 coincides with the Newton algorithm (N) associated with $F : \mathcal{A} \rightarrow L^2(Q_T)$, $F(y, v) := \partial_{tt}y - \partial_{xx}y + f(y)$.
25 In particular, this explains the super-linear convergence property obtained in Theorem 7 and numerically
26 illustrated in the first case (see Figure 1).

27 We consider $u_0 = 20(\cos(\pi x) - 1)$ and $u_1 = 0$. For several values of the parameter c_f , we compute
28 $(y_{k^*}, v_{k^*})_{k \in \mathbb{N}}$ with $\lambda_k = 1$ for all $k \in \mathbb{N}$. Table 4 collects some norms with respect to k . With this set of data,
29 we observe that the Newton method converges in fewer iterations than least-squares algorithm (see Table 4
30 and Table 2).

31 Now, we fix the parameter $c_f = -2$ and we consider the initial state $(u_0, u_1) = \left(50(\cos(\pi x) - 1), 100 \cdot \mathbf{1}_{(\frac{1}{2}, 1]}\right)$.
32 In particular, we observe that the Newton algorithm (when $\lambda_k = 1$ for all $k \in \mathbb{N}$) diverges: the norms of the
33 solution and control are blowing up as number of iterations are increasing. Remark that the least-squares
34 algorithm converges in $k^* = 673$ iterations.

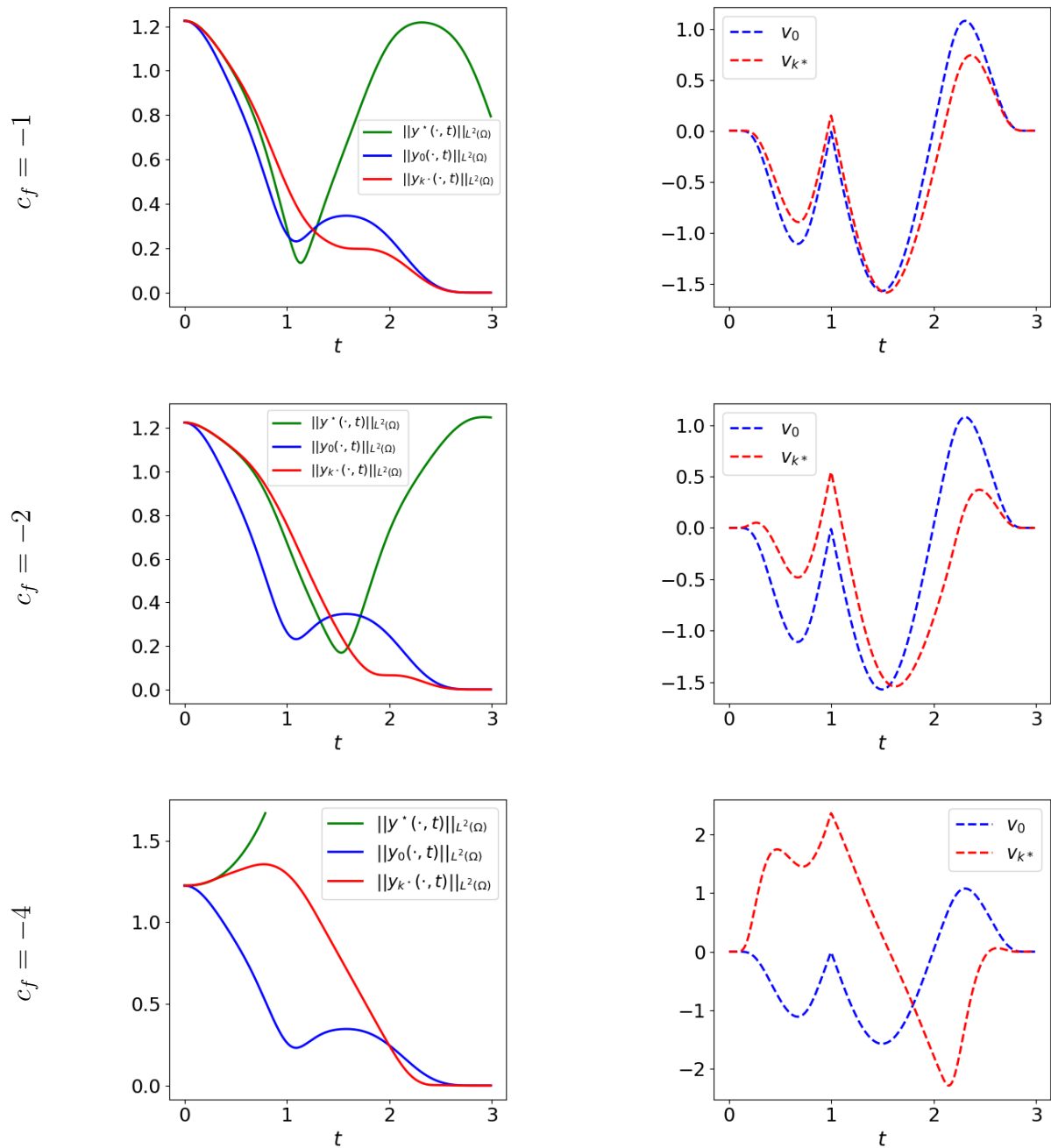


Figure 3: $c_{u_0} = 1$. **Right:** Representation of the $\|\cdot\|_{L^2(\Omega)}$ -norm of the uncontrolled solution $y^*(\cdot, t)$, the linear controlled solution $y_0(\cdot, t)$ used to the initialization and the solution y_{k^*} obtained by the least-squares algorithm with respect to t , for $c_f \in \{-1, -2, -4\}$. **Left:** Representation of the linear control v_0 associated with y_0 and the control v_{k^*} associated with y_{k^*} with respect to t , for $c_f \in \{-1, -2, -4\}$.

II-b. Fixed-point methods. We consider

$$u_0(x) = 100(x - \frac{1}{2}) \cdot \mathbf{1}_{(\frac{1}{2}, 1]}, \quad u_1(x) = 100 \cdot \mathbf{1}_{(\frac{1}{2}, 1]}.$$

c_f	$\sqrt{2E}(y_{k^*}, v_{k^*})$	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\frac{\ y_{k^*} - y_{k^*-1}\ _{L^2(Q_T)}}{\ y_{k^*-1}\ _{L^2(Q_T)}}$	$\frac{\ v_{k^*} - v_{k^*-1}\ _{L^2(0,T)}}{\ v_{k^*-1}\ _{L^2(0,T)}}$	k^*
10	1.209×10^{-9}	4.062×10^1	1.883×10^3	3.655×10^{-7}	3.869×10^{-7}	11
5	1.881×10^{-11}	3.447×10^1	8.185×10^2	7.377×10^{-8}	4.675×10^{-8}	8
2	2.549×10^{-8}	2.100×10^1	9.062×10^1	8.275×10^{-6}	3.373×10^{-5}	6
1	6.094×10^{-10}	2.087×10^1	5.947×10^1	1.891×10^{-6}	1.130×10^{-5}	5
-0.5	4.455×10^{-6}	4.381×10^1	1.418×10^2	1.281×10^{-4}	2.377×10^{-4}	4
-1	4.011×10^{-6}	1.118×10^2	5.879×10^2	4.144×10^{-5}	1.074×10^{-4}	5
-1.5	2.321×10^{-12}	3.298×10^2	2.590×10^3	1.967×10^{-9}	7.702×10^{-9}	6
-2	9.727×10^{-12}	9.451×10^2	1.083×10^4	3.816×10^{-10}	1.076×10^{-9}	7

Table 4: $c_{u_0} = 20$ and $\lambda_k = 1$ for all $k \in \mathbb{N}$. Some norms of the solution y_{k^*} obtained with the least-squares algorithm as well as the associated control v_{k^*} with respect to the parameter c_f .

- 1 For several values of c_f , we compare the least-squares algorithm with two fixed-point methods. The first
2 one is the fixed-point method associated with the operator K defined by (23) in Section 4.2. This leads to
3 the algorithm (PF1). The second one is associated with the operator $\tilde{K} : L^\infty(Q_T) \rightarrow L^\infty(Q_T)$ defined by
4 $y = \tilde{K}(\xi)$ where y is a controlled solution of

$$\begin{cases} \partial_{tt}y - \partial_{xx}y = -f(\xi), & Q_T, \\ y(0, \cdot) = 0, \quad \partial_x y(1, \cdot) = v, & (0, T), \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1), & \Omega, \end{cases} \quad (33)$$

- 5 given by Theorem 6. This leads to the following algorithm:

$$y_0 \in L^\infty(Q_T), \quad y_{k+1} = \tilde{K}(y_k), \quad k \geq 0. \quad (\text{PF2})$$

- 6 Table 5, Table 6 and Table 7 collect some norms of the sequence $(y_k, v_k)_{k \in \mathbb{N}}$ computed by the three algorithms
7 (LS), (PF1) and (PF2) for respectively $c_f = -0.5$, $c_f = -1$ and $c_f = -2$. Figure 4 represents the evolution
8 of $\sqrt{2E}(y_k, v_k)$ with respect to the iterations k . In particular, (PF1) and (PF2) do not usually converge, but
9 if they do, convergence is linear.

	$\sqrt{2E}(y_k, v_k)$	$\ y_k\ _{L^2(Q_T)}$	$\ v_k\ _{L^2(0,T)}$	$\frac{\ y_k - y_{k-1}\ _{L^2(Q_T)}}{\ y_{k-1}\ _{L^2(Q_T)}}$	$\frac{\ v_k - v_{k-1}\ _{L^2(0,T)}}{\ v_{k-1}\ _{L^2(0,T)}}$	k^*
(LS)	2.305×10^{-8}	8.394×10^1	3.800×10^2	2.457×10^{-7}	1.446×10^{-6}	5
(PF1)	7.941×10^{-6}	9.614×10^1	4.227×10^2	2.228×10^{-8}	1.878×10^{-8}	16
(PF2)	9.938×10^{-6}	2.338×10^2	1.027×10^3	2.244×10^{-9}	3.254×10^{-9}	79

Table 5: $c_f = -0.5$. Some norms obtained by algorithms (LS), (PF1) and (PF2).

	$\sqrt{2E}(y_k, v_k)$	$\ y_k\ _{L^2(Q_T)}$	$\ v_k\ _{L^2(0,T)}$	$\frac{\ y_k - y_{k-1}\ _{L^2(Q_T)}}{\ y_{k-1}\ _{L^2(Q_T)}}$	$\frac{\ v_k - v_{k-1}\ _{L^2(0,T)}}{\ v_{k-1}\ _{L^2(0,T)}}$	k^*
(LS)	1.518×10^{-8}	2.243×10^2	1.625×10^3	1.344×10^{-7}	2.183×10^{-7}	10
(PF1)	3.917×10^{-6}	2.623×10^2	1.931×10^3	1.478×10^{-9}	6.534×10^{-10}	28
(PF2)	—	—	—	—	—	$+\infty$

Table 6: $c_f = -1$. Some norms obtained by algorithms (LS) and (PF1).

10 6 Conclusion and perspectives

- 11 Following [Zua93], we give a generalized observability inequality with a constant expressed as an exponential
12 of the potential. Assuming an optimal growth assumption at infinity on the non-linearity of the type $s \ln^2 s$,

	$\sqrt{2E(y_k, v_k)}$	$\ y_k\ _{L^2(Q_T)}$	$\ v_k\ _{L^2(0,T)}$	$\frac{\ y_k - y_{k-1}\ _{L^2(Q_T)}}{\ y_{k-1}\ _{L^2(Q_T)}}$	$\frac{\ v_k - v_{k-1}\ _{L^2(0,T)}}{\ v_{k-1}\ _{L^2(0,T)}}$	k^*
(LS)	2.482×10^{-8}	1.408×10^3	1.548×10^4	3.769×10^{-8}	5.090×10^{-8}	76
(PF1)	—	—	—	—	—	$+\infty$
(PF2)	—	—	—	—	—	$+\infty$

Table 7: $c_f = -2$. Some norms obtained by algorithms (LS).

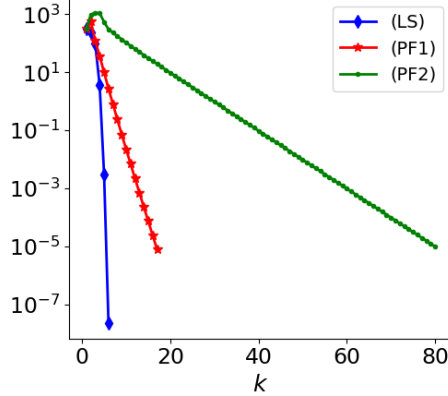


Figure 4: $c_f = -0.5$. Evolution of $\sqrt{2E(y_k, v_k)}$ for (LS), (PF1) and (PF2) with respect to the iterations k .

1 this leads to the existence of a control $v \in L^2(0, T)$ steering the semi-linear system $(\star\star)$ from an initial state
2 $(u_0, u_1) \in H^1_{(0)}(\Omega) \times L^2(\Omega)$ to the target $(z_0, z_1) \in H^1_{(0)}(\Omega) \times L^2(\Omega)$ within time T . Under an additional
3 regularity assumption, we adapt the least-squares approach introduced in [MT22] to boundary case leading
4 to a convergent algorithm. In particular, the convergence is super-linear after a finite number of iterations.

5 Numerical experiments are in agreement with the theoretical results. More precisely, the experiments
6 confirm the change in convergence speed for the least-squares algorithm and suggest that the fixed-point
7 operator K (defined by (23)) is not contracting in general.

8 We conclude with some comments:

1. **Construction of a contracting operator.** Numerical simulations show that the fixed-point algorithm (PF2) associated with \tilde{K} diverges in general. By introducing a Carleman parameter $s > 0$ large enough and weight functions ρ, ρ_1 , we can expect a contraction property for the operator $\mathcal{K} : L^\infty(Q_T) \rightarrow L^\infty(Q_T)$ where $y := \mathcal{K}(\xi)$ is the optimal controlled solution of (33) for the cost

$$\tilde{\mathcal{J}}(y, v) = \|\rho^{-1}(s)y\|_{L^2(Q_T)}^2 + s^{-1} \|\rho_1^{-1}(s)v\|_{L^2(0,T)}^2.$$

9 We refer to [BLM23a] and [CLM24]. This requires a Carleman inequality with a Neumann-type obser-
10 vation.

2. **The multi-dimensional case.** The generalization of these results in the multi-dimensional case is open. First, in dimension $d > 1$, the regularity of the solutions of (2) depends on the domain Ω (we refer to [Lio88, Chapitre III, Section 2 p.179-180] or also [LT89b, Theorem 1.1 p.52]). Secondly, to expect a generalization of all these results, we need an observability inequality which holds in any dimensions. Our estimate (Obs) is based on specific argument related to the one dimension and to our knowledge, there is no estimate like (Obs) valid in any dimension.

3. **Inverse problems.** On account of the duality between controllability and observability, it would be interesting to analyze the potential of the least-squares approach for solving inverse problems.

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