



HAL
open science

Some fixed points theorems of the Leray-Schauder type for multivalued mappings

Abdul-Majeed Al-Izери, Khalid Latrach

► **To cite this version:**

Abdul-Majeed Al-Izери, Khalid Latrach. Some fixed points theorems of the Leray-Schauder type for multivalued mappings. Bulletin of the Iranian Mathematical Society, In press. hal-04693982

HAL Id: hal-04693982

<https://hal.science/hal-04693982>

Submitted on 11 Sep 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Some fixed points theorems of the Leray-Schauder type for multivalued mappings

Abdul-Majeed Al-izeri¹ and Khalid Latrach^{1†}

¹Université Clermont Auvergne, CNRS, LMBP F-63000, Clermont-Ferrand, France.

Contributing authors: Abdul_Majeed.Al_izeri@uca.fr; khalid.latrach@uca.fr;

[†]These authors contributed equally to this work.

Abstract

In this note, we establish some fixed point theorems of the Leray-Schauder type for upper semicontinuous and weakly sequentially upper semicontinuous multivalued mappings in Banach spaces. The cases of condensing and nonexpansive multivalued maps were also considered.

Keywords: Fixed point theorems, upper semicontinuous and weakly sequentially upper semicontinuous multivalued mappings, measure of weak noncompactness, condensing and nonexpansive multivalued maps

MSC Classification: 47H10 , 47H04

1 Introduction and preliminaries

In the last years, fixed point theory for single and multivalued mappings, under the weak topology, has known many development. In particular, under various conditions, several works were dedicated to derive theorems of Schauder's type (Himmelberg's theorem [16]), Sadovskii's type, Krasnosel'skii's type and Leray-Schauder's type (see, for example, [1, 3–7, 9, 10, 13–15, 17–21] and the references therein). Our objective in this work is to establish some fixed point results of Leray-Schauder's type in Banach spaces for upper semicontinuous and weakly sequentially upper semicontinuous mappings. This work is essentially motivated by Theorems 1.1 and 1.2, obtained in [2], and conditions (H1) and (H2) which were introduced in [17] (see below).

Now we introduce notations and definitions which are required in the paper. Let X be Banach space and define the sets

$$\begin{aligned} \mathbf{P}(X) &= \{M \subset X : M \text{ is nonempty}\}, \\ \mathbf{P}_{bd}(X) &= \{M \subset X : M \text{ is nonempty and bounded}\}, \\ \mathbf{P}_{cv}(X) &= \{M \subset X : M \text{ is nonempty and convex}\}, \\ \mathbf{P}_{cl,cv}(X) &= \{M \subset X : M \text{ is nonempty, convex and closed}\}, \\ \mathbf{W}(X) &= \{M \subset X : M \text{ is nonempty weakly compact}\} \end{aligned}$$

We shall now give the notion of a measure of weak noncompactness on a Banach space [8].

Definition 1.1. A map $\mu : \mathbf{P}_{bd}(X) \rightarrow [0, +\infty[$ is said to be a measure of weak noncompactness on X if it satisfies the following conditions

- (1) The family $\ker \mu := \{M \in \mathbf{P}_{bd}(X) : \mu(M) = 0\}$ is non-empty and $\ker \mu$ is contained in the set of relatively weakly compact subsets of X .
- (2) *Monotonicity:* $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$ for all $M_1, M_2 \in \mathbf{P}_{bd}(X)$.
- (3) *Invariance under passage to the closed convex hull:* $\mu(\overline{\text{co}}(M)) = \mu(M)$ where $\overline{\text{co}}$ denotes the closed convex hull of M .
- (4) *Homogeneity:* $\mu(\lambda M) = |\lambda| \mu(M) \quad \forall \lambda \in \mathbb{R}$.
- (5) *Subadditivity:* $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$ for all $M_1, M_2 \in \mathbf{P}_{bd}(X)$.
- (6) *Maximum Property:* $\mu(M_1 \cup M_2) = \max(\mu(M_1), \mu(M_2))$ for all $M_1, M_2 \in \mathbf{P}_{bd}(X)$.
- (7) *Fullness:* $\mu(M) = 0$ if and only if M is a relatively weakly compact set.

The family $\ker \mu$ given in first assertion is called the kernel of the measure μ . It should be noticed that the inclusions $M \subseteq \overline{M^w} \subseteq \overline{\text{co}}(M)$ together with the item (3) of Definition 1.1 imply

$$(8) \quad \mu(\overline{M^w}) = \mu(M).$$

Note that if $\mu(\cdot)$ is a full measure of weak noncompactness having the maximum property, then it is non-singular, that is:

$$(9) \quad \mu(M \cup \{x\}) = \mu(M), \text{ for all } M \in \mathbf{P}_{bd}(X) \text{ and } x \in X.$$

Before going further we recall the following definitions required below.

Definition 1.2. Let X and Y be two normed spaces and let $F : X \rightarrow \mathbf{P}_{cl,cv}(Y)$ be a multivalued map.

- We say that F is upper semicontinuous (u.s.c. for short) if, for every open set U of Y , the set $F^{-1}(U)$ is open in X , where $F^{-1}(U) = \{x \in X : F(x) \subset U\}$.
- F is called weakly upper semicontinuous (w.u.s.c. for short) if F is upper semi-continuous with respect to the weak topologies of X and Y .
- F is called weakly sequentially upper semicontinuous (w.s.u.s.c. for short) if for any weakly closed set G of Y , $F^{-1}(G)$ is weakly sequentially closed. \square

Definition 1.3. Let X and Y be two normed spaces and $F : X \rightarrow \mathcal{P}_{cl,cv}(Y)$ be a multivalued map. A single valued map $f : X \rightarrow Y$ is called a selection of F if for every $x \in X$, $f(x) \in F(x)$.

Let M be a subset of a normed space X and let $F : M \rightarrow \mathcal{P}_{cl,cv}(M)$ be a multivalued map. We recall that F has a fixed point means that if there exists $x \in M$ such that $x \in F(x)$.

Definition 1.4. Let M be a nonempty closed, convex subset of a Banach space X , $\mu(\cdot)$ a measure of weak noncompactness on X and $F : M \rightarrow \mathcal{P}(X)$ a multivalued mapping. We say that :

- (a) F is μ -condensing if $\mu(F(D)) < \mu(D)$, for all bounded subset D of M with $\mu(D) > 0$.
- (b) F is μ -nonexpansive map if $\mu(F(D)) \leq \mu(D)$, for all bounded subset D of M .

Let $\mathcal{J} : D(\mathcal{J}) \subset X \rightarrow X$ be a single valued mapping. We recall the following conditions introduced in [17, p. 260]:

(H1) For each weakly convergent sequence $(x_n)_{n \in \mathbb{N}}$ of $D(\mathcal{J})$, the sequence $(\mathcal{J}(x_n))_{n \in \mathbb{N}}$ has a strongly convergent subsequence.

(H2) For each weakly convergent sequence $(x_n)_{n \in \mathbb{N}}$ of $D(\mathcal{J})$, the sequence $(\mathcal{J}(x_n))_{n \in \mathbb{N}}$ has a weakly convergent subsequence.

Now, we recall the following results established in [2].

Theorem 1.1. Let M be a nonempty closed, convex subset of a Banach space X and let $F : M \rightarrow \mathcal{P}_{cl,cv}(M)$ be a u.s.c. multivalued map. Suppose that all selections of F satisfy the condition (H1) and $F(M)$ is relatively weakly compact. Then there exists $x \in M$ such that $x \in F(x)$.

Theorem 1.2. Let X be a Banach space, M a nonempty closed, convex subset of X . Let $F : M \rightarrow \mathcal{P}_{cl,cv}(M)$ be a w.s.u.s.c. multivalued map and $F(M)$ is relatively weakly compact. Then there exists $x \in M$ such that $x \in F(x)$.

Theorem 1.3. Let X be a Banach space and M a nonempty bounded, closed and convex subset of X . Assume that $\mu(\cdot)$ a measure of weak noncompactness on X . If $F : M \rightarrow \mathcal{P}_{cl,cv}(M)$ is a w.s.u.s.c. μ -condensing multivalued map, then there exists $x \in M$ such that $x \in F(x)$.

In the remainder of this paper, if A be a subset of a normed space X , we denote by \overline{A}^w the closure of A in the weak topology of X .

2 Main results

In this section, we shall gather our results of Leray-Schauder's type for a class of multivalued mapping. The first one deals with mappings satisfying condition (H1).

Theorem 2.1. *Let X be a Banach space and M a nonempty closed, convex subset of X . Let $U \subset M$ be an open subset of M , and $p \in U$. Assume that $F : \overline{U} \rightarrow \mathcal{P}_{cl,cv}(M)$ is a continuous multivalued mapping and each selection of F satisfies condition (H1). If $F(\overline{U})$ is relatively weakly compact. Then, either*

(a) *there exists $x \in U$ such that $x \in F(x)$, or*

(b) *there exists $x \in \partial U$ and $\lambda \in (0, 1)$ such that $x \in \lambda F(x) + (1 - \lambda)p$.*

Proof. Suppose (b) does not hold and F does not have a fixed point in $x \in \partial U$ (otherwise, we are finished, i.e. (a) occurs). Then

$$x \notin \lambda F(x) + (1 - \lambda)p \text{ for all } x \in \partial U \text{ and } \lambda \in [0, 1].$$

Let A be the set

$$A := \{x \in \overline{U} : x \in tF(x) + (1 - t)p \text{ for some } t \in [0, 1]\}.$$

It is clear that A is nonempty because $0 \in A$ (take $t = 0$). Furthermore, we have $A \cap \partial U = \emptyset$ and the continuity of F imply that A is closed. So by Uryshon's theorem (see, for example, [11, p.15]), there exists a continuous function $\zeta : M \rightarrow [0, 1]$, such that $\zeta(x) = 1$ if $x \in A$ and $\zeta(x) = 0$ if $x \in \partial U$. We can define the multivalued mapping $S : M \rightarrow \mathcal{P}_{cl,cv}(M)$ by:

$$S(x) = \begin{cases} \zeta(x)F(x) + (1 - \zeta(x))p, & x \in \overline{U}, \\ \{p\}, & x \in M \setminus \overline{U}. \end{cases} \quad (1)$$

It is clear that $S(M)$ is bounded, $[0, 1]$ is compact, ζ and F are continuous multivalued maps, hence S is a continuous multi-valued map. So, it is enough to show that S satisfies the conditions of Theorem 1.1, that is, all selections of S satisfy the condition (H1) and $S(M)$ is relatively weakly compact. To do so, let $(x_n)_{n \in \mathbb{N}}$ be a weakly convergent sequence in M . According to either or neither $(x_n)_{n \in \mathbb{N}}$ lies in \overline{U} , for n large enough, we distinguish two cases:

(i) There exists some $n_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, ($n \geq n_0 \implies x_n \in \overline{U}$). In this case, the sequence $(x_n)_{n \geq n_0}$ is contained in \overline{U} and converges weakly to some $x \in \overline{U}$. Since all selections of S satisfy (H1), we infer that there exists a selection f of F (then of S) and a sequence $(y_n)_{n \geq n_0}$, such that for each $n \geq n_0$, $y_n = f(x_n)$. So, $(y_n)_{n \geq n_0}$ has a strongly convergent subsequence $(y_{n_k})_{k \in \mathbb{N}}$ to some $y \in S(\overline{U})$. Because $[0, 1]$ is compact, we can extract from $\zeta(x_{n_k})_{k \geq 0}$ a strongly convergent subsequence $\zeta(x_{n_{k_j}})_{j \geq 0}$. We note that the sequence $\zeta(x_{n_{k_j}})_{j \geq 0}$ satisfies

$$S(x_{n_{k_j}}) = \zeta(x_{n_{k_j}})F(x_{n_{k_j}}) + (1 - \zeta(x_{n_{k_j}}))p \text{ for all } j \in \mathbb{N}.$$

Thus, if we denote by \mathcal{H} the Hausdorff distance on $\mathbf{B}(X)$, it is not difficult to check that, for $t \in [0, 1]$, we have

$$\lim_{j \rightarrow +\infty} \mathcal{H}(\{S(x_{n_{k_j}})\}, \{ty + (1 - t)p\}) = 0.$$

(ii) If $(x_n)_{n \in \mathbb{N}}$ is such that for all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $x_{n_k} \notin \overline{U}$, then we may consider a subsequence $(x_{n_k})_{k \in \mathbb{N}} \subset M \setminus \overline{U}$. Therefore, we can consider a subsequence

$(x_{n_{k_j}})_{j \in \mathbb{N}}$ of $M \setminus \bar{U}$ such that, for each j , we have $S(x_{n_{k_j}}) = \{p\}$. This proves that, for j large enough, we have $\mathcal{H}(S(x_{n_{k_j}}), \{p\}) = \mathcal{H}(\{p\}, \{p\}) = 0$.

Hence, according to (i)-(ii) we conclude that all selections of S satisfy (H1).

To see that $S(\bar{U})$ is relatively weakly compact, it suffices to observe that $S(\bar{U}) \subset F(\bar{U})$ and $F(\bar{U})$ is relatively weakly compact, so $S(\bar{U})$ is relatively weakly compact. According to Equ. (1), for all $x \in \bar{U}$ we have $S(x) = \zeta(x)F(x) - (1 - \zeta(x))p$. Now Theorem 1.1 ensures the existence of a point $z \in M$ such that $z \in S(z)$. Since $p \in U$, we have $z \in U$ and so $z \in (\zeta(z)F(z) + (1 - \zeta(z))p)$. As a result, $z \in A$ and so $\zeta(z) = 1$. Hence $z \in F(z)$. QED

Notice that, in Theorem 2.1, the set M is not necessarily bounded. In case it is bounded, we obtain a more precise result:

Theorem 2.2. *Let X be a Banach space, M a nonempty closed, bounded, convex subset of X and $\mu(\cdot)$ a measure of weak non-compactness on X . Let $U \subset M$ be an open subset of M , and $p \in U$. Let $F : \bar{U} \rightarrow \mathcal{P}_{cl,cv}(M)$ is a continuous multivalued mapping and each selection of F satisfies condition (H1). If F is μ -condensing, then, either*

- (a) *there exists $x \in U$ such that $x \in F(x)$, or*
- (b) *there exists $x \in \partial U$ and $\lambda \in (0, 1)$ such that $x \in \lambda F(x) + (1 - \lambda)p$.*

This theorem is the nonlinear alternative version of the following fixed point theorem established in [2, Theorem 3.1].

Theorem 2.3. *Let M be a nonempty bounded, closed, convex subset of a Banach space X and let $F : M \rightarrow \mathcal{P}_{cl,cv}(M)$ be a u.s.c. multivalued map and $\mu(\cdot)$ a measure of weak non-compactness on X . Suppose that all selections of F satisfy condition (A). If F is μ -condensing, then there exists $x \in M$ such that $x \in F(x)$.*

Proof of Theorem 2.2 Using arguments similar to those used in proving Theorem 2.1, we can see that the operator S defined by (1) maps continuously M into itself and satisfies the condition (H1). By Theorem 2.3, it is enough to check that S is an μ -contraction. To this end, let $C \subset M$. For all $x \in C$, we either have $S(x) = \zeta(x)F(x) + (1 - \zeta(x))p$, $\zeta(x) \in [0, 1]$ or $S(x) = p$. We infer that $S(C) \subset \text{co}(F(C) \cup \{p\})$. Hence, using that fact that μ is non-singular, we can write

$$\begin{aligned} \mu(S(C)) &\leq \mu(\text{co}(F(C) \cup \{p\})) = \mu(F(C) \cup \{p\}) \\ &\leq \max(\mu(F(C)), \mu(\{p\})) = \mu(F(C)) \\ &< \mu(C). \end{aligned}$$

Since F is continuous, then it is u.s.c. multi-valued map, so Theorem 2.3 gives the desired result. QED

Now we are concerned with the existence of fixed point theorems for w.s.u.s.c. multivalued maps by applying Theorem 1.2 without assuming condition (H1).

Theorem 2.4. *Let X be a Banach space, M a nonempty closed, convex subset of X and let $U \subset M$ be weakly open relative to M , and $p \in U$. Suppose \overline{U}^w is a weakly compact subset of M and $F : \overline{U}^w \rightarrow \mathcal{P}_{cl,cv}(M)$ is a w.s.u.s.c. multivalued map. Then, either*

- (a) *there exists $x \in \overline{U}^w$ such that $x \in F(x)$, or*
- (b) *there exists $x \in \partial_M U$ (the weakly boundary of U in M) and $\lambda \in]0, 1[$ with $x \in \lambda F(x) + (1 - \lambda)p$.*

In this theorem we suppose that U weakly open relative to M , which means that U is open for the topology induced on M by the weak topology of X . In other words, there exists a weakly open subset O of X such that $U = O \cap M$. The hypothesis \overline{U}^w is a weakly compact means that there exists a weakly open subset O of X such that $U = O \cap M$ is relatively weakly compact.

Proof. Suppose (b) does not hold and F does not have a fixed point in $x \in \partial_M U$ (otherwise, we are finished, i.e. (a) occurs). Then

$$x \notin \lambda F(x) + (1 - \lambda)p \text{ for all } x \in \partial_M U \text{ and } \lambda \in [0, 1].$$

Let A be the set

$$A := \left\{ x \in \overline{U}^w : x \in tF(x) + (1 - t)p \text{ for some } t \in [0, 1] \right\}.$$

So, A is nonempty because $0 \in A$ (take $t = 0$). and $A \cap \partial_M U = \emptyset$. Moreover, since $A \subset \overline{U}^w$, we conclude that A is relatively weakly compact. Let $(x_n)_{n \in \mathbb{N}}$ a sequence of A such that $x_n \rightharpoonup x$ (it is clear that $x \in \overline{U}^w$). By the definition of A , for each $n \in \mathbb{N}$, there exists $t_n \in [0, 1]$ such that $x_n \in t_n F(x_n) + (1 - t_n)p$. Since $[0, 1]$ is compact, we can extract a subsequence $(t_{n_k})_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} t_{n_k} = t \in [0, 1]$. Using the fact that $F : \overline{U}^w \rightarrow \mathcal{P}_{cl,cv}(M)$ is a w.s.u.s.c. multivalued mapping, so the graph of F is weakly sequential closed. Therefore, we get $x_{n_k} \in t_{n_k} F(x_{n_k}) + (1 - t_{n_k})p \rightharpoonup tF(x) + (1 - t)p$ and so $x \in A$. This yields that A is weakly sequentially closed.

Let $x \in \overline{U}^w$ be adherent to A for the weak topology $\sigma(X, X^*)$. Since \overline{A}^w is weakly compact, by the Eberlein-Šmulian theorem (see [15, Theorem 1.7.3, p. 31]), there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in A such that $x_n \rightharpoonup x$. Now the use of the fact A is weakly sequentially closed, we conclude that $x \in A$. Hence $\overline{A}^w = A$ which proves that A is weakly closed. Since \overline{U}^w is weakly compact and $A \subset \overline{U}^w$, we deduce that A is weakly compact.

Since $(X, \sigma(X, X^*))$ (X endowed with its weak topology) is a Hausdorff locally convex topological vector space, it is completely regular and the assertion (b) is not satisfied, so we have $A \cap (M \setminus U) = \emptyset$. Again by Uryshon's theorem [15, Theorem 1.1.2], there is a weakly continuous function $\zeta : M \rightarrow [0, 1]$, such that $\zeta(x) = 1$ for $x \in A$ and $\zeta(x) = 0$ for $x \in M \setminus U$. We can define the multivalued map $S : M \rightarrow \mathcal{P}_{cl,cv}(M)$ by:

$$S(x) = \begin{cases} \zeta(x)F(x) + (1 - \zeta(x))p, & x \in \overline{U}^w, \\ \{p\}, & x \in M \setminus \overline{U}^w. \end{cases}$$

Since M is convex and closed, by Mazur's theorem, M is weakly closed and therefore $\partial_M U = \partial_M \overline{U^w}$. Using the fact that ζ is weakly continuous and F is w.s.u.s.c., we infer that S is w.s.u.s.c. multivalued mapping.

Furthermore, we have $S(M) \subseteq \overline{\text{co}}(F(\overline{U^w}) \cup \{p\})$. Putting $H := \overline{\text{co}}(F(\overline{U^w}) \cup \{p\})$. By Krein-Smulian's theorem (see [11, p. 434]), H is a weakly compact subset of M and $S(H) \subseteq H$ and therefore $S(H)$ is relatively weakly compact and consequently $S : H \rightarrow \mathcal{P}_{cl,cv}(H)$ is w.s.u.s.c. multivalued mapping (see above). Hence S satisfies all assumptions of Theorem 1.2, so there exists $z \in M$ such that $z \in S(z)$. If $z \notin U$, $\zeta(z) = 0$ and so $z = p$, which contradicts the hypothesis $p \in U$. This shows that $z \in U$ and $z \in (\zeta(z)F(z) + (1 - \zeta(z))p)$ which implies that $z \in A$, thus we have $\zeta(z) = 1$ and so $z \in F(z)$ which completes the proof. QED

The following corollary is an easy consequence of Theorem 2.4.

Corollary 2.1. *Let X be a Banach space, M a nonempty closed, convex subset of X and U a weakly open subset of M with $p \in U$. Assume that $\overline{U^w}$ is weakly compact and $F : \overline{U^w} \rightarrow \mathcal{P}_{cl,cv}(M)$ is a w.s.u.s.c. multivalued map. In addition, suppose that F satisfies the Leray-Schauder boundary conditions, that is,*

$$x \notin \lambda F(x) + (1 - \lambda)p \text{ for all } x \in \partial_M U \text{ and } \lambda \in]0, 1[.$$

Then F has a fixed point in $\overline{U^w}$.

QED

Now, we are in a position to establish the next result of the Leray-Schauder type for w.s.u.s.c. μ -condensing multivalued mappings.

Theorem 2.5. *Let X be a Banach space, M a nonempty closed, convex subset of X and $U \subset M$ a weakly open subset of M with $p \in U$. Assume $F : \overline{U^w} \rightarrow \mathcal{P}_{cl,cv}(M)$ is a w.s.u.s.c. μ -condensing multivalued map where $\mu(\cdot)$ is a measure of weak noncompactness on X . If $F(\overline{U^w})$ is bounded, then, either*

- (a) *there exists $x \in \overline{U^w}$ such that $x \in F(x)$, or*
- (b) *there exists $x \in \partial_M U$ (the weakly boundary of U in M) and $\lambda \in]0, 1[$ with $x \in \lambda F(x) + (1 - \lambda)p$.*

Proof. Suppose (b) does not hold and F does not have a fixed point in $x \in \partial_M U$ (otherwise, we are finished, i.e. (a) occurs). Then

$$x \notin \lambda F(x) + (1 - \lambda)p \text{ for all } x \in \partial_M U \text{ and } \lambda \in [0, 1].$$

Define the set A by

$$A := \{x \in \overline{U^w} : x \in tF(x) + (1 - t)p \text{ for some } t \in [0, 1]\}.$$

As in the proofs of theorems above, A is nonempty. Furthermore, we have $A \subset \overline{co}(F(A) \cup \{p\})$ and so

$$\mu(A) \leq \mu(\overline{co}(F(A) \cup \{p\})) = \mu(F(A)) < \mu(A),$$

which is a contradiction. Hence, $\mu(A) = 0$ and so A is relatively weakly compact. Using the same arguments as in Theorem 2.4, we see that A is weakly closed and so A is weakly compact. Now, We can define the multivalued map $S : M \rightarrow \mathcal{P}_{cl,cv}(M)$ by:

$$S(x) = \begin{cases} \zeta(x)F(x) + (1 - \zeta(x))p & x \in \overline{U^w}, \\ \{p\}, & x \in M \setminus \overline{U^w} \end{cases}$$

where $\zeta : M \rightarrow [0, 1]$ is a weakly continuous map satisfying $\zeta(x) = 1$ for $x \in A$ and $\zeta(x) = 0$ for $x \in M \setminus U$. Let V be a bounded subset of M . According to the definition of S , we have $S(V) \subseteq co(F(V) \cup \{p\})$ and so

$$\mu(S(V)) \leq \mu(F(V \cap U)) \leq \mu(F(V)) < \mu(V).$$

Hence, S is μ -condensing. Since ζ is weakly continuous and F is a w.s.u.s.c. μ -condensing multivalued map, we deduce that S is a w.s.u.s.c. μ -condensing multivalued map. Accordingly, S satisfies all assumptions of Theorem 1.3. Hence, there exists $z \in M$ with $z \in S(z)$. If $z \notin U$, $\zeta(z) = 0$ and so $z = p$, which contradicts the hypothesis $p \in U$. This shows that $z \in U$ and $z \in (\zeta(z)F(z) + (1 - \zeta(z))p)$ which implies that $z \in A$, thus we have $\zeta(z) = 1$ and so $z \in F(z)$ which completes the proof.

QED

The following corollary is an easy consequence of the theorem above.

Corollary 2.2. *Let X be a Banach space, M a nonempty closed, convex subset of X and $U \subset M$ a weakly open subset of M with $p \in U$. Assume $F : \overline{U^w} \rightarrow \mathcal{P}_{cl,cv}(M)$ is a w.s.u.s.c. μ -condensing multivalued map where $\mu(\cdot)$ is a measure of weak noncompactness on X . In addition, suppose that $F(\overline{U^w})$ is bounded and satisfies the Leray-Schauder boundary conditions:*

$$x \notin \lambda F(x) + (1 - \lambda)p \quad \text{for every } x \in \partial_M \overline{U^w} \quad \text{and } \lambda \in]0, 1[.$$

Then F has a fixed point in $\overline{U^w}$.

We conclude this paper by the following result.

Theorem 2.6. *Let X be a Banach space, M a nonempty closed, convex subset of X and $U \subset M$ a weakly open subset of M with $0 \in U$. Assume $F : \overline{U^w} \rightarrow \mathcal{P}_{cl,cv}(M)$ is a w.s.u.s.c. μ -nonexpansive multivalued map and each selection of F satisfies condition (H2) where $\mu(\cdot)$*

is a measure of weak noncompactness on X . In addition, suppose that $F(\overline{U}^w)$ is bounded and F satisfies the Leray-Schauder boundary conditions:

$$x \notin \lambda F(x) + (1 - \lambda)p \quad \text{for every } x \in \partial_M U \quad \text{and } \lambda \in]0, 1[. \quad (2)$$

Then F there exists $x \in \overline{U}^w$ such that $x \in F(x)$.

Proof. Let $F_n = \frac{n}{n+1}F$, for $n = 1, 2, \dots$. Set $\zeta_n = \frac{n}{n+1}$. It is clear that $(\zeta)_{n \in \mathbb{N}}$ is contained in $(0, 1)$ and $\zeta_n \rightarrow 1$ as n goes to $+\infty$. Since $\zeta_n < 1$, the map F_n is a w.s.u.s.c. μ -condensing multivalued mapping.

According to Theorem 2.5, there exists $z_n \in \overline{U}^w$ such that $z_n \in \zeta_n F(z_n)$ for some $z_n \in \partial_M U$ and $\lambda_n \in (0, 1)$. This yields that $z_n \in \lambda_n \zeta_n F(z_n)$ which contradicts the result of Corollary 2.2 because $\lambda_n \zeta_n F$ is μ -condensing. Hence, for each $x \in \partial U$ and $\lambda_n \in (0, 1)$, we have $x \notin \lambda_n F_n(x)$ and consequently F_n has a fixed point in \overline{U}^w .

Next, set $K = \overline{\text{co}}(F_n(M))$. It is clear that K is contained in M and, by Krien-Šmulian's theorem, K is a convex weakly compact subset of X .

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points of K . The weak compactness of K yields that there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that $x_{n_k} \rightharpoonup x$ as $n \rightarrow +\infty$. Let $(y_{n_k})_{k \in \mathbb{N}}$ be a sequence in $F_n(K)$ such that, for each $k \in \mathbb{N}$, $y_{n_k} \in \zeta_{n_k} F(x_{n_k})$. Hence, there is a selection f of F such that, for each $k \in \mathbb{N}$, $y_{n_k} = f(x_{n_k})$. Since f satisfies (H2), we infer that $(y_{n_k})_{k \in \mathbb{N}}$ has a weakly convergent subsequence in K , say $(y_{n_{k_j}})_{j \in \mathbb{N}}$, that is $y_{n_{k_j}} \rightharpoonup y$ as $j \rightarrow +\infty$. We note that, when n goes to $+\infty$, $\zeta_n \rightarrow 1$. Thus, we deduce that there exists $x \in K$ with $x \in F(x)$. QED

References

- [1] Abdallah MY , Al-Izeri A, Latrach K (2021) *Some remarks on fixed sets for perturbed multivalued mappings*. J. Fixed Point Theory Appl. **23** (3) :Paper No. 38, 14 pp
- [2] Al-Izeri A, Latrach K (2023) *A note on fixed point theory for multivalued mappings*. Fixed Point Theory **24**(1):233-239
- [3] Al-Izeri A, Latrach K *Krasnosel'skii-type fixed point results for weakly sequentially upper semicontinuous multivalued mappings*. Journal of Mathematics and Applications (to appear).
- [4] Agarwal RP, O'Regan D (2002) *Fixed point theory for weakly sequentially upper semicontinuous maps with applications to differential inclusions*. Nonlinear Oscil. (N. Y.) **5**(3):277-286
- [5] Agarwal RP, O'Regan D (2011) *Fixed point theorems for condensing multivalued mappings under weak topology features*. Fixed Point Theory **12** (2): 247-254
- [6] Al-Thagafi M A, Shahzad N (2010) *Krasnosel'skii-type fixed-set results*. Fixed Point Theory. Article ID **394139**:9 pages

- [7] Basoc I, Cardinali T (2015) A hybrid nonlinear alternative theorem and some hybrid fixed point theorems for multimaps. *J. Fixed Point Theory Appl* **17**:413-424
- [8] Banaś J, Rivero J (1988) *On measures of weak noncompactness*. *Ann. Mat. Pura Appl.* **151**(4):213-224
- [9] Cardinali T, Rubbioni P (2012) *Countably condensing multimaps and fixed points*. *Electron. J. Qual. Theory Differ. Equ.* **83**: 1-9
- [10] Dhage BC (2006) *Mutli-valued mappings and fixed points II* . *Tamkang J. Math* **37** :27-46
- [11] Dunford N, Schwartz JT (1958) *Linear Operators: Part I, Intersciences*. New York
- [12] Edwards RE (1965) *Functional Analysis, Theory and Applications*. Rinehart and Winston, New York-Toronto-London
- [13] Garcia-Falset J (2009) *Existence of fixed points and measure of weak noncompactness*. *Nonlinear Anal.* **71**:2625-2633
- [14] Graef JR, Henderson J, Ouahab A (2017) *Multivalued versions of a Krasnosel'skii-type fixed point theorem*. *J. Fixed Point Theory Appl.* **19**:1059-1082
- [15] Garcia-Falset J, Latrach K (2023) *Nonlinear Functional Analysis and Applications*. De Gruyter Series in Nonlinear Analysis and Applications, **41**
- [16] Himmelberg CJ (1972) *Fixed points of compact multifunctions*. *J. Math. Anal. Appl.* **38**:205-207

- [17] Latrach K, Taoudi MA, Zeghal A (2006) *Some fixed point theorems of the Schauder and the Krasnosel'skii type and application to nonlinear transport equations*. *J. Differential Equations* **221** (1):256-271
- [18] Ok EA (2009) Fixed set theorems of Krasnoselskii type. *Proc. Amer. Math. Soc* **137**:511-518
- [19] O'Regan D (1998) *Fixed point theory for weakly contractive maps with applications to operator inclusions in Banach spaces relative to the weak topology*. *Z. Anal. Anwend* **17** (2):281-296
- [20] O'Regan D (2000) *Fixed point theorems for weakly sequentially closed maps*. *Arch. Math. (Brno)* **36** (1):61-70
- [21] Somyot P, Thammathiwat T (2013) Fixed point theorems of Krasnosel'skii type for the sum of two multivalued mappings in Banach spaces. *J. Nonlinear Convex Anal* **14**:183-191