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A Reynolds-semi-robust and pressure robust Hybrid High-Order method for the time dependent incompressible Navier–Stokes equations on general meshes

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Abstract

In this work we develop and analyze a Reynolds-semi-robust and pressure-robust Hybrid High-Order (HHO) discretization of the incompressible Navier–Stokes equations. Reynolds-semi-robustness refers to the fact that, under suitable regularity assumptions, the right-hand side of the velocity error estimate does not depend on the inverse of the viscosity. This property is obtained here through a penalty term which involves a subtle projection of the convective term on a subgrid space constructed element by element. The estimated convergence order for the $L^\infty(L^2)$ - and L^2 (energy)-norm of the velocity is $h^{k+\frac{1}{2}}$, which matches the best results for continuous and discontinuous Galerkin methods and corresponds to the one expected for HHO methods in convection-dominated regimes. Two-dimensional numerical results on a variety of polygonal meshes complete the exposition.

Key words: Hybrid High-Order methods, time-dependent incompressible flow, general meshes, Re-semi-robust error estimates, pressure robustness

MSC 2010: 65N08, 65N30, 65N12, 35Q30, 76D05

1 Introduction

We consider the time-dependent incompressible Navier–Stokes equations:

$$\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } (0, t_F] \times \Omega, \quad (1a)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } (0, t_F] \times \Omega, \quad (1b)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } [0, t_F] \times \partial\Omega, \quad (1c)$$

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0(\cdot) \quad \text{in } \Omega, \quad (1d)$$

where $\Omega \subset \mathbb{R}^d$, for $d \in \{2, 3\}$, denotes an open, bounded, simply connected polyhedral domain with Lipschitz boundary $\partial\Omega$ and $t_F > 0$ denotes the final time. In addition, $\mathbf{u} : [0, t_F] \times \Omega \rightarrow \mathbb{R}^d$ is the fluid velocity field, $p : (0, t_F] \times \Omega \rightarrow \mathbb{R}$ is the (zero-average) kinematic pressure, $\mathbf{u}_0 : \Omega \rightarrow \mathbb{R}^d$

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represents a given initial condition, and $\mathbf{f} : (0, t_F] \times \Omega \rightarrow \mathbb{R}^d$ represents a given body force. The fluid is assumed to be Newtonian with constant kinematic viscosity $\nu > 0$.

In this work we propose a ‘‘Reynolds-semi robust’’ and pressure-robust Hybrid-High Order (HHO) method on general meshes to approximate the solution of the weak form of problem (1). A numerical scheme is considered ‘‘Reynolds-semi robust’’ [45] if its velocity error estimates are independent of the Reynolds number (or ν^{-1}). On the other hand, pressure robustness means that the velocity error estimates are independent of the pressure, a property whose relevance was first emphasized in [38, 40]. Recently, a detailed analysis of vortex-dominated flows has been carried out in [36], where the authors conclude that pressure robustness is a crucial prerequisite for the accurate discretisation of non-trivial Navier–Stokes flows; numerical tests where pressure robust methods significantly outperform non-pressure robust methods for transient incompressible flows at high Reynolds numbers are provided, focusing on simplicial meshes.

Reynolds-semi robust numerical schemes have recently made the object of several works. To obtain this kind of estimates, it is standard to assume additional regularity condition on $\nabla \mathbf{u}$. For instance, in the analysis of a continuous interior penalty finite element method [14], assuming $\mathbf{u} \in L^\infty(0, t_F; \mathbf{W}^{1,\infty}(\Omega))$ a velocity error estimate in the $L^\infty(0, t_F; \mathbf{L}^2(\Omega))$ -norm was obtained. Other Reynolds-semi robust numerical schemes using simplicial meshes have been proposed [2, 4, 18, 31, 32, 34, 37]. In particular, we refer to [45] for an outstanding review and further insight into the importance of Reynolds-semi robust numerical methods. At the time of the writing of this work, the best known velocity error estimate in the $L^\infty(0, t_F; \mathbf{L}^2(\Omega))$ -norm is of order $h^{k+\frac{1}{2}}$ (where k denotes the order of the polynomial approximation and h is the mesh size), for instance see the works of [4, 34, 37], where Reynolds-semi robust numerical methods are proposed which additionally satisfy the pressure robustness property.

In recent years, the mathematical community has become interested in developing numerical schemes that can make use of general polygonal and polyhedral meshes, as opposed to more standard triangular/quadrilateral (tetrahedral/hexahedral) meshes. A representative but by far non exhaustive sample concerning incompressible flow problems includes [5, 7, 10, 24, 25, 27, 35, 48]; see also [12, 17] concerning non-Newtonian fluids and [1] concerning the coupling with the heat equation. Regarding pressure robust methods on general meshes for the Stokes and Navier–Stokes equations, some work has recently been done using the Virtual Element method, generalized barycentric coordinates, the staggered Discontinuous Galerkin method, HHO methods, and Discrete de Rham methods; see, e.g., [11, 16, 19, 22, 30, 39, 41, 46, 47, 49]. Pressure-robust HHO methods for the Stokes and Navier–Stokes problem have been proposed in [15, 16, 26]; see also [9] for variants with hybrid pressure. To the authors’ knowledge, the development of Reynolds-semi robust and pressure-robust numerical schemes on general meshes has not yet been addressed. The present work fills this gap. To achieve this, the proposed method uses a slight variation of the divergence-preserving velocity reconstruction proposed by the same authors in [16]. Contrary to [16], however, we use here the convective form of the nonlinear term in the equation (1a) in the spirit of [37, 45], where \mathbf{H}_{div} -conforming spaces are used. Reynolds-semi robustness is obtained through a new term which penalizes the jumps of a potential operator within a decomposition of some discrete piecewise polynomial spaces (cf. (14) and (15) below). With these ingredients, we prove a velocity error estimate in the $L^\infty(0, t_F; \mathbf{L}^2(\Omega))$ -norm of order $h^{k+\frac{1}{2}}$, which equals the best known velocity error estimate on simplicial meshes and corresponds to the typical order of convergence of HHO methods in convectoin-dominated regimes [21].

The rest of the paper is organised as follows. In Section 2 we introduce some notations and present the weak form of problem (1). In Section 3 we present the discrete setting. In Section 4, which contains the statement of the discrete problem, particular emphasis is put on a novel scalar potential reconstruction that is used to stabilize the convective term and achieve Reynolds semi-robustness.

Section 5 contains the velocity error analysis and the main results. Finally, in Section 6 we present numerical experiments to verify our theoretical results.

2 Continuous setting

Throughout the paper, given an open bounded set $D \subset \mathbb{R}^d$ as well as two integers $m \geq 0$ and $p \geq 1$, we use the Sobolev space $W^{m,p}(D)$ for scalar-valued functions with associated norm $\|\cdot\|_{W^{m,p}(D)}$ and seminorm $|\cdot|_{W^{m,p}(D)}$. Spaces of vector- and tensor-valued functions are indicated with bold letters. In the case $m = 0$, we obtain the Lebesgue space $L^p(D) := W^{0,p}(D)$ and, when $p = 2$, the Hilbert space $H^m(D) := W^{m,2}(D)$. Additionally, the closed subspaces $H_0^1(D)$ consisting of $H^1(D)$ -functions with vanishing trace on ∂D , and the set $L_0^2(D)$ of $L^2(D)$ -functions with zero average in D are used in what follows. Given a Banach space X and a real number $t > 0$, we denote by $L^p(0, t; X)$, $p \in [1, \infty]$, the classical Bochner space. In the case $t = t_F$, we often use the abbreviation $L^p(X) := L^p(0, t_F; X)$.

Letting $U := \mathbf{H}_0^1(\Omega)$ and $P := L_0^2(\Omega)$, we consider the following weak form of problem (1): Find $\mathbf{u} : [0, t_F] \rightarrow U$ and $p : (0, t_F] \rightarrow P$ with $\mathbf{u}(0) = \mathbf{u}_0 \in U$, such that it holds, for all $(\mathbf{v}, q) \in U \times P$ and almost every $t \in (0, t_F)$,

$$(d_t \mathbf{u}(t), \mathbf{v}) + va(\mathbf{u}(t), \mathbf{v}) + t(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) + b(\mathbf{v}, p(t)) - b(\mathbf{u}(t), q) = \ell(\mathbf{f}(t), \mathbf{v}), \quad (2)$$

with (\cdot, \cdot) denoting the standard $L^2(\Omega)$ -product, while the bilinear forms $a : U \times U \rightarrow \mathbb{R}$, $b : U \times P \rightarrow \mathbb{R}$, and $\ell : L^2(\Omega) \times U \rightarrow \mathbb{R}$ are defined by

$$a(\mathbf{w}, \mathbf{v}) := \int_{\Omega} \nabla \mathbf{w} : \nabla \mathbf{v}, \quad b(\mathbf{v}, q) := - \int_{\Omega} (\nabla \cdot \mathbf{v}) q, \quad \ell(\mathbf{f}, \mathbf{v}) := \int_{\Omega} \mathbf{f} \cdot \mathbf{v},$$

and the trilinear form $t : U \times U \times U \rightarrow \mathbb{R}$ is such that

$$t(\mathbf{w}, \mathbf{v}, \mathbf{z}) := \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{v}) \cdot \mathbf{z}.$$

3 Discrete setting

3.1 Mesh

In what follows, for the sake of simplicity, we will systematically use the term polyhedral instead of polygonal and face instead of edge also when $d = 2$.

Following [20, Definition 1.4], we consider a polyhedral mesh defined as a couple $\mathcal{M}_h := (\mathcal{T}_h, \mathcal{F}_h)$, where \mathcal{T}_h is a finite collection of polyhedral elements, while \mathcal{F}_h is a finite collection of planar faces. We assume that every element $T \in \mathcal{T}_h$ is star-shaped with respect to a ball [16, Remark 2.5]. For any mesh element or face $X \in \mathcal{T}_h \cup \mathcal{F}_h$, we denote by $|X|$ its Hausdorff measure and by h_X its diameter, so that the meshsize satisfies $h = \max_{T \in \mathcal{T}_h} h_T$. Boundary faces lying on $\partial \Omega$ and internal faces contained in Ω are collected in the sets \mathcal{F}_h^b and \mathcal{F}_h^i , respectively. For each mesh element $T \in \mathcal{T}_h$, we denote by \mathcal{F}_T the set collecting the faces that lie on the boundary ∂T of T and, for all $F \in \mathcal{F}_T$, we denote by \mathbf{n}_{TF} the (constant) unit vector normal to F and pointing out of T .

It is assumed that \mathcal{M}_h belongs to a regular mesh sequence $(\mathcal{M}_h)_h$ in the sense of [20, Definition 1.9]. This assumption entails the existence of a matching simplicial submesh $\mathfrak{M}_h := (\mathfrak{T}_h, \mathfrak{F}_h)$ of \mathcal{M}_h with the following properties: \mathfrak{T}_h is a finite collection of simplicial elements; for any simplex $\tau \in \mathfrak{T}_h$, there is a unique mesh element $T \in \mathcal{T}_h$ such that $\tau \subset T$; for any simplicial face $\sigma \in \mathfrak{F}_h$ and any mesh face $F \in \mathcal{F}_h$, either $\sigma \cap F = \emptyset$ or $\sigma \subset F$. We additionally assume that, for any element $T \in \mathcal{T}_h$, its submesh is constructed in such way that all simplices contained in T and collected in the set \mathfrak{T}_T (see Figure 1a) have at least one common vertex \mathbf{x}_T . This assumption is directly used in Lemmas 3 and 6 below. Regarding the Lemma 3, we refer to [16, Remarks 2.1 and A.1] for further insight into this assumption.

We decompose the set of simplicial faces as $\mathfrak{F}_h = \mathfrak{F}_h^i \cup \mathfrak{F}_h^b$ where \mathfrak{F}_h^i and \mathfrak{F}_h^b respectively collect interior and boundary simplicial faces. For any $T \in \mathcal{T}_h$, we define \mathfrak{F}_T^i as the set of simplicial faces of \mathfrak{F}_h that lie in the interior of T . For any face $F \in \mathcal{F}_h$ lying on the boundary of $T \in \mathcal{T}_h$, \mathfrak{F}_F denotes the set of simplicial faces σ for which $\sigma \subset F$, and we let $\mathbf{n}_\sigma := \mathbf{n}_{TF}$, and $\mathbf{n}_{\tau\sigma} := \mathbf{n}_\sigma$ for the unique element $\tau \in \mathfrak{T}_T$, which contains σ . Additional notations for mesh elements and faces are introduced at the beginning of Section 4.5 below and illustrated in Figure 1b. For future use, we notice that, by [20, Lemma 1.12], mesh regularity implies the existence of an integer $N \geq 0$ depending only on the mesh regularity parameter such that

$$\max_h \max_{T \in \mathcal{T}_h} \text{card}(\mathfrak{T}_T) \leq N \quad \text{and} \quad \max_h \max_{T \in \mathcal{T}_h} \text{card}(\mathfrak{F}_T) \leq N. \quad (3)$$

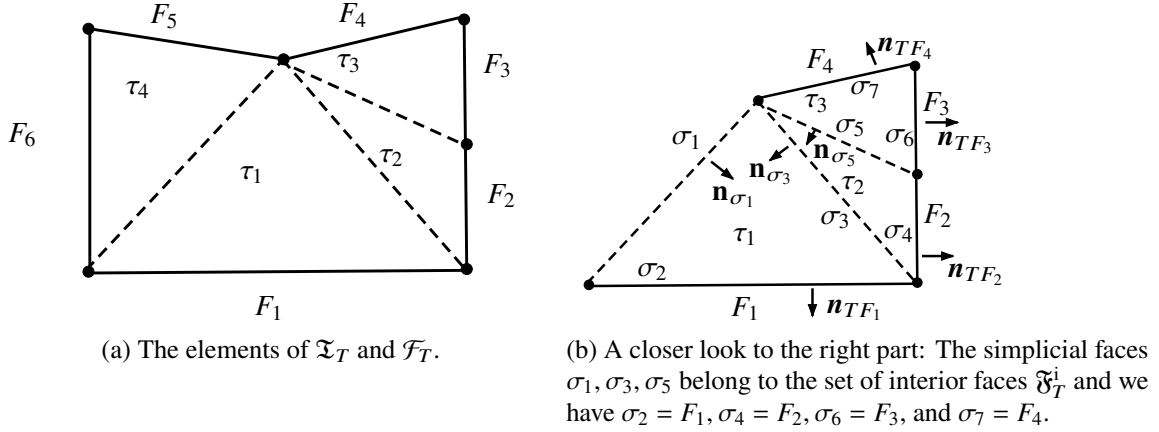


Figure 1: An illustration of the sets $\mathfrak{T}_T, \mathfrak{F}_T$ and \mathfrak{F}_T^i for a given element $T \in \mathcal{T}_h$ in \mathbb{R}^2 .

In order to prevent the proliferation of generic constants we write, whenever possible, $a \lesssim b$ in place of $a \leq Cb$ with $C > 0$ independent of ν, h and, for local inequalities, also on the mesh element or face. The dependencies of the hidden constant will be further specified when relevant. Moreover, we write $a \simeq b$, when both $a \lesssim b$ and $b \lesssim a$ hold.

3.2 Local and broken spaces and projectors

Let X denote a mesh element or face and, for an integer $l \geq 0$, denote by $\mathcal{P}^l(X)$ the space spanned by the restrictions to X of polynomials in the space variables of total degree $\leq l$. The L^2 -orthogonal projector $\pi_X^l : L^1(X) \rightarrow \mathcal{P}^l(X)$ is such that, for all $\zeta \in L^1(X)$,

$$\int_X (\zeta - \pi_X^l \zeta) w = 0 \quad \forall w \in \mathcal{P}^l(X).$$

Vector and matrix versions of the L^2 -orthogonal projector are obtained by applying π_X^l component-wise, and are both denoted with the bold symbol $\boldsymbol{\pi}_X^l$ in what follows. Optimal approximation properties for the L^2 -orthogonal projector are proved in [28, Appendix A.2]; see also [20, Chapter 1], where these estimates are extended to non-star shaped elements. Specifically, let $s \in \{0, \dots, l+1\}$ and $r \in [1, +\infty]$. Then, it holds, with hidden constant only depending on l, s, r , and the mesh regularity parameter: For all $T \in \mathcal{T}_h$, all $\zeta \in W^{s,r}(T)$, and all $m \in \{0, \dots, s\}$,

$$|\zeta - \pi_T^l \zeta|_{W^{m,r}(T)} \lesssim h_T^{s-m} |\zeta|_{W^{s,r}(T)}, \quad (4)$$

and, if $s \geq 1$,

$$h_T^{\frac{1}{r}} \|\zeta - \pi_T^l \zeta\|_{L^r(\partial T)} \lesssim h_T^s |\zeta|_{W^{s,r}(T)}. \quad (5)$$

At the global level, the space of broken polynomial functions on \mathcal{T}_h of total degree $\leq l$ is denoted by $\mathcal{P}^l(\mathcal{T}_h)$, and π_h^l is the corresponding L^2 -orthogonal projector such that, for all $\zeta \in L^1(\Omega)$, $(\pi_h^l \zeta)|_T := \pi_T^l \zeta|_T$ for all $T \in \mathcal{T}_h$. Regularity requirements in error estimates will be expressed in terms of the broken Sobolev spaces $W^{s,r}(\mathcal{T}_h)$, spanned by functions in $L^r(\Omega)$ the restriction of which to every $T \in \mathcal{T}_h$ is in $W^{s,r}(T)$. We additionally set, as usual, $H^s(\mathcal{T}_h) := W^{s,2}(\mathcal{T}_h)$.

3.3 Discrete spaces and norms

Let a polynomial degree $k \geq 0$ be fixed and set

$$k^\star := \begin{cases} k & \text{if } k \in \{0, 1\}, \\ k + 1 & \text{otherwise.} \end{cases}$$

We then define a variant of the usual HHO space as follows

$$\underline{U}_h^k := \{ \underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T \in \mathcal{T}_h}, (\mathbf{v}_F)_{F \in \mathcal{F}_h}) : \mathbf{v}_T \in \mathcal{P}^{k^\star}(T) \text{ for all } T \in \mathcal{T}_h \text{ and } \mathbf{v}_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_h \}. \quad (6)$$

The restrictions of \underline{U}_h^k and $\underline{\mathbf{v}}_h \in \underline{U}_h^k$ to a generic mesh element $T \in \mathcal{T}_h$ are respectively denoted by \underline{U}_T^k and $\underline{\mathbf{v}}_T = (\mathbf{v}_T, (\mathbf{v}_F)_{F \in \mathcal{F}_T})$. The vector of polynomials corresponding to a smooth function over Ω is obtained via the global interpolation operator $\underline{I}_h^k : \mathbf{H}^1(\Omega) \rightarrow \underline{U}_h^k$ such that, for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$,

$$\underline{I}_h^k \mathbf{v} := ((\pi_T^{k^\star} \mathbf{v}|_T)_{T \in \mathcal{T}_h}, (\pi_F^k \mathbf{v}|_F)_{F \in \mathcal{F}_h}).$$

Its restriction to a generic mesh element $T \in \mathcal{T}_h$, collecting the components on T and its faces, is denoted by \underline{I}_T^k . We furnish \underline{U}_h^k with the discrete H^1 -like seminorm such that, for all $\underline{\mathbf{v}}_h \in \underline{U}_h^k$,

$$\|\underline{\mathbf{v}}_h\|_{1,h} := \left(\sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{v}}_T\|_{1,T}^2 \right)^{1/2}, \quad (7)$$

where, for all $T \in \mathcal{T}_h$,

$$\|\underline{\mathbf{v}}_T\|_{1,T} := \left(\|\nabla \mathbf{v}_T\|_{\mathbf{L}^2(T)}^2 + |\underline{\mathbf{v}}_T|_{1,\partial T}^2 \right)^{1/2} \text{ with } |\underline{\mathbf{v}}_T|_{1,\partial T} := \left(\sum_{F \in \mathcal{F}_T} h_F^{-1} \|\mathbf{v}_F - \mathbf{v}_T\|_{\mathbf{L}^2(F)}^2 \right)^{1/2}. \quad (8)$$

The discrete spaces for the velocity and the pressure, respectively accounting for the wall boundary condition and the zero-average condition, are

$$\underline{U}_{h,0}^k := \{ \underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T \in \mathcal{T}_h}, (\mathbf{v}_F)_{F \in \mathcal{F}_h}) \in \underline{U}_h^k : \mathbf{v}_F = \mathbf{0} \quad \forall F \in \mathcal{F}_h^b \}, \quad P_h^k := \mathcal{P}^k(\mathcal{T}_h) \cap P.$$

For all $\underline{\mathbf{v}}_h \in \underline{U}_h^k$, we denote by $\mathbf{v}_h \in \mathcal{P}^{k^\star}(\mathcal{T}_h)$ the vector-valued broken polynomial function obtained patching element-based unknowns, that is $(\mathbf{v}_h)|_T := \mathbf{v}_T$ for all $T \in \mathcal{T}_h$. The following discrete Sobolev embeddings in $\underline{U}_{h,0}^k$ have been proved in [28, Proposition 5.4] for the standard HHO space (the modifications required to treat the variation considered here are straightforward): For all $r \in [1, 6]$ it holds, for all $\underline{\mathbf{v}}_h \in \underline{U}_{h,0}^k$,

$$\|\mathbf{v}_h\|_{\mathbf{L}^r(\Omega)} \lesssim \|\underline{\mathbf{v}}_h\|_{1,h}. \quad (9)$$

where the hidden constant is independent of both h and $\underline{\mathbf{v}}_h$, but possibly depends on Ω , k , r , and the mesh regularity parameter. It follows from (9) that the map $\|\cdot\|_{1,h}$ defines a norm on $\underline{U}_{h,0}^k$. Classically, the corresponding dual norm of a linear form $\mathcal{L}_h : \underline{U}_{h,0}^k \rightarrow \mathbb{R}$ is given by

$$\|\mathcal{L}_h\|_{1,h,*} := \sup_{\underline{\mathbf{v}}_h \in \underline{U}_{h,0}^k, \|\underline{\mathbf{v}}_h\|_{1,h}=1} |\mathcal{L}_h(\underline{\mathbf{v}}_h)|. \quad (10)$$

4 A pressure-robust and Reynolds semi-robust HHO scheme

4.1 Divergence-preserving local velocity reconstruction

Following [23], for any element $T \in \mathcal{T}_h$ we define the discrete divergence operator $D_T^k : \underline{U}_T^k \rightarrow \mathcal{P}^k(T)$ such that, for all $\underline{v}_T \in \underline{U}_T^k$ and all $q \in \mathcal{P}^k(T)$,

$$\int_T D_T^k \underline{v}_T q = - \int_T \mathbf{v}_T \cdot \nabla q + \sum_{F \in \mathcal{F}_T} \int_F (\mathbf{v}_F \cdot \mathbf{n}_{TF}) q. \quad (11)$$

The operator D_T^k satisfies the following crucial commutation property (see [20, Eq. (8.21)]):

$$D_T^k \mathbf{I}_T^k \mathbf{v} = \pi_T^k(\nabla \cdot \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(T). \quad (12)$$

To achieve pressure robustness, we proceed similarly to [16, Section 2.4], constructing divergence-preserving velocity test functions which are then used for the discretization of the body force and the nonlinear term. Let an element $T \in \mathcal{T}_h$ be fixed and, for $\tau \in \mathfrak{T}_T$, denote by $\mathcal{RT}^k(\tau)$ the classical Raviart–Thomas–Nédélec space of degree k on τ [42, 43]. We recall that a function in $\mathcal{RT}^k(\tau)$ is uniquely determined by its polynomial moments of degree up to $(k-1)$ inside τ and the polynomial moments of degree k of its normal components on the simplicial faces of τ , collected in the set \mathfrak{F}_τ . We introduce the Raviart–Thomas–Nédélec space of degree k on the matching simplicial submesh \mathfrak{T}_T of T defined as follows:

$$\mathcal{RT}^k(\mathfrak{T}_T) := \{ \mathbf{w} \in \mathbf{H}_{\text{div}}(T) : \mathbf{w}|_\tau \in \mathcal{RT}^k(\tau) \text{ for all } \tau \in \mathfrak{T}_T \},$$

where $\mathbf{H}_{\text{div}}(T) := \{ \mathbf{w} \in L^2(T) : \nabla \cdot \mathbf{w} \in L^2(T) \}$.

Recalling from Section 3.1 that, for a given element $T \in \mathcal{T}_h$, we denote by \mathbf{x}_T the common vertex of all simplices in \mathfrak{T}_T , we introduce the following spaces generated by the Koszul operator (see [3, Section 7.2]):

$$\begin{aligned} \mathcal{G}^{c,k}(T) &:= (\mathbf{x} - \mathbf{x}_T) \perp \mathcal{P}^{k-1}(T), & \mathcal{G}^{c,k}(\mathfrak{T}_T) &:= (\mathbf{x} - \mathbf{x}_T) \perp \mathcal{P}^{k-1}(\mathfrak{T}_T) \quad \text{for } k \geq 1, d = 2, \\ \mathcal{G}^{c,k}(T) &:= (\mathbf{x} - \mathbf{x}_T) \times \mathcal{P}^{k-1}(T), & \mathcal{G}^{c,k}(\mathfrak{T}_T) &:= (\mathbf{x} - \mathbf{x}_T) \times \mathcal{P}^{k-1}(\mathfrak{T}_T) \quad \text{for } k \geq 1, d = 3, \end{aligned} \quad (13)$$

where $\mathbf{y} \perp \alpha := \alpha[-y_2, y_1]^T \in \mathbb{R}^2$ for $\mathbf{y} \in \mathbb{R}^2$ and a scalar α , and \times is the usual cross product in \mathbb{R}^3 . In addition, we define $\mathcal{G}^{c,-1}(T) := \mathcal{G}^{c,0}(T) := \mathcal{G}^{c,-1}(\mathfrak{T}_T) := \mathcal{G}^{c,0}(\mathfrak{T}_T) := \{\mathbf{0}\}$. Observe that we have the following direct decompositions (see [3, Corollary 7.4]):

$$\mathcal{P}^k(T) = \mathcal{G}^k(T) \oplus \mathcal{G}^{c,k}(T) \quad \text{and} \quad \mathcal{P}^k(\mathfrak{T}_T) = \mathcal{G}^k(\mathfrak{T}_T) \oplus \mathcal{G}^{c,k}(\mathfrak{T}_T), \quad (14)$$

where $\mathcal{P}^k(\mathfrak{T}_T)$ is the broken polynomial spaces of total degree $\leq k$ on \mathfrak{T}_T and

$$\mathcal{G}^k(T) := \nabla \mathcal{P}^{k+1}(T) \quad \text{and} \quad \mathcal{G}^k(\mathfrak{T}_T) := \nabla \mathcal{P}^{k+1}(\mathfrak{T}_T), \quad (15)$$

and the direct sums in (14) are not necessarily orthogonal. Notice that, with a little abuse of notation, in the definition (15) of $\mathcal{G}^k(\mathfrak{T}_T)$ we have used the symbol ∇ for the piecewise gradient on \mathfrak{T}_T . This abuse of notation will be kept throughout the rest of the paper.

We denote the L^2 -orthogonal projectors onto the spaces $\mathcal{G}^k(T)$, $\mathcal{G}^k(\mathfrak{T}_T)$, $\mathcal{G}^{c,k}(T)$, and $\mathcal{G}^{c,k}(\mathfrak{T}_T)$ by $\pi_{\mathcal{G},T}^k$, $\pi_{\mathcal{G},\mathfrak{T}_T}^k$, $\pi_{\mathcal{G},T}^{c,k}$ and $\pi_{\mathcal{G},\mathfrak{T}_T}^{c,k}$, respectively. Then the *divergence-preserving velocity reconstruction* $\mathbf{R}_T^k : \underline{U}_T^k \rightarrow \mathcal{RT}^k(\mathfrak{T}_T)$ is defined, for all $\underline{v}_T \in \underline{U}_T^k$, as the first component of the solution of the

following mixed problem: Find $(\mathbf{R}_T^k \underline{\mathbf{v}}_T, \psi, \boldsymbol{\theta}) \in \mathcal{RT}^k(\mathfrak{T}_T) \times \mathcal{P}^k(\mathfrak{T}_T) \times \mathcal{G}^{c,k-1}(\mathfrak{T}_T)$ such that

$$(\mathbf{R}_T^k \underline{\mathbf{v}}_T)|_\sigma \cdot \mathbf{n}_\sigma = (\mathbf{v}_F \cdot \mathbf{n}_{TF})|_\sigma \quad \forall \sigma \in \mathfrak{F}_F, \forall F \in \mathcal{F}_T, \quad (16a)$$

$$\int_T (\nabla \cdot \mathbf{R}_T^k \underline{\mathbf{v}}_T) \phi = \int_T (D_T^k \underline{\mathbf{v}}_T) \phi \quad \forall \phi \in \mathcal{P}^k(\mathfrak{T}_T), \quad (16b)$$

$$\int_T \mathbf{R}_T^k \underline{\mathbf{v}}_T \cdot \boldsymbol{\xi} = \int_T \mathbf{v}_T \cdot \boldsymbol{\xi} \quad \forall \boldsymbol{\xi} \in \mathcal{G}^{c,k-1}(\mathfrak{T}_T), \quad (16c)$$

$$\int_T \mathbf{R}_T^k \underline{\mathbf{v}}_T \cdot \mathbf{w} + \int_T (\nabla \cdot \mathbf{w}) \psi + \int_T \mathbf{w} \cdot \boldsymbol{\theta} = \int_T \mathbf{v}_T \cdot \mathbf{w} \quad \forall \mathbf{w} \in \mathcal{RT}_0^k(\mathfrak{T}_T), \quad (16d)$$

with $\mathcal{RT}_0^k(\mathfrak{T}_T)$ denoting the subspace of $\mathcal{RT}^k(\mathfrak{T}_T)$ spanned by functions whose normal component vanishes on ∂T .

Remark 1 (Comparison with the $\mathbf{H}_{\text{div}}(\Omega)$ -conforming reconstruction of [16]). The above definition of the operator \mathbf{R}_T^k is a variant of the one originally proposed in [16, Section 2.4]. Specifically, there are two differences: the first one consists in changing the last component of the trial and test spaces from $\mathcal{G}^{c,k-1}(T)$ to $\mathcal{G}^{c,k-1}(\mathfrak{T}_T)$; the second one is the use of the HHO space variant defined in (6).

Lemma 2 (Properties of \mathbf{R}_T^k). *For all $T \in \mathcal{T}_h$, it holds:*

- (i) Well-posedness. *For a given $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$, there exists a unique $\mathbf{R}_T^k \underline{\mathbf{v}}_T \in \mathcal{RT}^k(\mathfrak{T}_T)$ that solves (16) and satisfies*

$$\|\mathbf{v}_T - \mathbf{R}_T^k \underline{\mathbf{v}}_T\|_{L^2(T)} \lesssim h_T \|\underline{\mathbf{v}}_T\|_{1,T}. \quad (17)$$

- (ii) Approximation in $\mathbf{W}^{s,p}$. *Let an integer $p \in [1, \infty]$ be given. Then, for all $s \in \{1, \dots, k+1\}$, $m \in \{0, 1\}$, and all $\mathbf{v} \in \mathbf{W}^{s,p}(T)$, it holds*

$$|\mathbf{v} - \mathbf{R}_T^k(\underline{\mathbf{I}}_T^k \mathbf{v})|_{\mathbf{W}^{m,p}(\mathfrak{T}_T)} \lesssim h_T^{s-m} |\mathbf{v}|_{\mathbf{W}^{s,p}(T)}. \quad (18)$$

- (iii) Consistency. *For all $k \geq 1$ and all $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$, it holds*

$$\boldsymbol{\pi}_T^{k-1}(\mathbf{R}_T^k \underline{\mathbf{v}}_T) = \boldsymbol{\pi}_T^{k-1}(\mathbf{v}_T). \quad (19)$$

The proof makes use of the following Lemma, whose proof is given in [16, Appendix]

Lemma 3 (Raviart–Thomas–Nédélec lifting of the projection on $\mathcal{G}^{c,k-1}(\mathfrak{T}_T)$). *Let $T \in \mathcal{T}_h$ and a function $\mathbf{v} \in L^2(T)$ be given. Then, for any integer $k \geq 0$, there exists $\widetilde{\mathbf{R}}_T^k(\mathbf{v}) \in \mathcal{RT}_0^k(\mathfrak{T}_T)$ such that*

$$\begin{aligned} \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{T}_T}^{c,k-1} \widetilde{\mathbf{R}}_T^k(\mathbf{v}) &= \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{T}_T}^{c,k-1} \mathbf{v}, \\ \nabla \cdot \widetilde{\mathbf{R}}_T^k(\mathbf{v}) &= 0, \\ \widetilde{\mathbf{R}}_T^k(\mathbf{v}) \cdot \mathbf{n}_\sigma &= 0 \quad \forall \sigma \in \mathfrak{F}_T^i, \\ \|\widetilde{\mathbf{R}}_T^k(\mathbf{v})\|_{L^2(T)} &\lesssim \|\mathbf{v}\|_{L^2(T)}, \end{aligned}$$

where we remind the reader that \mathfrak{F}_T^i is the set of the interior faces of the submesh of T .

Proof of Lemma 2. (i) Existence, uniqueness, and boundedness. The proof of existence and uniqueness of $\mathbf{R}_T^k \underline{\mathbf{v}}_T$ are the same as in [16, Lemma 2.i] using Lemma 3 above. We now prove the bound (17). Using essentially the same steps as in [16], it is inferred that

$$\|\mathbf{R}_T^k \underline{\mathbf{v}}_T\|_{L^2(T)} \lesssim \|\mathbf{v}_T\|_{L^2(T)} + h_T \|\underline{\mathbf{v}}_T\|_{1,T} + \sum_{F \in \mathcal{F}_T} h_F^{\frac{1}{2}} \|\mathbf{v}_F\|_{L^2(F)}. \quad (20)$$

Using a triangle inequality after inserting $\pm(\pi_T^k \mathbf{v}_T - \mathbf{R}_T^k(\underline{\mathbf{I}}_T^k \pi_T^k \mathbf{v}_T))$ into the norm in the left-hand side, we have

$$\begin{aligned} \|\mathbf{v}_T - \mathbf{R}_T^k \underline{\mathbf{v}}_T\|_{\mathbf{L}^2(T)} &\leq \|\mathbf{v}_T - \pi_T^k \mathbf{v}_T\|_{\mathbf{L}^2(T)} + \|\pi_T^k \mathbf{v}_T - \mathbf{R}_T^k(\underline{\mathbf{I}}_T^k \pi_T^k \mathbf{v}_T)\|_{\mathbf{L}^2(T)} \\ &\quad + \|\mathbf{R}_T^k(\underline{\mathbf{I}}_T^k \pi_T^k \mathbf{v}_T) - \mathbf{R}_T^k \underline{\mathbf{v}}_T\|_{\mathbf{L}^2(T)} =: \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3. \end{aligned} \quad (21)$$

To bound \mathfrak{I}_1 , we use the approximation properties (4) of π_T^k with $(l, m, r, s) = (k, 0, 2, 1)$ to get

$$\mathfrak{I}_1 \lesssim h_T |\mathbf{v}_T|_{\mathbf{H}^1(T)} \stackrel{(8)}{\lesssim} h_T \|\underline{\mathbf{v}}_T\|_{1,T}. \quad (22)$$

Letting $\hat{\mathbf{v}}_T := \mathbf{R}_T^k(\underline{\mathbf{I}}_T^k \pi_T^k \mathbf{v}_T)$ and using the same arguments as in [16], we obtain

$$\mathfrak{I}_2 = 0. \quad (23)$$

By linearity, $\mathfrak{I}_3 = \|\mathbf{R}_T^k(\underline{\mathbf{I}}_T^k \pi_T^k \mathbf{v}_T - \underline{\mathbf{v}}_T)\|_{\mathbf{L}^2(T)}$. Thus, using the bound (20) with $\underline{\mathbf{v}}_T$ replaced by $\underline{\mathbf{I}}_T^k \pi_T^k \mathbf{v}_T - \underline{\mathbf{v}}_T$, the fact that $(\underline{\mathbf{I}}_T^k \pi_T^k \mathbf{v}_T - \underline{\mathbf{v}}_T)_T = (\pi_T^k \mathbf{v}_T - \mathbf{v}_T)$ and $(\underline{\mathbf{I}}_T^k \pi_T^k \mathbf{v}_T - \underline{\mathbf{v}}_T)_F = (\pi_T^k \mathbf{v}_T - \mathbf{v}_F)$ for all $F \in \mathcal{F}_T$, and recalling the definition (8) of the $\|\cdot\|_{1,T}$ -norm, we can write

$$\begin{aligned} \mathfrak{I}_3 &\lesssim \|\pi_T^k \mathbf{v}_T - \mathbf{v}_T\|_{\mathbf{L}^2(T)} + h_T \|\underline{\mathbf{I}}_T^k \pi_T^k \mathbf{v}_T - \underline{\mathbf{v}}_T\|_{1,T} + \sum_{F \in \mathcal{F}_T} h_F^{\frac{1}{2}} \|\pi_T^k \mathbf{v}_T - \mathbf{v}_F\|_{\mathbf{L}^2(F)} \\ &\lesssim h_T |\mathbf{v}_T|_{\mathbf{H}^1(T)} + h_T |\pi_T^k \mathbf{v}_T - \mathbf{v}_T|_{\mathbf{H}^1(T)} + \sum_{F \in \mathcal{F}_T} h_F^{\frac{1}{2}} \left(\|\pi_T^k \mathbf{v}_T - \mathbf{v}_T\|_{\mathbf{L}^2(F)} + \|\pi_T^k \mathbf{v}_T - \mathbf{v}_F\|_{\mathbf{L}^2(F)} \right) \\ &\lesssim h_T |\mathbf{v}_T|_{\mathbf{H}^1(T)} + \sum_{F \in \mathcal{F}_T} h_F^{\frac{1}{2}} \|\pi_T^k \mathbf{v}_T - \mathbf{v}_T\|_{\mathbf{L}^2(F)} + \sum_{F \in \mathcal{F}_T} h_F^{\frac{1}{2}} \|\mathbf{v}_T - \mathbf{v}_F\|_{\mathbf{L}^2(F)} \\ &\lesssim h_T |\mathbf{v}_T|_{\mathbf{H}^1(T)} + h_T |\underline{\mathbf{v}}_T|_{1,\partial T} \lesssim h_T \left(|\mathbf{v}_T|_{\mathbf{H}^1(T)}^2 + |\underline{\mathbf{v}}_T|_{1,\partial T}^2 \right)^{\frac{1}{2}} \end{aligned}$$

where, in the second step, we have used the approximation properties (4) of π_T^k with $(l, m, r, s) = (k, 0, 2, 1)$ for the first addend, expanded the second addend using the definition (8) of $\|\cdot\|_{1,T}$, and used a triangle inequality in the boundary term; in the third step, we have used (4) with $(l, m, r, s) = (k, 1, 2, 1)$ for the second addend, added $\pm \mathbf{v}_T$ inside the norm of the last term, and used a triangle inequality; in the fourth step, we have used for the second addend the trace approximation properties (5) of π_T^k with $(l, r, s) = (k, 2, 1)$ along with $h_F \simeq h_T$ and the definition (8) of $|\cdot|_{1,\partial T}$ for the third addend; finally, in the last step, we have used the inequality

$$\left(\sum_{i=1}^n a_i \right)^2 \simeq \sum_{i=1}^n a_i^2, \quad (24)$$

valid for any integer $n \geq 1$ and non-negative real numbers a_i . This gives

$$\mathfrak{I}_3 \lesssim h_T \|\underline{\mathbf{v}}_T\|_{1,T}, \quad (25)$$

Plugging (22), (23), and (25) into (21), and using the definition (8) of the $\|\cdot\|_{1,T}$ -norm, we obtain (17).

(ii) *Approximation in $\mathbf{W}^{s,p}$.* For the sake of brevity, set $\mathbf{\Pi}_T^k := \mathbf{R}_T^k \circ \underline{\mathbf{I}}_T^k$. To prove (18), we use [20, Theorem 1.43], which requires to prove the following relations:

(a) *Polynomial consistency.* $\mathbf{\Pi}_T^k \mathbf{q} = \mathbf{q}$ for all $\mathbf{q} \in \mathcal{P}^k(T)$.

(b) *Boundedness.* For all integer $p \in [1, \infty]$ and all $\mathbf{v} \in \mathbf{W}^{1,p}(T)$, it holds

$$\|\mathbf{\Pi}_T^k \mathbf{v}\|_{\mathbf{L}^p(T)} \lesssim \|\mathbf{v}\|_{\mathbf{L}^p(T)} + h_T |\mathbf{v}|_{\mathbf{W}^{1,p}(T)}, \quad (26)$$

$$|\mathbf{\Pi}_T^k \mathbf{v}|_{\mathbf{W}^{1,p}(\mathfrak{T}_T)} \lesssim |\mathbf{v}|_{\mathbf{W}^{1,p}(T)}. \quad (27)$$

We prove Point (a) using the same arguments that lead to (23) along with the fact that π_T^k is a projection onto $\mathcal{P}^k(T)$. To prove (26), let $\mathbf{v} \in \mathbf{W}^{1,p}(T)$ and set, for the sake of brevity, $\hat{\mathbf{v}}_T := \underline{\mathbf{I}}_T^k \mathbf{v}$. We have that

$$\begin{aligned} \|\mathbf{\Pi}_T^k \mathbf{v}\|_{\mathbf{L}^p(T)}^2 &= \|\mathbf{R}_T^k \hat{\mathbf{v}}_T\|_{\mathbf{L}^p(T)}^2 \\ &\lesssim h_T^{2d\left(\frac{1}{p}-\frac{1}{2}\right)} \|\mathbf{R}_T^k \hat{\mathbf{v}}_T\|_{\mathbf{L}^2(T)}^2 \\ &\lesssim h_T^{2d\left(\frac{1}{p}-\frac{1}{2}\right)} \left(\|\hat{\mathbf{v}}_T\|_{\mathbf{L}^2(T)}^2 + h_T^2 \|\hat{\mathbf{v}}_T\|_{1,T}^2 \right) \\ &\lesssim \|\hat{\mathbf{v}}_T\|_{\mathbf{L}^p(T)}^2 + h_T^2 h_T^{2d\left(\frac{1}{p}-\frac{1}{2}\right)} \|\hat{\mathbf{v}}_T\|_{1,T}^2 \\ &\lesssim \|\hat{\mathbf{v}}_T\|_{\mathbf{L}^p(T)}^2 + h_T^2 \|\nabla \hat{\mathbf{v}}_T\|_{\mathbf{L}^p(T)}^2 + h_T^2 \sum_{F \in \mathcal{F}_T} h_T^{2d\left(\frac{1}{p}-\frac{1}{2}\right)} h_F^{-1} \|\hat{\mathbf{v}}_F - \hat{\mathbf{v}}_T\|_{\mathbf{L}^2(F)}^2, \end{aligned} \quad (28)$$

where: in the second step, we have used the discrete local Lebesgue embeddings (93) below with $(\alpha, \beta, X) = (p, 2, T)$; in the third step, we have inserted $\pm \hat{\mathbf{v}}_T$ into the norm and used a triangle inequality followed by (17) with $\mathbf{v}_T = \hat{\mathbf{v}}_T$; in the fourth step, we have used again the local discrete Lebesgue embeddings (93) below, this time with $(\alpha, \beta, X) = (2, p, T)$, for the first term; the last step follows from the definition (8) of $\|\cdot\|_{1,T}$ along with (93) with $(\alpha, \beta, X) = (2, p, T)$ for the gradient term. To bound the last term in the right hand side of (28), we observe that, for all $F \in \mathcal{F}_T$, it holds

$$\begin{aligned} h_T^{2d\left(\frac{1}{p}-\frac{1}{2}\right)} h_F^{-1} \|\hat{\mathbf{v}}_F - \hat{\mathbf{v}}_T\|_{\mathbf{L}^2(F)}^2 &\lesssim h_T^{2d\left(\frac{1}{p}-\frac{1}{2}\right)} h_F^{-1} |F|^{2\left(\frac{1}{2}-\frac{1}{p}\right)} \|\hat{\mathbf{v}}_F - \hat{\mathbf{v}}_T\|_{\mathbf{L}^p(F)}^2 \\ &\lesssim h_F^{2\left(\frac{1-p}{p}\right)} \|\hat{\mathbf{v}}_F - \hat{\mathbf{v}}_T\|_{\mathbf{L}^p(F)}^2, \end{aligned} \quad (29)$$

where in the first step we have used (94) below with $(\alpha, \beta, X) = (2, p, F)$, and in the last step the relations $|F| \simeq h_F^{(d-1)}$ and $h_T \simeq h_F$ valid for regular meshes. Moreover, using (24) with $n = 2$ along with the idempotency of π_F^k , we have

$$\begin{aligned} \|\hat{\mathbf{v}}_F - \hat{\mathbf{v}}_T\|_{\mathbf{L}^p(F)}^2 &\lesssim \|\pi_F^k(\mathbf{v} - \pi_T^k \mathbf{v})\|_{\mathbf{L}^p(F)}^2 + \|\pi_T^k \mathbf{v} - \hat{\mathbf{v}}_T\|_{\mathbf{L}^p(F)}^2 \\ &\lesssim \|\mathbf{v} - \pi_T^k \mathbf{v}\|_{\mathbf{L}^p(F)}^2 + \|\mathbf{v} - \hat{\mathbf{v}}_T\|_{\mathbf{L}^p(F)}^2, \end{aligned}$$

where, in the second line, we have used the \mathbf{L}^p -boundedness of π_F^k (cf. [20, Lemma 1.44]) for the first term, inserted $\pm \mathbf{v}$ and used a triangle inequality followed by (24) with $n = 2$ for the second term. Plugging the inequality above into (29), then using the result into (28), taking the square root, and then using (24), we obtain

$$\|\mathbf{\Pi}_T^k \mathbf{v}\|_{\mathbf{L}^p(T)} \lesssim \|\hat{\mathbf{v}}_T\|_{\mathbf{L}^p(T)} + h_T \|\nabla \hat{\mathbf{v}}_T\|_{\mathbf{L}^p(T)} + h_T \sum_{F \in \mathcal{F}_T} h_F^{\frac{1-p}{p}} \left(\|\mathbf{v} - \pi_T^k \mathbf{v}\|_{\mathbf{L}^p(F)} + \|\mathbf{v} - \hat{\mathbf{v}}_T\|_{\mathbf{L}^p(F)} \right).$$

Therefore, using, respectively, the \mathbf{L}^p - and $\mathbf{W}^{1,p}$ -boundedness of π_T^{k*} for the first and second term, a triangle inequality along with $h_T \lesssim 1$ followed by the trace approximation properties (5) of the L^2 -orthogonal projector with $(l, r, s) = (k, p, 1)$ for the first term in brackets, and with $(l, r, s) =$

$(k^*, p, 1)$ for the second term in brackets, and, finally, using the geometric bound (3) for \mathcal{F}_T , we obtain (26).

Let us now prove (27). First of all we notice that, by the discrete inverse inequality (95) below, it holds

$$\|\nabla \mathbf{R}_T^k \underline{\mathbf{v}}_T\|_{\mathbf{L}^2(T)} \lesssim h_T^{-1} \|\mathbf{R}_T^k \underline{\mathbf{v}}_T\|_{\mathbf{L}^2(T)}. \quad (30)$$

Thus, recalling that $\mathbf{\Pi}_T^k := \mathbf{R}_T^k \circ \mathbf{I}_T^k$, we infer

$$\begin{aligned} |\mathbf{\Pi}_T^k \underline{\mathbf{v}}|_{\mathbf{W}^{1,p}(T)} &= |\mathbf{\Pi}_T^k (\underline{\mathbf{v}} - \boldsymbol{\pi}_T^0 \underline{\mathbf{v}})|_{\mathbf{W}^{1,p}(T)} \stackrel{(30)}{\lesssim} h_T^{-1} \|\mathbf{\Pi}_T^k (\underline{\mathbf{v}} - \boldsymbol{\pi}_T^0 \underline{\mathbf{v}})\|_{\mathbf{L}^p(T)} \\ &\stackrel{(26)}{\lesssim} h_T^{-1} \|\underline{\mathbf{v}} - \boldsymbol{\pi}_T^0 \underline{\mathbf{v}}\|_{\mathbf{L}^p(T)} + |\underline{\mathbf{v}} - \boldsymbol{\pi}_T^0 \underline{\mathbf{v}}|_{\mathbf{W}^{1,p}(T)} \lesssim |\underline{\mathbf{v}}_T|_{\mathbf{W}^{1,p}(T)}, \end{aligned}$$

where in the last step we have used the approximation properties (4) of $\boldsymbol{\pi}_T^0$ with $(l, m, r, s) = (0, 0, p, 1)$ for the first term and with $(l, m, r, s) = (0, 1, p, 1)$ for the second term along with a triangle inequality and the fact that $h_T \lesssim 1$.

(iii) *Consistency.* The proof follows the same steps as in [16] along with Lemma 3 above and the fact that $\boldsymbol{\pi}_{\mathcal{G},T}^{c,k-1} \circ \boldsymbol{\pi}_{\mathcal{G},\mathcal{T}_T}^{c,k-1} = \boldsymbol{\pi}_{\mathcal{G},T}^{c,k-1}$, since $\mathcal{G}^{c,k-1}(T) \subset \mathcal{G}^{c,k-1}(\mathcal{T}_T)$. \square

Let now $\mathcal{RT}^k(\mathcal{T}_h)$ denote the global ($\mathbf{H}_{\text{div}}(\Omega)$ -conforming) Raviart–Thomas–Nédélec space on \mathcal{T}_h . We define the *global velocity reconstruction* $\mathbf{R}_h^k : \underline{\mathbf{U}}_h^k \rightarrow \mathcal{RT}^k(\mathcal{T}_h)$ patching the local contributions: For all $\underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_h^k$,

$$(\mathbf{R}_h^k \underline{\mathbf{v}}_h)|_T := \mathbf{R}_T^k \underline{\mathbf{v}}_T \quad \forall T \in \mathcal{T}_h.$$

Notice that $\mathbf{R}_h^k \underline{\mathbf{v}}_h$ is well-defined, since its normal components across mesh interfaces are continuous as a consequence of (16a) combined with the single-valuedness of interface unknowns.

To discretize the unsteady term in (2), for all $T \in \mathcal{T}_h$ we introduce the bilinear form $a_{R,T} : \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \rightarrow \mathbb{R}$ defined as follows:

$$a_{R,T}(\underline{\mathbf{v}}_T, \underline{\mathbf{w}}_T) := \int_T \mathbf{R}_T^k \underline{\mathbf{v}}_T \cdot \mathbf{R}_T^k \underline{\mathbf{w}}_T + s_{R,T}(\underline{\mathbf{v}}_T, \underline{\mathbf{w}}_T), \quad (31)$$

where the second term is the following stabilisation bilinear form:

$$s_{R,T}(\underline{\mathbf{v}}_T, \underline{\mathbf{w}}_T) := \int_T \boldsymbol{\delta}_{R,T}^k \underline{\mathbf{v}}_T \cdot \boldsymbol{\delta}_{R,T}^k \underline{\mathbf{w}}_T + \sum_{F \in \mathcal{F}_T} h_F \int_F \boldsymbol{\delta}_{R,TF}^k \underline{\mathbf{v}}_T \cdot \boldsymbol{\delta}_{R,TF}^k \underline{\mathbf{w}}_T,$$

with difference operators $\boldsymbol{\delta}_{R,T}^k : \underline{\mathbf{U}}_T^k \rightarrow \mathcal{P}^{k^*}(T)$ and, for all $F \in \mathcal{F}_T$, $\boldsymbol{\delta}_{R,TF}^k : \underline{\mathbf{U}}_T^k \rightarrow \mathcal{P}^k(F)$ such that, for all $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$,

$$\boldsymbol{\delta}_{R,T}^k(\underline{\mathbf{v}}_T) := \boldsymbol{\pi}_T^{k^*}(\mathbf{R}_T^k \underline{\mathbf{v}}_T - \underline{\mathbf{v}}_T) \quad \text{and} \quad \boldsymbol{\delta}_{R,TF}^k(\underline{\mathbf{v}}_T) := \boldsymbol{\pi}_F^k(\mathbf{R}_T^k \underline{\mathbf{v}}_T - \underline{\mathbf{v}}_F) \quad \forall F \in \mathcal{F}_T. \quad (32)$$

We also introduce the \mathbf{L}^2 -like norm $\|\cdot\|_{R,T} : \underline{\mathbf{U}}_T^k \rightarrow \mathbb{R}$ induced by $a_{R,T}(\cdot, \cdot)$, i.e.,

$$\|\underline{\mathbf{v}}_T\|_{R,T} := a_{R,T}(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T)^{\frac{1}{2}}. \quad (33)$$

Finally, we introduce the global bilinear form $a_{R,h} : \underline{\mathbf{U}}_h^k \times \underline{\mathbf{U}}_h^k \rightarrow \mathbb{R}$, and the corresponding global norm $\|\cdot\|_{R,h} : \underline{\mathbf{U}}_h^k \rightarrow \mathbb{R}$ defined, respectively, by: For all $(\underline{\mathbf{v}}_h, \underline{\mathbf{w}}_h) \in \underline{\mathbf{U}}_h^k \times \underline{\mathbf{U}}_h^k$,

$$a_{R,h}(\underline{\mathbf{v}}_h, \underline{\mathbf{w}}_h) := \sum_{T \in \mathcal{T}_h} a_{R,T}(\underline{\mathbf{v}}_T, \underline{\mathbf{w}}_T), \quad \text{and} \quad \|\underline{\mathbf{v}}_h\|_{R,h}^2 := \sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{v}}_T\|_{R,T}^2.$$

The following Lemma summarizes some key properties relevant for the analysis.

Lemma 4 (Properties of $\|\cdot\|_{\mathbb{R},T}$). For all $T \in \mathcal{T}_h$ and all $\underline{v}_T \in \underline{U}_T^k$, it holds

$$\|\underline{v}_T\|_{L^2(T)} + h_T \|\underline{v}_T\|_{1,T} \lesssim \|\underline{v}_T\|_{\mathbb{R},T}. \quad (34)$$

Remark 5 (Norm $\|\cdot\|_{\mathbb{R},h}$). A straightforward consequence of the above results is that the map $\|\cdot\|_{\mathbb{R},h} : \underline{U}_h^k \rightarrow \mathbb{R}$ is a norm in the space $\underline{U}_{h,0}^k$.

Proof of Lemma 4. Using the fact that $\underline{v}_T \in \mathcal{P}^{k^*}(T)$ along with (24) for $n = 2$, we get

$$\begin{aligned} \|\underline{v}_T\|_{L^2(T)}^2 &= \|\pi_T^{k^*} \underline{v}_T\|_{L^2(T)}^2 \lesssim \|\pi_T^{k^*} (\underline{v}_T - \mathbf{R}_T^k \underline{v}_T)\|_{L^2(T)}^2 + \|\pi_T^{k^*} \mathbf{R}_T^k \underline{v}_T\|_{L^2(T)}^2 \\ &\leq \|\pi_T^{k^*} (\underline{v}_T - \mathbf{R}_T^k \underline{v}_T)\|_{L^2(T)}^2 + \|\mathbf{R}_T^k \underline{v}_T\|_{L^2(T)}^2 \leq \|\underline{v}_T\|_{\mathbb{R},T}^2, \end{aligned}$$

where we have used the L^2 -boundedness of $\pi_T^{k^*}$ in the second line and the definition (33) of $\|\cdot\|_{\mathbb{R},T}$ to conclude.

To bound the second term in the left-hand side of (34) we preliminarily observe that, for any face $F \in \mathcal{F}_T$, it holds

$$\begin{aligned} \|\underline{v}_F - \underline{v}_T\|_{L^2(F)} &= \|\underline{v}_F - \pi_F^k \mathbf{R}_T^k \underline{v}_T + \pi_F^k \mathbf{R}_T^k \underline{v}_T - \underline{v}_T\|_{L^2(F)} \\ &\leq \|\pi_F^k (\underline{v}_F - \mathbf{R}_T^k \underline{v}_T)\|_{L^2(F)} + \|\pi_F^k \mathbf{R}_T^k \underline{v}_T - \underline{v}_T\|_{L^2(F)} =: \mathfrak{S}_1 + \mathfrak{S}_2, \end{aligned}$$

where, in the second step, we have used triangle inequalities and then $\pi_F^k \underline{v}_F = \underline{v}_F$ (since $\underline{v}_F \in \mathcal{P}^k(F)$) in the first term. Squaring the above inequality, using (24) with $n = 2$ in the right hand side, and multiplying by h_F both sides, we get

$$h_F \|\underline{v}_F - \underline{v}_T\|_{L^2(F)}^2 \lesssim h_F \mathfrak{S}_1^2 + h_F \mathfrak{S}_2^2. \quad (35)$$

The first term above is bounded as follows:

$$h_F \mathfrak{S}_1^2 \stackrel{(32)}{=} h_F \|\delta_{\mathbb{R},TF}^k(\underline{v}_T)\|_{L^2(F)}^2 \stackrel{(33)}{\leq} \|\underline{v}_T\|_{\mathbb{R},T}^2. \quad (36)$$

For the second term in (35), we begin as follows:

$$\begin{aligned} \mathfrak{S}_2 &= \|\pi_F^k \mathbf{R}_T^k \underline{v}_T + \pi_T^{k^*} \mathbf{R}_T^k \underline{v}_T - \pi_T^{k^*} \mathbf{R}_T^k \underline{v}_T - \underline{v}_T\|_{L^2(F)} \\ &\lesssim \|\mathbf{R}_T^k \underline{v}_T\|_{L^2(F)} + \|\pi_T^{k^*} \mathbf{R}_T^k \underline{v}_T\|_{L^2(F)} + \|\delta_{\mathbb{R},T}^k(\underline{v}_T)\|_{L^2(F)}, \end{aligned}$$

where in the last step, we have used triangle inequalities, the L^2 -boundedness of π_F^k , and the definition (32) of $\delta_{\mathbb{R},T}^k$. Denoting by τ_σ the simplex in \mathfrak{T}_T which contains $\sigma \in \mathfrak{F}_F$ and using a discrete trace inequality along with $h_F \leq h_T \lesssim h_{\tau_\sigma}$ (the second bound being a consequence of mesh regularity), we get

$$h_F \|\mathbf{R}_T^k \underline{v}_T\|_{L^2(F)}^2 \lesssim \sum_{\sigma \in \mathfrak{F}_F} h_{\tau_\sigma} \|\mathbf{R}_T^k \underline{v}_T\|_{L^2(\sigma)}^2 \lesssim \sum_{\sigma \in \mathfrak{F}_F} \|\mathbf{R}_T^k \underline{v}_T\|_{L^2(\tau_\sigma)}^2 \leq \|\mathbf{R}_T^k \underline{v}_T\|_{L^2(T)}^2,$$

where, in the last step, we have used the fact that $\bigcup_{\sigma \in \mathfrak{F}_F} \tau_\sigma \subset T$. Using the same process, we get $h_F \|\pi_T^{k^*} \mathbf{R}_T^k \underline{v}_T\|_{L^2(F)}^2 \lesssim \|\pi_T^{k^*} \mathbf{R}_T^k \underline{v}_T\|_{L^2(T)}^2 \leq \|\mathbf{R}_T^k \underline{v}_T\|_{L^2(T)}^2$, where we have used the L^2 -boundedness of $\pi_T^{k^*}$ to conclude. Thus,

$$h_F \mathfrak{S}_2^2 \lesssim \|\mathbf{R}_T^k \underline{v}_T\|_{L^2(T)}^2 + h_F \|\delta_{\mathbb{R},T}^k(\underline{v}_T)\|_{L^2(F)}^2 \lesssim \|\mathbf{R}_T^k \underline{v}_T\|_{L^2(T)}^2 + \|\delta_{\mathbb{R},T}^k(\underline{v}_T)\|_{L^2(T)}^2 \stackrel{(33)}{\leq} \|\underline{v}_T\|_{\mathbb{R},T}^2,$$

where, in the second step, we have used a discrete trace inequality. Using this bound along with (36) in (35), we get

$$h_F \|\mathbf{v}_F - \mathbf{v}_T\|_{\mathbf{L}^2(F)}^2 \lesssim \|\underline{\mathbf{v}}_T\|_{\mathbf{R},T}^2. \quad (37)$$

Recalling the definition (8) of $\|\underline{\mathbf{v}}_T\|_{1,T}$ and using a discrete inverse inequality, we have that

$$h_T^2 \|\underline{\mathbf{v}}_T\|_{1,T}^2 \lesssim \|\mathbf{v}_T\|_{\mathbf{L}^2(T)}^2 + \sum_{F \in \mathcal{F}_T} h_F \|\mathbf{v}_F - \mathbf{v}_T\|_{\mathbf{L}^2(F)}^2 \lesssim \|\underline{\mathbf{v}}_T\|_{\mathbf{R},T}^2$$

where, in the second step, we have used additionally the inequality $h_T \lesssim h_F$ valid for regular meshes while, in the last, step we have used (34) to bound the first term above and (37) for the second. Taking the square root, we conclude. \square

4.2 Viscous term and pressure-velocity coupling

Let an element $T \in \mathcal{T}_h$ be fixed. We define the local gradient reconstruction $\mathbf{G}_T^k : \underline{\mathbf{U}}_T^k \rightarrow \mathcal{P}^k(T)$ such that, for all $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$ and all $\boldsymbol{\tau} \in \mathcal{P}^k(T)$,

$$\int_T \mathbf{G}_T^l \underline{\mathbf{v}}_T : \boldsymbol{\tau} = - \int_T \mathbf{v}_T \cdot (\nabla \cdot \boldsymbol{\tau}) + \sum_{F \in \mathcal{F}_T} \int_F \mathbf{v}_F \cdot \boldsymbol{\tau} \mathbf{n}_{TF}.$$

A global gradient reconstruction $\mathbf{G}_h^k : \underline{\mathbf{U}}_h^k \rightarrow \mathcal{P}^k(\mathcal{T}_h)$ can be defined setting, for all $\underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_h^k$, $(\mathbf{G}_h^k \underline{\mathbf{v}}_h)|_T := \mathbf{G}_T^k \underline{\mathbf{v}}_T$ for all $T \in \mathcal{T}_h$.

The viscous term and the pressure-velocity coupling are essentially the same as in the standard HHO method; see, e.g., [10, 27]. We briefly recall them here to make the exposition self-contained. The viscous bilinear form $a_h : \underline{\mathbf{U}}_h^k \times \underline{\mathbf{U}}_h^k \rightarrow \mathbb{R}$ is such that, for all $\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_h^k$,

$$a_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h) := \sum_{T \in \mathcal{T}_h} \left[\int_T \mathbf{G}_T^k \underline{\mathbf{w}}_T : \mathbf{G}_T^k \underline{\mathbf{v}}_T + s_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) \right],$$

where, for any $T \in \mathcal{T}_h$, $s_T : \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \rightarrow \mathbb{R}$ denotes a local stabilization bilinear form designed according to the principles of [20, Assumption 2.4], so that the following properties hold:

- (i) *Stability and boundedness.* There exists $C_a > 0$ independent of h (and, clearly, also of ν) such that, for all $\underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_h^k$,

$$C_a \|\underline{\mathbf{v}}_h\|_{1,h}^2 \leq a_h(\underline{\mathbf{v}}_h, \underline{\mathbf{v}}_h) \leq C_a^{-1} \|\underline{\mathbf{v}}_h\|_{1,h}^2. \quad (38)$$

- (ii) *Consistency.* For all $\mathbf{w} \in \mathbf{U} \cap \mathbf{H}^{k+2}(\mathcal{T}_h)$ such that $\Delta \mathbf{w} \in \mathbf{L}^2(\Omega)$, it holds

$$\|\mathcal{E}_{a,h}(\mathbf{w}; \cdot)\|_{1,h,*} \lesssim h^{k+1} |\mathbf{w}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)}, \quad (39)$$

where the linear form $\mathcal{E}_{a,h}(\mathbf{w}; \cdot) : \underline{\mathbf{U}}_{h,0}^k \rightarrow \mathbb{R}$ representing the consistency error is such that

$$\mathcal{E}_{a,h}(\mathbf{w}; \underline{\mathbf{v}}_h) := - \int_{\Omega} \Delta \mathbf{w} \cdot \mathbf{v}_h - a_h(\underline{\mathbf{I}}_h^k \mathbf{w}, \underline{\mathbf{v}}_h) \quad \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_{h,0}^k. \quad (40)$$

A classical example of stabilization bilinear form along with the proofs of properties (38) and (39) can be found in [20, Section 5.1], to which we refer for further details.

Recalling the definition (11) of the local divergence D_T^k , the global pressure-velocity coupling bilinear form $b_h : \underline{\mathbf{U}}_{h,0}^k \times \mathcal{P}^k(\mathcal{T}_h) \rightarrow \mathbb{R}$ is such that, for all $(\underline{\mathbf{v}}_h, q_h) \in \underline{\mathbf{U}}_{h,0}^k \times \mathcal{P}^k(\mathcal{T}_h)$,

$$b_h(\underline{\mathbf{v}}_h, q_h) := - \sum_{T \in \mathcal{T}_h} \int_T D_T^k \underline{\mathbf{v}}_T q_T, \quad (41)$$

where $q_T := q_h|_T$. The properties of b_h relevant for the analysis can be found in [20, Lemma 8.12].

4.3 Body force

The discretization of the body force uses the divergence-preserving velocity reconstruction introduced in Section 4.1. Specifically, we define the bilinear form $\ell_h : L^2(\Omega) \times \underline{U}_h^k \rightarrow \mathbb{R}$ such that, for any $\boldsymbol{\phi} \in L^2(\Omega)$ and any $\underline{\mathbf{v}}_h \in \underline{U}_h^k$,

$$\ell_h(\boldsymbol{\phi}, \underline{\mathbf{v}}_h) := \int_{\Omega} \boldsymbol{\phi} \cdot \mathbf{R}_h^k \underline{\mathbf{v}}_h. \quad (42)$$

Proceeding as in the proof of [16, Lemma 4] (the only difference being the definition of \mathbf{R}_T^k , as discussed in Remark 1), we can prove the following consistency property: For all $\boldsymbol{\phi} \in L^2(\Omega) \cap \mathbf{H}^k(\mathcal{T}_h)$,

$$\|\mathcal{E}_{\ell,h}(\boldsymbol{\phi}; \cdot)\|_{1,h,*} \lesssim h^{k+1} |\boldsymbol{\phi}|_{\mathbf{H}^k(\mathcal{T}_h)},$$

where the linear form $\mathcal{E}_{\ell,h}(\boldsymbol{\phi}; \cdot) : \underline{U}_h^k \rightarrow \mathbb{R}$, representing the consistency error associated with ℓ_h , is such that

$$\mathcal{E}_{\ell,h}(\boldsymbol{\phi}; \underline{\mathbf{v}}_h) := \ell_h(\boldsymbol{\phi}, \underline{\mathbf{v}}_h) - \int_{\Omega} \boldsymbol{\phi} \cdot \mathbf{v}_h = \sum_{T \in \mathcal{T}_h} \int_T \boldsymbol{\phi} \cdot (\mathbf{R}_T^k \underline{\mathbf{v}}_T - \mathbf{v}_T).$$

4.4 Scalar potential operator for convective stabilization

Recall the fact that, for a given element $T \in \mathcal{T}_h$, we denote by \mathbf{x}_T the common vertex of all simplices in \mathfrak{T}_T , as well as the definitions (13) and (15) of $\mathcal{G}^{c,k}(\mathfrak{T}_T)$ and $\mathcal{G}^k(\mathfrak{T}_T)$ along with the respective L^2 -orthogonal projectors $\pi_{\mathcal{G},T}^{c,k}$ and $\pi_{\mathcal{G},\mathfrak{T}_T}^k$. We introduce, for any polynomial degree $l \geq 0$, the potential operator $\varrho_{\mathfrak{T}_T}^{l+1} : \mathcal{P}^l(\mathfrak{T}_T) \rightarrow \mathcal{P}^{l+1}(\mathfrak{T}_T)$ such that, for all $\mathbf{q} \in \mathcal{P}^l(\mathfrak{T}_T)$,

$$\begin{aligned} \nabla \varrho_{\mathfrak{T}_T}^{l+1} \mathbf{q} &= (\text{Id} - \pi_{\mathcal{G},\mathfrak{T}_T}^l \pi_{\mathcal{G},\mathfrak{T}_T}^{c,l})^{-1} (\pi_{\mathcal{G},\mathfrak{T}_T}^l \mathbf{q} - \pi_{\mathcal{G},\mathfrak{T}_T}^l \pi_{\mathcal{G},\mathfrak{T}_T}^{c,l} \mathbf{q}) \\ &\text{and } (\varrho_{\mathfrak{T}_T}^{l+1} \mathbf{q})|_{\tau}(\mathbf{x}_T) = 0 \text{ for all } \tau \in \mathfrak{T}_T, \end{aligned} \quad (43)$$

where Id is the identity operator. Observe that, by definition (15), we have $\nabla \varrho_{\mathfrak{T}_T}^{l+1} \mathbf{q} \in \mathcal{G}^l(\mathfrak{T}_T)$. The relevant properties of the potential operator $\varrho_{\mathfrak{T}_T}^{l+1}$ are stated in Lemma 6 below.

Lemma 6 (Properties of $\varrho_{\mathfrak{T}_T}^{l+1}$). *The potential operator $\varrho_{\mathfrak{T}_T}^{l+1}$ defined in (43) is a linear operator with the following properties:*

(i) For all $\mathbf{b} \in \mathcal{G}^l(\mathfrak{T}_T)$ and all $\mathbf{c} \in \mathcal{G}^{c,l}(\mathfrak{T}_T)$,

$$\nabla \varrho_{\mathfrak{T}_T}^{l+1} \mathbf{b} = \mathbf{b} \quad \text{and} \quad \varrho_{\mathfrak{T}_T}^{l+1} \mathbf{c} \equiv 0. \quad (44)$$

(ii) Boundedness. For all $\mathbf{q} \in \mathcal{P}^l(\mathfrak{T}_T)$,

$$\|\varrho_{\mathfrak{T}_T}^{l+1} \mathbf{q}\|_{L^2(T)} + h_T \|\nabla \varrho_{\mathfrak{T}_T}^{l+1} \mathbf{q}\|_{L^2(T)} \lesssim h_T \|\mathbf{q}\|_{L^2(T)}. \quad (45)$$

(iii) For all $\mathbf{q} \in \mathcal{P}^l(\mathfrak{T}_T)$, it holds,

$$(\text{Id} - \nabla \varrho_{\mathfrak{T}_T}^{l+1}) \mathbf{q} \in \mathcal{G}^{c,l}(\mathfrak{T}_T). \quad (46)$$

(iv) For all $\mathbf{q} \in \mathcal{P}^l(T)$,

$$\varrho_{\mathfrak{T}_T}^{l+1} \mathbf{q} \in \mathcal{P}^{l+1}(T). \quad (47)$$

The proof of Lemma 6 requires some preliminary results. We start by introducing the *recovery operator* (see [19, Section 2.5]) $\mathfrak{R}_{\mathcal{G}, \mathfrak{I}_T}^l : \mathcal{G}^l(\mathfrak{I}_T) \times \mathcal{G}^{c,l}(\mathfrak{I}_T) \rightarrow \mathcal{P}^l(\mathfrak{I}_T)$ defined by

$$\mathfrak{R}_{\mathcal{G}, \mathfrak{I}_T}^l(\mathbf{b}, \mathbf{c}) := \mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^l(\mathbf{b} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l(\mathbf{c})) + \mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^{c,l}(\mathbf{c} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l}(\mathbf{b})), \quad (48)$$

where $\mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^l$ and $\mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^{c,l}$ are linear functions $\mathcal{P}^l(\mathfrak{I}_T) \rightarrow \mathcal{P}^l(\mathfrak{I}_T)$ defined by

$$\mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^l := (\text{Id} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l})^{-1} \quad \text{and} \quad \mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} := (\text{Id} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l)^{-1}. \quad (49)$$

By definition, $\nabla \varrho_{\mathfrak{I}_T}^{l+1} \mathbf{q} = \mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^l(\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \mathbf{q} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \mathbf{q})$.

Lemma 7 (Properties of $\mathfrak{R}_{\mathcal{G}, \mathfrak{I}_T}^l$, $\mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^l$, and $\mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^{c,l}$). *The following holds:*

(i) For all $\mathbf{q} \in \mathcal{P}^l(\mathfrak{I}_T)$,

$$\mathbf{q} = \mathfrak{R}_{\mathcal{G}, \mathfrak{I}_T}^l(\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \mathbf{q}, \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \mathbf{q}). \quad (50)$$

(ii) For all $\mathbf{b} \in \mathcal{G}^l(\mathfrak{I}_T)$, $\mathbf{c} \in \mathcal{G}^{c,l}(\mathfrak{I}_T)$, and $\mathbf{q} \in \mathcal{P}^l(\mathfrak{I}_T)$,

$$\mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^l(\mathbf{b} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \mathbf{q}) \in \mathcal{G}^l(\mathfrak{I}_T) \quad \text{and} \quad \mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^{c,l}(\mathbf{c} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \mathbf{q}) \in \mathcal{G}^{c,l}(\mathfrak{I}_T). \quad (51)$$

(iii) For all $\mathbf{b} \in \mathcal{G}^l(\mathfrak{I}_T)$, and $\mathbf{c} \in \mathcal{G}^{c,l}(\mathfrak{I}_T)$,

$$\mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^l(\mathbf{b} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l}(\mathbf{b})) = \mathbf{b} \quad \text{and} \quad \mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^{c,l}(\mathbf{c} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l(\mathbf{c})) = \mathbf{c}. \quad (52)$$

(iv) For all $\mathbf{q} \in \mathcal{P}^l(\mathfrak{I}_T)$,

$$\|\mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^l \mathbf{q}\|_{L^2(T)} \lesssim \|\mathbf{q}\|_{L^2(T)} \quad \text{and} \quad \|\mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \mathbf{q}\|_{L^2(T)} \lesssim \|\mathbf{q}\|_{L^2(T)}. \quad (53)$$

Proof. (i-ii) The identity (50) is a straightforward consequence of [19, Lemma 1]. We now proceed to prove the first relation in (51). The second one follows using entirely similar arguments. By [19, Lemma 1], it holds $\|\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l}\|_{\mathcal{L}(\mathcal{P}^l(\mathfrak{I}_T), \mathcal{P}^l(\mathfrak{I}_T))} < 1$ for the norm induced by the $L^2(T)$ -norm, so we have the following well-defined expansion:

$$\begin{aligned} \mathbf{\Gamma}_{\mathcal{G}, \mathfrak{I}_T}^l(\mathbf{b} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \mathbf{q}) &= \sum_{n=0}^{\infty} \left(\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \right)^n (\mathbf{b} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \mathbf{q}) \\ &= (\mathbf{b} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \mathbf{q}) + \left\{ \sum_{n=1}^{\infty} \left(\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \right)^n (\mathbf{b} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \mathbf{q}) \right\} =: \mathfrak{I}_1 + \mathfrak{I}_2. \end{aligned}$$

Since $\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l$ is a projector into the space $\mathcal{G}^l(\mathfrak{I}_T)$, we have that $\mathfrak{I}_1 \in \mathcal{G}^l(\mathfrak{I}_T)$. Observe that the infinite sum contained in \mathfrak{I}_2 is well-defined and, to compute it, let $\tilde{\mathbf{q}} := \mathbf{b} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \mathbf{q} \in \mathcal{G}^l(\mathfrak{I}_T)$ and write

$$\begin{aligned} \mathfrak{I}_2 &= \sum_{n=1}^{\infty} \left(\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \right)^n \tilde{\mathbf{q}} = \sum_{n=1}^{\infty} \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \left(\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \right)^{n-1} \tilde{\mathbf{q}} \\ &= \sum_{n=1}^{\infty} \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \left(\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \right)^{n-1} \tilde{\mathbf{q}} = \sum_{n=1}^{\infty} \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \left(\left(\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{I}_T}^{c,l} \right)^n \tilde{\mathbf{q}} \right), \end{aligned}$$

where in the third step we have used the idempotency property $\pi_{\mathcal{G}, \mathfrak{X}_T}^l = \pi_{\mathcal{G}, \mathfrak{X}_T}^l \circ \pi_{\mathcal{G}, \mathfrak{X}_T}^l$. Thus, \mathfrak{X}_2 is equal to a convergent infinite sum where each term of this sum is in $\mathcal{G}^l(\mathfrak{X}_T)$, but this subspace is closed (since it is finite dimensional), thus the limit of the partial sum is in $\mathcal{G}^l(\mathfrak{X}_T)$, i.e., $\mathfrak{X}_2 \in \mathcal{G}^l(\mathfrak{X}_T)$. Therefore, $\mathfrak{X}_1 + \mathfrak{X}_2 \in \mathcal{G}^l(\mathfrak{X}_T)$ and we conclude.

(iii) We now proceed to prove the first relation in (52). The second one is proved similarly. For any $\mathbf{b} \in \mathcal{G}^l(\mathfrak{X}_T)$, expanding $\Gamma_{\mathcal{G}, \mathfrak{X}_T}^l$ as in the previous point, we have that

$$\begin{aligned} \Gamma_{\mathcal{G}, \mathfrak{X}_T}^l (\mathbf{b} - \pi_{\mathcal{G}, \mathfrak{X}_T}^l \pi_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \mathbf{b}) &= \sum_{n=0}^{\infty} \left(\pi_{\mathcal{G}, \mathfrak{X}_T}^l \pi_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \right)^n (\mathbf{b} - \pi_{\mathcal{G}, \mathfrak{X}_T}^l \pi_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \mathbf{b}) \\ &= \sum_{n=0}^{\infty} \left(\pi_{\mathcal{G}, \mathfrak{X}_T}^l \pi_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \right)^n \mathbf{b} - \sum_{n=0}^{\infty} \left(\pi_{\mathcal{G}, \mathfrak{X}_T}^l \pi_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \right)^n \pi_{\mathcal{G}, \mathfrak{X}_T}^l \pi_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \mathbf{b} \\ &= \mathbf{b} + \left[\sum_{n=1}^{\infty} \left(\pi_{\mathcal{G}, \mathfrak{X}_T}^l \pi_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \right)^n \mathbf{b} - \sum_{n=0}^{\infty} \left(\pi_{\mathcal{G}, \mathfrak{X}_T}^l \pi_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \right)^{n+1} \mathbf{b} \right] = \mathbf{b}. \end{aligned}$$

(iv) We prove the first relation in (53). The second one follows from a similar reasoning. Observe first that, for all $\mathbf{q} \in \mathcal{P}^l(\mathfrak{X}_T)$, we can express $\pi_{\mathcal{G}, \mathfrak{X}_T}^l \mathbf{q}$ as $\pi_{\mathcal{G}, \mathfrak{X}_T}^l \mathbf{q} = \sum_{\tau \in \mathfrak{X}_T} \chi_{\tau} \pi_{\mathcal{G}, \tau}^l \mathbf{q}|_{\tau}$, where χ_{τ} is the characteristic function of $\tau \in \mathfrak{X}_T$; similarly, we have $\pi_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \mathbf{q} = \sum_{\tau \in \mathfrak{X}_T} \chi_{\tau} \pi_{\mathcal{G}, \tau}^{c,l} \mathbf{q}|_{\tau}$. Thus,

$$\begin{aligned} \pi_{\mathcal{G}, \mathfrak{X}_T}^l \pi_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \mathbf{q} &= \pi_{\mathcal{G}, \mathfrak{X}_T}^l \left(\sum_{\tau \in \mathfrak{X}_T} \chi_{\tau} \pi_{\mathcal{G}, \tau}^{c,l} \mathbf{q}|_{\tau} \right) \\ &= \sum_{\tau' \in \mathfrak{X}_T} \chi_{\tau'} \pi_{\mathcal{G}, \tau'}^l \left(\sum_{\tau \in \mathfrak{X}_T} \chi_{\tau} \pi_{\mathcal{G}, \tau}^{c,l} \mathbf{q}|_{\tau} \right)_{|\tau'} = \sum_{\tau \in \mathfrak{X}_T} \chi_{\tau} \pi_{\mathcal{G}, \tau}^l \pi_{\mathcal{G}, \tau}^{c,l} \mathbf{q}|_{\tau}, \end{aligned}$$

and, using the same procedure recursively, we obtain

$$\left(\pi_{\mathcal{G}, \mathfrak{X}_T}^l \pi_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \right)^n \mathbf{q} = \sum_{\tau \in \mathfrak{X}_T} \chi_{\tau} \left(\pi_{\mathcal{G}, \tau}^l \pi_{\mathcal{G}, \tau}^{c,l} \right)^n \mathbf{q}|_{\tau},$$

thus it is inferred that

$$\Gamma_{\mathcal{G}, \mathfrak{X}_T}^l \mathbf{q} = \sum_{n=0}^{\infty} \left(\pi_{\mathcal{G}, \mathfrak{X}_T}^l \pi_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \right)^n \mathbf{q} = \sum_{n=0}^{\infty} \sum_{\tau \in \mathfrak{X}_T} \chi_{\tau} \left(\pi_{\mathcal{G}, \tau}^l \pi_{\mathcal{G}, \tau}^{c,l} \right)^n \mathbf{q}|_{\tau} = \sum_{\tau \in \mathfrak{X}_T} \chi_{\tau} \sum_{n=0}^{\infty} \left(\pi_{\mathcal{G}, \tau}^l \pi_{\mathcal{G}, \tau}^{c,l} \right)^n \mathbf{q}|_{\tau},$$

where, in the last step, we have used Fubini's theorem to exchange the order of the sums and the fact that the infinite expansion of $\Gamma_{\mathcal{G}, \mathfrak{X}_T}^l$ is well-defined. Now, using a triangle inequality and the fact that $|\chi_{\tau}(\mathbf{x})| = 1$ for all $\mathbf{x} \in \tau$, we get

$$\begin{aligned} \|\Gamma_{\mathcal{G}, \mathfrak{X}_T}^l \mathbf{q}\|_{L^2(T)} &\leq \sum_{\tau \in \mathfrak{X}_T} \left\| \sum_{n=0}^{\infty} \left(\pi_{\mathcal{G}, \tau}^l \pi_{\mathcal{G}, \tau}^{c,l} \right)^n \mathbf{q}|_{\tau} \right\|_{L^2(\tau)} \\ &\lesssim \sum_{\tau \in \mathfrak{X}_T} \|\mathbf{q}|_{\tau}\|_{L^2(\tau)} \leq \sum_{\tau \in \mathfrak{X}_T} \|\mathbf{q}\|_{L^2(T)} \stackrel{(3)}{\lesssim} \|\mathbf{q}\|_{L^2(T)}, \end{aligned}$$

where we have used the inequality $\|(\text{Id} - \pi_{\mathcal{G}, \tau}^l \pi_{\mathcal{G}, \tau}^{c,l})^{-1}\|_{\mathcal{L}(\mathcal{P}^l(\tau), \mathcal{P}^l(\tau))} \lesssim 1$ (see [19, Proof of Lemma 2]) in the second step and the fact that $\tau \subset T$ for $\tau \in \mathfrak{X}_T$ in third step. \square

The second intermediate result needed in the proof of Lemma 6 is the following local Poincaré inequality.

Lemma 8 (Poincaré inequality over a simplex). *Let a mesh element $T \in \mathcal{T}_h$, a simplex $\tau \in \mathfrak{X}_T$, and integer $l \geq 0$ be given. Then, for any $q \in \mathcal{P}^l(\tau)$ vanishing at one of the vertices of τ , the following holds:*

$$\|q\|_{L^2(\tau)} \lesssim h_\tau \|\nabla q\|_{L^2(\tau)}. \quad (54)$$

Proof. The case $l = 0$ is trivial. Then we separate the remaining cases when $l = 1$, and $l > 1$. We begin with $l = 1$. By assumption, there exists a vertex of τ denoted by $\mathbf{x}_\tau \in \mathbb{R}^d$ such that $q(\mathbf{x}_\tau) = 0$. Thus, q can be expressed as $q(\mathbf{x}) = \sum_{i=1}^d a_i(x_i - x_{\tau,i})$ where $a_i \in \mathbb{R}$, and x_i and $x_{\tau,i}$ are the Cartesian coordinates of \mathbf{x} , and \mathbf{x}_τ , respectively. We write

$$\|q\|_{L^2(\tau)}^2 = \int_\tau \left(\sum_{i=1}^d a_i(x_i - x_{\tau,i}) \right)^2 \stackrel{(24)}{\lesssim} \sum_{i=1}^d \int_\tau a_i^2 (x_i - x_{\tau,i})^2 \leq h_\tau^2 \sum_{i=1}^d \int_\tau a_i^2 = h_\tau \|\nabla q\|_{L^2(\tau)}^2.$$

Passing to the square root, we get (54). Now, to prove the case $l > 1$, let $q \in \mathcal{P}^l(\tau)$ and denote by \hat{q} the standard nodal interpolate of q on $\mathcal{P}^1(\tau)$ (see, e.g., [29, Eq. (1.36)]). Then, it is inferred that

$$\|q\|_{L^2(\tau)} \leq \|q - \hat{q}\|_{L^2(\tau)} + \|\hat{q}\|_{L^2(\tau)} \lesssim h_\tau |q|_{H^1(\tau)} + \|\hat{q}\|_{L^2(\tau)} \lesssim h_\tau \|\nabla q\|_{L^2(\tau)},$$

where in the first step we have used a triangle inequality, in the second step standard approximation properties of \hat{q} ([29, Theorem (1.103)]) followed by an inverse inequality to write $\|q - \hat{q}\|_{L^2(\tau)} \lesssim h_\tau^2 |q|_{H^2(\tau)} \lesssim h_\tau |q|_{H^1(\tau)}$, and in the last step the fact that $\hat{q} \in \mathcal{P}^1(\tau)$ and $\hat{q}(\mathbf{x}_\tau) = 0$, so we can use (54), since this case has already been proved. \square

Proof of Lemma 6. We first prove that $\varrho_{\mathfrak{X}_T}^{l+1}$ is well defined. By the definitions (43) of $\varrho_{\mathfrak{X}_T}^{l+1}$ and (49) of $\mathbf{\Gamma}_{\mathcal{G}, \mathfrak{X}_T}^l$, it holds, for all $\mathbf{q} \in \mathcal{P}^l(\mathfrak{X}_T)$,

$$\nabla \varrho_{\mathfrak{X}_T}^{l+1} \mathbf{q} = \mathbf{\Gamma}_{\mathcal{G}, \mathfrak{X}_T}^l (\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^l \mathbf{q} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \mathbf{q}). \quad (55)$$

By the first equation of (51), $\mathbf{\Gamma}_{\mathcal{G}, \mathfrak{X}_T}^l (\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^l \mathbf{q} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \mathbf{q}) =: \mathbf{g} \in \mathcal{G}^l(\mathfrak{X}_T)$. Thus, there exists $g \in \mathcal{P}^{l+1}(\mathfrak{X}_h)$ such that $\nabla g = \mathbf{g}$. Let $\varrho_{\mathfrak{X}_T}^{l+1} \mathbf{q} := g - g(\mathbf{x}_T)$. Clearly, $\varrho_{\mathfrak{X}_T}^{l+1} \mathbf{q}$ satisfies (55) as well as the condition at the common vertex \mathbf{x}_T required by (43). Using this process and the fact that the operators $\mathbf{\Gamma}_{\mathcal{G}, \mathfrak{X}_T}^l$, $\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^l$ and $\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^{c,l}$ are all linear, it is easy to prove the linearity of $\varrho_{\mathfrak{X}_T}^{l+1}$.

(i) *Proof of (44).* The first identity in (44) comes from (55) taking $\mathbf{q} = \mathbf{b} \in \mathcal{G}^l(\mathfrak{X}_T) \subset \mathcal{P}^l(\mathfrak{X}_T)$ and using the first identity in (52) and the fact that $\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^l \mathbf{b} = \mathbf{b}$. To prove the second identity in (44), we write (55) with $\mathbf{q} = \mathbf{c} \in \mathcal{G}^{c,1}(\mathfrak{X}_T) \subset \mathcal{P}^l(\mathfrak{X}_T)$ to get

$$\nabla \varrho_{\mathfrak{X}_T \mathfrak{X}_T}^{l+1} \mathbf{c} = \mathbf{\Gamma}_{\mathcal{G}, \mathfrak{X}_T}^l (\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^l \mathbf{c} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^l \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \mathbf{c}) = \mathbf{\Gamma}_{\mathcal{G}, \mathfrak{X}_T}^l (\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^l \mathbf{c} - \boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^l \mathbf{c}) = \mathbf{0},$$

where, in the third step, we have used the fact that $\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^{c,k-1}$ is a projection onto $\mathcal{G}^{c,k-1}(\mathfrak{X}_T)$ (so that $\boldsymbol{\pi}_{\mathcal{G}, \mathfrak{X}_T}^{c,l} \mathbf{c} = \mathbf{c}$) and, in the last step, the linearity of $\mathbf{\Gamma}_{\mathcal{G}, \mathfrak{X}_T}^l$. The conclusion follows from the condition at the common vertex \mathbf{x}_T required by (43).

(ii) *Proof of (45).* We first prove the bound for the first term in the left-hand side of (45). We observe that

$$\|\varrho_{\mathfrak{X}_T}^{l+1} \mathbf{q}\|_{L^2(T)}^2 = \sum_{\tau \in \mathfrak{X}_T} \|\varrho_{\mathfrak{X}_T}^{l+1} \mathbf{q}\|_{L^2(\tau)}^2 \lesssim \sum_{\tau \in \mathfrak{X}_T} h_\tau^2 \|\nabla \varrho_{\mathfrak{X}_T}^{l+1} \mathbf{q}\|_{L^2(\tau)}^2 \lesssim \sum_{\tau \in \mathfrak{X}_T} h_\tau^2 \|\mathbf{q}\|_{L^2(\tau)}^2 \leq h_T^2 \|\mathbf{q}\|_{L^2(T)}^2, \quad (56)$$

where, in the second step, we have used the local Poincaré inequality (54) for all $\tau \in \mathfrak{T}_T$ (this is made possible by the condition at the common vertex x_T in (43)), in the third step we have used first (55), then the first bound in (53) along with a triangle inequality and then the L^2 -boundedness of $\pi_{\mathcal{G}, \mathfrak{T}_T}^{c,l}$ and $\pi_{\mathcal{G}, \mathfrak{T}_T}^l$, and in the last step the fact that $\tau \subset T$ (so that, by mesh regularity, $h_\tau^{-1} \lesssim h_T^{-1}$). Taking the square root yields the bound for the first term in the left-hand side of (45). The bound for the second term in the left-hand side of (45) is a straightforward consequence of (56) combined with the discrete inverse inequality (95) below with $p = 2$.

(iii) *Proof of (46).* To prove (46), we use the identities $\mathbf{q} = \nabla \varrho_{\mathfrak{T}_T}^{l+1} \mathbf{q} + (\text{Id} - \nabla \varrho_{\mathfrak{T}_T}^{l+1}) \mathbf{q}$ and (50) along with the definitions (48) of $\mathfrak{R}_{\mathcal{G}, \mathfrak{T}_T}^l$ and (49) of $\Gamma_{\mathcal{G}, \mathfrak{T}_T}^l$ and $\Gamma_{\mathcal{G}, \mathfrak{T}_T}^{c,l}$, and use (55) to write

$$(\text{Id} - \nabla \varrho_{\mathfrak{T}_T}^{l+1}) \mathbf{q} = \mathbf{q} - \nabla \varrho_{\mathfrak{T}_T}^{l+1} \mathbf{q} \stackrel{(50), (48), (55)}{=} \Gamma_{\mathcal{G}, \mathfrak{T}_T}^{c,l} (\pi_{\mathcal{G}, \mathfrak{T}_T}^{c,l} \mathbf{q} - \pi_{\mathcal{G}, \mathfrak{T}_T}^l \pi_{\mathcal{G}, \mathfrak{T}_T}^l \mathbf{q}).$$

Using (51) with $(\mathbf{c}, \mathbf{q}) = (\pi_{\mathcal{G}, \mathfrak{T}_T}^{c,l} \mathbf{q}, \pi_{\mathcal{G}, \mathfrak{T}_T}^l \mathbf{q})$, we obtain $(\text{Id} - \nabla \varrho_{\mathfrak{T}_T}^{l+1}) \mathbf{q} \in \mathcal{G}^{c,l}(\mathfrak{T}_T)$.

(iv) *Proof of (47).* Let $\mathbf{q} \in \mathcal{P}^l(T)$ and write $\mathbf{q} \stackrel{(14)}{=} \mathbf{g}_T + \mathbf{g}_T^c$, where $\mathbf{g}_T \in \mathcal{G}^l(T)$ and $\mathbf{g}_T^c \in \mathcal{G}^{c,l}(T)$. Using the fact that $\mathcal{G}^l(T) \subset \mathcal{G}^l(\mathfrak{T}_T)$ and $\mathcal{G}^{c,l}(T) \subset \mathcal{G}^{c,l}(\mathfrak{T}_T)$ along with the definition (43) of $\varrho_{\mathfrak{T}_T}^{l+1}$, (44), and its linearity, it is readily seen that

$$\mathcal{G}^{k-1}(\mathfrak{T}_T) \ni \nabla \varrho_{\mathfrak{T}_T}^{l+1} \mathbf{q} = \nabla \varrho_{\mathfrak{T}_T}^{l+1} (\mathbf{g}_T + \mathbf{g}_T^c) = \nabla \varrho_{\mathfrak{T}_T}^{l+1} \mathbf{g}_T + \nabla \varrho_{\mathfrak{T}_T}^{l+1} \mathbf{g}_T^c = \mathbf{g}_T + \mathbf{0} = \mathbf{g}_T \in \mathcal{G}^{k-1}(T),$$

thus $\varrho_{\mathfrak{T}_T}^{l+1}(\mathbf{q}) = g_T + g_{0, \mathfrak{T}_T}$, where $g_T \in \mathcal{P}^{l+1}(T)$ and $g_{0, \mathfrak{T}_T} \in \mathcal{P}^0(\mathfrak{T}_T)$. But, using the second condition in (43), we have that $g_T(x_T) = -g_{0, \mathfrak{T}_T}(x_T)$ for all τ in \mathfrak{T}_T , thus $g_{0, \mathfrak{T}_T} \in \mathcal{P}^0(T)$, since $g_T(x_T)$ is independent of τ , and we conclude. \square

4.5 Convective term

Let an element $T \in \mathcal{T}_h$ be fixed. For every simplicial face $\sigma \in \mathfrak{F}_h$, we introduce an arbitrary but fixed ordering of the simplicial elements τ_1 and τ_2 such that $\sigma \subset \partial\tau_1 \cap \partial\tau_2$, and let $\mathbf{n}_\sigma := \mathbf{n}_{\tau_1\sigma} = -\mathbf{n}_{\tau_2\sigma}$, where $\mathbf{n}_{\tau_i\sigma}, i \in \{1, 2\}$, denotes the unit vector normal to σ pointing out of τ_i (see Figure 1b). With this convention, for every scalar-valued function ζ admitting a possibly two-valued trace on σ , we define the jump and average of ζ across σ respectively as

$$[[\zeta]]_\sigma := \zeta|_{\tau_1} - \zeta|_{\tau_2} \quad \text{and} \quad \{\zeta\}_\sigma := \frac{1}{2} (\zeta|_{\tau_1} + \zeta|_{\tau_2}). \quad (57)$$

For any boundary simplicial face $\sigma \subset F \in \mathcal{F}_h^b$, we set $[[\zeta]]_\sigma := \{\zeta\}_\sigma := \zeta$. When applied to vector- or tensor-valued functions, the jump and average operators act component-wise.

We introduce the global function $t_h : [\underline{U}_h^k]^3 \rightarrow \mathbb{R}$ such that, for all $(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h, \underline{\mathbf{z}}_h) \in [\underline{U}_h^k]^3$,

$$\begin{aligned} t_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h, \underline{\mathbf{z}}_h) &:= \int_\Omega (\mathbf{R}_h^k \underline{\mathbf{w}}_h \cdot \nabla) \mathbf{R}_h^k \underline{\mathbf{v}}_h \cdot \mathbf{R}_h^k \underline{\mathbf{z}}_h - \sum_{\sigma \in \mathfrak{F}_h^i} \int_\sigma (\mathbf{R}_h^k \underline{\mathbf{w}}_h \cdot \mathbf{n}_\sigma) [[\mathbf{R}_h^k \underline{\mathbf{v}}_h]]_\sigma \cdot \{\mathbf{R}_h^k \underline{\mathbf{z}}_h\}_\sigma \\ &+ \sum_{\sigma \in \mathfrak{F}_h^i} \int_\sigma \frac{1}{2} |\mathbf{R}_h^k \underline{\mathbf{w}}_h \cdot \mathbf{n}_\sigma| [[\mathbf{R}_h^k \underline{\mathbf{v}}_h]]_\sigma \cdot [[\mathbf{R}_h^k \underline{\mathbf{z}}_h]]_\sigma + \sum_{T \in \mathcal{T}_h} \sum_{\sigma \in \mathfrak{F}_T^i} t_{T, \sigma}^k(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T, \underline{\mathbf{z}}_T). \end{aligned} \quad (58)$$

The first term in the second line is the usual upwind stabilization. The form $t_{T, \sigma}^k : \underline{U}_T^k \times \underline{U}_T^k \times \underline{U}_T^k \rightarrow \mathbb{R}$ also works as a penalty term and, setting $\mathbf{w}_T^0 := \pi_T^0 \mathbf{w}_T$, it is defined as follows:

$$t_{T, \sigma}^k(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T, \underline{\mathbf{z}}_T) := \begin{cases} 0 & \text{if } k = 0, \\ \int_\sigma [[\varrho_{\mathfrak{T}_T}^k (\pi_{\mathfrak{T}_T}^{k-1} ((\mathbf{w}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{v}}_T))]]_\sigma [[\varrho_{\mathfrak{T}_T}^k (\pi_{\mathfrak{T}_T}^{k-1} ((\mathbf{w}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{z}}_T))]]_\sigma & \text{if } k \geq 1 \end{cases}. \quad (59)$$

Notice that $t_{T, \sigma}^k$ in (59) is linear only in its second and third arguments.

Remark 9 (Comparison with Virtual Elements). In [6], the authors propose a Virtual Element discretization of the Oseen equation which includes a penalization term somewhat similar to $t_{T,\sigma}^k$ (see, in particular, [6, Eq. (4.18)]). The penalization term $t_{T,\sigma}^k$ used here appears, however, more subtle, since the factor $(\mathbf{w}_T^0 \cdot \nabla) \mathbf{R}_T^k \mathbf{v}_T$ is in $\mathcal{P}^k(\mathfrak{T}_T)$, and does not necessarily belong to $\mathcal{P}^k(T)$. How to deal with this difficulty is one of the main contributions of this paper, as detailed in Section 5 below.

4.6 Discrete problem

The HHO discrete formulation of problem (2) then reads: Find $\underline{\mathbf{u}}_h : [0, t_F] \rightarrow \underline{\mathbf{U}}_{h,0}^k$ with $\underline{\mathbf{u}}_h(0) = \underline{\mathbf{I}}_h^k \mathbf{u}_0 \in \underline{\mathbf{U}}_{h,0}^k$ and $p_h : (0, t_F] \rightarrow P_h^k$ such that it holds, for all $(\underline{\mathbf{v}}_h, q_h) \in \underline{\mathbf{U}}_{h,0}^k \times \mathcal{P}^k(\mathcal{T}_h)$ and almost every $t \in (0, t_F)$,

$$a_{R,h}(d_t \underline{\mathbf{u}}_h(t), \underline{\mathbf{v}}_h) + \nu a_h(\underline{\mathbf{u}}_h(t), \underline{\mathbf{v}}_h) + t_h(\underline{\mathbf{u}}_h(t), \underline{\mathbf{u}}_h(t), \underline{\mathbf{v}}_h) + b_h(\underline{\mathbf{v}}_h, p_h(t)) - b_h(\underline{\mathbf{u}}_h(t), q_h) = \ell_h(\mathbf{f}(t), \underline{\mathbf{v}}_h). \quad (60)$$

5 Velocity error analysis

Most of the relations that we will write in this section hold for almost every $t \in (0, t_F)$. To simplify the notation, we omit the dependence on t and write, e.g., \mathbf{u} and $\underline{\mathbf{u}}_h$ instead of $\mathbf{u}(t)$ and $\underline{\mathbf{u}}_h(t)$. Let \mathbf{u} solve the continuous problem (2) and $\underline{\mathbf{u}}_h$ the discrete problem (60). We define the velocity error as

$$\underline{\mathbf{e}}_h := \underline{\mathbf{u}}_h - \hat{\underline{\mathbf{u}}}_h, \quad (61)$$

where $\hat{\underline{\mathbf{u}}}_h = \underline{\mathbf{I}}_h^k \mathbf{u}$. It will be also useful to define $\boldsymbol{\eta}_h \in \mathbf{L}^2(\mathfrak{T}_h)$ such that

$$(\boldsymbol{\eta}_h)|_T := \boldsymbol{\eta}_T := \mathbf{R}_T^k \hat{\underline{\mathbf{u}}}_T - \mathbf{u} \text{ for all } T \in \mathcal{T}_h. \quad (62)$$

We now present the two main results of this manuscript, which bound the velocity error (61).

Theorem 10 (Velocity error estimate). *Let the pair (\mathbf{u}, p) be the solution of the continuous problem (2), and let the pair $(\underline{\mathbf{u}}_h, p_h)$ be the solution of the discrete problem (60). Suppose that $\mathbf{u} \in L^2(\mathbf{W}^{1,\infty}(\Omega)) \cap L^\infty(\mathbf{H}^1(\Omega)) \cap L^2(\mathbf{H}^2(\Omega))$, $d_t \mathbf{u} \in L^2(\mathbf{H}^1(\Omega))$, and $p : (0, t_F) \rightarrow P \cap H^1(\Omega)$. Then, we have the following error estimate:*

$$\|\underline{\mathbf{e}}_h\|_{L^\infty(\|\cdot\|_{R,h})}^2 + \nu \int_0^{t_F} \|\underline{\mathbf{e}}_h\|_{1,h}^2 \lesssim e^{G(\mathbf{u}, t_F)} \int_0^{t_F} H(\mathbf{u}), \quad (63)$$

where $G(\mathbf{u}, t_F) := t_F + \int_0^{t_F} \left(\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\Omega)} + h\gamma_k \|\mathbf{u}\|_{L^\infty(\Omega)} + h\tilde{\delta}_{0k} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\Omega)}^2 \right)$ and

$$H(\mathbf{u}) := \sum_{T \in \mathcal{T}_h} \nu h_T^2 \|\Delta \mathbf{u} - \boldsymbol{\pi}_T^{k-1} \Delta \mathbf{u}\|_{L^2(T)}^2 + \nu \|\mathcal{E}_{a,h}(\mathbf{u}; \cdot)\|_{1,h,*}^2 + \mathfrak{R}_1 + \mathfrak{R}_2, \quad (64)$$

where, denoting by δ_{0k} the Kronecker delta, we have let

$$\tilde{\delta}_{0k} := 1 - \delta_{0k}, \quad (65)$$

while γ_k is defined by

$$\gamma_k := \begin{cases} 0, & \text{if } k \in \{0, 1\} \\ 1, & \text{otherwise} \end{cases}, \quad (66)$$

and the terms \mathfrak{R}_1 and \mathfrak{R}_2 are respectively such that, for almost every $t \in (0, t_F)$,

$$\begin{aligned} \mathfrak{R}_1 := & \sum_{T \in \mathcal{T}_h} \left(\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(T)} \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{\mathbf{L}^2(T)}^2 + \gamma_k h_T^{-3} \|\mathbf{u}\|_{\mathbf{L}^\infty(T)} \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{\mathbf{L}^2(T)}^2 \right) \\ & + \sum_{T \in \mathcal{T}_h} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(T)} \left(h_T^{-1} \|\boldsymbol{\eta}_T\|_{\mathbf{L}^2(T)}^2 + h_T |\boldsymbol{\eta}_T|_{\mathbf{H}^1(\mathfrak{X}_T)}^2 + \|\boldsymbol{\eta}_T\|_{\mathbf{L}^2(T)}^2 \right) \\ & + \tilde{\delta}_{0k} \sum_{T \in \mathcal{T}_h} h_T \|\mathbf{u}\|_{\mathbf{L}^\infty(T)}^2 \left(\|\nabla \boldsymbol{\eta}_T\|_{\mathbf{L}^2(T)}^2 + \|\nabla(\mathbf{u} - \boldsymbol{\pi}_T^k \mathbf{u})\|_{\mathbf{L}^2(T)}^2 \right), \end{aligned} \quad (67)$$

and

$$\begin{aligned} \mathfrak{R}_2 := & \sum_{T \in \mathcal{T}_h} \left(\|\mathbf{R}_T^k(\widehat{d_t \mathbf{u}}_T) - d_t \mathbf{u}\|_{\mathbf{L}^2(T)}^2 + \|d_t \mathbf{u} - (\widehat{d_t \mathbf{u}})_T\|_{\mathbf{L}^2(T)}^2 \right) \\ & + \sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla((\widehat{d_t \mathbf{u}})_T - d_t \mathbf{u})\|_{\mathbf{L}^2(T)}^2 + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \|d_t \mathbf{u} - (\widehat{d_t \mathbf{u}})_F\|_{\mathbf{L}^2(F)}^2, \end{aligned} \quad (68)$$

where $\widehat{d_t \mathbf{u}}_T := \mathbf{I}_T^k(d_t \mathbf{u})$ for all $T \in \mathcal{T}_h$.

Using the theorem above, the approximation properties of the local operator \mathbf{R}_T^k presented in Lemma 2, and the approximation properties (4) of $\boldsymbol{\pi}_T^{k*}$ for all $T \in \mathcal{T}_h$, along with the following standard property of the L^2 -projector $\boldsymbol{\pi}_F^k$ on $F \in \mathcal{F}_h$:

$$\|\boldsymbol{\pi}_F^k \mathbf{v} - \mathbf{v}\|_{\mathbf{L}^2(F)} = \inf_{\mathbf{w} \in \mathcal{P}^k(F)} \|\mathbf{w} - \mathbf{v}\|_{\mathbf{L}^2(F)} \leq \|\boldsymbol{\pi}_T^k \mathbf{v} - \mathbf{v}\|_{\mathbf{L}^2(F)},$$

we have the following corollary.

Corollary 11 (Velocity convergence rates). *Under the notations and the assumptions of the previous theorem, and additionally assuming $\mathbf{u} \in L^\infty(\mathbf{H}^{k+1}(\mathcal{T}_h)) \cap L^2(\mathbf{H}^{k+2}(\mathcal{T}_h))$ for $k \in \{0, 1\}$, $\mathbf{u} \in L^\infty(\mathbf{H}^{k+2}(\mathcal{T}_h))$ for $k > 1$, and $d_t \mathbf{u} \in L^2(\mathbf{H}^{k+1}(\mathcal{T}_h))$, it holds:*

$$\|\underline{\mathbf{e}}_h\|_{L^\infty(\|\cdot\|_{R,h})}^2 + \nu \int_0^{t_F} \|\underline{\mathbf{e}}_h\|_{1,h}^2 \lesssim e^{G_1(\mathbf{u}, t_F)} H_1(\mathbf{u}, t_F), \quad (69)$$

where $G_1(\mathbf{u}, t_F) := t_F + \|\nabla \mathbf{u}\|_{\mathbf{L}^1(\mathbf{L}^\infty)} + h\gamma_k \|\mathbf{u}\|_{\mathbf{L}^1(\mathbf{L}^\infty)} + h\tilde{\delta}_{0k} \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\mathbf{L}^\infty)}^2$ and

$$\begin{aligned} H_1(\mathbf{u}, t_F) := & \nu h^{(2k+2)} \|\mathbf{u}\|_{\mathbf{L}^2(\mathbf{H}^{k+2}(\mathcal{T}_h))}^2 + h^{(2k+1)} \|\mathbf{u}\|_{\mathbf{L}^1(\mathbf{W}^{1,\infty})} \left(\|\mathbf{u}\|_{\mathbf{L}^\infty(\mathbf{H}^{k+1}(\mathcal{T}_h))}^2 + \gamma_k \|\mathbf{u}\|_{\mathbf{L}^\infty(\mathbf{H}^{k+2}(\mathcal{T}_h))}^2 \right) \\ & + \tilde{\delta}_{0k} h^{(2k+1)} \|\mathbf{u}\|_{\mathbf{L}^2(\mathbf{L}^\infty)}^2 \|\mathbf{u}\|_{\mathbf{L}^\infty(\mathbf{H}^{k+1}(\mathcal{T}_h))}^2 + h^{(2k+2)} \|d_t \mathbf{u}\|_{\mathbf{L}^2(\mathbf{H}^{k+1}(\mathcal{T}_h))}^2. \end{aligned}$$

Remark 12 (No dependency on ν^{-1} or p). The right-hand sides of (63) and (69) do not depend on ν^{-1} or on the pressure p , thus we have indeed a Reynolds-semi-robust and pressure-robust method.

Remark 13 (Formulation using only $\mathbf{H}_{\text{div}}(\mathfrak{X}_h)$). One may wonder why we do not use a formulation completely based on a discrete conforming subspace of $\mathbf{H}_{\text{div}}(\mathfrak{X}_h)$. The main reason is that the functions in this space are not fully continuous at interfaces, hence jump penalization for the discretization of the viscous term would be required for stability. This would, in turn, result in a stronger coupling among the degrees of freedom of neighbouring elements and, thus, in a larger stencil.

Remark 14 (Estimate including the upwind-norm). Following the same steps in the proof of Theorem 10, and similar steps in the proof of Lemma 16, we can include the upwind norm in the left hand side of the inequalities (63) and (69), and have the following estimate instead

$$\|\underline{\mathbf{e}}_h\|_{L^\infty(\|\cdot\|_{R,h})}^2 + \int_0^{t_F} \left(\nu \|\underline{\mathbf{e}}_h\|_{1,h}^2 + \frac{1-\epsilon}{2} \sum_{\sigma \in \mathfrak{D}_h^i} \int_\sigma |\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_\sigma| |\llbracket \mathbf{R}_h^k \underline{\mathbf{e}}_h \rrbracket_\sigma|^2 \right) \lesssim e^{G_1(\mathbf{u}, t_F)} H_1(\mathbf{u}, t_F),$$

where ϵ is a real number such that $0 < \epsilon \leq 1$.

Remark 15 (Error in the $L^\infty(\mathbf{L}^2)$ -norm). Using the inequality (34) for $T \in \mathcal{T}_h$, we obtain an order of convergence of $k + \frac{1}{2}$ in the $L^\infty(0, t_F; \mathbf{L}^2(\Omega))$ -norm for the velocity in (69). This order of convergence equals, e.g., the one obtained in [37, Corollary 5.7] on simplicial meshes. However, in Corollary 11 above, we do not need to require dominating convection, whereas in [37] the condition $\nu \lesssim h$ is necessary. Specifically, observe that in the definition (64) of $H(\mathbf{u})$ in Theorem 10 includes the consistency error (39) which, under the regularity conditions stated in Corollary 11, gives an order of convergence of $k + 1$, thus no further assumptions on ν are needed, in order to make the error $\|\underline{\mathbf{e}}_h\|_{L^\infty(\|\cdot\|_{R,h})}$ in (69) of order $k + \frac{1}{2}$. In contrast, the consistency error of the diffusion term using \mathbf{H}_{div} -conforming discontinuous Galerkin elements, such as the ones used in [37], is of order k , which increases to $k + \frac{1}{2}$ assuming dominating convection, thus making this assumption essential to obtain a convergence error of $k + \frac{1}{2}$ for the overall numerical scheme.

Proof of Theorem 10. Recalling that $\underline{\mathbf{e}}_h = \underline{\mathbf{u}}_h - \hat{\underline{\mathbf{u}}}_h$ (cf. (61)), we take $(\underline{\mathbf{v}}_h, q_h) = (\underline{\mathbf{e}}_h, 0)$ in (60) and, after subtracting the quantity $(a_{R,h}(d_t \hat{\underline{\mathbf{u}}}_h, \underline{\mathbf{e}}_h) + \nu a_h(\hat{\underline{\mathbf{u}}}_h, \underline{\mathbf{e}}_h))$ from both sides, we get, for almost every $t \in (0, t_F)$,

$$a_{R,h}(d_t \underline{\mathbf{e}}_h, \underline{\mathbf{e}}_h) + \nu a_h(\underline{\mathbf{e}}_h, \underline{\mathbf{e}}_h) = \ell_h(\mathbf{f}, \underline{\mathbf{e}}_h) - t_h(\underline{\mathbf{u}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{e}}_h) - b_h(\underline{\mathbf{e}}_h, p_h) - a_{R,h}(d_t \hat{\underline{\mathbf{u}}}_h, \underline{\mathbf{e}}_h) - \nu a_h(\hat{\underline{\mathbf{u}}}_h, \underline{\mathbf{e}}_h).$$

Recalling that $\mathbf{f} = d_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p$ almost everywhere in $(0, t_F) \times \Omega$, we go on writing

$$\begin{aligned} a_{R,h}(d_t \underline{\mathbf{e}}_h, \underline{\mathbf{e}}_h) &= \underbrace{(\ell_h((\mathbf{u} \cdot \nabla) \mathbf{u}, \underline{\mathbf{e}}_h) - t_h(\underline{\mathbf{u}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{e}}_h))}_{\mathfrak{S}_1} + \underbrace{(\ell_h(d_t \mathbf{u}, \underline{\mathbf{e}}_h) - a_{R,h}(d_t \hat{\underline{\mathbf{u}}}_h, \underline{\mathbf{e}}_h))}_{\mathfrak{S}_2} \\ &\quad - \underbrace{(\nu \ell_h(\Delta \mathbf{u}, \underline{\mathbf{e}}_h) + \nu a_h(\hat{\underline{\mathbf{u}}}_h, \underline{\mathbf{e}}_h) + \nu a_h(\underline{\mathbf{e}}_h, \underline{\mathbf{e}}_h))}_{\mathfrak{S}_3} + \underbrace{(\ell_h(\nabla p, \underline{\mathbf{e}}_h) - b_h(\underline{\mathbf{e}}_h, p_h))}_{\mathfrak{S}_4}. \end{aligned}$$

We next bound the terms in the right-hand side.

(i) *Estimate of \mathfrak{S}_1 .* As we will see, the first term is the most difficult to estimate, so the details of the following bound are provided in the separate Lemma 16 below:

$$\mathfrak{S}_1 \lesssim \sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{e}}_T\|_{\mathbf{R},T}^2 \left(\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(T)} + h_T \gamma_k \|\mathbf{u}\|_{L^\infty(T)} + h_T \tilde{\delta}_{0k} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(T)}^2 \right) + \mathfrak{R}_1. \quad (70)$$

(ii) *Estimate of \mathfrak{S}_2 .* Let us now prove that

$$\mathfrak{S}_2 \lesssim \sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{e}}_T\|_{\mathbf{R},T}^2 + \mathfrak{R}_2. \quad (71)$$

Using the definition of the L^2 -orthogonal projections $\pi_T^{k^*}$ and π_F^k , it can be seen that $d_t \hat{\underline{\mathbf{u}}}_T = \underline{\mathbf{I}}_T^k(d_t \mathbf{u}) =: \widehat{d_t \mathbf{u}}_T$ for all $T \in \mathcal{T}_h$ (assuming that $d_t \mathbf{u} : (0, t_F) \rightarrow \mathbf{H}^1(T)$) and, by the definitions (31)–(32) of $a_{R,T}$ and (42) of the linear form ℓ_h , we get

$$\begin{aligned} \mathfrak{S}_2 &= \sum_{T \in \mathcal{T}_h} \int_T (d_t \mathbf{u} - \mathbf{R}_T^k(\widehat{d_t \mathbf{u}}_T)) \cdot \mathbf{R}_T^k \underline{\mathbf{e}}_T \\ &\quad - \sum_{T \in \mathcal{T}_h} \int_T \pi_T^{k^*}(\mathbf{R}_T^k(\widehat{d_t \mathbf{u}}_T) - (\widehat{d_t \mathbf{u}})_T) \cdot \pi_T^{k^*}(\mathbf{R}_T^k \underline{\mathbf{e}}_T - \mathbf{e}_T) \\ &\quad - \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \int_F \pi_F^k(\mathbf{R}_T^k(\widehat{d_t \mathbf{u}}_T) - (\widehat{d_t \mathbf{u}})_F) \cdot \pi_F^k(\mathbf{R}_T^k \underline{\mathbf{e}}_T - \mathbf{e}_F) =: \mathfrak{S}_{2,1} + \mathfrak{S}_{2,2} + \mathfrak{S}_{2,3}. \end{aligned} \quad (72)$$

To bound $\mathfrak{I}_{2,1}$ we use Cauchy–Schwarz and Young inequalities followed by the definition (33) of $\|\cdot\|_{\mathbb{R},T}$ to write

$$\mathfrak{I}_{2,1} \lesssim \sum_{T \in \mathcal{T}_h} \|d_t \mathbf{u} - \mathbf{R}_T^k(\widehat{d_t \mathbf{u}}_T)\|_{\mathbf{L}^2(T)}^2 + \sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2 \stackrel{(68)}{\leq} \mathfrak{N}_2 + \sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2.$$

We bound $\mathfrak{I}_{2,2}$ in (72) using similar steps along with the \mathbf{L}^2 -boundedness of $\boldsymbol{\pi}_T^{k^*}$, (33), and (34) to write $\|\mathbf{R}_T^k \underline{\mathbf{e}}_T - \mathbf{e}_T\|_{\mathbf{L}^2(T)} \leq \|\mathbf{R}_T^k \underline{\mathbf{e}}_T\|_{\mathbf{L}^2(T)} + \|\mathbf{e}_T\|_{\mathbf{L}^2(T)} \lesssim \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}$, and then adding and subtracting $d_t \mathbf{u}$ and using (24) with $n = 2$ to infer

$$\begin{aligned} \mathfrak{I}_{2,2} &\lesssim \sum_{T \in \mathcal{T}_h} \left(\|\mathbf{R}_T^k(\widehat{d_t \mathbf{u}}_T) - d_t \mathbf{u}\|_{\mathbf{L}^2(T)}^2 + \|d_t \mathbf{u} - (\widehat{d_t \mathbf{u}})_T\|_{\mathbf{L}^2(T)}^2 \right) + \sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2 \\ &\stackrel{(68)}{\leq} \mathfrak{N}_2 + \sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2. \end{aligned}$$

The term $\mathfrak{I}_{2,3}$ in (72) is treated similarly to obtain

$$\mathfrak{I}_{2,3} \lesssim \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \|\mathbf{R}_T^k \underline{\mathbf{e}}_T - \mathbf{e}_F\|_{\mathbf{L}^2(F)}^2 + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \|\mathbf{R}_T^k(\widehat{d_t \mathbf{u}}_T) - (\widehat{d_t \mathbf{u}})_F\|_{\mathbf{L}^2(F)}^2 =: \mathfrak{I}_{2,3,1} + \mathfrak{I}_{2,3,2}.$$

By definition (33) of the $\|\cdot\|_{\mathbb{R},T}$ -norm, we have

$$\mathfrak{I}_{2,3,1} \leq \sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2.$$

To bound $\mathfrak{I}_{2,3,2}$, we use repeatedly (24) with $n = 2$, the discrete trace inequality (96) below valid for piecewise polynomial functions, and the bound (3) for $\text{card}(\mathcal{F}_T)$ as follows:

$\mathfrak{I}_{2,3,2}$

$$\begin{aligned} &\lesssim \sum_{T \in \mathcal{T}_h} \left(\sum_{F \in \mathcal{F}_T} h_F \|\mathbf{R}_T^k(\widehat{d_t \mathbf{u}}_T) - (\widehat{d_t \mathbf{u}})_F\|_{\mathbf{L}^2(F)}^2 + \sum_{F \in \mathcal{F}_T} h_F \|(\widehat{d_t \mathbf{u}})_T - (\widehat{d_t \mathbf{u}})_F\|_{\mathbf{L}^2(F)}^2 \right) \\ &\lesssim \sum_{T \in \mathcal{T}_h} \left[\|\mathbf{R}_T^k(\widehat{d_t \mathbf{u}}_T) - (\widehat{d_t \mathbf{u}})_T\|_{\mathbf{L}^2(T)}^2 + \sum_{F \in \mathcal{F}_T} \left(h_F \|(\widehat{d_t \mathbf{u}})_T - d_t \mathbf{u}\|_{\mathbf{L}^2(F)}^2 + h_F \|d_t \mathbf{u} - (\widehat{d_t \mathbf{u}})_F\|_{\mathbf{L}^2(F)}^2 \right) \right] \\ &\lesssim \sum_{T \in \mathcal{T}_h} \left(\|\mathbf{R}_T^k(\widehat{d_t \mathbf{u}}_T) - d_t \mathbf{u}\|_{\mathbf{L}^2(T)}^2 + \|d_t \mathbf{u} - (\widehat{d_t \mathbf{u}})_T\|_{\mathbf{L}^2(T)}^2 + h_T^2 \|\nabla((\widehat{d_t \mathbf{u}})_T - d_t \mathbf{u})\|_{\mathbf{L}^2(T)}^2 \right) \\ &\quad + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \|d_t \mathbf{u} - (\widehat{d_t \mathbf{u}})_F\|_{\mathbf{L}^2(F)}^2 \stackrel{(68)}{=} \mathfrak{N}_2, \end{aligned}$$

where, in the last bound, we have used a continuous trace inequality. Plugging this last estimate along with those obtained before for $\mathfrak{I}_{2,1}$, $\mathfrak{I}_{2,2}$, and $\mathfrak{I}_{2,3,1}$ into (72) concludes the proof of (71).

(iii) *Estimate of \mathfrak{I}_3 .* We next show that

$$\mathfrak{I}_3 \leq CH(\mathbf{u}) - \frac{1}{2} \nu C_a \|\underline{\mathbf{e}}_h\|_{1,h}^2, \quad (73)$$

where C is a constant independent of ν and h and C_a is the coercivity constant of a_h (cf. (38)). We

use the definition (42) of ℓ_h with $\phi = \Delta \mathbf{u}$ and add and subtract $\nu \int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{e}_h$ to get

$$\begin{aligned}
\mathfrak{I}_3 &= -\nu \int_{\Omega} \Delta \mathbf{u} \cdot (\mathbf{R}_h^k \underline{\mathbf{e}}_h - \mathbf{e}_h) - \nu \int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{e}_h - \nu a_h(\hat{\mathbf{u}}_h, \underline{\mathbf{e}}_h) - \nu a_h(\underline{\mathbf{e}}_h, \underline{\mathbf{e}}_h) \\
&\stackrel{(40)}{=} -\nu \sum_{T \in \mathcal{T}_h} \int_T \cancel{\pi_T^{k-1} \Delta \mathbf{u} \cdot (\mathbf{R}_T^k \underline{\mathbf{e}}_T - \mathbf{e}_T)} - \nu \sum_{T \in \mathcal{T}_h} \int_T (\Delta \mathbf{u} - \pi_T^{k-1} \Delta \mathbf{u}) \cdot (\mathbf{R}_T^k \underline{\mathbf{e}}_T - \mathbf{e}_T) \\
&\quad + \mathcal{E}_{a,h}(\mathbf{u}; \underline{\mathbf{e}}_h) - \nu a_h(\underline{\mathbf{e}}_h, \underline{\mathbf{e}}_h) \\
&\leq \nu \sum_{T \in \mathcal{T}_h} h_T \|\Delta \mathbf{u} - \pi_T^{k-1} \Delta \mathbf{u}\|_{L^2(T)} \|\underline{\mathbf{e}}_T\|_{1,T} + \|\mathcal{E}_{a,h}(\mathbf{u}; \cdot)\|_{1,h,*} \|\underline{\mathbf{e}}_h\|_{1,h} - \nu C_a \|\underline{\mathbf{e}}_h\|_{1,h}^2 \\
&\leq C \nu \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|\Delta \mathbf{u} - \pi_T^{k-1} \Delta \mathbf{u}\|_{L^2(T)}^2 + \|\mathcal{E}_{a,h}(\mathbf{u}; \cdot)\|_{1,h,*}^2 \right) - \frac{1}{2} \nu C_a \|\underline{\mathbf{e}}_h\|_{1,h}^2,
\end{aligned}$$

where we have used (19) in the cancellation, Cauchy–Schwarz inequalities along with (17) for the first term, the definition (10) of $\|\cdot\|_{1,h,*}$ for the second term, and the coercivity (38) for the last term, while the conclusion follows using the generalized Young inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2 \quad (74)$$

for the first and second term with ϵ selected so as to make the contribution $\frac{1}{2} C_a \|\underline{\mathbf{e}}_h\|_{1,h}^2$ appear. Observe that the constant C appearing in the last inequality is indeed independent of ν and h and recall the definition (64) of $H(\mathbf{u})$ to obtain (73).

(iv) Estimate of \mathfrak{I}_4 . We finally show that

$$\mathfrak{I}_4 = 0. \quad (75)$$

Using the definitions (41) of b_h and (42) of ℓ_h , we get

$$\mathfrak{I}_4 = \sum_{T \in \mathcal{T}_h} \left\{ \int_T \nabla p \cdot \mathbf{R}_T^k \underline{\mathbf{e}}_T + \int_T D_T^k \underline{\mathbf{e}}_T p_T \right\} = \sum_{T \in \mathcal{T}_h} \left\{ - \int_T p (\nabla \cdot \mathbf{R}_T^k \underline{\mathbf{e}}_T) + \int_T D_T^k \underline{\mathbf{e}}_T p_T \right\} = 0,$$

where, in the second step, we have integrated by parts element by element the first term and used the fact that, for all $\sigma \in \mathfrak{F}_h^i$, $[[p]]_{\sigma} = 0$ (because $p : (0, t_F) \rightarrow H^1(\Omega) \cap P$) and $[[\mathbf{R}_h^k \underline{\mathbf{e}}_h]]_{\sigma} \cdot \mathbf{n}_{\sigma} = 0$, as well as $\mathbf{R}_h^k \underline{\mathbf{e}}_h \cdot \mathbf{n}_{\sigma} = 0$ on $\partial\Omega$ (consequence of (16a)), while, to conclude, we have used the fact that $\nabla \cdot \mathbf{R}_T^k \underline{\mathbf{e}}_T = D_T^k \underline{\mathbf{e}}_T$ by (16b) and $D_T^k \underline{\mathbf{e}}_T \stackrel{(61)}{=} D_T^k \underline{\mathbf{u}}_T - D_T^k \hat{\mathbf{u}}_T = 0$, the conclusion being a consequence of (60) (test with $(\mathbf{0}, q_h)$ and let q_h span $\mathcal{P}^k(\mathcal{T}_h)$ to infer $D_T^k \underline{\mathbf{u}}_T = 0$ for all $T \in \mathcal{T}_h$) and of (12) along with $\nabla \cdot \mathbf{u} = 0$ (which give $D_T^k \hat{\mathbf{u}}_T = \pi_T^k(\nabla \cdot \mathbf{u}) = 0$ for all $T \in \mathcal{T}_h$).

(v) Conclusion. Using the fact that $\frac{1}{2} d_t \|\underline{\mathbf{e}}_h\|_{R,h}^2 = a_{R,h}(d_t \underline{\mathbf{e}}_h, \underline{\mathbf{e}}_h)$ and gathering the estimates (70), (71), (73), and (75), we obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\underline{\mathbf{e}}_h\|_{R,h}^2 + \frac{1}{2} \nu C_a \|\underline{\mathbf{e}}_h\|_{1,h}^2 \\
&\lesssim \sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{e}}_T\|_{R,T}^2 \left(1 + |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} + h_T \gamma_k \|\mathbf{u}\|_{L^{\infty}(T)} + h_T \tilde{\delta}_{0k} |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)}^2 \right) \\
&\quad + \nu \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|\Delta \mathbf{u} - \pi_T^{k-1} \Delta \mathbf{u}\|_{L^2(T)}^2 + \|\mathcal{E}_{a,h}(\mathbf{u}; \cdot)\|_{1,h,*}^2 \right) + \mathfrak{R}_1 + \mathfrak{R}_2.
\end{aligned}$$

Applying a Gronwall inequality (see, e.g., [29, Lemma 6.9]) and observing that $\underline{\mathbf{u}}_h(0) = \underline{\mathbf{I}}_h^k \mathbf{u}_0$ implies $\underline{\mathbf{e}}_h(0) = \mathbf{0}$ concludes the proof of (63). \square

The rest of this section is devoted to the proof of (70).

Lemma 16 (Estimate of \mathfrak{S}_1). *Estimate (70) holds.*

Proof. We begin by expanding ℓ_h and t_h according to the respective definitions (42) and (58), then add and subtract $\sum_{T \in \mathcal{T}_h} \int_T (\mathbf{R}_T^k \hat{\mathbf{u}}_T \cdot \nabla) \mathbf{u} \cdot \mathbf{R}_T^k \underline{\mathbf{e}}_T + \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{R}_T^k \underline{\mathbf{u}}_T \cdot \nabla) \mathbf{u} \cdot \mathbf{R}_T^k \underline{\mathbf{e}}_T$ and recall the definitions (61) of $\underline{\mathbf{e}}_h$ and (62) of η_T to write

$$\begin{aligned}
\mathfrak{S}_1 &= \ell_h((\mathbf{u} \cdot \nabla) \mathbf{u}, \underline{\mathbf{e}}_h) - t_h(\underline{\mathbf{u}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{e}}_h) \\
&= - \sum_{T \in \mathcal{T}_h} \int_T (\eta_T \cdot \nabla) \mathbf{u} \cdot \mathbf{R}_T^k \underline{\mathbf{e}}_T - \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \nabla) \mathbf{u} \cdot \mathbf{R}_T^k \underline{\mathbf{e}}_T \\
&\quad - \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{R}_T^k \underline{\mathbf{u}}_T \cdot \nabla) (\mathbf{R}_T^k \underline{\mathbf{u}}_T - \mathbf{u}) \cdot \mathbf{R}_T^k \underline{\mathbf{e}}_T \\
&\quad + \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} (\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}) \llbracket \mathbf{R}_h^k \underline{\mathbf{u}}_h - \mathbf{u} \rrbracket_{\sigma} \cdot \{ \mathbf{R}_h^k \underline{\mathbf{e}}_h \}_{\sigma} \\
&\quad - \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} \frac{1}{2} | \mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma} | \llbracket \mathbf{R}_h^k \underline{\mathbf{u}}_h - \mathbf{u} \rrbracket_{\sigma} \cdot \llbracket \mathbf{R}_h^k \underline{\mathbf{e}}_h \rrbracket_{\sigma} - \sum_{T \in \mathcal{T}_h} \sum_{\sigma \in \mathfrak{F}_T^i} t_{T,\sigma}^k(\underline{\mathbf{u}}_T, \underline{\mathbf{u}}_T, \underline{\mathbf{e}}_T) \\
&=: \mathfrak{S}_{1,1} + \mathfrak{S}_{1,2} + \mathfrak{S}_{1,3} + \mathfrak{S}_{1,4} + \mathfrak{S}_{1,5} + \mathfrak{S}_{1,6},
\end{aligned} \tag{76}$$

where we have additionally used the fact that $\llbracket \mathbf{u} \rrbracket_{\sigma} = \mathbf{0}$ for all $\sigma \in \mathfrak{F}_h^i$ to insert this quantity into the fourth and fifth terms.

(i) Estimate of $\mathfrak{S}_{1,1} + \mathfrak{S}_{1,2}$. Using a Hölder inequality with exponents $(2, \infty, 2)$ followed by Young's inequality, we obtain

$$\mathfrak{S}_{1,1} \lesssim \sum_{T \in \mathcal{T}_h} |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\eta_T\|_{L^2(T)}^2 + \sum_{T \in \mathcal{T}_h} |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2$$

where we have additionally used the definition (33) of $\|\cdot\|_{\mathbb{R},T}$ in the second addend. We bound $|\mathfrak{S}_{1,2}|$ using a Hölder inequality with exponents $(2, \infty, 2)$ to get

$$\mathfrak{S}_{1,2} \leq \sum_{T \in \mathcal{T}_h} |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2.$$

Gathering the above bounds and recalling the definition (67) of \mathfrak{N}_1 , we obtain

$$\mathfrak{S}_{1,1} + \mathfrak{S}_{1,2} \lesssim \mathfrak{N}_1 + \sum_{T \in \mathcal{T}_h} |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2. \tag{77}$$

(ii) Estimate of $\mathfrak{S}_{1,3} + \mathfrak{S}_{1,4} + \mathfrak{S}_{1,5} + \mathfrak{S}_{1,6}$. Using an element-by-element integration by parts along with the fact that $\nabla \cdot \mathbf{R}_T^k \underline{\mathbf{u}}_T = 0$ for all $T \in \mathcal{T}_h$ (use $(\mathbf{0}, q_h)$ with $q_h \in \mathcal{P}^k(\mathcal{T}_h)$ in (60) and recall the definition (41) of b_h to infer $D_T^k \underline{\mathbf{u}}_T = 0$ for all $T \in \mathcal{T}_h$, and plug this result into (16b) to conclude), we get

$$\begin{aligned}
\mathfrak{S}_{1,3} + \mathfrak{S}_{1,4} &= \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{R}_T^k \underline{\mathbf{u}}_T \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot (\mathbf{R}_T^k \underline{\mathbf{u}}_T - \mathbf{u}) \\
&\quad - \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} (\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}) \llbracket \mathbf{R}_h^k \underline{\mathbf{e}}_h \rrbracket_{\sigma} \cdot \{ \mathbf{R}_h^k \underline{\mathbf{u}}_h - \mathbf{u} \}_{\sigma}.
\end{aligned}$$

Adding and subtracting

$$\sum_{T \in \mathcal{T}_h} \int_T (\mathbf{R}_T^k \underline{\mathbf{u}}_T \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \mathbf{R}_T^k \hat{\underline{\mathbf{u}}}_T + \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} (\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}) \llbracket \mathbf{R}_h^k \underline{\mathbf{e}}_h \rrbracket_{\sigma} \cdot \{\mathbf{R}_h^k \hat{\underline{\mathbf{u}}}_h\}_{\sigma},$$

recalling the definitions (61) of $\underline{\mathbf{e}}_h$ and (62) of $\boldsymbol{\eta}_h$, the following is obtained:

$$\begin{aligned} \mathfrak{S}_{1,3} + \mathfrak{S}_{1,4} &= \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{R}_T^k \underline{\mathbf{u}}_T \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \mathbf{R}_T^k \underline{\mathbf{e}}_T - \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} (\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}) \llbracket \mathbf{R}_h^k \underline{\mathbf{e}}_h \rrbracket_{\sigma} \cdot \{\boldsymbol{\eta}_h\}_{\sigma} \\ &\quad - \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} (\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}) \llbracket \mathbf{R}_h^k \underline{\mathbf{e}}_h \rrbracket_{\sigma} \cdot \{\mathbf{R}_h^k \underline{\mathbf{e}}_h\}_{\sigma} + \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{R}_T^k \underline{\mathbf{u}}_T \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \boldsymbol{\eta}_T, \end{aligned}$$

where the first and third terms are cancelled using the integration by parts formula [10, Eq. (2)] and the fact that $\nabla \cdot \mathbf{R}_T^k \underline{\mathbf{e}}_T = 0$. Summing $\mathfrak{S}_{1,5}$ to the previous expression, adding and subtracting $\frac{1}{2} \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} |\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}| \llbracket \mathbf{R}_h^k \hat{\underline{\mathbf{u}}}_h \rrbracket_{\sigma} \cdot \llbracket \mathbf{R}_h^k \underline{\mathbf{e}}_h \rrbracket_{\sigma}$, and rearranging, it is inferred that

$$\begin{aligned} &\mathfrak{S}_{1,3} + \mathfrak{S}_{1,4} + \mathfrak{S}_{1,5} \\ &= \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{R}_T^k \underline{\mathbf{u}}_T \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \boldsymbol{\eta}_T - \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} (\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}) \llbracket \mathbf{R}_h^k \underline{\mathbf{e}}_h \rrbracket_{\sigma} \cdot \{\boldsymbol{\eta}_h\}_{\sigma} \\ &\quad - \frac{1}{2} \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} |\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}| \llbracket \mathbf{R}_h^k \underline{\mathbf{e}}_h \rrbracket_{\sigma}^2 - \frac{1}{2} \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} |\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}| \llbracket \boldsymbol{\eta}_h \rrbracket_{\sigma} \cdot \llbracket \mathbf{R}_h^k \underline{\mathbf{e}}_h \rrbracket_{\sigma} \\ &\leq \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{R}_T^k \underline{\mathbf{u}}_T \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \boldsymbol{\eta}_T + \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} |\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}| \llbracket \mathbf{R}_h^k \underline{\mathbf{e}}_h \rrbracket_{\sigma} \cdot \{\boldsymbol{\eta}_h\}_{\sigma} \\ &\quad - \frac{1}{2} \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} |\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}| \llbracket \mathbf{R}_h^k \underline{\mathbf{e}}_h \rrbracket_{\sigma}^2 + \frac{1}{2} \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} |\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}| \llbracket \boldsymbol{\eta}_h \rrbracket_{\sigma} \cdot \llbracket \mathbf{R}_h^k \underline{\mathbf{e}}_h \rrbracket_{\sigma}. \end{aligned}$$

Using the generalized Young inequality (74) for the second and fourth terms with ϵ respectively equal to 1 and $\frac{1}{2}$, so that the third term cancels out, we finally get

$$\begin{aligned} \mathfrak{S}_{1,3} + \mathfrak{S}_{1,4} + \mathfrak{S}_{1,5} &\leq \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{R}_T^k \underline{\mathbf{u}}_T \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \boldsymbol{\eta}_T + \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} |\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}| \llbracket \boldsymbol{\eta}_h \rrbracket_{\sigma}^2 \\ &\quad + \frac{1}{4} \sum_{\sigma \in \mathfrak{F}_h^i} \int_{\sigma} |\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}| \llbracket \boldsymbol{\eta}_h \rrbracket_{\sigma}^2, \end{aligned} \tag{78}$$

Denote by $\mathcal{T}_{\sigma} \subset \mathcal{T}_h$ the set collecting the (one or two) mesh elements that share $\sigma \in \mathfrak{F}_h^i$. To bound the second term in the right hand side of (78), we recall the definition (61) of $\underline{\mathbf{e}}_h$ and use a triangle inequality to write

$$\begin{aligned} &\int_{\sigma} |\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_{\sigma}| \llbracket \boldsymbol{\eta}_h \rrbracket_{\sigma}^2 \\ &\leq \int_{\sigma} |\mathbf{R}_h^k \underline{\mathbf{e}}_h \cdot \mathbf{n}_{\sigma}| \llbracket \boldsymbol{\eta}_h \rrbracket_{\sigma}^2 + \int_{\sigma} |\mathbf{R}_h^k \hat{\underline{\mathbf{u}}}_h \cdot \mathbf{n}_{\sigma}| \llbracket \boldsymbol{\eta}_h \rrbracket_{\sigma}^2 \\ &\lesssim \|\boldsymbol{\eta}_h\|_{L^{\infty}(\mathcal{T}_{\sigma})} \|\mathbf{R}_h^k \underline{\mathbf{e}}_h \cdot \mathbf{n}_{\sigma}\|_{L^2(\sigma)} \|\llbracket \boldsymbol{\eta}_h \rrbracket_{\sigma}\|_{L^2(\sigma)} + \|\mathbf{R}_h^k \hat{\underline{\mathbf{u}}}_h\|_{L^{\infty}(\mathcal{T}_{\sigma})} \|\llbracket \boldsymbol{\eta}_h \rrbracket_{\sigma}\|_{L^2(\sigma)}^2 \\ &\lesssim h_{\sigma} |\mathbf{u}|_{\mathbf{W}^{1,\infty}(\mathcal{T}_{\sigma})} \|\mathbf{R}_h^k \underline{\mathbf{e}}_h \cdot \mathbf{n}_{\sigma}\|_{L^2(\sigma)}^2 + \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_{\sigma})} \|\llbracket \boldsymbol{\eta}_h \rrbracket_{\sigma}\|_{L^2(\sigma)}^2, \end{aligned}$$

where we have used Hölder inequalities with exponents $(\infty, 2, 2)$ along with the fact that $\|\mathbf{n}_\sigma\|_{\mathbf{L}^\infty(\sigma)} \leq 1$ in the second step and Young's inequality for the first term together with the approximation properties (18) of $\mathbf{R}_T^k \circ \underline{\mathbf{l}}_T^k$ with $(m, p, s) = (0, \infty, 1)$ and the fact that $h_T \lesssim 1$ for $T \in \mathcal{T}_\sigma$ in the third step. We continue using a discrete trace inequality on σ to write $\|\mathbf{R}_h^k \underline{\mathbf{e}}_h \cdot \mathbf{n}_\sigma\|_{L^2(\sigma)}^2 \lesssim h_\sigma^{-1} \|\mathbf{R}_h^k \underline{\mathbf{e}}_h\|_{L^2(\mathcal{T}_\sigma)}^2 \stackrel{(33)}{\lesssim} h_\sigma^{-1} \|\underline{\mathbf{e}}_h\|_{\mathbf{R}, \mathcal{T}_\sigma}^2$ and a continuous trace inequality followed by $h_\sigma^{-1} \lesssim h_T^{-1}$ (consequence of mesh regularity) to infer

$$\begin{aligned} \|\{\boldsymbol{\eta}_{T_\sigma}\}_\sigma\|_{L^2(\sigma)}^2 &\stackrel{(57), (24)}{\lesssim} \sum_{\tau \in \mathfrak{I}_\sigma} \|(\boldsymbol{\eta}_h)|_\tau\|_{L^2(\sigma)}^2 \\ &\lesssim \sum_{\tau \in \mathfrak{I}_\sigma} \left(h_\sigma^{-1} \|\boldsymbol{\eta}_\tau\|_{L^2(\tau)}^2 + h_\sigma |\boldsymbol{\eta}_\tau|_{\mathbf{H}^1(\tau)}^2 \right) \\ &\lesssim \sum_{T \in \mathcal{T}_\sigma} \left(h_T^{-1} \|\boldsymbol{\eta}_T\|_{L^2(T)}^2 + h_T |\boldsymbol{\eta}_T|_{\mathbf{H}^1(\mathfrak{I}_T)}^2 \right). \end{aligned} \quad (79)$$

This gives

$$\int_\sigma |\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_\sigma| |\{\boldsymbol{\eta}_h\}_\sigma|^2 \lesssim |\mathbf{u}|_{\mathbf{W}^{1,\infty}(\mathcal{T}_\sigma)} \|\underline{\mathbf{e}}_h\|_{\mathbf{R}, \mathcal{T}_\sigma}^2 + \sum_{T \in \mathcal{T}_\sigma} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(T)} \left(h_T^{-1} \|\boldsymbol{\eta}_T\|_{L^2(T)}^2 + h_T |\boldsymbol{\eta}_T|_{\mathbf{H}^1(\mathfrak{I}_T)}^2 \right).$$

Plugging this bound into (78), observing that $\text{card}(\mathfrak{F}_T^i) + \text{card}(\mathcal{F}_T) \lesssim \text{card}(\mathfrak{I}_T) \lesssim 1$ by (3), and bounding the last term in the right hand side of (78) similarly as we just did with the second term, it is seen that

$$\begin{aligned} \mathfrak{S}_{1,3} + \mathfrak{S}_{1,4} + \mathfrak{S}_{1,5} &\lesssim \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{R}_T^k \underline{\mathbf{u}}_T \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \boldsymbol{\eta}_T + \sum_{T \in \mathcal{T}_h} |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\underline{\mathbf{e}}_T\|_{\mathbf{R}, T}^2 \\ &\quad + \sum_{T \in \mathcal{T}_h} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(T)} \left(h_T^{-1} \|\boldsymbol{\eta}_T\|_{L^2(T)}^2 + h_T |\boldsymbol{\eta}_T|_{\mathbf{H}^1(\mathfrak{I}_T)}^2 \right). \end{aligned} \quad (80)$$

We use the following bound, proved at the end of this section, for the first term in the right hand side of (80):

$$\begin{aligned} \left| \int_T (\mathbf{R}_T^k \underline{\mathbf{u}}_T \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \boldsymbol{\eta}_T \right| &\lesssim \|\underline{\mathbf{e}}_T\|_{\mathbf{R}, T}^2 \left(|\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} + h_T \gamma_k \|\mathbf{u}\|_{L^\infty(T)} \right) + |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\boldsymbol{\eta}_T\|_{L^2(T)}^2 \\ &\quad + \left(|\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} + \gamma_k h_T^{-3} \|\mathbf{u}\|_{L^\infty(T)} \right) \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{L^2(T)}^2 \\ &\quad + \tilde{\delta}_{0k} \sum_{\sigma \in \mathfrak{F}_T^i} \|\llbracket \varrho_{\mathfrak{I}_T}^k((\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T) \rrbracket_\sigma\|_{L^2(\sigma)} \|\boldsymbol{\eta}_T \cdot \mathbf{n}_\sigma\|_{L^2(\sigma)}, \end{aligned} \quad (81)$$

to finally get

$$\begin{aligned} \mathfrak{S}_{1,3} + \mathfrak{S}_{1,4} + \mathfrak{S}_{1,5} &\lesssim \sum_{T \in \mathcal{T}_h} \left[\|\underline{\mathbf{e}}_T\|_{\mathbf{R}, T}^2 \left(|\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} + h_T \gamma_k \|\mathbf{u}\|_{L^\infty(T)} \right) + |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\boldsymbol{\eta}_T\|_{L^2(T)}^2 \right] \\ &\quad + \sum_{T \in \mathcal{T}_h} \left(|\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} + \gamma_k h_T^{-3} \|\mathbf{u}\|_{L^\infty(T)} \right) \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{L^2(T)}^2 \\ &\quad + \sum_{T \in \mathcal{T}_h} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(T)} \left(h_T^{-1} \|\boldsymbol{\eta}_T\|_{L^2(T)}^2 + h_T |\boldsymbol{\eta}_T|_{\mathbf{H}^1(\mathfrak{I}_T)}^2 \right) \\ &\quad + \tilde{\delta}_{0k} \sum_{T \in \mathcal{T}_h} \sum_{\sigma \in \mathfrak{F}_T^i} \|\llbracket \varrho_{\mathfrak{I}_T}^k((\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T) \rrbracket_\sigma\|_{L^2(\sigma)} \|\boldsymbol{\eta}_T \cdot \mathbf{n}_\sigma\|_{L^2(\sigma)}. \end{aligned} \quad (82)$$

To bound $\mathfrak{I}_{1,6}$ in (76), we use the definition (61) of \underline{e}_h and the linearity of $t_{T,\sigma}^k$ in its second argument to write

$$\begin{aligned}
\mathfrak{I}_{1,6} &= - \sum_{T \in \mathcal{T}_h} \sum_{\sigma \in \mathfrak{F}_T^i} t_{T,\sigma}^k(\underline{\mathbf{u}}_T, \underline{\mathbf{u}}_T, \underline{\mathbf{e}}_T) = - \sum_{T \in \mathcal{T}_h} \sum_{\sigma \in \mathfrak{F}_T^i} \left(t_{T,\sigma}^k(\underline{\mathbf{u}}_T, \hat{\underline{\mathbf{u}}}_T, \underline{\mathbf{e}}_T) + t_{T,\sigma}^k(\underline{\mathbf{u}}_T, \underline{\mathbf{e}}_T, \underline{\mathbf{e}}_T) \right) \\
&\leq \sum_{T \in \mathcal{T}_h} \sum_{\sigma \in \mathfrak{F}_T^i} \left(|t_{T,\sigma}^k(\underline{\mathbf{u}}_T, \hat{\underline{\mathbf{u}}}_T, \underline{\mathbf{e}}_T)| - t_{T,\sigma}^k(\underline{\mathbf{u}}_T, \underline{\mathbf{e}}_T, \underline{\mathbf{e}}_T) \right) \\
&\leq \frac{\tilde{\delta}_{0k}}{2} \sum_{T \in \mathcal{T}_h} \sum_{\sigma \in \mathfrak{F}_T^i} \left\| \llbracket \varrho_{\mathfrak{X}_T}^k((\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \hat{\underline{\mathbf{u}}}_T) \rrbracket_{\sigma} \right\|_{L^2(\sigma)}^2 \\
&\quad \boxed{- \frac{\tilde{\delta}_{0k}}{2} \sum_{T \in \mathcal{T}_h} \sum_{\sigma \in \mathfrak{F}_T^i} \left\| \llbracket \varrho_{\mathfrak{X}_T}^k((\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T) \rrbracket_{\sigma} \right\|_{L^2(\sigma)}^2},
\end{aligned} \tag{83}$$

where, in the last step, we have used the definitions (59) of $t_{T,\sigma}^k$ and (65) of $\tilde{\delta}_{0k}$, Young's inequality, and the fact that $\mathbf{R}_T^k \hat{\underline{\mathbf{u}}}_T$ and $\mathbf{R}_T^k \underline{\mathbf{e}}_T$ both belong to $\mathcal{P}^k(\mathfrak{X}_T)$ (as proved in the item (iv.b) below), thus $\pi_{\mathfrak{X}_T}^{k-1}((\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \hat{\underline{\mathbf{u}}}_T) = (\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \hat{\underline{\mathbf{u}}}_T$, and $\pi_{\mathfrak{X}_T}^{k-1}((\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T) = (\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T$. Using the generalized Young inequality (74) in the last term of (82) with ϵ selected so as to compensate the boxed term in (83) and summing the resulting relation to (83), we obtain

$$\begin{aligned}
&\mathfrak{I}_{1,3} + \mathfrak{I}_{1,4} + \mathfrak{I}_{1,5} + \mathfrak{I}_{1,6} \\
&\lesssim \sum_{T \in \mathcal{T}_h} \left[\|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2 \left(|\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} + h_T \gamma_k \|\mathbf{u}\|_{\mathbf{L}^\infty(T)} \right) + |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\boldsymbol{\eta}_T\|_{\mathbf{L}^2(T)}^2 \right] \\
&\quad + \sum_{T \in \mathcal{T}_h} \left(|\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} + \gamma_k h_T^{-3} \|\mathbf{u}\|_{\mathbf{L}^\infty(T)} \right) \|\hat{\underline{\mathbf{u}}}_T - \mathbf{u}\|_{\mathbf{L}^2(T)}^2 \\
&\quad + \sum_{T \in \mathcal{T}_h} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(T)} \left(h_T^{-1} \|\boldsymbol{\eta}_T\|_{\mathbf{L}^2(T)}^2 + h_T |\boldsymbol{\eta}_T|_{\mathbf{H}^1(\mathfrak{X}_T)}^2 \right) \\
&\quad + \tilde{\delta}_{0k} \sum_{T \in \mathcal{T}_h} \sum_{\sigma \in \mathfrak{F}_T^i} \left[\underbrace{\|\boldsymbol{\eta}_T \cdot \mathbf{n}_\sigma\|_{\mathbf{L}^2(\sigma)}^2}_{=: \mathfrak{A}(\sigma)} + \underbrace{\left\| \llbracket \varrho_{\mathfrak{X}_T}^k((\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \hat{\underline{\mathbf{u}}}_T) \rrbracket_{\sigma} \right\|_{\mathbf{L}^2(\sigma)}^2}_{=: \mathfrak{B}(\sigma)} \right].
\end{aligned} \tag{84}$$

We now proceed to bound the last term in the the right-hand side of (84). We bound the first contribution as we did with $\|\{\boldsymbol{\eta}_{T_\sigma}\}_\sigma\|_{\mathbf{L}^2(\sigma)}^2$ in (79), that is,

$$\mathfrak{A}(\sigma) \lesssim \sum_{T \in \mathcal{T}_\sigma} \left(h_T^{-1} \|\boldsymbol{\eta}_T\|_{\mathbf{L}^2(T)}^2 + h_T |\boldsymbol{\eta}_T|_{\mathbf{H}^1(\mathfrak{X}_T)}^2 \right). \tag{85}$$

To bound $\mathfrak{B}(\sigma)$ in (84), observe that, since $(\mathbf{u}_T^0 \cdot \nabla) \pi_T^k \mathbf{u} \in \mathcal{P}^{k-1}(T)$, using (47), we have $\varrho_{\mathfrak{X}_T}^k((\mathbf{u}_T^0 \cdot \nabla) \pi_T^k \mathbf{u}) \in \mathcal{P}^k(T)$, and then $\llbracket \varrho_{\mathfrak{X}_T}^k((\mathbf{u}_T^0 \cdot \nabla) \pi_T^k \mathbf{u}) \rrbracket_{\sigma} \equiv 0$ for all simplicial faces $\sigma \in \mathfrak{F}_T^i$ internal to any mesh element $T \in \mathcal{T}_h$. Therefore, denoting by τ_1 and τ_2 the simplices in \mathfrak{X}_T that share

$\sigma \in \mathfrak{F}_T^i$, it is inferred, letting for the sake of brevity $\varrho_{\tau_i}^k := (\varrho_{\mathfrak{F}_T^i}^k)|_{\tau_i}$,

$$\begin{aligned}
\mathfrak{B}(\sigma) &= \|\llbracket \varrho_{\mathfrak{F}_T^i}^k((\mathbf{u}_T^0 \cdot \nabla)(\mathbf{R}_T^k \hat{\mathbf{u}}_T - \boldsymbol{\pi}_T^k \mathbf{u})) \rrbracket_\sigma\|_{L^2(\sigma)}^2 \\
&\stackrel{(57), (24)}{\lesssim} \sum_{i=1}^2 \|\varrho_{\tau_i}^k((\mathbf{u}_T^0 \cdot \nabla)(\mathbf{R}_T^k \hat{\mathbf{u}}_T - \boldsymbol{\pi}_T^k \mathbf{u}))\|_{L^2(\sigma)}^2 \\
&\lesssim \sum_{i=1}^2 h_{\tau_i}^{-1} \|\varrho_{\tau_i}^k((\mathbf{u}_T^0 \cdot \nabla)(\mathbf{R}_T^k \hat{\mathbf{u}}_T - \boldsymbol{\pi}_T^k \mathbf{u}))\|_{L^2(\tau_i)}^2 \\
&\lesssim h_T^{-1} \|\varrho_{\mathfrak{F}_T^i}^k((\mathbf{u}_T^0 \cdot \nabla)(\mathbf{R}_T^k \hat{\mathbf{u}}_T - \boldsymbol{\pi}_T^k \mathbf{u}))\|_{L^2(T)}^2 \\
&\stackrel{(45)}{\lesssim} h_T \|((\mathbf{u}_T^0 \cdot \nabla)(\mathbf{R}_T^k \hat{\mathbf{u}}_T - \boldsymbol{\pi}_T^k \mathbf{u}))\|_{L^2(T)}^2 \\
&\stackrel{(61), (24)}{\lesssim} h_T \|\mathbf{e}_T^0 \cdot \nabla(\mathbf{R}_T^k \hat{\mathbf{u}}_T - \boldsymbol{\pi}_T^k \mathbf{u})\|_{L^2(T)}^2 + h_T \|\hat{\mathbf{u}}_T^0 \cdot \nabla(\mathbf{R}_T^k \hat{\mathbf{u}}_T - \boldsymbol{\pi}_T^k \mathbf{u})\|_{L^2(T)}^2,
\end{aligned}$$

where we have used a discrete trace inequality in the third step and the fact that $h_{\tau_i}^{-1} \lesssim h_T^{-1}$ by mesh regularity along with $\tau_1 \cup \tau_2 \subset T$ in the fourth step. We continue adding and subtracting \mathbf{u} and using (24) with $n = 2$ to write

$$\begin{aligned}
\mathfrak{B}(\sigma) &\lesssim h_T \|\mathbf{e}_T^0 \cdot \nabla(\mathbf{R}_T^k \hat{\mathbf{u}}_T - \mathbf{u})\|_{L^2(T)}^2 + h_T \|\mathbf{e}_T^0 \cdot \nabla(\mathbf{u} - \boldsymbol{\pi}_T^k \mathbf{u})\|_{L^2(T)}^2 \\
&\quad + h_T \|\hat{\mathbf{u}}_T^0 \cdot \nabla(\mathbf{R}_T^k \hat{\mathbf{u}}_T - \mathbf{u})\|_{L^2(T)}^2 + h_T \|\hat{\mathbf{u}}_T^0 \cdot \nabla(\mathbf{u} - \boldsymbol{\pi}_T^k \mathbf{u})\|_{L^2(T)}^2 \\
&\lesssim h_T \|\nabla(\mathbf{R}_T^k \hat{\mathbf{u}}_T - \mathbf{u})\|_{L^\infty(T)} \|\mathbf{e}_T\|_{L^2(T)}^2 + h_T \|\nabla(\mathbf{u} - \boldsymbol{\pi}_T^k \mathbf{u})\|_{L^\infty(T)} \|\mathbf{e}_T\|_{L^2(T)}^2 \\
&\quad + h_T \|\hat{\mathbf{u}}_T\|_{L^\infty(T)}^2 \|\nabla(\mathbf{R}_T^k \hat{\mathbf{u}}_T - \mathbf{u})\|_{L^2(T)}^2 + h_T \|\hat{\mathbf{u}}_T\|_{L^\infty(T)}^2 \|\nabla(\mathbf{u} - \boldsymbol{\pi}_T^k \mathbf{u})\|_{L^2(T)}^2 \\
&\lesssim h_T |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)}^2 \|\mathbf{e}_T\|_{\mathbf{R},T}^2 + h_T \|\mathbf{u}\|_{L^\infty(T)}^2 |\boldsymbol{\eta}_T|_{\mathbf{H}^1(T)} + h_T \|\mathbf{u}\|_{L^\infty(T)}^2 \|\nabla(\mathbf{u} - \boldsymbol{\pi}_T^k \mathbf{u})\|_{L^2(T)}^2,
\end{aligned} \tag{86}$$

where in the second step we have used Hölder inequalities with exponents $(2, \infty)$ for the first two terms and with exponents $(\infty, 2)$ for last two terms, then the L^2 -boundedness of $\boldsymbol{\pi}_T^0$ in all terms. In the last step we have used: for the first term the approximation properties (18) of $\mathbf{R}_T^k \circ \mathbf{I}_T^k$ with $(m, p, s) = (1, \infty, 1)$ along with (34); for the second term, the \mathbf{H}^1 -boundedness of $\boldsymbol{\pi}_T^k$ and again (34); for the third term, the definition (62) of $\boldsymbol{\eta}_T$, and invoked the L^∞ -boundedness of $\boldsymbol{\pi}_T$ for the third and fourth terms.

(iii) *Estimate of \mathfrak{S}_1 .* Plugging (86) along with (85) into (84), then using the fact that $\text{card}(\mathfrak{F}_T^i) \lesssim \frac{\text{card}(\mathfrak{T}_T)}{\bar{T}} \lesssim 1$ for $\bar{T} \in \mathcal{T}_h$ by (3), and using additionally the bound (77), we obtain in (76)

$$\mathfrak{S}_1 \lesssim \sum_{T \in \mathcal{T}_h} \|\mathbf{e}_T\|_{\mathbf{R},T}^2 \left(|\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} + h_T \gamma_k \|\mathbf{u}\|_{L^\infty(T)} + h_T \tilde{\delta}_{0k} |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)}^2 \right) + \mathfrak{R}_1,$$

which is precisely (70).

(iv) *Proof of (81).* To conclude the proof of Lemma 16, it only remains to prove (81), which we do next. We begin using a triangle inequality as follows:

$$\begin{aligned}
&\left| \int_T (\mathbf{R}_T^k \hat{\mathbf{u}}_T \cdot \nabla) \mathbf{R}_T^k \mathbf{e}_T \cdot \boldsymbol{\eta}_T \right| \\
&\leq \left| \int_T ((\mathbf{R}_T^k \hat{\mathbf{u}}_T - \mathbf{u}_T) \cdot \nabla) \mathbf{R}_T^k \mathbf{e}_T \cdot \boldsymbol{\eta}_T \right| + \left| \int_T (\mathbf{u}_T \cdot \nabla) \mathbf{R}_T^k \mathbf{e}_T \cdot \boldsymbol{\eta}_T \right| =: \mathfrak{Z}_1 + \mathfrak{Z}_2. \tag{87}
\end{aligned}$$

(iv.a) *Estimate of \mathfrak{I}_1 .* Adding the terms $\pm(\mathbf{R}_T^k \hat{\mathbf{u}}_T - \hat{\mathbf{u}}_T)$ in the inner parentheses of \mathfrak{I}_1 , then recalling that $\underline{\mathbf{e}}_h = \underline{\mathbf{u}}_h - \hat{\underline{\mathbf{u}}}_h$ (cf. (61)) and using a triangle inequality, we get

$$\mathfrak{I}_1 \leq \left| \int_T \left((\mathbf{R}_T^k \underline{\mathbf{e}}_T - \mathbf{e}_T) \cdot \nabla \right) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \boldsymbol{\eta}_T \right| + \left| \int_T \left((\mathbf{R}_T^k \hat{\mathbf{u}}_T - \hat{\mathbf{u}}_T) \cdot \nabla \right) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \boldsymbol{\eta}_T \right| =: \mathfrak{I}_{1,1} + \mathfrak{I}_{1,2}.$$

To bound $\mathfrak{I}_{1,1}$, we use a Hölder inequality with exponents $(2, 2, \infty)$ to obtain

$$\begin{aligned} \mathfrak{I}_{1,1} &\leq \|\mathbf{R}_T^k \underline{\mathbf{e}}_T - \mathbf{e}_T\|_{\mathbf{L}^2(T)} \|\nabla \mathbf{R}_T^k \underline{\mathbf{e}}_T\|_{\mathbf{L}^2(T)} \|\boldsymbol{\eta}_T\|_{\mathbf{L}^\infty(T)} \\ &\lesssim h_T^2 \|\underline{\mathbf{e}}_T\|_{1,T} \|\nabla \mathbf{R}_T^k \underline{\mathbf{e}}_T\|_{\mathbf{L}^2(T)} |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)}, \end{aligned}$$

where, in the second step, we have used (17) and (18) with $(m, p, s) = (0, \infty, 1)$ for the first and third factors, respectively, after recalling, for the latter, that $\boldsymbol{\eta}_T = \mathbf{R}_T^k \hat{\mathbf{u}}_T - \mathbf{u}$ by (62). To proceed, we bound $\|\underline{\mathbf{e}}_T\|_{1,T}$ with (34), then use the inverse inequality (30) along with the definition (33) of $\|\cdot\|_{\mathbf{R},T}$ to write $\|\nabla \mathbf{R}_T^k \underline{\mathbf{e}}_T\|_{\mathbf{L}^2(T)} \lesssim h_T^{-1} \|\mathbf{R}_T^k \underline{\mathbf{v}}_T\|_{\mathbf{L}^2(T)} \leq h_T^{-1} \|\underline{\mathbf{e}}_T\|_{\mathbf{R},T}$. Thus, we get

$$\mathfrak{I}_{1,1} \lesssim \|\underline{\mathbf{e}}_T\|_{\mathbf{R},T}^2 |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)}.$$

We bound $\mathfrak{I}_{1,2}$ using a Hölder inequality with exponents $(2, 2, \infty)$, then using the inverse inequality (30) along with the definition (33) of $\|\cdot\|_{\mathbf{R},T}$ for the second factor, and recalling that $\boldsymbol{\eta}_T = \mathbf{R}_T^k \hat{\mathbf{u}}_T - \mathbf{u}$ by (62) to use the approximation properties (18) of $\mathbf{R}_T^k \circ \mathbf{I}_T^k$ with $(m, p, s) = (0, \infty, 1)$ for the third factor, thus obtaining

$$\begin{aligned} \mathfrak{I}_{1,2} &\lesssim \|\mathbf{R}_T^k \hat{\mathbf{u}}_T - \hat{\mathbf{u}}_T\|_{\mathbf{L}^2(T)} \|\underline{\mathbf{e}}_T\|_{\mathbf{R},T} |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \\ &\lesssim |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\mathbf{R}_T^k \hat{\mathbf{u}}_T - \hat{\mathbf{u}}_T\|_{\mathbf{L}^2(T)}^2 + |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\underline{\mathbf{e}}_T\|_{\mathbf{R},T}^2 \\ &\lesssim |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \left(\|\boldsymbol{\eta}_T\|_{\mathbf{L}^2(T)}^2 + \|\mathbf{u} - \hat{\mathbf{u}}_T\|_{\mathbf{L}^2(T)}^2 \right) + |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\underline{\mathbf{e}}_T\|_{\mathbf{R},T}^2, \end{aligned}$$

where in the second step we have used Young's inequality while, in the third step, we have added $\pm \mathbf{u}$ into the second term, used (24) with $n = 2$, and recalled (62). Gathering the above bounds for $\mathfrak{I}_{1,1}$ and $\mathfrak{I}_{1,2}$, we obtain

$$\mathfrak{I}_1 \lesssim |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \left(\|\boldsymbol{\eta}_T\|_{\mathbf{L}^2(T)}^2 + \|\mathbf{u} - \hat{\mathbf{u}}_T\|_{\mathbf{L}^2(T)}^2 \right) + |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\underline{\mathbf{e}}_T\|_{\mathbf{R},T}^2. \quad (88)$$

(iv.b) *Estimate of \mathfrak{I}_2 and conclusion.* To bound \mathfrak{I}_2 in (87), recalling that $\mathbf{u}_T^0 := \boldsymbol{\pi}_T^0 \mathbf{u}_T$ and using a triangle inequality, we have that

$$\mathfrak{I}_2 \leq \left| \int_T \left((\mathbf{u}_T - \mathbf{u}_T^0) \cdot \nabla \right) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \boldsymbol{\eta}_T \right| + \left| \int_T \left(\mathbf{u}_T^0 \cdot \nabla \right) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \boldsymbol{\eta}_T \right| =: \mathfrak{I}_{2,1} + \mathfrak{I}_{2,2}. \quad (89)$$

To treat $\mathfrak{I}_{2,1}$, we add and subtract $(\hat{\mathbf{u}}_T - \hat{\mathbf{u}}_T^0)$, use (61), and apply a triangle inequality to obtain

$$\mathfrak{I}_{2,1} \leq \left| \int_T \left((\mathbf{e}_T - \mathbf{e}_T^0) \cdot \nabla \right) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \boldsymbol{\eta}_T \right| + \left| \int_T \left((\hat{\mathbf{u}}_T - \hat{\mathbf{u}}_T^0) \cdot \nabla \right) \mathbf{R}_T^k \underline{\mathbf{e}}_T \cdot \boldsymbol{\eta}_T \right| =: \mathfrak{I}_{2,1,1} + \mathfrak{I}_{2,1,2}. \quad (90)$$

To bound $\mathfrak{I}_{2,1,1}$, we use a Hölder inequality with exponents $(2, 2, \infty)$, the approximation properties (4) of $\boldsymbol{\pi}_T^0$ with $(l, m, r, s) = (0, 0, 2, 1)$ followed by the definition (8) of $\|\cdot\|_{1,T}$ to write $\|\mathbf{e}_T - \mathbf{e}_T^0\|_{\mathbf{L}^2(T)} \lesssim h_T \|\nabla \mathbf{e}_T\|_{\mathbf{L}^2(T)} \leq h_T \|\underline{\mathbf{e}}_T\|_{1,T}$, the inverse inequality (30), and the approximation properties (18) of $\mathbf{R}_T^k \circ \mathbf{I}_T^k$ with $(m, p, s) = (0, \infty, 1)$ to write $\|\boldsymbol{\eta}_T\|_{\mathbf{L}^\infty(T)} \lesssim h_T |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)}$ and thus get

$$\mathfrak{I}_{2,1,1} \lesssim h_T \|\underline{\mathbf{e}}_T\|_{1,T} \|\mathbf{R}_T^k \underline{\mathbf{e}}_T\|_{\mathbf{L}^2(T)} |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \stackrel{(34)}{\lesssim} \|\underline{\mathbf{e}}_T\|_{\mathbf{R},T}^2 |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)}.$$

Similarly, using a Hölder inequality with exponents $(\infty, 2, 2)$ followed by a triangle inequality, we obtain

$$\begin{aligned} \mathfrak{I}_{2,1,2} &\leq \left(\|\hat{\mathbf{u}}_T - \mathbf{u}\|_{\mathbf{L}^\infty(T)} + \|\mathbf{u} - \hat{\mathbf{u}}_T^0\|_{\mathbf{L}^\infty(T)} \right) \|\nabla \mathbf{R}_T^k \underline{\mathbf{e}}_T\|_{\mathbf{L}^2(T)} \|\boldsymbol{\eta}_T\|_{\mathbf{L}^2(T)} \\ &\lesssim |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\mathbf{R}_T^k \underline{\mathbf{e}}_T\|_{\mathbf{L}^2(T)} \|\boldsymbol{\eta}_T\|_{\mathbf{L}^2(T)} \lesssim |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\underline{\mathbf{e}}_T\|_{\mathbf{R},T}^2 + |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\boldsymbol{\eta}_T\|_{\mathbf{L}^2(T)}^2, \end{aligned}$$

where, in the second step, we have used the approximation properties (4) of $\pi_T^{k^*}$ and π_T^0 with $(l, m, r, s) = (k^*, 0, \infty, 1)$ and $(l, m, r, s) = (0, 0, \infty, 1)$, respectively, and then the inverse inequality (30) while, in the last step, we have used the definition (33) of $\|\cdot\|_{\mathbf{R},T}$ and Young's inequality.

To bound $\mathfrak{I}_{2,2}$ in (89), we use the fact that $\nabla \cdot \mathbf{u} = 0$, the commutation property (12), and (16b) to infer that $\nabla \cdot \mathbf{R}_T^k \hat{\mathbf{u}}_T = 0$. In addition, we have $\nabla \cdot \mathbf{R}_T^k \underline{\mathbf{u}}_T = 0$ (take $\underline{\mathbf{v}}_h = \mathbf{0}$ and $q_h \in \mathcal{P}^k(\mathcal{T}_h)$ supported in T in (60) and use (41) and (16b)). By (61), the above relations yield $\nabla \cdot \mathbf{R}_T^k \underline{\mathbf{e}}_T = 0$; this implies that $\mathbf{R}_T^k \underline{\mathbf{e}}_T \in \mathcal{P}^k(\mathfrak{T}_T)$ (see [8, Corollary 2.3.1, p. 90]), thus $(\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T \in \mathcal{P}^{k-1}(\mathfrak{T}_T)$, and the proof is finished for $k = 0$, since $\mathbf{R}_T^0 \underline{\mathbf{e}}_T \in \mathcal{P}^0(\mathfrak{T}_T)$ and thus $(\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^0 \underline{\mathbf{e}}_T = \mathbf{0}$, i.e., $\mathfrak{I}_{2,2} = 0$. If $k \geq 1$, on the other hand, we recall the definition (43) of $\varrho_{\mathfrak{T}_T}^k$ and observe that

$$\mathcal{P}^{k-1}(\mathfrak{T}_T) \ni (\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T = \nabla g + \mathbf{c},$$

with $g := \varrho_{\mathfrak{T}_T}^k((\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T)$ and $\mathbf{c} := (\text{Id} - \nabla \varrho_{\mathfrak{T}_T}^k)((\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T)$. Using a triangle inequality, we split $\mathfrak{I}_{2,2}$ as follows:

$$\mathfrak{I}_{2,2} \leq \left| \int_T \nabla g \cdot \boldsymbol{\eta}_T \right| + \left| \int_T \mathbf{c} \cdot \boldsymbol{\eta}_T \right| =: \mathfrak{I}_{2,2,1} + \mathfrak{I}_{2,2,2}. \quad (91)$$

Applying an integration by parts to the integral contained in $\mathfrak{I}_{2,2,1}$ and using the fact that $\boldsymbol{\eta}_T$ has continuous normal trace across simplicial faces $\sigma \in \mathfrak{F}_T^i$, we obtain

$$\begin{aligned} \mathfrak{I}_{2,2,1} &\leq \sum_{\sigma \in \mathfrak{F}_T^i} \left| \int_\sigma \llbracket g \rrbracket_\sigma (\boldsymbol{\eta}_T \cdot \mathbf{n}_\sigma) \right| + \sum_{F \in \mathcal{F}_T} \sum_{\sigma \in \mathfrak{F}_F} \left| \int_\sigma g (\boldsymbol{\eta}_T \cdot \mathbf{n}_\sigma) \right| + \left| \int_T g (\nabla \cdot \boldsymbol{\eta}_T) \right| \\ &\leq \sum_{\sigma \in \mathfrak{F}_T^i} \|\llbracket g \rrbracket_\sigma\|_{L^2(\sigma)} \|\boldsymbol{\eta}_T \cdot \mathbf{n}_\sigma\|_{L^2(\sigma)} \\ &= \tilde{\delta}_{0k} \sum_{\sigma \in \mathfrak{F}_T^i} \|\llbracket \varrho_{\mathfrak{T}_T}^k((\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T) \rrbracket_\sigma\|_{L^2(\sigma)} \|\boldsymbol{\eta}_T \cdot \mathbf{n}_\sigma\|_{L^2(\sigma)}, \end{aligned}$$

where the cancellations in the first step follow using the fact that $\nabla \cdot \boldsymbol{\eta}_T = \nabla \cdot (\mathbf{R}_T^k \hat{\mathbf{u}}_T - \mathbf{u}) = 0$ (since $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{R}_T^k \hat{\mathbf{u}}_T = 0$) and the boundary condition (16a), while the second step follows from a Cauchy–Schwarz inequality. We now proceed to bound $\mathfrak{I}_{2,2,2}$ in (91). By (46) with $l = k - 1$, we have that $\mathbf{c} \in \mathcal{G}^{c,k-1}(\mathfrak{T}_T)$ and, recalling that, by definition, $\mathcal{G}^{c,k-1}(\mathfrak{T}_T)$ is the trivial space when $k = 1$, we get $\mathfrak{I}_{2,2,2} = 0$. For $k \geq 2$, on the other hand, recalling the definition (66) of γ_k , the fact that $\boldsymbol{\eta}_T := \mathbf{R}_T^k \hat{\mathbf{u}}_T - \mathbf{u}$, and using the condition (16c), we obtain

$$\begin{aligned} \mathfrak{I}_{2,2,2} &= \gamma_k \left| \int_T \mathbf{c} \cdot (\hat{\mathbf{u}}_T - \mathbf{u}) \right| \\ &\leq \gamma_k \|\mathbf{c}\|_{\mathbf{L}^2(T)} \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{\mathbf{L}^2(T)} \\ &\lesssim \gamma_k \|(\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T\|_{\mathbf{L}^2(T)} \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{\mathbf{L}^2(T)} \\ &\leq \gamma_k \left(\|\mathbf{e}_T^0\|_{\mathbf{L}^\infty(T)} + \|\hat{\mathbf{u}}_T^0\|_{\mathbf{L}^\infty(T)} \right) \|\nabla \mathbf{R}_T^k \underline{\mathbf{e}}_T\|_{\mathbf{L}^2(T)} \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{\mathbf{L}^2(T)} \\ &\lesssim \gamma_k h_T^{-\frac{d}{2}} \|\mathbf{e}_T\|_{\mathbf{L}^2(T)} \|\nabla \mathbf{R}_T^k \underline{\mathbf{e}}_T\|_{\mathbf{L}^2(T)} \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{\mathbf{L}^2(T)} \\ &\quad + \gamma_k \|\mathbf{u}\|_{\mathbf{L}^\infty(T)} h_T^{\frac{3}{2}} \|\nabla \mathbf{R}_T^k \underline{\mathbf{e}}_T\|_{\mathbf{L}^2(T)} h_T^{-\frac{3}{2}} \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{\mathbf{L}^2(T)}, \end{aligned} \quad (92)$$

where we have used the Cauchy–Schwarz inequality in the second step, the fact that $\|(\text{Id} - \nabla \varrho_{\mathfrak{T}_T}^k) \mathbf{q}\|_{L^2(T)} \leq \|\mathbf{q}\|_{L^2(T)} + \|\nabla \varrho_{\mathfrak{T}_T}^k \mathbf{q}\|_{L^2(T)}$ for $\mathbf{q} \in \mathcal{P}^{k-1}(\mathfrak{T}_T)$ along with the bound (45) with $l = k - 1$ in the third step, a Hölder inequality with exponents $(\infty, 2)$ for the first factor along with (61) and then a triangle inequality in the fourth step, and, for the fifth step, we have used the discrete Lebesgue inequality (93) with $(\alpha, \beta, X) = (\infty, 2, T)$ and the L^2 -boundness of π_T^0 for the first addend and the L^∞ -boundness of π_T^0 for the second addend. To further bound the first term in the right-hand side of (92), we use the inverse inequality (30) along with the definition (33) of $\|\cdot\|_{\mathbb{R},T}$ to write

$$\begin{aligned} \gamma_k h_T^{-\frac{d}{2}} \|\mathbf{e}_T\|_{L^2(T)} \|\nabla \mathbf{R}_T^k \underline{\mathbf{e}}_T\|_{L^2(T)} \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{L^2(T)} \\ \lesssim \gamma_k h_T^{-\frac{d}{2}-1} \|\mathbf{e}_T\|_{L^2(T)} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T} \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{L^2(T)} \lesssim \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2 |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)}, \end{aligned}$$

where, in the last step, we have used the fact that $\gamma_k \leq 1$, the inequality (34) for the first factor, and the approximation properties (4) of $\pi_T^{k^*}$ with $(l, m, r, s) = (k^*, 0, 2, 1)$ along with $\|\nabla \mathbf{u}\|_{L^2(T)} \lesssim h_T^{\frac{d}{2}} \|\nabla \mathbf{u}\|_{L^\infty(T)}$ for the last factor. For the second term in the right-hand side of (92), we use the inverse inequality (30) along with the definition (33) of $\|\cdot\|_{\mathbb{R},T}$, and then Young’s inequality, thus we obtain

$$\begin{aligned} \gamma_k \|\mathbf{u}\|_{L^\infty(T)} h_T^{\frac{3}{2}} \|\nabla \mathbf{R}_T^k \underline{\mathbf{e}}_T\|_{L^2(T)} h_T^{-\frac{3}{2}} \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{L^2(T)} \\ \lesssim \gamma_k h_T \|\mathbf{u}\|_{L^\infty(T)} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2 + \gamma_k h_T^{-3} \|\mathbf{u}\|_{L^\infty(T)} \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{L^2(T)}^2. \end{aligned}$$

Using the above estimates in (92), we get

$$\mathfrak{I}_{2,2,2} \lesssim |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2 + \gamma_k h_T \|\mathbf{u}\|_{L^\infty(T)} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2 + \gamma_k h_T^{-3} \|\mathbf{u}\|_{L^\infty(T)} \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{L^2(T)}^2.$$

Therefore plugging the bounds for $\mathfrak{I}_{2,2,1}$ and $\mathfrak{I}_{2,2,2}$ into (91), the bounds $\mathfrak{I}_{2,1,1}$ and $\mathfrak{I}_{2,1,2}$ into (90), and then using the result in (89), we get

$$\begin{aligned} \mathfrak{I}_2 \lesssim |\mathbf{u}|_{\mathbf{W}^{1,\infty}(T)} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2 + \|\mathbf{u}\|_{L^\infty(T)} \|\boldsymbol{\eta}_T\|_{L^2(T)}^2 + \gamma_k h_T \|\mathbf{u}\|_{L^\infty(T)} \|\underline{\mathbf{e}}_T\|_{\mathbb{R},T}^2 \\ + \gamma_k h_T^{-3} \|\mathbf{u}\|_{L^\infty(T)} \|\hat{\mathbf{u}}_T - \mathbf{u}\|_{L^2(T)}^2 + \tilde{\delta}_{0k} \sum_{\sigma \in \mathfrak{F}_T^i} \|\llbracket \varrho_{\mathfrak{T}_T}^k ((\mathbf{u}_T^0 \cdot \nabla) \mathbf{R}_T^k \underline{\mathbf{e}}_T) \rrbracket_\sigma\|_{L^2(\sigma)} \|\boldsymbol{\eta}_T \cdot \mathbf{n}_\sigma\|_{L^2(\sigma)}. \end{aligned}$$

Thus, plugging the above estimate along with (88) into (87), we conclude the proof of (81). \square

6 Numerical tests

In this section we verify numerically the proposed method for general meshes of the unit square domain $\Omega = (0, 1)^2$. The analytical solution (\mathbf{u}, p) in (1) is taken from [33, 37], with velocity components and pressure as follows

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &:= \frac{6 + 4 \cos(4t)}{10} \begin{pmatrix} 8 \sin^2(\pi x_1) (2x_2(1-x_2)(1-2x_2)) \\ -8\pi \sin(2\pi x_1) (x_2(1-x_2))^2 \end{pmatrix}, \\ p(\mathbf{x}) &:= \frac{6 + 4 \cos(4t)}{10} \sin(\pi x_1) \cos(\pi x_2). \end{aligned}$$

For each element $T \in \mathcal{T}_h$, we construct its simplicial submesh \mathfrak{T}_T by adding an internal node which corresponds to the geometrical center \mathbf{x}_T of the element T and construct the simplicial mesh \mathfrak{T}_T in such a way that all simplices in \mathfrak{T}_T have the vertex \mathbf{x}_T in common (this construction fulfills the assumptions made in Section 3.1). We set $k = 1$ and consider computations over three h -refined mesh families (Cartesian, hexagonal and Voronoi type). Figure 2 shows the coarsest mesh for each family. As for the temporal discretization, an implicit/explicit (IMEX) BDF2 scheme is used, in

which $t_h(2\underline{\mathbf{u}}_h^{n-1} - \underline{\mathbf{u}}_h^{n-2}, \underline{\mathbf{u}}_h^n, \underline{\mathbf{v}}_h)$ is used in the convective term (58). The interpolate of the analytical solution is used in the first two steps as the initialization of the algorithm. We set as a time step $\Delta t = 10^{-3}$, and the final time as $t_F = 2$. This choice has been numerically verified to make the error in time negligible with respect to the error in space for all the considered meshes. Our implementation is based on the HARDCore library¹ which makes extensive use of the linear algebra Eigen open-source library (see <http://eigen.tuxfamily.org>), and we use the linear direct solver Pardiso [44]. We monitor the following quantities in Table 1: N_{dof} denoting the number of discrete unknowns and nonzero entries of the global system; $\|\underline{\mathbf{e}}_h\|_{L^\infty(\mathbf{L}^2)}$, the discrete infinity $L^2(\Omega)$ -norm of the velocity error $\underline{\mathbf{e}}_h := \underline{\mathbf{u}}_h - \underline{\mathbf{I}}_h^k \mathbf{u}$; $\|\underline{\mathbf{e}}_h\|_{\sharp, \Delta t}$, the discrete L^2 -energy-upwind-norm of the velocity error defined as follows

$$\|\underline{\mathbf{e}}_h\|_{\sharp, \Delta t}^2 := \Delta t \sum_{n=2}^{N_{t_F}} \left(\nu \|\underline{\mathbf{e}}_h^n\|_{1,h}^2 + \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_h^i} \int_{\sigma} |\mathbf{R}_h^k(2\underline{\mathbf{u}}_h^{n-1} - \underline{\mathbf{u}}_h^{n-2}) \cdot \mathbf{n}_\sigma| |[\![\mathbf{R}_h^k \underline{\mathbf{e}}_h^n]\!]_\sigma|^2 \right),$$

where N_{t_F} is the total number of time steps. The error norms are accompanied by their corresponding spatial Estimated Order of Convergence (EOC) computed using successive spatial refinement steps. To confirm the independence of the velocity error on ν^{-1} , we solve numerically the problem for $\nu \in \{10^{-2}, 10^{-4}, 10^{-6}, 10^{-10}\}$. Recalling the Corollary 11, and the Remarks 14 and 15, we expect to get a convergence rate of 1.5 in both norms (since we have set $k = 1$). The results collected in Table 1 show that the convergence rate of $\|\underline{\mathbf{e}}_h\|_{L^\infty(\mathbf{L}^2)}$ is always greater than 1.5 for all values of ν , and the convergence rate of $\|\underline{\mathbf{e}}_h\|_{\sharp, \Delta t}$ is asymptotically close to 1.5. Moreover for small values of ν , the velocity errors hold practically unchanged, that is to say, they are independent of ν^{-1} . This is consistent with our theoretical results, see Theorem 10 and its Corollary 11.

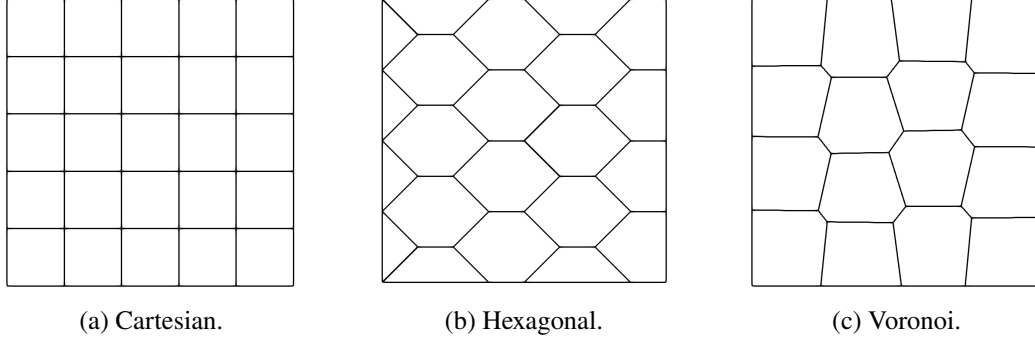


Figure 2: Coarsest meshes used in Section 6.

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¹<https://github.com/jdrioniou/HARDCore>

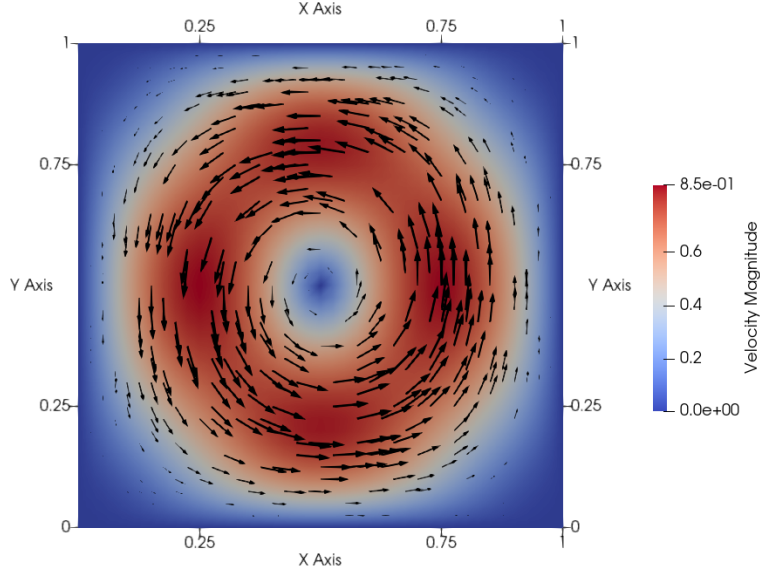


Figure 3: Velocity solution at $t = t_F$ of Section 6.

A Local inequalities for piecewise polynomials on a submesh

Lemma 17 (Local discrete Lebesgue embedding for piecewise polynomials). *Let $T \in \mathcal{T}_h$ and, for an integer $l \geq 0$, let $q \in \mathcal{P}^l(\mathfrak{I}_T)$. Then, for all $(\alpha, \beta) \in [1, +\infty]$, it holds, with the convention that $\frac{1}{\infty} := 0$,*

$$\|q\|_{L^\alpha(T)} \simeq h_T^{d\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)} \|q\|_{L^\beta(T)}, \quad (93)$$

where the hidden constant is independent of h_T , T , and q , but possibly depends on l , α , β , and the mesh regularity parameter.

Proof. We first recall the discrete Lebesgue embedding proved in [20, Lemma 1.25], which establishes that for all $(\alpha, \beta) \in [1, +\infty]$, all $X \in \mathcal{T}_h \cup \mathfrak{I}_h \cup \mathcal{F}_h \cup \mathfrak{F}_h$, and all $\zeta \in \mathcal{P}^l(X)$

$$\|\zeta\|_{L^\alpha(X)} \simeq |X|^{\frac{1}{\alpha} - \frac{1}{\beta}} \|\zeta\|_{L^\beta(X)}, \quad (94)$$

where $|X|$ is the Lebesgue measure of X , and the hidden constant is independent of X , and ζ , but possibly depends on l , α , β , and the mesh regularity parameter. Then, for $\alpha \in [1, +\infty)$, we obtain that

$$\begin{aligned} \|q\|_{L^\alpha(T)}^\alpha &= \sum_{\tau \in \mathfrak{I}_T} \|q\|_{L^\alpha(\tau)}^\alpha \simeq \sum_{\tau \in \mathfrak{I}_T} |\tau|^{\alpha\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)} \|q\|_{L^\beta(\tau)}^\alpha \\ &\simeq \sum_{\tau \in \mathfrak{I}_T} |\tau|^{\alpha\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)} \|q\|_{L^\beta(T)}^\alpha \\ &\simeq h_T^{\alpha d\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)} \|q\|_{L^\beta(T)}^\alpha, \end{aligned}$$

where in the second step we have used (94) with $X = \tau$, in the third step the fact that $\tau \subset T$, and in the fourth step the fact that $|\tau| \simeq h_\tau^d \simeq h_T^d$ by mesh regularity along with (3) for \mathfrak{I}_T . Raising to the power

of $\frac{1}{\alpha}$ the above inequality, we obtain (93). If $\alpha = +\infty$, we use similar steps as before to obtain that

$$\|q\|_{L^\infty(T)} = \sup_{\tau \in \mathfrak{T}_T} \|q\|_{L^\infty(\tau)} \lesssim \sup_{\tau \in \mathfrak{T}_T} |\tau|^{-\frac{1}{\beta}} \|q\|_{L^\beta(\tau)} \lesssim h_T^{-\frac{d}{\beta}} \|q\|_{L^\beta(T)}. \quad \square$$

Lemma 18 (Local inverse inequalities for piecewise polynomials). *Let $T \in \mathcal{T}_h$ and, for an integer $l \geq 0$, let $q \in \mathcal{P}^l(\mathfrak{T}_T)$. Then, for all $p \in [1, +\infty]$, it holds*

$$\|q\|_{L^p(T)} \lesssim h_T^{-1} \|\nabla q\|_{L^p(T)}, \quad (95)$$

where we remind the reader that here ∇ stands for the piecewise gradient on \mathfrak{T}_T .

Proof. For $p \in [1, +\infty)$, we write

$$\|q\|_{L^p(T)}^p = \sum_{\tau \in \mathfrak{T}_T} \|q\|_{L^p(\tau)}^p \lesssim \sum_{\tau \in \mathfrak{T}_T} h_\tau^{-p} \|\nabla q\|_{L^p(\tau)}^p \lesssim h_T^{-p} \|\nabla q\|_{L^p(T)}^p,$$

where we have used a standard discrete inverse inequality on simplices (cf., e.g., [13]) in the second passage and the fact that $h_\tau \simeq h_T$ by mesh regularity to conclude. If $p = +\infty$, the proof is similar with sums over $\tau \in \mathfrak{T}_T$ replaced by maximums over $\tau \in \mathfrak{T}_T$. \square

Lemma 19 (Local trace inequality for piecewise polynomials). *Let $T \in \mathcal{T}_h$, $F \in \mathcal{F}_T$, and, for an integer $l \geq 0$, let $q \in \mathcal{P}^l(\mathfrak{T}_T)$. Then, for all $p \in [1, +\infty]$, it holds*

$$\|q\|_{L^p(F)} \lesssim h_F^{-\frac{1}{p}} \|q\|_{L^p(T)}. \quad (96)$$

Proof. Recalling that \mathfrak{F}_F collects the simplicial faces contained in F and denoting, for all $\sigma \in \mathfrak{F}_F$, by $\tau_\sigma \in \mathfrak{T}_T$ the simplex to which F belongs, we write

$$\|q\|_{L^p(F)}^p = \sum_{\sigma \in \mathfrak{F}_F} \|q\|_{L^p(\sigma)}^p \lesssim \sum_{\sigma \in \mathfrak{F}_F} h_\sigma^{-1} \|q\|_{L^p(\tau_\sigma)}^p \lesssim h_T^{-1} \|q\|_{L^p(T)}^p,$$

where we have used a standard discrete trace inequality on simplices in the second step and the fact that $h_\sigma^{-1} \lesssim h_T^{-1}$, $\tau_\sigma \subset T$ for all $\sigma \in \mathfrak{F}_F$, and $\text{card}(\mathfrak{F}_F) \lesssim 1$ to conclude. Taking the q -th root yields (96). \square

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N_{dof}	$\ \underline{e}_h\ _{L^\infty(\mathbf{L}^2)}$	EOC	$\ \underline{e}_h\ _{\sharp, \Delta t}$	EOC	$\ \underline{e}_h\ _{L^\infty(\mathbf{L}^2)}$	EOC	$\ \underline{e}_h\ _{\sharp, \Delta t}$	EOC	
Cartesian, $\nu = 10^{-2}$					Cartesian, $\nu = 10^{-4}$				
385	1.88E-01	–	2.58E-01	–	3.69E-01	–	2.09E-01	–	
1620	2.64E-02	2.82	1.04E-01	1.30	7.21E-02	2.34	9.86E-02	1.08	
6640	3.24E-03	3.04	3.74E-02	1.48	1.30E-02	2.48	3.87E-02	1.36	
26880	3.95E-04	3.04	1.30E-02	1.53	2.25E-03	2.54	1.43E-02	1.43	
108160	4.88E-05	3.02	4.36E-03	1.57	3.85E-04	2.54	5.17E-03	1.47	
Cartesian, $\nu = 10^{-6}$					Cartesian, $\nu = 10^{-10}$				
385	3.72E-01	–	2.07E-01	–	3.72E-01	–	2.07E-01	–	
1620	7.45E-02	2.31	9.82E-02	1.07	7.45E-02	2.31	9.82E-02	1.07	
6640	1.51E-02	2.31	3.85E-02	1.35	1.51E-02	2.31	3.85E-02	1.35	
26880	3.22E-03	2.23	1.43E-02	1.43	3.25E-03	2.22	1.43E-02	1.43	
108160	6.85E-04	2.23	5.18E-03	1.46	7.16E-04	2.19	5.18E-03	1.46	
Hexagonal, $\nu = 10^{-2}$					Hexagonal, $\nu = 10^{-4}$				
386	3.01E-01	–	4.14E-01	–	5.77E-01	–	3.54E-01	–	
1436	5.68E-02	2.42	1.88E-01	1.14	1.48E-01	1.97	1.79E-01	0.99	
5560	8.67E-03	2.70	7.17E-02	1.39	2.48E-02	2.57	7.05E-02	1.34	
21872	1.14E-03	2.93	2.54E-02	1.50	3.48E-03	2.83	2.57E-02	1.46	
Hexagonal, $\nu = 10^{-6}$					Hexagonal, $\nu = 10^{-10}$				
386	5.83E-01	–	3.53E-01	–	5.83E-01	–	3.53E-01	–	
1436	1.51E-01	1.95	1.79E-01	0.98	1.51E-01	1.95	1.79E-01	0.98	
5560	2.64E-02	2.51	7.03E-02	1.34	2.64E-02	2.51	7.03E-02	1.34	
21872	4.25E-03	2.63	2.56E-02	1.46	4.27E-03	2.63	2.56E-02	1.46	
Voronoi, $\nu = 10^{-2}$					Voronoi, $\nu = 10^{-4}$				
276	3.61E-01	–	3.63E-01	–	5.66E-01	–	3.02E-01	–	
1228	5.11E-02	3.29	1.61E-01	1.37	1.15E-01	2.68	1.49E-01	1.19	
5136	6.19E-03	2.95	5.99E-02	1.38	1.94E-02	2.49	5.67E-02	1.35	
21032	7.60E-04	2.94	2.11E-02	1.46	3.01E-03	2.62	2.02E-02	1.45	
Voronoi, $\nu = 10^{-6}$					Voronoi, $\nu = 10^{-10}$				
276	5.70E-01	–	3.01E-01	–	5.70E-01	–	3.01E-01	–	
1228	1.17E-01	2.66	1.48E-01	1.19	1.17E-01	2.66	1.48E-01	1.19	
5136	2.05E-02	2.44	5.64E-02	1.35	2.05E-02	2.44	5.64E-02	1.35	
21032	3.55E-03	2.46	2.00E-02	1.45	3.56E-03	2.46	2.00E-02	1.45	

Table 1: Convergence rates for the numerical test of Section 6 for $k = 1$ using the Cartesian, hexagonal and the Voronoi mesh families for values of $\nu \in \{10^{-2}, 10^{-4}, 10^{-6}, 10^{-10}\}$.