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Analytical formula for the roots of the general complex quartic polynomial

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Abstract

We present a new method to calculate analytically the roots of the general complex polynomial of degree four. This method is based on the approach of appropriate changes of variable involving arbitrary parameters. The advantage of this method is to calculate the roots of the quartic polynomial as closed formula using the standard convention of the square and cubic roots. In contrast, the reference methods for this problem, as Ferrari, Descartes, Lagrange and Euler, give the roots of the quartic polynomial as expressions required a root of a cubic equation, called resolvent cubic of the fourth degree equation, given in terms of cubic roots with case distinctions which require a convention of roots depending on coefficients.

keyword: quartic polynomial roots, appropriated change of variable, analytical closed formula.

1 Introduction

The problem of solution formulas of polynomial equations is fundamental in algebra [1, 2]. It is useful, by example, for the eigenvalues perturbation [3] and the solution of a system of nonlinear equations [4]. Moreover, it has many physical applications, as the study of the singularities of the surfaces of refractive indices in crystal optics [5] and the diagonal eigenvalues perturbation [6], or the resonance oscillation and transition to chaos in a complex oscillator [7]. Another example is Alhazen's problem in optics which leads to a quartic equation: given a light source and a spherical mirror, find the point on the mirror where the light will be reflected to the eye of an observer [8]. In Timoshenko-Rayleigh theory of beam bending, the eigenvalues of a 4×4 matrix are the roots of a quartic polynomial which is the characteristic polynomial of the matrix [9].

For a quartic equation, the formulas of Ferrari and Lagrange are given by expressions in terms of a root of a cubic equation. So, without a closed formula of this cubic equation, the expressions of Ferrari and Lagrange themselves are not a closed formulas. The classical result to solve the cubic equations was the formulas of Cardan-Tartaglia and Lagrange which are subject to different interpretations depending on the choices for the values of the square and cubic roots of the formulas [10, 11]. The correct interpretation is generally derived from conditions between the roots and the coefficients of the cubic polynomial. It was a fundamental question to be able to define closed formula that yields correct solutions for all cubic equations. Ting Zhao's thesis and related work recently brought an important advance on the formulas of Lagrange-Tartaglia with closed solution formulas for cubic and quartic equations with real coefficients. The results were obtained by introducing a new convention for the argument of the cubic root of a complex number [12]. This open problem since 1545 [1], the date when Cardan had published his method in his work *Ars Magna*, was recently resolved in 2024 by [13]. Defined by the standard convention of the square and cubic roots, [13] introduced a closed formula for the cubic equations. For quartic equations, we present in this paper a closed formula that is valid with complex coefficients. This more general result is uniquely defined by the standard convention of the square and cubic roots.

To illustrate the problem, we introduce the formula of Ferrari and afterward we give two examples. Indeed, Ferrari calculate the four solutions of the equation

$$(\mathcal{F}) : y^4 + py^2 + qy + r = 0$$

if $q \neq 0$ as

$$\begin{aligned} y_1 &= \frac{1}{2} \left(-\sqrt{2\lambda_0 - p} + \sqrt{-2\lambda_0 - p + \frac{2q}{\sqrt{2\lambda_0 - p}}} \right), \\ y_2 &= \frac{1}{2} \left(-\sqrt{2\lambda_0 - p} - \sqrt{-2\lambda_0 - p + \frac{2q}{\sqrt{2\lambda_0 - p}}} \right), \\ y_3 &= \frac{1}{2} \left(\sqrt{2\lambda_0 - p} + \sqrt{-2\lambda_0 - p + \frac{2q}{\sqrt{2\lambda_0 - p}}} \right), \\ y_4 &= \frac{1}{2} \left(\sqrt{2\lambda_0 - p} - \sqrt{-2\lambda_0 - p + \frac{2q}{\sqrt{2\lambda_0 - p}}} \right), \end{aligned}$$

where λ_0 is a root of the cubic equation

$$8\lambda^3 - 4p\lambda^2 - 8r\lambda + 4pr - q^2 = 0.$$

Lagrange (1795-1796), Descartes (1596-1650) and Euler (1707-1783) have proposed another equivalent methods, to solve (\mathcal{F}) whose tricks resemble that of Ferrari (1522-1565), using a root of a cubic equation, called resolvent cubic of the fourth degree equation [11, 14, 15, 16].

So, without the closed expression of λ_0 and the resolvent cubics of the fourth degree equation, we can not use any formulas from those of Ferrari, Descartes and Euler. Indeed, to get explicitly the root of a cubic equation, we need to use the formula of [13] which is for now the only closed formula for the general cubic equations.

Therefore, a natural question is how we can calculate the roots of (\mathcal{F}) using uniquely closed formulas?

The response to this question is the objective of this paper.

We present a closed analytical formula for the roots of the general complex quartic polynomial. Our method is based on the approach of changes of variable involving arbitrary parameters, which rests on two principal ideas to diagonalize the particular 4×4 complex matrix such that its characteristic polynomial will coincide with the quartic polynomial.

In fact, to calculate the roots of the general complex quartic polynomial, we construct this particular matrix and we calculate its spectrum. Consequently, the eigenvalues of this particular matrix become identical to the roots of this cubic polynomial.

Firstly, we introduce one change of variable involving arbitrary parameters. So, the characteristic equation of this matrix can be replaced by another non-polynomial equation in terms of the first new variable.

Secondly, we introduce another change of variable involving another arbitrary parameters. The second variable replaces the first and the second arbitrary parameter can be chosen as required to make the form of the new equation as a second degree polynomial of the unknown second variable.

Therefore, we can solve easily this new polynomial equation and consequently deduce the solution of the original equation, which is the roots of the quartic polynomial.

The rest of the paper is structured as follows. In section 2, we derive the approach of appropriate changes of variable involving arbitrary parameters. Consequently, in section 3, we apply it on the general complex quartic polynomial.

2 Approach of appropriate changes of variable involving arbitrary parameters

Let the general complex quartic polynomial $\mathcal{P}(Z)$:

$$\mathcal{P}(Z) := Z^4 + b_Z Z^3 + c_Z Z^2 + d_Z Z + e_Z = 0; \quad b_Z, c_Z, d_Z, e_Z \in \mathbb{C}.$$

Indeed, $\mathcal{P}(Z)$ is the characteristic polynomial of the 4×4 complex matrix \mathbf{A}

$$\mathbf{A} := \begin{pmatrix} -b_Z & 1 & 0 & c_Z/e_Z \\ 0 & 0 & 1 & 0 \\ -d_Z & 0 & 0 & 1 \\ -e_Z & 0 & 0 & 0 \end{pmatrix}.$$

then, the spectrum of \mathbf{A} is identical to the set of the roots of this polynomial $\mathcal{P}(Z)$.

In this section, we aim to calculate the spectrum of \mathbf{A} with the approach of appropriate changes of variable involving arbitrary parameters. We detail it by the following four steps. The first step consists of performing a change of variable for the eigenvalues of \mathbf{A} so that we get a non-polynomial equation. In the second step, we will deal with this equation in order to simplify it by some algebraic manipulations which leads to a non-polynomial equation. The third step will introduce a second change of variable in order to replace the non-polynomial equation by another polynomial equation. Finally, in the fourth step, we will solve the latter polynomial equation to deduce the roots of the quartic polynomial.

We start the first step by introducing the following definition which will be useful in the sequel:

Definition 2.1. We associate to \mathbf{A} the following expressions in terms of its components and arbitrary parameters $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ and \tilde{c}_4 :

$$\begin{aligned} A &= 8\tilde{c}_4^4 \\ B &= 4\tilde{c}_4^3 (b_Z + 4\tilde{c}_3 + 2\tilde{c}_1\tilde{c}_4) \\ C &= -4\tilde{c}_4^3 (b_Z + 4\tilde{c}_3 + \tilde{c}_1\tilde{c}_4) \\ D &= \tilde{c}_4^2 [(6\tilde{c}_3 + 3\tilde{c}_1\tilde{c}_4) b_Z + 2c_Z + \tilde{c}_1^2\tilde{c}_4^2 + 12\tilde{c}_1\tilde{c}_3\tilde{c}_4 + 12\tilde{c}_3^2 + 8\tilde{c}_2\tilde{c}_4^2] \\ E &= -\tilde{c}_4^2 [(6\tilde{c}_3 + \tilde{c}_1\tilde{c}_4) b_Z + 2c_Z + 4(3\tilde{c}_3^2 + \tilde{c}_1\tilde{c}_3\tilde{c}_4 + \tilde{c}_2\tilde{c}_4^2)] \\ F &= \tilde{c}_4 [3(\tilde{c}_3^2 + \tilde{c}_1\tilde{c}_3\tilde{c}_4 + \tilde{c}_2\tilde{c}_4^2) b_Z + (2\tilde{c}_3 + \tilde{c}_1\tilde{c}_4) c_Z + d_Z + 2(2\tilde{c}_3^3 + 3\tilde{c}_1\tilde{c}_3^2\tilde{c}_4 + 6\tilde{c}_2\tilde{c}_3\tilde{c}_4^2 + \tilde{c}_1\tilde{c}_2\tilde{c}_4^3)] \\ G &= -\tilde{c}_4 [(3\tilde{c}_3^2 + \tilde{c}_2\tilde{c}_4^2) b_Z + 2c_Z\tilde{c}_3 + d_Z + 4\tilde{c}_3(\tilde{c}_3^2 + \tilde{c}_2\tilde{c}_4^2)] \\ H &= \tilde{c}_3(\tilde{c}_3^2 + 3\tilde{c}_2\tilde{c}_4^2) b_Z + (\tilde{c}_3^2 + \tilde{c}_2\tilde{c}_4^2) c_Z + d_Z\tilde{c}_3 + e_Z + \tilde{c}_3^2\tilde{c}_4^4 + 6\tilde{c}_2\tilde{c}_3\tilde{c}_4^2 + \tilde{c}_3^4. \end{aligned}$$

Now, we introduce the first change of variable involving arbitrary parameters so that the roots of $\mathcal{P}(Z)$ can be rewritten in terms of this new variable \tilde{b} such that \tilde{b} satisfies a non-polynomial equation like the following proposition:

Proposition 2.2. A root of the general complex quartic polynomial $\mathcal{P}(Z)$ can be written in terms of an unknown \tilde{b} and arbitrary parameters $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ and \tilde{c}_4 as follows:

$$k := \tilde{c}_3 + \tilde{c}_4 \left(\tilde{b} - \sqrt{\Delta_{\tilde{b}}} \right), \quad (1)$$

where $\Delta_{\tilde{b}}$ is defined by

$$\Delta_{\tilde{b}} := \tilde{b}^2 + \tilde{c}_1\tilde{b} + \tilde{c}_2 \quad (2)$$

and \tilde{b} is governed by the following non-polynomial equation:

$$A\tilde{b}^4 - A\tilde{b}^3\sqrt{\Delta_{\tilde{b}}} + B\tilde{b}^3 + C\tilde{b}^2\sqrt{\Delta_{\tilde{b}}} + D\tilde{b}^2 + E\tilde{b}\sqrt{\Delta_{\tilde{b}}} + F\tilde{b} + G\sqrt{\Delta_{\tilde{b}}} + H = 0, \quad (3)$$

here, $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ and \tilde{c}_4 can be chosen arbitrarily provided that \tilde{c}_1 and \tilde{c}_2 are not null simultaneously and $\tilde{c}_1^2 - 4\tilde{c}_2 \neq 0$. The coefficients of Eq. (3) are given by definition 2.1.

Proof:

Since $\mathcal{P}(Z)$ is the characteristic polynomial of the matrix \mathbf{A} , then a root of $\mathcal{P}(Z)$ is an eigenvalue of matrix \mathbf{A} . Consequently, it is enough to prove that k is an eigenvalue of \mathbf{A} if Eq. (3) is verified.

Firstly, we decompose the matrix \mathbf{A} as the sum of two matrixes \mathbf{B} and \mathbf{C} as follows:

$$\mathbf{A} := \begin{pmatrix} -bz & 1 & 0 & cz/ez \\ 0 & 0 & 1 & 0 \\ -d_Z & 0 & 0 & 1 \\ -ez & 0 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & m & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\mathbf{B}} + \underbrace{\begin{pmatrix} -bz & 1 & 0 & cz/ez \\ 0 & -m & 0 & 0 \\ -d_Z & 0 & 0 & 1 \\ -ez & 0 & 0 & 0 \end{pmatrix}}_{\mathbf{C}}.$$

The matrix \mathbf{C} has

$$X(X+m)(X^2+b_ZX+cz)=0$$

as characteristic polynomial. Consequently, \mathbf{C} has 0, $-m$ and $k_{\mathbf{C}}$ as eigenvalues according to the eigenvectors

$$\mathbf{v}_0 := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_m := \begin{pmatrix} 1 \\ b_Z - m - cz/m \\ -(ez - d_Z m)/m^2 \\ ez/m \end{pmatrix} \quad \text{and} \quad \mathbf{v}_{k_{\mathbf{C}}} := \begin{pmatrix} 1 \\ 0 \\ -(ez + d_Z k_{\mathbf{C}})/k_{\mathbf{C}}^2 \\ -ez/k_{\mathbf{C}} \end{pmatrix}$$

respectively where:

$$k_{\mathbf{C}} := \frac{\sqrt{b_Z^2 - 4cz} - b_Z}{2}.$$

The advantage of this decomposition of \mathbf{A} is to construct, according to a suitable choice of m , an eigenvector of \mathbf{A} in terms of these of \mathbf{C} which will be elaborated in the sequel.

Secondly, we search an eigenvector for \mathbf{A} of the form

$$\mathbf{v}_k := \mathbf{v}_0 + v_m \mathbf{v}_m + v_{k_{\mathbf{C}}} \mathbf{v}_{k_{\mathbf{C}}}; \quad v_m, v_{k_{\mathbf{C}}} \in \mathbb{C}.$$

Indeed, we take

$$v_m := \frac{kk_{\mathbf{C}}m^2}{ez(k+m)(k_{\mathbf{C}}+m)}$$

$$v_{k_{\mathbf{C}}} := \frac{-kk_{\mathbf{C}}^2m}{ezk_{\mathbf{C}}^2 - ezkk_{\mathbf{C}} - e_zkm + e_zk_{\mathbf{C}}m},$$

here m is given as

$$m := -\frac{k^2 + cz + b_Zk_{\mathbf{C}} + \sqrt{(k^2 + cz + b_Zk_{\mathbf{C}})^2 - 4(k - k_{\mathbf{C}})(czk + k^2k_{\mathbf{C}} + b_Zkk_{\mathbf{C}})}}{2(k - k_{\mathbf{C}})},$$

where m is a root of the following equation:

$$e_m := (k - k_{\mathbf{C}})m^2 + (k^2 + cz + b_Zk_{\mathbf{C}})m + czk + k^2k_{\mathbf{C}} + b_Zkk_{\mathbf{C}} = 0.$$

Then, the matrix \mathbf{A} get k as eigenvalue according to the eigenvector \mathbf{v}_k if k verify $\mathcal{P}(k) = 0$; because:

$$\mathbf{A}\mathbf{v}_k - k\mathbf{v}_k = \frac{1}{ez(k+m)(k_{\mathbf{C}}+m)(k-k_{\mathbf{C}})} \begin{pmatrix} -kk_{\mathbf{C}}me_m \\ -k_{\mathbf{C}}m[(k_{\mathbf{C}}+m)\mathcal{P}(k) - k^2e_m] \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \mathbf{A}\mathbf{v}_k = k\mathbf{v}_k$$

so, it is enough to check that the equation $\mathcal{P}(k) = 0$ is equivalent to Eq. (3).

Thirdly, to end the proof, we justify the above conditions on \tilde{c}_1 and \tilde{c}_2 by simple reason that: otherwise, k no longer depends on \tilde{b} and consequently Eq. (3) will govern another unknown among \tilde{c}_3 or \tilde{c}_4 . \square

In the second step, we aim to simplify Eq. (3) of the unknown \tilde{b} , where $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ and \tilde{c}_4 can be chosen as required. So, we derive from Eq. (3) two equations of \tilde{b} given by the following two propositions:

Proposition 2.3. For fixed $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ and \tilde{c}_4 , \tilde{b} satisfies the following equation:

$$d_1\tilde{b}^3 - d_1\tilde{b}^2\Delta_{\tilde{b}} + d_2\tilde{b}^2 + d_3\tilde{b}\Delta_{\tilde{b}} + d_4\tilde{b} + d_5\Delta_{\tilde{b}} + d_6 = 0, \quad (4)$$

where the coefficients of Eq. (4) are given by:

$$d_1 = 2D + 2E + \tilde{c}_1B + 2\tilde{c}_1C - 2\tilde{c}_2A$$

$$d_2 = 2F + 2G + \tilde{c}_1D + 2\tilde{c}_1E + 2\tilde{c}_2C$$

$$d_3 = 2F + 2G + \tilde{c}_1E$$

$$d_4 = 2H + \tilde{c}_1F + 2\tilde{c}_1G + 2\tilde{c}_2E$$

$$d_5 = 2H + \tilde{c}_1G$$

$$d_6 = \tilde{c}_1H + 2\tilde{c}_2G$$

Proof:

Eq. (4) is deduced by multiplying Eq. (3) by $\tilde{c}_1 + 2(\tilde{b} + \sqrt{\Delta_{\tilde{b}}})$ where $\Delta_{\tilde{b}}$ is given by Eq. (2). \square

Proposition 2.4. For fixed $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ and \tilde{c}_4, \tilde{b} satisfies the following equation:

$$f_1 \tilde{b}^2 + f_2 \tilde{b} \Delta_{\tilde{b}} + f_3 \tilde{b} + f_4 \Delta_{\tilde{b}} + f_5 = 0, \quad (5)$$

where the coefficients of Eq. (5) are given by:

$$\begin{aligned} f_1 &= -8\tilde{c}_1^2 \tilde{c}_2 \tilde{c}_4^4 + D\tilde{c}_1^2 + 4d_3 \tilde{c}_1 + 32\tilde{c}_2^2 \tilde{c}_4^4 - 4D\tilde{c}_2 + 4d_5 \\ f_2 &= E\tilde{c}_1^2 + 4d_4 - 4\tilde{c}_2 E \\ f_3 &= F\tilde{c}_1^2 + 4d_5 \tilde{c}_1 + 4\tilde{c}_2 d_3 - 4\tilde{c}_2 F \\ f_4 &= G\tilde{c}_1^2 + 4d_6 - 4\tilde{c}_2 G \\ f_5 &= H\tilde{c}_1^2 + 4\tilde{c}_2 d_5 - 4\tilde{c}_2 H \end{aligned}$$

Proof:

Similarly to the proof of proposition 2.3, on one hand we multiply Eq. (3) by d_1 and on the other hand we multiply Eq. (4) by $A\sqrt{\Delta_{\tilde{b}}}$. Consequently, we add the obtained equations to deduce Eq. (5). \square

Due to a classic change of variable, it is enough to calculate the roots of the quartic polynomial $\mathcal{P}_0(z)$:

$$\mathcal{P}_0(z) := z^4 + cz^2 + dz + e = 0; \quad c, d, e \in \mathbb{C}.$$

indeed

$$\begin{aligned} \mathcal{P}(Z) &:= Z^4 + b_Z Z^3 + c_Z Z^2 + d_Z Z + e_Z \\ &= \left(Z + \frac{b_Z}{4}\right)^4 + \left(c_Z - \frac{3}{8}b_Z^2\right) \left(Z + \frac{b_Z}{4}\right)^2 + \left(\frac{b_Z^3}{8} - \frac{c_Z b_Z}{2} + d_Z\right) \left(Z + \frac{b_Z}{4}\right) - \frac{3}{256}b_Z^4 + \frac{c_Z b_Z^2}{16} - \frac{d_Z b_Z}{4} + e_Z \\ &= z^4 + cz^2 + dz + e; \quad \text{for: } \begin{cases} z := Z + \frac{b_Z}{4} \\ c := c_Z - \frac{3}{8}b_Z^2 \\ d := \frac{b_Z^3}{8} - \frac{c_Z b_Z}{2} + d_Z \\ e := -\frac{3}{256}b_Z^4 + \frac{c_Z b_Z^2}{16} - \frac{d_Z b_Z}{4} + e_Z \end{cases} \end{aligned} \quad (6)$$

so, the general case of $\mathcal{P}(Z)$ comes down to the particular case $\mathcal{P}_0(z)$ according to the change of variable given by Eq. (6). In the sequel, we aim to calculate the roots of $\mathcal{P}_0(z)$.

In the third step, we introduce the second change of variable

$$l := -o_2 \tilde{b} - o_1 \sqrt{\Delta_{\tilde{b}}} \quad (7)$$

in order to deduce from the intractable equations of \tilde{b} another polynomial equation of the new variable l , where o_1 and o_2 are arbitrary parameters that can be chosen as required:

Proposition 2.5. Let $o_1, o_2 \in \mathbb{C}$. Then l satisfies the following equation:

$$A_l \tilde{c}_4^4 l^4 + B_l \tilde{c}_4^3 l^3 + C_l \tilde{c}_4^2 l^2 + D_l \tilde{c}_4 l + E_l = 0 \quad (8)$$

where the coefficients of Eq. (8) are given by:

$$\begin{aligned} A_l &:= 4\tilde{c}_{134}^2 c - 8\tilde{c}_{134} d + 16e + \tilde{c}_{134}^4 \\ B_l &:= -2g_1 o_1 - 2g_2 o_2 \\ C_l &:= g_3 o_1^2 - g_4 o_2 o_1 + g_5 o_2^2 \\ D_l &:= g_6 o_1^3 - g_7 o_2 o_1^2 + g_8 o_2^2 o_1 + g_9 o_2^3 \\ E_l &:= g_{10} o_1^4 - g_{11} o_2 o_1^3 + g_{12} o_2^2 o_1^2 + g_{13} o_2^3 o_1 + g_{14} o_2^4 \end{aligned}$$

here the coefficients of B_l are

$$\begin{aligned} g_1 &:= \tilde{c}_{134} (\tilde{c}_1^2 \tilde{c}_4^2 - 8\tilde{c}_1 \tilde{c}_3 \tilde{c}_4 + 8\tilde{c}_3^2 + 4\tilde{c}_2 \tilde{c}_4^2) c - (3\tilde{c}_1^2 \tilde{c}_4^2 - 16\tilde{c}_1 \tilde{c}_3 \tilde{c}_4 + 16\tilde{c}_3^2 + 4\tilde{c}_2 \tilde{c}_4^2) d + 8\tilde{c}_{134} e + 2\tilde{c}_{134}^3 \tilde{c}_{1234} \\ g_2 &:= \tilde{c}_{134} (\tilde{c}_1^2 \tilde{c}_4^2 - 8\tilde{c}_3^2 + 4\tilde{c}_2 \tilde{c}_4^2) c - (3\tilde{c}_1^2 \tilde{c}_4^2 - 16\tilde{c}_3^2 + 4\tilde{c}_2 \tilde{c}_4^2) d + 8(2\tilde{c}_3 + \tilde{c}_1 \tilde{c}_4) e - 2\tilde{c}_{134}^3 \tilde{c}_{234} \end{aligned}$$

the coefficients of C_l are

$$\begin{aligned} g_3 &:= \tilde{c}_{134}^2 c^2 + (4e - 2\tilde{c}_{134} d + \tilde{c}_1^4 \tilde{c}_4^4 - 10\tilde{c}_1^3 \tilde{c}_3 \tilde{c}_4^3 + 2\tilde{c}_1^2 \tilde{c}_2 \tilde{c}_4^4 + 38\tilde{c}_1^2 \tilde{c}_3^2 \tilde{c}_4^2 - 16\tilde{c}_1 \tilde{c}_2 \tilde{c}_3 \tilde{c}_4^3 - 56\tilde{c}_1 \tilde{c}_3^3 \tilde{c}_4 + 4\tilde{c}_2^2 \tilde{c}_4^4 + 16\tilde{c}_2 \tilde{c}_3^2 \tilde{c}_4^2 + 28\tilde{c}_3^4) c \\ &\quad - 3\tilde{c}_{134}^3 d + 2(5\tilde{c}_1^2 \tilde{c}_4^2 - 12\tilde{c}_1 \tilde{c}_3 \tilde{c}_4 + 12\tilde{c}_3^2 - 8\tilde{c}_2 \tilde{c}_4^2) e + 6\tilde{c}_{134}^2 \tilde{c}_{1234}^2 \\ g_4 &:= 2\tilde{c}_{134}^2 c^2 + [8e - 4\tilde{c}_{134} d + 2(\tilde{c}_1^3 \tilde{c}_3 \tilde{c}_4^3 - 4\tilde{c}_1^2 \tilde{c}_2 \tilde{c}_4^4 + 6\tilde{c}_1^2 \tilde{c}_3^2 \tilde{c}_4^2 + 12\tilde{c}_1 \tilde{c}_2 \tilde{c}_3 \tilde{c}_4^3 - 32\tilde{c}_1 \tilde{c}_3^3 \tilde{c}_4 - 4\tilde{c}_2^2 \tilde{c}_4^4 + 28\tilde{c}_3^4)] c \\ &\quad + 3(\tilde{c}_1^3 \tilde{c}_4^3 - 4\tilde{c}_1^2 \tilde{c}_3 \tilde{c}_4^2 - 8\tilde{c}_1 \tilde{c}_3^2 \tilde{c}_4 + 4\tilde{c}_2 \tilde{c}_1 \tilde{c}_4^3 + 16\tilde{c}_3^3) d - 12\tilde{c}_{134} (2\tilde{c}_3 + \tilde{c}_1 \tilde{c}_4) e + 12\tilde{c}_{134}^2 \tilde{c}_{234} \tilde{c}_{1234} \\ g_5 &:= \tilde{c}_{134}^2 c^2 + [4e - 2\tilde{c}_{134} d + 2(3\tilde{c}_1^2 \tilde{c}_2 \tilde{c}_4^4 - \tilde{c}_1^2 \tilde{c}_3^2 \tilde{c}_4^2 - 4\tilde{c}_1 \tilde{c}_2 \tilde{c}_3 \tilde{c}_4^3 - 4\tilde{c}_1 \tilde{c}_3^3 \tilde{c}_4 + 2\tilde{c}_2^2 \tilde{c}_4^4 - 8\tilde{c}_2 \tilde{c}_3^2 \tilde{c}_4^2 + 14\tilde{c}_3^4)] c \\ &\quad - 6(\tilde{c}_1^2 \tilde{c}_3 \tilde{c}_4^2 - 2\tilde{c}_1 \tilde{c}_3 \tilde{c}_4 + 2\tilde{c}_2 \tilde{c}_1 \tilde{c}_4^3 - 4\tilde{c}_3^3) d + 2(\tilde{c}_1^2 \tilde{c}_4^2 + 12\tilde{c}_1 \tilde{c}_3 \tilde{c}_4 + 12\tilde{c}_3^2 + 8\tilde{c}_2 \tilde{c}_4^2) e + 6\tilde{c}_{134}^2 \tilde{c}_{234}^2 \end{aligned}$$

the coefficients of D_l are

$$\begin{aligned}
g_6 &:= -2\tilde{c}_{134}\tilde{c}_{1234}c^2 + (\tilde{c}_1^2\tilde{c}_4^2 - 6\tilde{c}_1\tilde{c}_3\tilde{c}_4 + 6\tilde{c}_3^2 + 2\tilde{c}_2\tilde{c}_4^2)dc - 2\tilde{c}_{134}ec - 2\tilde{c}_{134}\tilde{c}_{1234}(\tilde{c}_1^2\tilde{c}_4^2 - 3\tilde{c}_1\tilde{c}_3\tilde{c}_4 + 3\tilde{c}_3^2 - \tilde{c}_2\tilde{c}_4^2)c \\
&\quad - \tilde{c}_{134}d^2 + (2e + \tilde{c}_1^4\tilde{c}_4^4 - 8\tilde{c}_1^3\tilde{c}_3\tilde{c}_4^3 + 18\tilde{c}_1^2\tilde{c}_3^2\tilde{c}_4^2 + 12\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^3 - 20\tilde{c}_1\tilde{c}_3^3\tilde{c}_4 - 6\tilde{c}_2^2\tilde{c}_4^4 - 12\tilde{c}_2\tilde{c}_3^2\tilde{c}_4^2 + 10\tilde{c}_3^4) d \\
&\quad - 4\tilde{c}_{134}(\tilde{c}_1^2\tilde{c}_4^2 - \tilde{c}_1\tilde{c}_3\tilde{c}_4 + \tilde{c}_3^2 - 3\tilde{c}_2\tilde{c}_4^2)e - 4\tilde{c}_{134}\tilde{c}_{1234}^3 \\
g_7 &:= (4\tilde{c}_1^2\tilde{c}_3\tilde{c}_4^2 - 14\tilde{c}_1\tilde{c}_3^2\tilde{c}_4 - 2\tilde{c}_2\tilde{c}_1\tilde{c}_4^3 + 12\tilde{c}_3^3 + 4\tilde{c}_2\tilde{c}_3\tilde{c}_4^2)c^2 + 2(\tilde{c}_1^2\tilde{c}_4^2 - 7\tilde{c}_1\tilde{c}_3\tilde{c}_4 + 9\tilde{c}_3^2 + \tilde{c}_2\tilde{c}_4^2)dc + 2(6\tilde{c}_3 - \tilde{c}_1\tilde{c}_4)ec \\
&\quad + 2(\tilde{c}_1^3\tilde{c}_2\tilde{c}_4^5 - 2\tilde{c}_1^2\tilde{c}_3^2\tilde{c}_4^3 - 6\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^4 + 16\tilde{c}_1^2\tilde{c}_3^3\tilde{c}_4^2 + \tilde{c}_1\tilde{c}_2^2\tilde{c}_4^5 + 2\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^3 - 31\tilde{c}_1\tilde{c}_3^4\tilde{c}_4 + 2\tilde{c}_2^2\tilde{c}_3\tilde{c}_4^4 + 4\tilde{c}_2\tilde{c}_3^3\tilde{c}_4^2 + 18\tilde{c}_3^5)c \\
&\quad - 3\tilde{c}_{134}d^2 - 6(\tilde{c}_1^2\tilde{c}_2\tilde{c}_4^4 - 2\tilde{c}_1^2\tilde{c}_3^2\tilde{c}_4^2 - 2\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^3 + 6\tilde{c}_1\tilde{c}_3^3\tilde{c}_4 - \tilde{c}_2^2\tilde{c}_4^4 + 2\tilde{c}_2\tilde{c}_3^2\tilde{c}_4^2 - 5\tilde{c}_3^4 - e)d \\
&\quad + 4(\tilde{c}_1^3\tilde{c}_4^3 - 3\tilde{c}_1\tilde{c}_3^2\tilde{c}_4 - \tilde{c}_2\tilde{c}_1\tilde{c}_4^3 + 6\tilde{c}_3^3 - 6\tilde{c}_2\tilde{c}_3\tilde{c}_4^2)e - 12\tilde{c}_{134}\tilde{c}_{234}\tilde{c}_{1234}^2 \\
g_8 &:= -2\tilde{c}_{134}(3\tilde{c}_3^2 - \tilde{c}_1\tilde{c}_3\tilde{c}_4 - \tilde{c}_2\tilde{c}_4^2)c^2 + (\tilde{c}_1^2\tilde{c}_4^2 - 10\tilde{c}_1\tilde{c}_3\tilde{c}_4 + 18\tilde{c}_3^2 - 2\tilde{c}_2\tilde{c}_4^2)dc + 2(6\tilde{c}_3 + \tilde{c}_1\tilde{c}_4)ec - 12\tilde{c}_{134}\tilde{c}_{234}^2\tilde{c}_{1234} \\
&\quad + 2(2\tilde{c}_1^2\tilde{c}_2\tilde{c}_3\tilde{c}_4^4 + 2\tilde{c}_1^2\tilde{c}_3^2\tilde{c}_4^2 - 5\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^5 + 6\tilde{c}_1\tilde{c}_2\tilde{c}_3^2\tilde{c}_4^3 - 17\tilde{c}_1\tilde{c}_3^4\tilde{c}_4 + 2\tilde{c}_2^2\tilde{c}_3\tilde{c}_4^4 - 4\tilde{c}_2\tilde{c}_3^3\tilde{c}_4^2 + 18\tilde{c}_3^5)c - 3\tilde{c}_{134}d^2 \\
&\quad + 6(\tilde{c}_1^2\tilde{c}_2\tilde{c}_4^4 - \tilde{c}_1^2\tilde{c}_3^2\tilde{c}_4^2 - 2\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^5 - 2\tilde{c}_1\tilde{c}_3^3\tilde{c}_4 + \tilde{c}_2^2\tilde{c}_4^4 + 2\tilde{c}_2\tilde{c}_3^2\tilde{c}_4^2 + 5\tilde{c}_3^4 + e)d - 12\tilde{c}_{134}(\tilde{c}_3^2 + \tilde{c}_1\tilde{c}_3\tilde{c}_4 + \tilde{c}_2\tilde{c}_4^2)e \\
g_9 &:= 2\tilde{c}_{134}\tilde{c}_{234}c^2 - [(6\tilde{c}_3^2 - 2\tilde{c}_1\tilde{c}_3\tilde{c}_4 - 2\tilde{c}_2\tilde{c}_4^2)d + 2(2\tilde{c}_3 + \tilde{c}_1\tilde{c}_4)e + 2\tilde{c}_{234}(6\tilde{c}_3^3 - \tilde{c}_1\tilde{c}_3^2\tilde{c}_4 + 2\tilde{c}_2\tilde{c}_3\tilde{c}_4^2 - 3\tilde{c}_1\tilde{c}_2\tilde{c}_4^3)]c + \tilde{c}_{134}d^2 \\
&\quad + 2(3\tilde{c}_2^2\tilde{c}_4^4 - 6\tilde{c}_2\tilde{c}_3^2\tilde{c}_4^2 + 6\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^3 - 5\tilde{c}_3^4 - 2\tilde{c}_1\tilde{c}_3^3\tilde{c}_4 - e)d - 4(2\tilde{c}_3^3 + 3\tilde{c}_1\tilde{c}_3^2\tilde{c}_4 + 6\tilde{c}_2\tilde{c}_3\tilde{c}_4^2 + \tilde{c}_1\tilde{c}_2\tilde{c}_4^3)e + 4\tilde{c}_{134}\tilde{c}_{234}^3
\end{aligned}$$

the coefficients of E_l are

$$\begin{aligned}
g_{10} &:= \tilde{c}_{1234}^2c^2 - \tilde{c}_{134}\tilde{c}_{1234}dc + (\tilde{c}_1^2\tilde{c}_4^2 - 2\tilde{c}_1\tilde{c}_3\tilde{c}_4 + 2\tilde{c}_3^2 - 2\tilde{c}_2\tilde{c}_4^2)ec + (\tilde{c}_1^2\tilde{c}_4^2 - 2\tilde{c}_1\tilde{c}_3\tilde{c}_4 + 2\tilde{c}_3^2 - 2\tilde{c}_2\tilde{c}_4^2)\tilde{c}_{1234}c + \tilde{c}_{1234}d^2 \\
&\quad + [(2\tilde{c}_3 - \tilde{c}_1\tilde{c}_4)e - \tilde{c}_{134}\tilde{c}_{1234}(\tilde{c}_1^2\tilde{c}_4^2 - \tilde{c}_1\tilde{c}_3\tilde{c}_4 + \tilde{c}_3^2 - 3\tilde{c}_2\tilde{c}_4^2)]d + e^2 \\
&\quad + (\tilde{c}_1^4\tilde{c}_4^4 - 4\tilde{c}_1^3\tilde{c}_3\tilde{c}_4^3 - 4\tilde{c}_1^2\tilde{c}_2\tilde{c}_4^4 + 6\tilde{c}_1^2\tilde{c}_3^2\tilde{c}_4^2 + 12\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^3 - 4\tilde{c}_1\tilde{c}_3^3\tilde{c}_4 + 2\tilde{c}_2^2\tilde{c}_4^4 - 12\tilde{c}_2\tilde{c}_3^2\tilde{c}_4^2 + 2\tilde{c}_3^4)e + \tilde{c}_{1234}^4 \\
g_{11} &:= -2\tilde{c}_3\tilde{c}_{134}\tilde{c}_{1234}c^2 + (2\tilde{c}_1^2\tilde{c}_3\tilde{c}_4^2 - 9\tilde{c}_1\tilde{c}_3^2\tilde{c}_4 - \tilde{c}_2\tilde{c}_1\tilde{c}_4^3 + 8\tilde{c}_3^3 + 4\tilde{c}_2\tilde{c}_3\tilde{c}_4^2)dc + 2(\tilde{c}_1^2\tilde{c}_4^2 - 3\tilde{c}_1\tilde{c}_3\tilde{c}_4 + 4\tilde{c}_3^2 - 2\tilde{c}_2\tilde{c}_4^2)ec \\
&\quad - 2\tilde{c}_{1234}(\tilde{c}_1^2\tilde{c}_2\tilde{c}_4^4 - 2\tilde{c}_1^2\tilde{c}_3^2\tilde{c}_4^2 - \tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^5 + 5\tilde{c}_1\tilde{c}_3^3\tilde{c}_4 - 2\tilde{c}_2^2\tilde{c}_4^4 + 2\tilde{c}_2\tilde{c}_3^2\tilde{c}_4^2 - 4\tilde{c}_3^4)c + (4\tilde{c}_3^2 - 3\tilde{c}_1\tilde{c}_3\tilde{c}_4 + 2\tilde{c}_2\tilde{c}_4^2)d^2 \\
&\quad + [(8\tilde{c}_3 - 3\tilde{c}_1\tilde{c}_4)e + \tilde{c}_1^3\tilde{c}_2\tilde{c}_4^5 - 4\tilde{c}_1^3\tilde{c}_3^2\tilde{c}_4^3 + 12\tilde{c}_1^2\tilde{c}_3^3\tilde{c}_4^2 - 3\tilde{c}_1\tilde{c}_2^2\tilde{c}_4^5 + 6\tilde{c}_1\tilde{c}_2\tilde{c}_3^2\tilde{c}_4^3 - 15\tilde{c}_1\tilde{c}_3^4\tilde{c}_4 - 8\tilde{c}_2\tilde{c}_3^3\tilde{c}_4^2 + 8\tilde{c}_3^5]d \\
&\quad + 4e^2 - 4\tilde{c}_3\tilde{c}_{134}(\tilde{c}_1^2\tilde{c}_4^2 - \tilde{c}_1\tilde{c}_3\tilde{c}_4 + \tilde{c}_3^2 - 3\tilde{c}_2\tilde{c}_4^2)e + 4\tilde{c}_{234}\tilde{c}_{1234}^3 \\
g_{12} &:= (\tilde{c}_1^2\tilde{c}_3\tilde{c}_4^2 + 2\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^3 - 6\tilde{c}_1\tilde{c}_3^3\tilde{c}_4 - 2\tilde{c}_2^2\tilde{c}_4^4 + 6\tilde{c}_3^4)c^2 + (\tilde{c}_1^2\tilde{c}_3\tilde{c}_4^2 - 9\tilde{c}_1\tilde{c}_3^2\tilde{c}_4 + \tilde{c}_2\tilde{c}_1\tilde{c}_4^3 + 12\tilde{c}_3^3)dc \\
&\quad + (\tilde{c}_1^2\tilde{c}_4^2 - 6\tilde{c}_1\tilde{c}_3\tilde{c}_4 + 12\tilde{c}_3^2)ec + (\tilde{c}_1^2\tilde{c}_2\tilde{c}_4^6 - 4\tilde{c}_1^2\tilde{c}_2\tilde{c}_3^2\tilde{c}_4^4 + 7\tilde{c}_1^2\tilde{c}_4^3\tilde{c}_4^2 - 2\tilde{c}_1\tilde{c}_2^2\tilde{c}_3\tilde{c}_4^5 + 4\tilde{c}_1\tilde{c}_2\tilde{c}_3^3\tilde{c}_4^3 - 18\tilde{c}_1\tilde{c}_3^5\tilde{c}_4 + 4\tilde{c}_2^2\tilde{c}_3\tilde{c}_4^4 + 12\tilde{c}_3^6)c \\
&\quad - 3\tilde{c}_3\tilde{c}_{134}d^2 + [3(4\tilde{c}_3 - \tilde{c}_1\tilde{c}_4)e + 3(2\tilde{c}_1^2\tilde{c}_3^2\tilde{c}_4^2 - \tilde{c}_1\tilde{c}_2\tilde{c}_4^5 - 2\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^3 - 5\tilde{c}_1\tilde{c}_3^4\tilde{c}_4 + 4\tilde{c}_2^2\tilde{c}_3\tilde{c}_4^4 + 4\tilde{c}_3^5)]d + 6e^2 \\
&\quad + 2(2\tilde{c}_1^2\tilde{c}_2\tilde{c}_4^4 + 3\tilde{c}_1^2\tilde{c}_3^2\tilde{c}_4^2 - 6\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^5 - 6\tilde{c}_1\tilde{c}_3^3\tilde{c}_4 - 2\tilde{c}_2^2\tilde{c}_4^4 + 6\tilde{c}_3^4)e + 6\tilde{c}_{234}^2\tilde{c}_{1234}^2 \\
g_{13} &:= 2\tilde{c}_3\tilde{c}_{134}\tilde{c}_{234}c^2 - (8\tilde{c}_3^3 - 3\tilde{c}_1\tilde{c}_3^2\tilde{c}_4 - 4\tilde{c}_2\tilde{c}_3\tilde{c}_4^2 + \tilde{c}_1\tilde{c}_2\tilde{c}_4^3)dc - 2(4\tilde{c}_3^2 - \tilde{c}_1\tilde{c}_3\tilde{c}_4 + 2\tilde{c}_2\tilde{c}_4^2)ec \\
&\quad - 2\tilde{c}_{234}(2\tilde{c}_2^2\tilde{c}_4^4 + 2\tilde{c}_2\tilde{c}_3^2\tilde{c}_4^2 - \tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^3 + 4\tilde{c}_3^4 - 3\tilde{c}_1\tilde{c}_3^3\tilde{c}_4)c - (4\tilde{c}_3^2 - \tilde{c}_1\tilde{c}_3\tilde{c}_4 - 2\tilde{c}_2\tilde{c}_4^2)d^2 \\
&\quad + [(\tilde{c}_1\tilde{c}_4 - 8\tilde{c}_3)e - 3\tilde{c}_1\tilde{c}_2^2\tilde{c}_4^5 - 8\tilde{c}_2\tilde{c}_3\tilde{c}_4^2 + 6\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4^3 - 8\tilde{c}_3^5 + 5\tilde{c}_1\tilde{c}_3^4\tilde{c}_4]d + 4\tilde{c}_3\tilde{c}_{134}(\tilde{c}_3^2 + 3\tilde{c}_2\tilde{c}_4^2)e - 4e^2 - 4\tilde{c}_{234}^3\tilde{c}_{1234} \\
g_{14} &:= \tilde{c}_{234}^2c^2 + [2\tilde{c}_3\tilde{c}_{234}d + 2(\tilde{c}_3^2 + \tilde{c}_2\tilde{c}_4^2)e + 2(\tilde{c}_3^2 + \tilde{c}_2\tilde{c}_4^2)\tilde{c}_{234}]c + \tilde{c}_{234}d^2 - 2\tilde{c}_3(3\tilde{c}_2^2\tilde{c}_4^4 - 2\tilde{c}_2\tilde{c}_3\tilde{c}_4^2 - \tilde{c}_3^4 - e)d \\
&\quad + e^2 + 2(\tilde{c}_2^2\tilde{c}_4^4 + 6\tilde{c}_2\tilde{c}_3\tilde{c}_4^2 + \tilde{c}_3^4)e + \tilde{c}_{234}^4
\end{aligned}$$

where:

$$\begin{aligned}
\tilde{c}_{134} &:= \tilde{c}_1\tilde{c}_4 - 2\tilde{c}_3 \\
\tilde{c}_{234} &:= \tilde{c}_3^2 - \tilde{c}_2\tilde{c}_4^2 \\
\tilde{c}_{1234} &:= \tilde{c}_3^2 - \tilde{c}_1\tilde{c}_3\tilde{c}_4 + \tilde{c}_2\tilde{c}_4^2.
\end{aligned}$$

Proof:

The first part of the proof consists in getting expressions for \tilde{b} and $\sqrt{\Delta_{\tilde{b}}}$ just in terms of l , o_1 and o_2 . Indeed, Eq. (7) is equivalent to the following equation (9) which implies two other equations:

$$o_2\tilde{b} + o_1\sqrt{\Delta_{\tilde{b}}} + l = 0 \Rightarrow \quad (9)$$

$$o_1\tilde{b}^2 + o_2\tilde{b}\sqrt{\Delta_{\tilde{b}}} + o_1\tilde{c}_1\tilde{b} + l\sqrt{\Delta_{\tilde{b}}} + \tilde{c}_2o_1 = 0 \Rightarrow \quad (10)$$

$$(o_2^2 - o_1^2)\tilde{b}\sqrt{\Delta_{\tilde{b}}} + o_1(\tilde{c}_1o_2 - l)\tilde{b} + lo_2\sqrt{\Delta_{\tilde{b}}} + \tilde{c}_2o_1o_2 = 0 \quad (11)$$

where Eq. (10) is equivalent to $\sqrt{\Delta_{\tilde{b}}}\times\text{Eq. (9)}$ and Eq. (11) is equivalent to $o_2\times\text{Eq. (10)} - o_1\tilde{b}\times\text{Eq. (9)}$.

Then, $(o_2^2 - o_1^2)(o_1\times\text{Eq. (5)} - f_1\times\text{Eq. (10)}) - (f_2o_1 - f_1o_2)\times\text{Eq. (11)}$ implies the following equation (12) in terms of \tilde{b} and $\sqrt{\Delta_{\tilde{b}}}$:

$$e_1\tilde{b} + e_2\sqrt{\Delta_{\tilde{b}}} + e_3 = 0, \quad (12)$$

where the coefficients of Eq. (12) are given by:

$$e_1 := f_3o_2^2 - f_3o_1^2 - f_1lo_2 + f_2lo_1 + \tilde{c}_1f_1o_1^2 - \tilde{c}_1f_2o_1o_2$$

$$e_2 := -f_4o_1^2 + f_1lo_1 + f_4o_2^2 - f_2lo_2$$

$$e_3 := f_5o_2^2 - f_5o_1^2 + \tilde{c}_2f_1o_1^2 - \tilde{c}_2f_2o_1o_2$$

So, Eq. (9) and Eq. (12) end the first part of the proof:

$$\begin{aligned}\tilde{b} &= -\frac{r_1 l^2 + r_2 l + r_3}{e_4 l + e_5} \\ \sqrt{\Delta_{\tilde{b}}} &= \frac{s_1 l^2 + s_2 l + s_3}{e_4 l + e_5}\end{aligned}\tag{13}$$

where

$$\begin{aligned}e_4 &:= f_2 o_1^2 - 2f_1 o_1 o_2 + f_2 o_2^2 \\ e_5 &:= f_3 o_1 o_2^2 - f_4 o_2^3 - f_3 o_1^3 + f_4 o_1^2 o_2 + \tilde{c}_1 f_1 o_1^3 - \tilde{c}_1 f_2 o_1^2 o_2 \\ r_1 &:= f_2 o_2 - f_1 o_1 \\ r_2 &:= f_4 (o_1^2 - o_2^2) \\ r_3 &:= o_1 (f_5 o_2^2 - f_5 o_1^2 + \tilde{c}_2 f_1 o_1^2 - \tilde{c}_2 f_2 o_1 o_2) \\ s_1 &:= f_1 o_2 - f_2 o_1 \\ s_2 &:= f_3 o_1^2 - f_3 o_2^2 - \tilde{c}_1 f_1 o_1^2 + \tilde{c}_1 f_2 o_1 o_2 \\ s_3 &:= o_2 (f_5 o_2^2 - f_5 o_1^2 + \tilde{c}_2 f_1 o_1^2 - \tilde{c}_2 f_2 o_1 o_2).\end{aligned}$$

The second part of the proof consists of using the previous expressions of \tilde{b} and $\sqrt{\Delta_{\tilde{b}}}$ in order to deduce Eq. (8). Indeed, by inserting Eq. (13) in Eq. (11), we replace \tilde{b} and $\sqrt{\Delta_{\tilde{b}}}$ by their expressions to get an equation in terms of l :

$$\begin{aligned}(o_2^2 - o_1^2) & \underbrace{\tilde{b}}_{-\frac{r_1 l^2 + r_2 l + r_3}{e_4 l + e_5}} \underbrace{\sqrt{\Delta_{\tilde{b}}}}_{\frac{s_1 l^2 + s_2 l + s_3}{e_4 l + e_5}} + o_1 (\tilde{c}_1 o_2 - l) \underbrace{\tilde{b}}_{-\frac{r_1 l^2 + r_2 l + r_3}{e_4 l + e_5}} + l o_2 \sqrt{\Delta_{\tilde{b}}} + \tilde{c}_2 o_1 o_2 = 0\end{aligned}$$

so, Eq. (8) is equivalent to this previous equation multiplying by $(e_4 l + e_5)^2 \left[o_1 o_2 (o_1^2 - o_2^2) (\tilde{c}_1^2 - 4\tilde{c}_2)^2 \right]^{-1}$. \square

In the fourth step, we can determine the set of solutions of l . Indeed, since o_1 and o_2 are arbitrary parameters, then we can choose them as required to solve equation (8) of the unknown l :

Lemma 2.6. *If o_1 satisfies the following polynomial equation of degree three:*

$$h_4 o_1^3 + h_5 o_2 o_1^2 + h_6 o_2^2 o_1 + h_7 o_2^3 = 0,\tag{14}$$

then the set of solutions of l is given by

$$l = \frac{1}{4A_l \tilde{c}_4} \left(m_2 \sqrt{\frac{-F_l + m_1 \sqrt{\Delta}}{2}} - B_l \right); \quad m_1, m_2 \in \{-1, 1\}\tag{15}$$

where:

$$\Delta := F_l^2 - 4H_l\tag{16}$$

Proof:

Equation (8) can be rewritten, by multiplying it by $256A_l^3$, as follows

$$(4A_l \tilde{c}_4 l + B_l)^4 + F_l (4A_l \tilde{c}_4 l + B_l)^2 + G_l (4A_l \tilde{c}_4 l + B_l) + H_l = 0\tag{17}$$

where

$$\begin{aligned}F_l &:= 2(8A_l C_l - 3B_l^2) = 8(h_1 o_1^2 - 2h_2 o_2 o_1 + h_3 o_2^2) \\ G_l &:= 2(32A_l^2 D_l - 2B_l^3) - 2B_l F_l = 64(h_4 o_1^3 + h_5 o_2 o_1^2 + h_6 o_2^2 o_1 + h_7 o_2^3) \\ H_l &:= 256A_l^3 E_l - B_l^4 - B_l^2 F_l - B_l G_l = 16(h_8 o_1^4 + 4h_9 o_2 o_1^3 + 2h_{10} o_2^2 o_1^2 + 4h_{11} o_2^3 o_1 + h_{12} o_2^4)\end{aligned}$$

here the coefficients of F_l are

$$\begin{aligned}h_1 &= 2A_l g_3 - 3g_1^2 \\ h_2 &= A_l g_4 + 3g_2 g_1 \\ h_3 &= 2A_l g_5 - 3g_2^2\end{aligned}$$

the coefficients of G_l are

$$\begin{aligned}h_4 &= A_l^2 g_6 + A_l g_3 g_1 - g_1^3 \\ h_5 &= -A_l^2 g_7 - A_l g_4 g_1 + A_l g_3 g_2 - 3g_2 g_1^2 \\ h_6 &= A_l^2 g_8 - A_l g_4 g_2 + A_l g_5 g_1 - 3g_1 g_2^2 \\ h_7 &= A_l^2 g_9 + A_l g_5 g_2 - g_2^3\end{aligned}$$

the coefficients of H_l are

$$\begin{aligned}
h_8 &= 16A_l^3 g_{10} + 8A_l^2 g_6 g_1 + 4A_l g_3 g_1^2 - 3g_1^4 \\
h_9 &= -4A_l^3 g_{11} - 2A_l^2 g_7 g_1 + 2A_l^2 g_6 g_2 - A_l g_4 g_1^2 + 2A_l g_3 g_2 g_1 - 3g_2 g_1^3 \\
h_{10} &= 8A_l^3 g_{12} - 4A_l^2 g_7 g_2 + 4A_l^2 g_8 g_1 + 2A_l g_3 g_2^2 - 4A_l g_4 g_2 g_1 + 2A_l g_5 g_1^2 - 9g_2^2 g_1^2 \\
h_{11} &= 4A_l^3 g_{13} + 2A_l^2 g_8 g_2 + 2A_l^2 g_9 g_1 - A_l g_4 g_2^2 + 2A_l g_5 g_1 g_2 - 3g_1 g_2^3 \\
h_{12} &= 16A_l^3 g_{14} + 8A_l^2 g_9 g_2 + 4A_l g_5 g_2^2 - 3g_2^4
\end{aligned}$$

Therefore, if o_1 satisfies Eq. (14), then $G_l = 0$ and consequently Eq. (17) becomes a polynomial equation of degree two in terms of the unknown $(4A_l \tilde{c}_4 l + B_l)^2$. So, Eq. (15) is easily deduced. \square

Lemma 2.6 gives the expressions for the set of solutions of l . Then, the expressions of \tilde{b} and $\sqrt{\Delta_{\tilde{b}}}$ given by Eq. (13), which depends only on the arbitrary parameters $\{\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4, o_2\}$ and the unknowns o_1 and l , lead to the roots of the polynomial $\mathcal{P}_0(z)$:

Corollary 2.7. *If o_1 satisfies Eq. (14) and the set of solutions of l is given by Eq. (15), then the roots of the polynomial $\mathcal{P}_0(z)$ are of the form:*

$$k = \tilde{c}_3 - \tilde{c}_4 \frac{(r_1 + s_1)l^2 + (r_2 + s_2)l + r_3 + s_3}{e_4 l + e_5}. \quad (18)$$

provided that \tilde{c}_1 and \tilde{c}_2 are not null simultaneously and $\tilde{c}_1^2 - 4\tilde{c}_2 \neq 0$.

Proof:

Proposition 2.2 requires the above conditions on \tilde{c}_1 and \tilde{c}_2 . So, by inserting Eq. (13) in Eq. (1) in order to substitute \tilde{b} and $\sqrt{\Delta_{\tilde{b}}}$ by their expressions, we get the roots of the polynomial $\mathcal{P}_0(z)$ as the following form:

$$\begin{aligned}
k &= \tilde{c}_3 + \tilde{c}_4 \underbrace{\tilde{b}}_{-\frac{r_1 l^2 + r_2 l + r_3}{e_4 l + e_5}} - \tilde{c}_4 \underbrace{\sqrt{\Delta_{\tilde{b}}}}_{\frac{s_1 l^2 + s_2 l + s_3}{e_4 l + e_5}}
\end{aligned}$$

this expression of k implies the equation 18. \square

Finally, using the approach of appropriate changes of variable involving arbitrary parameters presented in this section, we get its aim to calculate the roots of $\mathcal{P}_0(z)$ which lead to the roots of $\mathcal{P}(Z)$ according to the change of variable given by Eq. (6). In the following section, we will explain the formula of the roots.

3 Application on the quartic polynomial

To get the root of $\mathcal{P}(z)$ the general complex polynomial of degree four in $\mathbb{C}[z]$, we need firstly the expression of o_1 which should satisfy Eq. (14). But, the equation 14 is a cubic polynomial equation of degree three. So, to solve it, we use the formula of [13]. So, we introduce firstly this formula:

Theorem 3.1. *The roots of the general cubic polynomial equation*

$$\mathcal{C}(z) := az^3 + bz^2 + cz + d = 0; \quad a, b, c, d \in \mathbb{C}$$

are given by

$$z_\mu = \frac{1}{a} \left(\mu \alpha_1 \mathcal{R}_1 + \mu^2 \alpha_2 \mathcal{R}_2 - \frac{b}{3} \right); \quad \mu \in \left\{ -1, \frac{1 - i\sqrt{3}}{2}, \frac{1 + i\sqrt{3}}{2} \right\} \quad (19)$$

where the two cubic roots

$$\begin{aligned}
\mathcal{R}_1 &= \sqrt[3]{\frac{i\sqrt{3}}{9} \sqrt{-4c_c^3 - 27d_c^2} + dc} \\
\mathcal{R}_2 &= \sqrt[3]{\frac{i\sqrt{3}}{9} \sqrt{-4c_c^3 - 27d_c^2} - dc}
\end{aligned}$$

are given in terms of

$$\begin{aligned}
c_c &= ac - \frac{1}{3}b^2 \\
d_c &= \frac{2}{27}b^3 - \frac{1}{3}acb + a^2d
\end{aligned}$$

and the coefficients of the two cubic roots

$$\begin{aligned}
\alpha_1 &= \frac{\sqrt[3]{4}}{2} \exp \left\{ i \left[\arg \left(\mathcal{A}_1 \sqrt[3]{\delta_c} \right) - \arg \left(- (c_c^4 - 12c_c d_c^2) \mathcal{R}_1 \right) \right] \right\} \\
\alpha_2 &= \frac{\sqrt[3]{4}}{2} \exp \left\{ i \left[\arg \left(\mathcal{A}_2 \sqrt[3]{\delta_c^2} \right) - \arg \left((c_c^4 - 12c_c d_c^2)^2 \mathcal{R}_2 \right) \right] \right\}
\end{aligned}$$

are given in terms of

$$\begin{aligned} A_1 &= 2 \left(-\frac{i\sqrt{3}}{9} 15d_c c_c + c_c \sqrt{-4c_c^3 - 27d_c^2} \right) \\ A_2 &= 27d_c^3 - 21c_c^3 d_c - i\sqrt{3} (c_c^3 + 3d_c^2) \sqrt{-4c_c^3 - 27d_c^2} \\ \delta_c &= (2c_c^3 d_c + d_c^3) \sqrt{-4c_c^3 - 27d_c^2} + \frac{i\sqrt{3}}{9} (2c_c^6 + 72c_c^3 d_c^2 - 27d_c^4) \end{aligned}$$

here, i is the complex number: $i^2 = -1$. Notice that $\alpha_1^3 = \alpha_2^3 = 1/2$ and $\mu^3 = -1$.
Moreover, the roots of the cubic polynomial $C_0(z)$

$$C_0(z) := z^3 + c_c z + d_c = 0$$

are

$$az_\mu + \frac{b}{3} = \mu\alpha_1 \mathcal{R}_1 + \mu^2 \alpha_2 \mathcal{R}_2; \quad \mu \in \left\{ -1, \frac{1-i\sqrt{3}}{2}, \frac{1+i\sqrt{3}}{2} \right\}.$$

Proof:

The classical change of variable for the cubic polynomial equation implies that

$$\begin{aligned} \mathcal{C}(z) := a \left(z + \frac{b}{3a} \right)^3 + \frac{c_c}{a} \left(z + \frac{b}{3a} \right) + \frac{d_c}{a^2} = 0 &\Leftrightarrow C_0(Z) := Z^3 + \frac{c_c}{a^2} Z + \frac{d_c}{a^3} = 0; \\ \text{where: } Z &:= z + \frac{b}{3a}. \end{aligned}$$

Then, to end the proof, we apply the formula of [13] on $C_0(Z)$ using the following property in equation (20)

$$\exp \left\{ i \left[\arg \left(\frac{b}{d} \right) - \arg \left(\frac{c}{d} \right) \right] \right\} = \exp \{ i [\arg(b) - \arg(c)] \}; \quad \forall b, c \in \mathbb{C}; \quad \forall d \in \mathbb{C}^* \quad (20)$$

to get α_1 and α_2 . □

We aim to calculate analytically the roots of \mathcal{P}_0 by applying the results of section 2. So, we introduce the following definitions which will be useful to find the explicit expressions for the roots of \mathcal{P}_0 :
Indeed, since the roots of \mathcal{P}_0 are given by Eq. (18) in terms of o_1 and l , then we introduce for each one a set of expressions corresponding to it:

Definition 3.2. We associate to \mathcal{P}_0 the following expressions in terms of its coefficients:
expressions corresponding to o_1 :

$$\begin{aligned} \alpha_o &= 4ce^2 - 4ce + d^2 - 8e^2 + 8e^3 \\ \beta_o &= -(12ce + 4ce^2 - 3d^2 + 8e^2 + 24e^3) \\ c_o &= 4c^2 e^2 + 12c^2 e - 3cd^2 + 64ce^2 - 18d^2 e + 48e^3 + 16e^2 \\ d_o &= 16e^2 (e - 9) c^3 + 12e [3d^2 - 8e(5e + 3)] c^2 + 12e(3e + 5) (3d^2 - 16e^2) c - 27d^2 [d^2 - 8e^2(e + 1)] - 128e^3(9e - 1) \\ D_o &= 16c^4 e - 4c^3 d^2 - 128c^2 e^2 + 144cd^2 e - 27d^4 + 256e^3 \end{aligned}$$

expressions corresponding to l :

$$\begin{aligned} \lambda_1 &= 16c^2 e^4 - 32c^2 e^3 + 16c^2 e^2 - 16cd^2 e + 3d^4 - 48d^2 e^3 + 16d^2 e^2 - 64e^5 + 128e^4 - 64e^3 \\ \lambda_2 &= -2^2 (16c^2 e^4 - 16c^2 e^2 + 16cd^2 e - 3d^4 - 24d^2 e^3 - 8d^2 e^2 - 64e^5 + 64e^3) \\ \lambda_3 &= 2 (48c^2 e^4 + 32c^2 e^3 + 48c^2 e^2 - 48cd^2 e + 9d^4 - 192e^5 - 128e^4 - 192e^3) \\ \lambda_4 &= -2^2 (16c^2 e^4 - 16c^2 e^2 + 16cd^2 e - 3d^4 + 24d^2 e^3 + 8d^2 e^2 - 64e^5 + 64e^3) \\ \lambda_5 &= 16c^2 e^4 - 32c^2 e^3 + 16c^2 e^2 - 16cd^2 e + 3d^4 + 48d^2 e^3 - 16d^2 e^2 - 64e^5 + 128e^4 - 64e^3 \\ \lambda_6 &= 8ce + 8ce^2 - 3d^2 - 32e^2 \\ \lambda_7 &= 3d^2 + 8ce^2 - 8ce \\ \lambda_8 &= 8ce + 8ce^2 - 3d^2 + 32e^2 \end{aligned}$$

Definition 3.3. We associate to \mathcal{P}_0 the following expressions in terms of these given by definition 3.2:
expressions corresponding to o_1 :

$$\begin{aligned} A_1 &= \frac{5}{3^3} \sqrt{3} i d_o c_o - \alpha_o c_o \sqrt{D_o} \\ A_2 &= \frac{1}{3^2} \left(\frac{d_o^3}{3} + 7c_o^3 d_o \right) + \sqrt{3} i \alpha_o \left(c_o^3 - \frac{d_o^2}{3^2} \right) \sqrt{D_o} \\ \delta_o &= \alpha_o \left(\frac{d_o^3}{3^3} - 2c_o^3 d_o \right) \sqrt{D_o} + \frac{\sqrt{3} i}{9} \left(2c_o^6 - \frac{2^3}{3} c_o^3 d_o^2 - \frac{d_o^4}{3^3} \right) \\ \alpha_1 &= \frac{\sqrt[3]{4}}{2} \exp \left\{ i \left[\arg \left(2A_1 \sqrt[3]{\delta_o} \right) - \arg \left(- \left(c_o^4 + \frac{2^2}{3^2} c_o d_o^2 \right) \frac{R_1}{3^2} \right) \right] \right\} \\ \alpha_2 &= \frac{\sqrt[3]{4}}{2} \exp \left\{ i \left[\arg \left(A_2 \sqrt[3]{\delta_o} \right) - \arg \left(\left(c_o^4 + \frac{2^2}{3^2} c_o d_o^2 \right)^2 \frac{R_2}{3^2} \right) \right] \right\} \end{aligned}$$

expressions corresponding to l :

$$\begin{aligned}\delta_l &= \lambda_1 \Delta_o^4 + 3\alpha_o \lambda_2 \Delta_o^3 + 3^2 \alpha_o^2 \lambda_3 \Delta_o^2 + 3^3 \alpha_o^3 \lambda_4 \Delta_o + 3^4 \alpha_o^4 \lambda_5 \\ d_l &= \lambda_6 \Delta_o^2 - 6\lambda_7 \alpha_o \Delta_o + 3^2 \alpha_o^2 \lambda_8\end{aligned}$$

Now, we present the principal result of the paper which is the formula of the roots of \mathcal{P} . On the hand, this formula is closed, contrary to the formulas of Ferrari, Descartes and Euler which required a root of a cubic polynomial, whose the classical methods to calculate it as Cardan-Tartaglia and Lagrange do not give a closed formula [13]. On the other hand, this formula is in fact a Lagrange-type [11], where the square and cubic roots are given by the standard convention, contrary to the formulas of Cardan-Tartaglia and Lagrange which require a specific convention depending on coefficients. Indeed, the difference between this and the previous works is manifested in the two terms α_1 and α_2 of Δ_o given by definition 3.3 which were missed in the formulas of Cardan-Tartaglia and Lagrange required by the methods of Ferrari, Descartes and Euler methods:

Theorem 3.4. *Let the two cubic roots R_1 and R_2 :*

$$\begin{aligned}R_1 &= \sqrt[3]{3\sqrt{3i}\sqrt{D_o}\alpha_o + d_o} \\ R_2 &= \sqrt[3]{3\sqrt{3i}\sqrt{D_o}\alpha_o - d_o}.\end{aligned}$$

The set of solutions of \mathcal{P}_0

$$\mathcal{P}_0(z) := z^4 + cz^2 + dz + e = 0; \quad c, d, e \in \mathbb{C}$$

is given in terms of Δ_o and Δ_l

$$\begin{aligned}\Delta_o &= 4e(-\alpha_1 R_1 + \alpha_2 R_2) - \beta_o \\ \Delta_l &= m_2 \sqrt{m_1 2\sqrt{\delta_l} - d_l - d(\Delta_o + 3\alpha_o)}\end{aligned}$$

as follows:

$$z_{m_1, m_2} = \frac{(3\alpha_o - \Delta_o) \Delta_l^2 + 4d(9\alpha_o^2 - \Delta_o^2) \Delta_l + 16e(3\alpha_o + \Delta_o) [(e - c + 1) \Delta_o^2 - 6(e - 1) \alpha_o \Delta_o + 9(c + e + 1) \alpha_o^2]}{4[(e - 1) \Delta_o^2 - 6(e + 1) \alpha_o \Delta_o + 9(e - 1) \alpha_o^2] \Delta_l + 16ed(3\alpha_o + \Delta_o)(3\alpha_o - \Delta_o)^2},$$

where: $m_1, m_2 \in \{-1, 1\}$ (21)

using the expressions of definitions 3.2 and 3.3.

Proof:

The result of corollary 2.7 is valid for any choice of arbitrary parameters $\{\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4, o_2\}$ provided that \tilde{c}_1 and \tilde{c}_2 are not null simultaneously and $\tilde{c}_1^2 - 4\tilde{c}_2 \neq 0$. So, by taking $\tilde{c}_1 = 0$, $\tilde{c}_2 = 1$, $\tilde{c}_3 = 3$, $\tilde{c}_4 = 1$ and $o_2 = 1$, these conditions are verified. Consequently, we apply this corollary with this choice.

In the sequel, the proof rests on two principal sets. In the first step, we calculate o_1 explicitly using theorem 3.1. In the second step, we calculate l explicitly using lemma 2.6. Indeed:

In the first step, with the choice of the arbitrary parameters, the explicit form of Eq. (14) becomes equivalent to:

$$\alpha_o o_1^3 + \beta_o o_1^2 + \gamma_o o_1 + \zeta_o = 0; \quad \text{where: } \begin{cases} \gamma_o = -(12ce + 4ce^2 - 3d^2 - 8e^2 - 24e^3) \\ \zeta_o = 4ce^2 - 4ce + d^2 + 8e^2 - 8e^3 \end{cases}$$

So, by applying Eq. (19) for the unknown o_1 and the coefficients α_o , β_o , γ_o and ζ_o , we get:

$$\left. \begin{aligned} c_C &= -\frac{2^4}{3} e^2 c_o \\ d_C &= \frac{2^6}{3^3} e^3 d_o \\ -4c_C^3 - 27d_C^2 &= 2^{12} e^6 D_o \alpha_o^2 \end{aligned} \right\} \Rightarrow \begin{cases} \mathcal{R}_1 = \frac{4}{3} e R_1 \\ \mathcal{R}_2 = \frac{4}{3} e R_2 \end{cases}$$

and

$$\begin{aligned} \mathcal{A}_1 &= \frac{2^{11}}{3} e^5 A_1 \\ \mathcal{A}_2 &= \frac{2^{18}}{3^3} e^9 A_2 \\ \delta_C &= \left(\frac{2^8}{3^2} e^4 \right)^3 \delta_o \\ c_C^4 - 12c_C d_C^2 &= \frac{2^{16}}{3^4} e^8 \left(c_o^4 + \frac{2^2}{3^2} c_o d_o^2 \right) \end{aligned}$$

which implies, using the property of Eq. (20), that

$$\left. \begin{aligned} \alpha_1 &= \alpha_1 \\ \alpha_2 &= \alpha_2 \end{aligned} \right\} \Rightarrow o_1 = \frac{\Delta_o}{3\alpha_o}$$

In the second step, with the choice of the arbitrary parameters, the explicit forms of 16 and F_l become equivalent respectively to:

$$\begin{aligned}\Delta &= 2^{16}D_l; \quad \text{where: } D_l = \lambda_1 o_1^4 + \lambda_2 o_1^3 + \lambda_3 o_1^2 + \lambda_4 o_1 + \lambda_5 \\ F_l &= 2^7 (\lambda_6 o_1^2 - 2\lambda_7 o_1 + \lambda_8)\end{aligned}$$

the, the equation (15) implies that

$$l = \frac{m_2 \sqrt{m_1 2\sqrt{D_l} - (\lambda_6 o_1^2 - 2\lambda_7 o_1 + \lambda_8)} - d(o_1 + 1)}{2^3 e}.$$

Finally, to end the proof, we multiply both the numerator and the denominator in Eq. (18) by $e\alpha_o$ to get Eq. (21). \square

Now, we apply theorem 3.4 to deduce the roots $\mathcal{P}(Z)$:

Theorem 3.5. *The set of solutions of the general complex quartic polynomial \mathcal{P}*

$$\mathcal{P}(Z) := Z^4 + b_Z Z^3 + c_Z Z^2 + d_Z Z + e_Z$$

is given by

$$Z_{m_1, m_2} = z_{m_1, m_2} - \frac{b_Z}{4}; \quad \text{where: } m_1, m_2 \in \{-1, 1\}.$$

here:

$$\begin{cases} c := c_Z - \frac{3}{8}b_Z^2 \\ d := \frac{b_Z^3}{8} - \frac{c_Z b_Z}{2} + d_Z \\ e := -\frac{3}{256}b_Z^4 + \frac{c_Z b_Z^2}{16} - \frac{d_Z b_Z}{4} + e_Z. \end{cases}$$

Proof:

The proof is an immediate consequence of the change of variable given by Eq. (6). \square

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