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ON L^1-NORMS FOR NON-HARMONIC TRIGONOMETRIC POLYNOMIALS WITH SPARSE FREQUENCIES

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In honnor of Karlheinz Gröchenig on the occasion of his 60^{th} birthday.

Abstract. In this paper we show that, if an increasing sequence $\Lambda = (\lambda_k)_{k\in\mathbb{Z}}$ has gaps going to infinity $\lambda_{k+1} - \lambda_k \to +\infty$ when $k \to \pm\infty$, then for every $T > 0$ and every sequence $(a_k)_{k\in\mathbb{Z}}$ and every $N \geq 1$,

$$A \sum_{k=0}^{N} \frac{|a_k|}{1+k} \leq \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t$$

further, if $\displaystyle\sum_{k\in\mathbb{Z}} \frac{1}{1+|\lambda_k|} < +\infty$,

$$B \max_{|k|\leq N} |a_k| \leq \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t$$

where A, B are constants that depend on T and Λ only.

The first inequality was obtained by Nazarov for $T > 1$ and the second one by Ingham for $T \geq 1$ under the condition that $\lambda_{k+1} - \lambda_k \geq 1$. The main novelty is that if those gaps go to infinity, then T can be taken arbitrarily small. The result is new even when the λ_k's are integers where it extends a result of McGehee, Pigno and Smith.

The results are then applied to observability of Schrödinger equations with moving sensors.

1. Introduction

The aim of this paper is to establish a lower bound of L^1-norms of non-harmonic trigonometric polynomials with sparse frequencies. The results are then applied to obtain L^1-observability estimate of certain PDEs, including the free Schrödinger equation. We thus obtain L^1-analogues of a result of Kahane [Ka] and Haraux [Ha] on the L^2-norm of sparse trigonometric polynomials while the L^2-observability result was previously obtained by the first author together with Komornik [JK].

Let us now be more precise. We first describe the well-known results in the L^2-setting. The celebrated Ingham Inequality gives a lower and upper bound of $L^2([-T,T])$-norms of (non-harmonic) trigonometric polynomials and is stated as follows:

Theorem 1.1 (Ingham [I1])**.** *Let $\gamma > 0$ and $T > \dfrac{1}{\gamma}$. Then there exist constants $0 < A_2(T,\gamma) \leq B_2(T,\gamma)$ such that*

– for every sequence of real numbers $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\lambda_{k+1} - \lambda_k \geq \gamma$;

– for every sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$,

$$A_2(T,\gamma) \sum_{k\in\mathbb{Z}} |a_k|^2 \leq \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t} \right|^2 dt \leq B_2(T,\gamma) \sum_{k\in\mathbb{Z}} |a_k|^2 \tag{1.1}$$

Note that $A_2(T,\gamma), B_2(T,\gamma)$ are explicit constants (*see* [KL, JS]). Ingham has also shown that the upper bound is valid for any $T > 0$ while the lower bound may not be true for $T \leq \dfrac{1}{\gamma}$. In

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his seminal work on almost periodic functions [Ka], Kahane has shown that this condition can be lifted if $\lambda_{k+1}-\lambda_k \to +\infty$ when $k \to \pm\infty$:

Theorem 1.2 (Kahane)**.** *Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\lambda_{k+1}-\lambda_k \to +\infty$ when $k \to \pm\infty$. Then, for every $T>0$, there exist constants $0 < A_2(T,\Lambda) \le B_2(T,\Lambda)$ such that*

$$A_2(T,\Lambda)\sum_{k\in\mathbb{Z}}|a_k|^2 \le \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}}a_k e^{2i\pi\lambda_k t}\right|^2 dt \le B_2(T,\Lambda)\sum_{k\in\mathbb{Z}}|a_k|^2$$

holds for every sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$.

The constants are not explicit in [Ka], they were later obtained by Haraux [Ha] (but with constants that are difficult to compute explicitly, *see e.g.* [KL, JS]).

Those inequalities have found many applications in control theory. Among the numerous results (*see* the book [KL] for a good introduction to the subject), our starting point is a result of the first author with V. Komornik [JK]. To state it, let us introduce some notation. We write $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $H^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \sum_{k\in\mathbb{Z}}(1+|k|^2)^2|c_k(f)|^2 < \infty\}$, where the $c_k(f)$'s are the Fourier coefficient of f. Then the following holds:

Theorem 1.3. *Fix $(t_0,x_0)\in\mathbb{R}^2$, $a\in\mathbb{R}$ and $T>0$. For $u_0\in H^2(\mathbb{T})$, let u be the solution of*

$$\begin{cases} u_t = \dfrac{i}{2\pi}u_{xx} & \text{in } \mathbb{R}\times\mathbb{T},\\ u(0,x) = u_0(x) & \text{for } x\in\mathbb{T}. \end{cases}$$

(i) There exists $D_2(T,a)$ such that, for every $u_0\in H^2(\mathbb{T})$,

$$\int_0^T |u(t_0+t,x_0+at)|^2\,dt \le D_2(T,a)\|u_0\|_{L^2}^2$$

(ii) If $a\notin\mathbb{Z}$, then there exists $C_2(T,a)$ such that, for every $u_0\in H^2(\mathbb{T})$,

$$C_2(T,a)\|u_0\|_{L^2}^2 \le \int_0^T |u(t_0+t,x_0+at)|^2\,dt \tag{1.2}$$

also holds.

(iii) If $a\in\mathbb{Z}$, then there exists $u_0\neq 0$ such that $u(t_0+t,x_0+at)=0$ so that (1.2) fails.

Let us sketch the proof. If we write $u_0(x) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi kx}$ then the solution of the Schrödinger equation can be written as a Fourier series $u(t,x) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi(k^2t+kx)}$ and the fact that $u_0\in H^2(\mathbb{T})$ implies that this series is uniformly convergent. One can thus restrict it to a segment:

$$u(t_0+t,x_0+2at) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi k^2(t_0+t)+2i\pi k(x_0+2at)} := \sum_{k\in\mathbb{Z}} d_k e^{2i\pi\lambda_k t}.$$

Then one shows that the λ_k's are such that Kahane's Theorem applies (provided a is not an integer). Our aim is to extend this argument to the L^1-setting.

The first task is thus to obtain an L^1-version of Ingham's Inequality. An $L^1-\ell^\infty$ estimate was obtained by Ingham [I1] (and is an easy adaptation of the L^2 proof) and a much more evolved L^1 to weighted ℓ^1-inequality was obtained by Nazarov, inspired by the proof of Littlewood's conjecture by McGehee-Pigno-Smith. The results are the following:

Theorem 1.4. *Let $(\lambda_k)_{k\in\mathbb{Z}}$ be an increasing sequence of real numbers such that there exists $\gamma>0$ for which $\lambda_{k+1}-\lambda_k\ge\gamma$ for every k. Let $(a_k)_{k\in\mathbb{Z}}$ be a sequence of complex numbers.*

- Ingham [I2]: *For $T\ge\dfrac{1}{\gamma}$, there exists a constant $A_1(T,\gamma)>0$ such that, for every $N\ge 1$,*

$$A_1(T,\gamma)\max_{k=-N,\dots,N}|a_k| \le \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=-N}^{N}a_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t.$$

- Nazarov [Na]: *For $T > \dfrac{1}{\gamma}$, there exists a constant $\tilde{A}_1(T,\gamma) > 0$ such that, for every $N \geq 1$,*

$$\tilde{A}_1(T,\gamma) \sum_{k=0}^{N} \frac{|a_k|}{1+k} \leq \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t.$$

Ingham established the first inequality for $T > \dfrac{1}{\gamma}$ in [I1] and improved his result in [I2] showing that it holds when $T = \dfrac{1}{\gamma}$, and that one may take $A_1(T,\gamma) = \dfrac{1}{2}$. This was further improved by Mordell [Mo]. There is a major difference between the two inequalities: the right hand side in Ingham's Inequality is generally much smaller than in Nazarov's Inequality (*e.g.* take $|a_k| = 1$ for all k then Ingham provides a constant lower bound while Nazarov provides a logarithmic one). On the other hand, in Nazarov's inequality the sum starts at 0 and may fail for symmetric sums. Also its validity for $T = \dfrac{1}{\gamma}$ is an open question.

Further Nazarov did not provide an estimate of the constant $C(\gamma, T)$. However, his proof can be modified to establish quantitative bounds. This was done in [JKS] when γT is large enough and in [JS] for γT near to 1.

This result is sufficient to partially extend Theorem 1.3 to the L^1-setting. The only thing that would be missing is that in Theorem 1.3, there is no minimal time needed thanks to Kahane's extension of Ingham's inequality. However, so far this is unknown in the L^1-case and our first result is precisely to prove this:

Theorem 1.5. *Let $\Lambda = (\lambda_k)_{k\in\mathbb{Z}}$ be an increasing sequence with $\lambda_{k+1} - \lambda_k \to +\infty$ when $k \to \pm\infty$. Then, for every $T > 0$, there exists a constant $\tilde{A}_1(T,\Lambda) > 0$ such that, if $(a_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ is a sequence of complex numbers, and $N \geq 1$, then*

$$\tilde{A}_1(T,\Lambda) \sum_{k=0}^{\infty} \frac{|a_k|}{1+k} \leq \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t. \tag{1.3}$$

If further $\displaystyle\sum_{k\in\mathbb{Z}} \frac{1}{1+|\lambda_k|}$ converges, then there also exists a constant $A_1(T,\Lambda)$ such that, for every $(a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ and every $N \geq 1$,

$$A_1(T,\Lambda) \max_{k=-N,\dots,N} |a_k| \leq \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t. \tag{1.4}$$

The main difficulty in the proof of this result is that both Kahane's and Haraux's argument can not be adapted directly. Indeed, both use in a crucial way that in Ingham's Inequality the L^2 norm of a trigonometric polynomial is both lower and upper bounded by the ℓ^2-norm of its coefficients. In the L^1-case, the upper bound is in terms of the ℓ^1-norm of the coefficients and does not match the lower bound. Instead, our proof uses a compactness argument so that we don't obtain an estimate of $A_1(\gamma,T), \tilde{A}_1(\gamma,T)$ in this case. It would be interesting to obtain such an estimate.

Finally, we apply this result to an observability inequality for the Schrödinger equation with moving sensor. We show the following: take $u_0 \in H^2(\mathbb{T})$ and write its Fourier series $u_0(x) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi kt}$. Let u be the solution of

$$\begin{cases} u_t = \dfrac{i}{2\pi} u_{xx} & \text{in } \mathbb{R}\times\mathbb{T}, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{T}. \end{cases}$$

then, for every $a \notin \mathbb{Z}$ and every $T > 0$, there exists a constant $C(a,T) > 0$ such that

$$\frac{1}{T} \int_0^T |u(t_0+t, x_0+at)| \, \mathrm{d}t \geqslant C \sum_{k\in\mathbb{Z}} \frac{|c_k|}{1+|k|}.$$

Similar results are then obtained for higher order Schrödinger equations.

The remaining of the paper is organized as follows. In Section 2 we prove Theorem 1.5. We then devote section 3 to the free Schrödinger equation while the last section is devoted to higher order Schrödinger equations.

2. Proof of Theorem 1.5

First note that replacing the sequence $(\lambda_k)_{k\in\mathbb{Z}}$ with a translate $\lambda_k+\lambda$, leaves (1.3)-(1.4) unchanged. So there is no loss of generality in assuming that $\lambda_0>0>\lambda_{-1}$. We now fix $T>0$.

Define K to be an integer such that, if $|k|\geq K$, $\lambda_{k+1}-\lambda_k\geq\dfrac{2}{T}$. As a consequence, from Nazarov's inequality, the following holds for every sequence $(b_k)_{k\in\mathbb{N}}$ and every $N\geq 0$:

$$\begin{aligned}\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=K}^{K+N}b_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t &= \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N}b_{k+K}e^{2i\pi\lambda_{k+\mathcal{N}}t}\right|\mathrm{d}t\\ &\geq \tilde{A}_1\left(T,\frac{2}{T}\right)\sum_{k=0}^{N}\frac{|b_{k+K}|}{k+1}\geq\tilde{A}_1\left(T,\frac{2}{T}\right)\sum_{k=K}^{K+N}\frac{|b_k|}{k+1};\end{aligned}\tag{2.1}$$

while Ingham's inequality shows that

$$\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{K\leq|k|\leq K+N}b_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t\geq A_1\left(T,\frac{2}{T}\right)\max_{K\leq|k|\leq K+N}|b_k|.\tag{2.2}$$

We first prove (1.3). To do so, we will adopt the following convention. An element z of $\mathbb{C}^N$ will be indexed starting at 0, $z=(z_0,\dots,z_{N-1})$. We will identify it with a vector in $\mathbb{C}^M$, $M\geq N$ as well as with a sequence $(z_k)_{k\geqslant 0}$ by adding 0's at the end, *i.e.* setting $z_k=0$ for $k\geq N$. An element of $\mathbb{C}^N$ is thus called a vector or a sequence, which ever is the most convenient.

On $\mathbb{C}^N$, we introduce two norms through

$$\|(a_0,\dots,a_{N-1})\|_{\ell^{1,-1}_N}=\sum_{k=0}^{N-1}\frac{|a_k|}{1+k}$$

and

$$\|(a_0,\dots,a_{N-1})\|_{L^1_N}=\left\|\sum_{k=0}^{N-1}a_k e^{2i\pi\lambda_k t}\right\|_{L^1([-T/2,T/2])}:=\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N-1}a_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t.$$

The first one is clearly a norm while for the second one, it is enough to notice that the set $\{t\to e^{2i\pi\lambda t}\}_{\lambda\in\mathbb{R}}$ is linearly independent in $L^1([-T/2,T/2])$.

As $\|\cdot\|_{\ell^{1,-1}_N}$ and $\|\cdot\|_{L^1_N}$ are both norms on the finite dimensional space $\mathbb{C}^N$, they are equivalent. Thus there are $\kappa_N\leq 1\leq\Lambda_N$ such that, for every $a\in\mathbb{C}^N$,

$$\kappa_N\|a\|_{\ell^{1,-1}_N}\leq\|a\|_{L^1_N}\leq\Lambda_N\|a\|_{\ell^{1,-1}_N}.\tag{2.3}$$

Nazarov's theorem asserts that one may choose κ_N independent of N provided T is large enough. Our aim is to show that this is possible for every T under our additional condition on $(\lambda_k)_{k\geq 0}$.

Assume towards a contradiction that this is not the case. Then, for every integer $n\geq 1$, there exist an integer K_n and $a^{(n)}=(a^{(n)}_0,\dots,a^{(n)}_{K_n-1})\in\mathbb{C}^{K_n}$ such that $\|a^{(n)}\|_{\ell^{1,-1}_{K_n}}=1$ while $\|a^{(n)}\|_{L^1_{K_n}}\leq\dfrac{1}{n}$. The first observation is that $K_n\to+\infty$ otherwise, we would contradict (2.3) when n is large enough. Hence, without loss of generality, we will assume that $K_{n+1}>K_n\geq K$ for every n, where K was defined so that if $|k|\geq K$, $\lambda_{k+1}-\lambda_k\geq\dfrac{2}{T}$.

Next, we split $a^{(n)}$ into two vectors

$$a^{(n)}_-=(a^{(n)}_0,\dots,a^{(n)}_{K-1},0,\dots,0)\quad\text{and}\quad a^{(n)}_+=a^{(n)}-a^{(n)}_-.$$

With an obvious abuse of notation, we consider that $a_-^{(n)} \in \mathbb{C}^K$. In particular $\|a_-^{(n)}\|_{\ell^{1,-1}_K} \leq \|a_-^{(n)}\|_{\ell^{1,-1}_{K_n}} \leq 1$. Thus, up to taking a subsequence, we may assume that $a_-^{(n)} \to (a_0, \dots, a_{K-1})$.

Next, define the following functions:

(1) the functions $\varphi^{(n)}$ given by

$$\varphi^{(n)}(t) = \sum_{k=0}^{K_n} a_k^{(n)} e^{2i\pi\lambda_k t}$$

so that $\|\varphi^{(n)}\|_{L^1([-T/2,T/2])} \leq \dfrac{1}{n} \to 0$ *i.e.* $\varphi^{(n)} \to 0$ in $L^1([-T/2,T/2])$.

(2) The functions $\varphi_-^{(n)}, \varphi_-$ given by

$$\varphi_-^{(n)}(t) = \sum_{k=0}^{K-1} a_k^{(n)} e^{2i\pi\lambda_k t} \quad \text{and} \quad \varphi_-(t) = \sum_{k=0}^{K-1} a_k e^{2i\pi\lambda_k t}.$$

This functions are in a finite dimensional subspace of $L^1([-T/2,T/2])$ so that the convergence $a_k^{(n)} \to a_k$ for $k = 0, \dots, K-1$ implies that $\varphi_-^{(n)} \to \varphi_-$ in $L^1([-T/2,T/2])$.

(3) The functions

$$\varphi_+^{(n)} = \varphi^{(n)} - \varphi_-^{(n)} = \sum_{k=K}^{K_n} a_k^{(n)} e^{2i\pi\lambda_k t}.$$

Note that $\varphi_+^{(n)} \to -\varphi_-$ in $L^1([-T/2,T/2])$. On the other hand, for $n \geq m$ we can apply (2.1) to $\varphi_+^{(n)} - \varphi_+^{(m)}$ leading to

$$\begin{aligned} \frac{1}{T}\int_{-T/2}^{T/2} |\varphi_+^{(n)}(t) - \varphi_+^{(m)}(t)|\,\mathrm{d}t &= \frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k=K}^{K_n} \big(a_k^{(n)} - a_k^{(m)}\big) e^{2i\pi\lambda_k t}\right| \mathrm{d}t \\ &\geq \tilde{A}_1\left(T, \frac{2}{T}\right) \sum_{k=K}^{K_n} \frac{|a_k^{(n)} - a_k^{(m)}|}{k+1}. \end{aligned}$$

Using also that $a_k^{(n)} \to a_k$ for $k = 0, \dots, K-1$ this shows that $\big(a_k^{(n)}\big)_{k\geq 0}$ is a Cauchy sequence in the Banach space

$$\ell^{1,-1} = \left\{ (b_k)_{k\geq 0} \,:\, \|(b_k)\|_{\ell^{1,-1}} := \sum_{k=0}^{+\infty} \frac{|b_k|}{k+1} \right\}.$$

In particular, $\big(a_k^{(n)}\big)_{k\geq 0} \to (a_k)_{k\geq 0}$ in $\ell^{1,-1}$. This implies that, for all k, $a_k^{(n)} \to a_k$ and that

$$1 = \|a^{(n)}\|_{\ell^{1,-1}_{K_n}} = \|a^{(n)}\|_{\ell^{1,-1}} \to \|a\|_{\ell^{1,-1}}.$$

We will thus reach a contradiction if we show that $a_k = 0$ for all k.

To do so, we introduce further functions via

$$\Phi_\pm^{(n)}(x) = \int_0^x \varphi_\pm^{(n)}(t)\,\mathrm{d}t \quad \text{and} \quad \Phi_-(x) = \int_0^x \varphi_-(t)\,\mathrm{d}t = \frac{1}{2i\pi}\sum_{k=0}^{K-1} \frac{a_k}{\lambda_k}\big(e^{2i\pi\lambda_k x} - 1\big).$$

Note that as $\varphi_\pm^{(n)} \to \pm\varphi_-$ in $L^1([-T/2,T/2])$, $\Phi_\pm^{(n)} \to \pm\Phi_-$ uniformly over $[-T/2,T/2]$ thus also in $L^2([-T/2,T/2])$.

Next, as $(\lambda_n)_{n\in\mathbb{N}}$ is increasing with $\lambda_0 > 0$ and $\lambda_{n+1} - \lambda_n \to +\infty$, there exists $\alpha > 0$ such that $\lambda_n \geq \alpha(n+1)$. It follows that

$$\sum_{k=0}^{+\infty} \frac{|a_k|}{\lambda_k} \leq \frac{1}{\alpha}\sum_{k=0}^{+\infty} \frac{|a_k|}{k+1} < +\infty \quad \text{and} \quad \sum_{k=0}^{+\infty} \frac{|a_k^{(n)} - a_k|}{\lambda_k} \leq \frac{1}{\alpha}\sum_{k=0}^{+\infty} \frac{|a_k^{(n)} - a_k|}{k+1} \to 0.$$

As $|e^{2i\pi\lambda_k x}-1|\leq 2$, it follows that

$$\Phi_+^{(n)}=\frac{1}{2i\pi}\sum_{k=K}^{K_n}\frac{a_k^{(n)}}{\lambda_k}\big(e^{2i\pi\lambda_k x}-1\big)\to\Phi_+=\frac{1}{2i\pi}\sum_{k=K}^{\infty}\frac{a_k}{\lambda_k}\big(e^{2i\pi\lambda_k x}-1\big)$$

where the series defining Φ_+ is uniformly convergent over $[-T/2,T/2]$ and the convergence $\Phi_+^{(n)}\to\Phi_+$ is uniform over $[-T/2,T/2]$, thus also in $L^2([-T/2,T/2])$. But we also know that $\Phi_+^{(n)}\to-\Phi_-$ in $L^2([-T/2,T/2])$ thus $\Phi_++\Phi_-=0$.

It remains to apply Kahane's extension of Ingham's Inequality to obtain that

$$\begin{aligned}0=\frac{1}{T}\int_{-T/2}^{T/2}|\Phi_+(t)+\Phi_-(t)|^2\,\mathrm{d}t &= \frac{1}{T}\int_{-T/2}^{T/2}\left|-\frac{1}{2i\pi}\sum_{k=0}^{+\infty}\frac{a_k}{\lambda_k}+\sum_{k=0}^{+\infty}\frac{a_k}{2i\pi\lambda_k}e^{2i\pi\lambda_k t}\right|^2\mathrm{d}t\\ &\geq A_2(T,\Lambda)\left(\left|\frac{1}{2i\pi}\sum_{k=0}^{+\infty}\frac{a_k}{\lambda_k}\right|^2+\sum_{k=0}^{+\infty}\left|\frac{a_k}{2i\pi\lambda_k}\right|^2\right)\end{aligned}$$

thus $a_k=0$ for all k and we obtain the desired contradiction.

The proof of (1.4) is similar, so we give less detail. Elements of $\mathbb{C}^{2N+1}$ will be indexed from $-N$ to N, *i.e.* $z=(z_{-N},\dots,z_N)$ and will be considered as an element of $\mathbb{C}^{2M+1}$, $M\geq N$ and also as a sequence $(z_k)_{k\in\mathbb{Z}}$ by setting $z_k=0$ when $|k|>N$.

We again consider two norms on $\mathbb{C}^{2N+1}$, the ℓ^∞ norm and (with a small abuse of notation)

$$\|(a_{-N},\dots,a_N)\|_{L^1_K}=\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=-N}^{N}a_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t.$$

For every N there exists $\tilde\kappa_N$ such that, for every $a\in\mathbb{C}^{2N+1}$,

$$\tilde\kappa_N\|a\|_\infty\leq\|a\|_{L^1_K}.$$

Ingham's Theorem asserts that one may choose κ_N independently of N provided T is large enough. Our aim is again to show that this is possible for every T under our additional condition on λ_k. Assume towards a contradiction that this is not possible.

Then, for every integer $n\geq 1$, there exist an integer $K_n\to+\infty$ with $K_{n+1}>K_n\geq K$ and $a^{(n)}=(a^{(n)}_{-K_n},\dots,a^{(n)}_{K_n})\in\mathbb{C}^{2K_n+1}$ such that $\|a^{(n)}\|_\infty=1$ while $\|a^{(n)}\|_{L^1_{K_n}}\leq\dfrac{1}{n}$. So, without loss of generality, we will assume that $K_{n+1}>K_n\geq K$ for every n. Recall that we defined K so that if $|k|\geq K$, $\lambda_{k+1}-\lambda_k\geq\dfrac{2}{T}$.

We split $a^{(n)}$ into two vectors

$$a_-^{(n)}=(a^{(n)}_{-K+1},\dots,a^{(n)}_{K-1})\in\mathbb{C}^{2K-1}\quad\text{and}\quad a_+^{(n)}=a^{(n)}-a_-^{(n)}.$$

As $\|a_-^{(n)}\|_\infty\leq\|a^{(n)}\|_\infty=1$, there is no loss of generality in assuming that

$$a_-^{(n)}\to(a_{-K+1},\dots,a_{K-1}).$$

We again consider

$$\varphi^{(n)}(t)=\sum_{k=-K_n}^{K_n}a_k^{(n)}e^{2i\pi\lambda_k t}\to 0$$

in $L^1([-T/2,T/2])$,

$$\varphi_-^{(n)}(t)=\sum_{k=-K+1}^{K-1}a_k^{(n)}e^{2i\pi\lambda_k t}\to\varphi_-(t)=\sum_{k=-K+1}^{K-1}a_k e^{2i\pi\lambda_k t}.$$

in $L^1([-T/2,T/2])$ and

$$\varphi_+^{(n)}=\varphi^{(n)}-\varphi_-^{(n)}=\sum_{K\leq|k|\leq K_n}a_k^{(n)}e^{2i\pi\lambda_k t}\to-\varphi_-$$

in $L^1([-T/2,T/2])$.

Using (2.2) instead of (2.1) we get, for $n \geq m$

$$\frac{1}{T}\int_{-T/2}^{T/2} |\varphi_+^{(n)}(t) - \varphi_+^{(m)}(t)|\,\mathrm{d}t \geq A_1\left(T, \frac{2}{T}\right) \max_{K\leq|k|\leq K_n} \left|a_k^{(n)} - a_k^{(m)}\right|$$

so that $\left(a_k^{(n)}\right)_{k\in\mathbb{Z}}$ is a Cauchy sequence in ℓ^∞ and we call $a = (a_k)_{k\in\mathbb{Z}}$ its limit. Of course $\|a\|_\infty = 1$ so that we will again reach a contradiction if we show that $a_k = 0$ for all k.

To do so, we introduce again

$$\Phi_\pm^{(n)}(x) = \int_0^x \varphi_\pm^{(n)}(t)\,\mathrm{d}t \quad \text{and} \quad \Phi_-(x) = \int_0^x \varphi_-(t)\,\mathrm{d}t = \frac{1}{2i\pi}\sum_{k=-K+1}^{K-1} \frac{a_k}{\lambda_k}\left(e^{2i\pi\lambda_k x} - 1\right)$$

so that $\Phi_\pm^{(n)} \to \pm\Phi_-$ uniformly over $[-T/2,T/2]$ thus also in $L^2([-T/2,T/2])$.

Next, as $\lambda_k \neq 0$ and $\sum_{k\in\mathbb{Z}} \dfrac{1}{1+|\lambda_n|}$ converges so is $\sum_{k\in\mathbb{Z}} \dfrac{1}{|\lambda_n|}$. As $(a_k) \in \ell^\infty$ and $|a_k^{(n)} - a_k| \to 0$ in ℓ^∞ we get

$$\sum_{k\in\mathbb{Z}} \frac{|a_k|}{|\lambda_k|} < +\infty \quad \text{and} \quad \sum_{k\in\mathbb{Z}} \frac{|a_k^{(n)} - a_k|}{|\lambda_k|} \to 0.$$

As $|e^{2i\pi\lambda_k x} - 1| \leq 2$, it follows that

$$\Phi_+^{(n)} = \frac{1}{2i\pi}\sum_{K\leq|k|\leq K_n} \frac{a_k^{(n)}}{\lambda_k}\left(e^{2i\pi\lambda_k x} - 1\right) \to \Phi_+ = \frac{1}{2i\pi}\sum_{|k|\geq K} \frac{a_k}{\lambda_k}\left(e^{2i\pi\lambda_k x} - 1\right)$$

where the series defining Φ_+ is uniformly convergent over $[-T/2,T/2]$ and the convergence $\Phi_+^{(n)} \to \Phi_+$ is uniform over $[-T/2,T/2]$, thus also in $L^2([-T/2,T/2])$. But we also know that $\Phi_+^{(n)} \to -\Phi_-$ in $L^2([-T/2,T/2])$ thus $\Phi_+ + \Phi_- = 0$. Applying again Kahane's extension of Ingham's Inequality we obtain $a_k = 0$ for all k which is the desired contradiction.

3. 1-periodic Schrödinger equation

Recall that the Wiener algebra is defined as

$$A(\mathbb{T}) = \{f \in L^1(\mathbb{T}) \,:\, \|f\|_{A(\mathbb{T})} = \sum_{k\in\mathbb{Z}} |c_k(f)| < +\infty\}.$$

Theorem 3.1. *Let u be a weak solution of the Schrödinger equation*

$$\begin{cases} i\partial_t u(t,x) = \dfrac{1}{2\pi}\partial_x^2 u(t,x) & t\in\mathbb{R}, x\in\mathbb{T} \\ u_0 = u(0,x) & x\in\mathbb{R} \end{cases}, \tag{3.1}$$

with initial value $u_0 \in A(\mathbb{T})$. Let $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{T}$. Then

(1) For $a \in \mathbb{R}\setminus\mathbb{Z}$, for every $T > 0$ there exists a constant $C(a,T) > 0$ such that

$$\frac{1}{T}\int_0^T |u(t_0+t, x_0+at)|\,\mathrm{d}t \geqslant C\sum_{k\in\mathbb{Z}} \frac{|c_k(u_0)|}{1+|k|}. \tag{3.2}$$

(2) If $a \in \mathbb{Z}$, there exists $u_0 \neq 0$ such that $u(t_0+t, x_0+at) = 0$ for all t. In particular, (3.2) *fails.*

Remark 3.2. Recall also that if $u_0 \in H^s(\mathbb{T})$ with $s > \dfrac{1}{2}$ then, with Cauchy-Schwarz,

$$\|u_0\|_{A(\mathbb{T})} \leq \left(\sum_{k\in\mathbb{Z}} \frac{1}{(1+|k|^2)^s}\right)^{1/2}\left(\sum_{k\in\mathbb{Z}} (1+|k|^2)^s |c_k(u_0)|^2\right)^{1/2} < +\infty.$$

One may thus replace the condition $u_0 \in A(\mathbb{T})$ with a more familiar condition like $u_0 \in H^1(\mathbb{T})$.

Proof. Write $u_0(x) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi kx}$ so that $u(t,x) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi(k^2t+kx)}$. This series uniformly converges over $\mathbb{R}\times\mathbb{T}$ since $\sum|c_k|$ converges thus u is continuous. Further

$$v(t) = u(t_0+t, x_0+at) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi k^2(t_0+t)+2i\pi k(x_0+at)} = \sum_{k\in\mathbb{Z}} d_k e^{2i\pi\lambda_k t}$$

with

$$d_k = c_k e^{2i\pi(k^2t_0+kx_0)} \quad \text{and} \quad \lambda_k = k^2+ak.$$

Note that $|d_k| = |c_k|$. On the other hand

$$\begin{aligned}\lambda_k - \lambda_m &= k^2+2ak-(m^2+am) = k^2-m^2+a(k-m)\\ &= (k-m)(k+m+a). \end{aligned} \tag{3.3}$$

Assume first that $a\in\mathbb{Z}$. This case was already treated in [JK] but let us reproduce the proof here for completeness. In this case, the frequencies (λ_k) satisfy the symmetry property $\lambda_k = \lambda_{-a-k}$. Now fix $k\neq -a$ and notice that $-a-k\neq 0$ so that, if we fix $c_k\neq 0$ we can choose c_{-a-k} so that $d_{-a-k} = -d_k$ that is

$$c_{-a-k} = -c_k e^{2i\pi\left((k^2t_0+kx_0)-((-a-k)^2t_0+(-a-k)x_0)\right)} = -c_k e^{-2i\pi\left(a(a+2k)t_0+x_0)\right)}.$$

Setting

$$u_0(x) = c_k\left(e^{2i\pi kt} - e^{-2i\pi\left(a(a+2k)t_0+2ax_0)\right)}e^{-2i\pi(a+k)t}\right)$$

we obtain $u(t_0+t, x_0+at) = 0$.

From now on, we assume that $a\notin\mathbb{Z}$ so that from (3.3) we deduce that $\lambda_k\neq\lambda_m$ when $k\neq m$. It will be convenient to write $a = 2b$. We can then further split the sequence $(\lambda_k)_{k\in\mathbb{Z}}$ into a disjoint union, $(\lambda_k)_{k\in\mathbb{Z}} = (\lambda_k^+)_{k\geq 0}\cup(\lambda_k^-)_{k\geq 1}$ with

$$\lambda_k^+ := \lambda_{-[b]+k} = (-[b]+k)^2+2b(-[b]+k) \quad \text{for} \quad k\geqslant 0$$

and

$$\lambda_k^- := \lambda_{-[b]-k} = (-[b]-k)^2+2b(-[b]-k) \quad \text{for} \quad k\geqslant 1.$$

By definition

$$\lambda_0^+ = [b]^2-2b[b] \quad \text{and} \quad \lambda_1^- = \lambda_0^+ + 1 - 2(b-[b]).$$

We will now distinguish two cases:

First case: Assume that $\dfrac{1}{2} < b-[b] < 1$ so that $\lambda_1^- < \lambda_0^+$.

In this case, the frequencies interlace as follows:

$$\lambda_{k+1}^- < \lambda_k^+ < \lambda_{k+2}^- \quad \text{for all} \quad k\geqslant 0.$$

Indeed, for all $k\geqslant 0$

$$\lambda_k^+ - \lambda_{k+1}^- = 2(2k+1)\left(b-[b]-\frac{1}{2}\right) > 0 \quad \text{and} \quad \to\infty \quad \text{as} \quad k\to\infty$$

and

$$\lambda_{k+2}^- - \lambda_k^+ = 4(k+1)\big(1-(b-[b])\big) > 0 \quad \text{and} \quad \to\infty \quad \text{as} \quad k\to\infty$$

with our hypothesis on $b-[b]$. In particular, if we set $\mu_{2k} = \lambda_{k+1}^- = \lambda_{-[b]-k-1}$ and $\mu_{2k+1} = \lambda_k^+ = \lambda_{[-b]+k}$ for $k\geq 0$ then $0 < \mu_{2k+1}-\mu_{2k}\to+\infty$. Thus, from Theorem 1.5, we get that

$$\int_0^T |v(t)|\,\mathrm{d}t \geqslant C(T)\sum_{k=0}^{\infty}\left(\frac{|c_{-[a]-k-1}|}{2k+1}+\frac{|c_{-[a]+k}|}{2k+2}\right).$$

Finally, for $k\geq 0$, $2k+1\leq\alpha_a(|-[a]-k-1|+1)$ and $2k+2\leq\alpha_a(|-[a]+k|+1)$ with a constant α_a depending on a only, so that

$$\int_0^T |v(t)|\,\mathrm{d}t \geqslant \frac{C(T)}{\alpha_a}\sum_{k\in\mathbb{Z}}\frac{|c_k|}{|k|+1}$$

as claimed.

Second case: $0 < b - [b] < \dfrac{1}{2}$.

In this case, similar computations show that the frequencies interlace as

$$\lambda_k^+ < \lambda_{k+1}^- < \lambda_{k+1}^+ \quad \text{for all} \quad k \geqslant 0$$

with $\lambda_{k+1}^- - \lambda_k^+, \lambda_{k+1}^+ - \lambda_{k+1}^- \to +\infty$. The remaining of the proof is the same and is thus omitted.

Note that $b - [b] \neq 0, \dfrac{1}{2}, 1$ since $a = 2b \notin \mathbb{Z}$ so all cases are now covered. □

4. General Case

Let x in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $t \in \mathbb{R}^+$, we consider the following the equations

$$\begin{cases} i\partial_t u(t,x) = 2\pi P\left(\dfrac{\partial_x}{2i\pi}\right) u \\ u_0 = u(0,x) = \displaystyle\sum_{k\in\mathbb{Z}} c_k e^{2i\pi kx} \in \mathbb{A}(\mathbb{T}) \end{cases} . \tag{4.1}$$

where

$$P(X) = a_n X^n + a_{n-1}X^{n-1} + \ldots + a_1 X + a_0$$

with $n \geq 2$ and $a_n \neq 0$. There is no loss of generality in assuming that $a_n > 0$.

If $u_0(x) = \displaystyle\sum_{k\in\mathbb{Z}} c_k e^{2i\pi kx} \in A(\mathbb{T})$, then the solution to this system is given by

$$u(t,x) = \sum_{k\in\mathbb{Z}} c_k e^{-2i\pi P(k)t} e^{2i\pi kx}.$$

Again, this is a continuous function.

Let $a \in \mathbb{R}$ to be chosen later. For any $(t_0, x_0) \in \mathbb{R}\times\mathbb{T}$, we define

$$u(t_0+t, x_0+at) = \sum_{k\in\mathbb{Z}} c_k e^{-2i\pi P(k)(t_0+t)+2i\pi k(x_0+at)} = \sum_{k\in\mathbb{Z}} d_k e^{-2i\pi\lambda_k t}$$

with

$$d_k = c_k e^{-2i\pi(P(k)t_0 - kx_0)} \quad \text{and} \quad \lambda_k = P(k) - ak.$$

Note that $\lambda_k - \lambda_m = (k-m)\big(Q(k,m) - a\big)$ with

$$\begin{aligned} Q(k,m) &= a_n(k^{n-1} + k^{n-2}m + \ldots + m^{n-1}) + a_{n-1}(k^{n-2} + \ldots + m^{n-2}) + \ldots + a_1 \\ &= \sum_{\ell=1}^{n} a_\ell \sum_{j=0}^{\ell-1} k^{\ell-1-j} m^j. \end{aligned}$$

Define

$$E = \{Q(k,m),\ k,m \in \mathbb{Z} \text{ such that } k \neq m\}$$

which is countable (thus of measure 0).

Theorem 4.1. *Let u be any solution of the Schrödinger equation (4.1) with initial value $u_0 = \displaystyle\sum_{k\in\mathbb{Z}} c_k e^{2i\pi kx} \in A(\mathbb{T})$. Then*

(1) If $a \notin E$, for all $T > 0$ there exists a constant $C(a,T) > 0$ such that

$$\frac{1}{T}\int_0^T |u(t_0+t, x_0+at)|\,\mathrm{d}t \geqslant C \max_{k\in\mathbb{Z}} |c_k|. \tag{4.2}$$

If n is even, there also exists a constant $C(a,T) > 0$ such that

$$\frac{1}{T}\int_0^T |u(t_0+t, x_0+at)|\,\mathrm{d}t \geqslant C \sum_{k\in\mathbb{Z}} \frac{|c_k|}{1+|k|}. \tag{4.3}$$

(2) If $a \in E$ then both (4.2)-(4.3) fail.

An L^2 analogue of this result can be found in [WW].

Proof. The last part of the theorem is the same as for the Schrödinger equation in the previous section. Indeed, if $a \in E$, we can choose two indexes $k \neq m$ such that $\lambda_k = \lambda_m$ and then choose c_k, c_m such that $d_k = -d_m$. Taking $u_0 = c_k e^{2i\pi kt} + c_m e^{2i\pi mt}$, the corresponding solution u satisfies $u(t_0 + t, x_0 + at) = 0$.

We now assume that $a \notin E$ so that $\lambda_k \neq \lambda_m$ for all $k, m \in \mathbb{Z}$. We will further show that the $(\lambda_k)'s$ can be ordered as a sequence with gaps going to infinity. Here we need to distinguish between n even or odd. We start with the odd case.

If n is odd, then $\lambda_k = P(k) - ak \to \pm\infty$ when $k \to \pm\infty$. Note also that, as P has degree at least 3, $\sum \dfrac{1}{1+|\lambda_k|}$ converges.

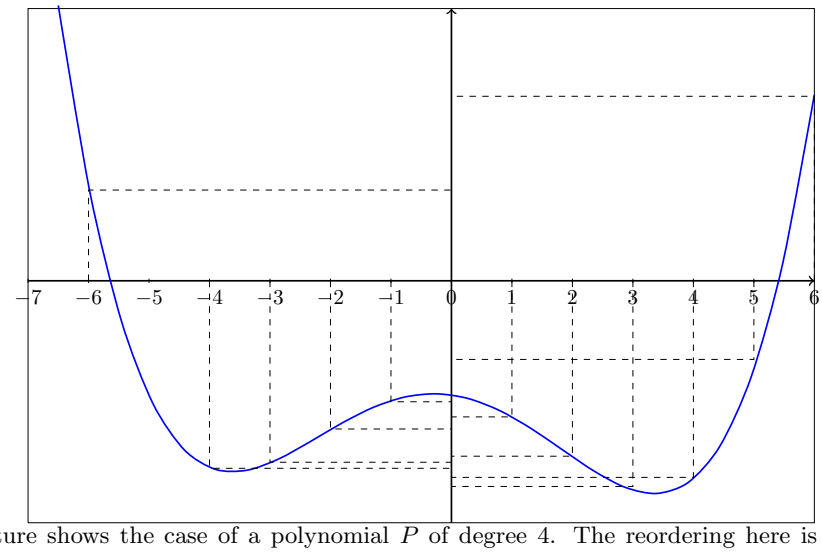
Further $\lambda_{k+1} - \lambda_k = Q(k+1, k) - a = a_n k^{n-1} + o(k^{n-1}) \to +\infty$ when $k \to \pm\infty$. so that, there exists K such that, for $k \geq K$, λ_k is increasing as well as for $k \leq -K$. There further exists $K' \geq K$ such that, if $k, \ell \geq K'$, then

$$\lambda_{-\ell} \leq \min_{|j|\leq K} \lambda_j \leq \max_{|j|\leq K} \lambda_j \leq \lambda_k.$$

We then define $(\mu_k)_{|k|\leq K'}$ as an ordering of $(\lambda_k)_{|k|\leq K'}$ and $\mu_k = \lambda_k$ for $|k| > K'$. Note that those λ_k's are not one of the $(\mu_k)_{|k|\leq K'}$'s. Then $(\mu_k)_{k\in\mathbb{Z}}$ is an increasing sequence with gaps $\mu_{k+1} - \mu_k \to +\infty$ when $k \to \pm\infty$. We can then apply (1.4) to conclude.

We now assume that $n = 2p$ is even. In this case $\lambda_k = P(k) - ak \to +\infty$ when $k \to \pm\infty$ and $\lambda_{k+1} - \lambda_k \to \pm\infty$ when $k \to \pm\infty$. In this case, the ordering needs to be made differently.

The idea is rather simple, there is an oscillating part and we are going to show that, for k, ℓ large, the λ_k's and $\lambda_{-\ell}$'s interlace. In the generic case we actually have $\lambda_{k+q_0} < \lambda_{-k} < \lambda_{k+q_0+1}$ for some fixed q_0 and large enough k. This shows that, for some K_0, $(\lambda_k)_{k\notin\{-K_0,\dots,K_0+q_0\}}$ can be rearranged in an increasing way as $\lambda_{K_0+q_0+1}, \lambda_{-K_0-1}, \lambda_{K_0+q_0+2}, \lambda_{-K_0-2}, \dots$. The finite number of remaining λ_k's are rearranged separately and, provided K_0 is large enough, they can be put at the start and the resulting sequence $(\mu_k)_{k\geq 0}$ is then increasing with gaps going to infinity. A key aspect of this construction is that each μ_k is a $\lambda_{k'}$ with $\left|k - |k'|\right| \leq C_\Lambda$ depending only on Λ. The idea is the same in the exceptional case.



The picture shows the case of a polynomial P of degree 4. The reordering here is $\mu_0 = \lambda_3$, $\mu_1 = \lambda_4$, $\mu_2 = \lambda_{-4}$, $\mu_3 = \lambda_{-3}$ $\mu_4 = \lambda_2$, $\mu_5 = \lambda{-}2$, $\mu_6 = \lambda_1$, $\mu_7 = \lambda_1$, $\mu_8 = \lambda_{-5}$ (not represented to keep the picture readable), $\mu_9 = \lambda_{-1}$, $\mu_{10} = \lambda_0$, $\mu_{11} = \lambda_5$, $\mu_{12} = \lambda_{-6}$, $\mu_{13} = \lambda_6$ and more generally $\mu_{12+2k} = \lambda_{-6-k}$ while $\mu_{13+2k} = \lambda_{6+k}$.

Let us now be more precise.

We again take K such that from $-\infty$ to K, λ_k is decreasing while from K to $+\infty$, λ_k is increasing and define K' such that if $k,\ell \geq K'$, then

$$\max_{|j|\leq K} \lambda_j \leq \lambda_{-\ell}, \lambda_k.$$

Next, an easy computation shows that

$$\sum_{j=0}^{\ell}(-1)^j = \begin{cases} 1 & \text{if } \ell \text{ is even} \\ 0 & \text{if } \ell \text{ is odd} \end{cases} \quad \text{and} \quad \sum_{j=0}^{\ell}(-1)^j j = \begin{cases} \ell/2 & \text{if } \ell \text{ is even} \\ -(\ell+1)/2 & \text{if } \ell \text{ is odd} \end{cases}$$

so that

$$\lambda_{k+q} - \lambda_{-k} = (2k+q)\left(\sum_{\ell=1}^{2p} a_\ell \sum_{j=0}^{\ell-1}(-1)^j (k+q)^{\ell-1-j}k^j - a\right).$$

But

$$\begin{aligned} a_{2p}\sum_{j=0}^{2p-1}(-1)^j(k+q)^{\ell-1-j}k^j &= a_{2p}k^{2p-1}\sum_{j=0}^{2p-1}(-1)^j + a_{2p}qk^{2p-2}\sum_{j=0}^{2p-2}(-1)^j j + o(k^{2p-2}) \\ &= (p-1)a_{2p}qk^{2p-2} + o(k^{2p-2}) \end{aligned}$$

and

$$a_{2p-1}\sum_{j=0}^{2p-2}(-1)^j(k+q)^{2p-2-j}k^j = a_{2p-1}k^{2p-2} + o(k^{2p-2})$$

so that

$$\lambda_{k+q} - \lambda_{-k} = \begin{cases} \big((p-1)a_{2p}q + a_{2p-1}\big)k^{2p-1} + o(k^{2p-1}) & \text{if } p \geq 2 \\ \big(a_2 q + a_1 - a\big)k + o(k) & \text{if } p = 1 \end{cases}. \tag{4.4}$$

Set $\alpha_q = \begin{cases} a_2 q + a_1 - a & \text{if } p = 1 \\ (p-1)a_{2p}q + a_{2p-1} & \text{if } p \geq 2 \end{cases}$ so that $\lambda_{k+q} - \lambda_{-k} = \alpha_q k^{2p-1} + o(k^{2p-1})$.

There are now two cases:

Case 2.1 *For every q, $\alpha_q \neq 0$*

Then there exists q_0 such that $\alpha_{q_0} > 0$ and $\alpha_{q_0-1} < 0$. But then, $\lambda_{k+q_0} - \lambda_{-k} \to +\infty$ while $\lambda_{k+q_0-1} - \lambda_{-k} \to -\infty$.

We now take $K'' > \max(K' - q_0, K')$ such that, for $k \geq K''$, $\lambda_{k+q_0} - \lambda_{-k} > 0$ and $\lambda_{k+q_0-1} - \lambda_{-k} < 0$, that is $\lambda_{k+q_0-1} < \lambda_{-k} < \lambda_{k+q_0}$. The choice of K'' also implies that $\lambda_{-K''+1}, \ldots, \lambda_{K''+q_0-1}$ are all $< \min(\lambda_{-K''}, \lambda_{K''+q_0})$. We can thus reorder those terms as an increasing sequence $(\mu_k)_{k=0,\cdots,\hat{K}}$ with $\hat{K} = 2K'' + q_0 - 2$, that we then complete into a sequence $(\mu_k)_{k\in\mathbb{N}}$ by adding successively a term $\lambda_{K''+k+q_0}$ and a term $\lambda_{-K''-k}$ and the resulting sequence is an increasing rearrangement of (λ_k) such that $\mu_k \to +\infty$ and $\mu_{k+1} - \mu_k \to +\infty$. Note that if we define σ the mapping $\mathbb{Z} \to \mathbb{N}$ defined by $\mu_k = \lambda_{\sigma(k)}$ then there is a constant C_Λ such that $\big||k| - \sigma(k)\big| \leq C_\Lambda$.

It follows from (1.3) that

$$\begin{aligned} \frac{1}{T}\int_{-T/2}^{T/2} |u(t_0+t, x_0+at)|\,\mathrm{d}t &= \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}} d_k e^{-2i\pi\lambda_k t}\right|\mathrm{d}t \\ &= \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{+\infty} d_{\sigma^{-1}(k)} e^{-2i\pi\mu_k t}\right|\mathrm{d}t \\ &\geq \tilde{A}_1(T,\Lambda)\sum_{k=0}^{+\infty}\frac{|d_{\sigma^{-1}(k)}|}{1+k} = \tilde{A}_1(T,\Lambda)\sum_{j\in\mathbb{Z}}\frac{|c_j|}{1+\sigma(j)} \\ &\geq \frac{\tilde{A}_1(T,\Lambda)}{1+C_\Lambda}\sum_{k\in\mathbb{Z}}\frac{|c_k|}{1+|k|}. \end{aligned}$$

Note that the series $\sum_{k\in\mathbb{Z}} d_k e^{-2i\pi\lambda_k t}$ is uniformly convergent so that we can re-order terms.

Case 2.2 *There exists q_0, such that $\alpha_{q_0} = 0$.*

The proof is essentially the same, but the interlacing of the λ_k and $\lambda_{-\ell}$ for k, ℓ large may be different. This comes from the fact that the leading term in (4.4) is now 0. Nevertheless, $\alpha_{q_0+1} > 0$ and $\alpha_{q_0-1} < 0$ so that, for k large enough $\lambda_{k+q_0+1} - \lambda_{-k} > 0$ while $\lambda_{k+q_0-1} - \lambda_{-k} < 0$. So, for each k, either $\lambda_{k+q_0-1} < \lambda_{k+q_0} < \lambda_{-k} < \lambda_{k+q_0+1}$ or $\lambda_{k+q_0-1} < \lambda_{-k} < \lambda_{k+q_0} < \lambda_{k+q_0+1}$ (actually only one can occur for k large enough) and we define the rearrangement μ_k accordingly. □

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