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# FROM CURVE GRAPHS TO FINE CURVE GRAPHS AND BACK

FEDERICA FANONI AND SEBASTIAN HENSEL

ABSTRACT. We first show that not all boundary points of the fine curve graph of a closed surface are seen via finite approximations, by which we mean via curve graphs of the surface punctured at finitely many points. We then use fine curve graph tools to prove that there exist parabolic isometries of graphs of curves associated to surfaces of infinite type.

## 1. INTRODUCTION

The fine curve graph was introduced by Bowden, the second author and Webb in [BHW22] as a tool to study homeomorphism and diffeomorphism groups of closed orientable surfaces of positive genus. The definition of the graph is heavily inspired by that of the curve graph. First introduced by Harvey in [Har81], the curve graph has been very useful to study mapping class groups of surfaces. In [BHW22], the authors used the fine curve graph to show that the identity component of the group of diffeomorphisms of a closed surface of positive genus admits an infinite-dimensional space of unbounded quasi-morphisms, answering a question of Burago, Ivanov and Polterovich [BIP08]. Since then, multiple authors have related dynamical properties of homeomorphisms to the dynamics of their action on the graph (see [BHM<sup>+</sup>22, GM23b, GM23a]). The boundary of the fine curve graph has also been investigated (see [BHW24] and [LT24]), and this study has for example led to prove a Tits-like alternative for certain subgroups of  $\text{Homeo}^+(S)$  (see [BHW24]).

One of the main tools in [BHW22] is a result ([BHW22, Lemma 3.4]) which establishes that the distance between two curves in the fine curve graph can be computed as a distance in the (surviving) curve graph of the surface with an appropriately chosen finite set of points removed. With this in mind, we can think of the fine curve graph as being approximated by usual curve graphs.

The first objective of this note is to explore if points in the boundary of the fine curve graph are *visible* in finite approximations. As the fine curve graph and its nonseparating version are quasi-isometric (see [BHW22, Corollary 3.9]), restricting to nonseparating curves has no impact on the study of the boundary. So, for convenience, from now on we will restrict to nonseparating (fine) curve graphs.

We define visibility of boundary points as follows:

**Definition 1.1.** A point  $\xi \in \partial_\infty \mathcal{NC}^\dagger(S)$  is *visible in*  $\mathcal{NC}(S \setminus P)$ , for some finite set  $P \subset S$ , if for any quasi-geodesic ray  $\{\alpha_i \mid i \in \mathbb{N}\} \subset \mathcal{NC}^\dagger(S)$  converging to  $\xi$ , the projection of the quasi-geodesic to  $\mathcal{NC}(S \setminus P)$  is unbounded.

Note that in the definition we consider nonseparating (fine) curve graphs.

The main upshot of our work is that not all boundary points of the fine curve graph are visible in some finite approximation:

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*Date:* September 9, 2024.

**Theorem A.** *Let  $S$  be a closed orientable surface of genus at least two. There are  $\xi \in \partial_\infty \mathcal{NC}^\dagger(S)$  and a quasi-geodesic  $\{\alpha_i \mid i \in \mathbb{N}\} \subset \mathcal{NC}^\dagger(S)$  converging to  $\xi$  such that the projection of the quasi-geodesic to  $\mathcal{NC}(S \setminus P)$ , for every finite  $P$ , is finite. In particular,  $\xi$  is not visible in  $\mathcal{NC}(S \setminus P)$  for any finite set  $P \subset S$ .*

Note that by fixing a hyperbolic structure on  $S \setminus P$  and choosing geodesic representatives of the classes of curves, we get a quasi-isometric embedding  $\mathcal{NC}(S \setminus P) \hookrightarrow \mathcal{NC}^\dagger(S)$ , inducing an embedding  $\partial_\infty \mathcal{NC}(S \setminus P) \hookrightarrow \partial_\infty \mathcal{NC}^\dagger(S)$ . So we can interpret Theorem A as saying that boundaries of fine curve graphs are “more complicated than” boundaries of curve graphs of finite-type surfaces.

The second objective is to construct a parabolic isometry of the nonseparating curve graph of an infinite-type surface of genus one:

**Theorem B.** *Let  $T$  be a torus and let  $S$  be  $T$  with a Cantor set  $C$  and a point  $p_0$  removed. Then there is a homeomorphism  $F$  of  $T$ , fixing  $C$  and  $p_0$ , such that  $F$  is a parabolic isometry of  $\mathcal{NC}^\dagger(T)$  and  $[F]$  is a parabolic isometry of  $\mathcal{NC}(S)$ .*

To the best of our knowledge, there are no other known examples of parabolic isometries acting on any graph associated to an infinite-type surface (see also [aim19, Problem 2.55]). To show the existence of such a map, we use techniques coming from the study of homeomorphism of surfaces and fine curve graphs.

**1.1. Defining visibility.** Let us discuss visibility of boundary points more in detail, to explain in particular why we consider Definition 1.1. At first, it might seem more natural to define visibility as follows:

A point  $\xi \in \partial_\infty \mathcal{NC}^\dagger(S)$  is *visible in  $\mathcal{NC}(S \setminus P)$* , where  $P \subset S$  is finite, if there is a sequence of curves  $\{\alpha_i \mid i \in \mathbb{N}\} \subset \mathcal{NC}^\dagger(S)$  converging to  $\xi$  and such that  $\{[\alpha_i] \in \mathcal{NC}(S \setminus P) \mid i \in \mathbb{N}\}$  is not a bounded set.

We can prove (see Lemma 3.1) that according to this definition, any boundary point is visible in  $\mathcal{NC}(S)$  (i.e. we can always choose  $P = \emptyset$ ). On the other hand, the lemma shows that the sequence of curves doesn’t contain any geometric information about the boundary point. In particular, such a notion doesn’t seem to be useful in practice to further our understanding of  $\partial_\infty \mathcal{NC}^\dagger(S)$ . To try and fix this issue, we could strengthen the definition and consider *all* sequences converging to the boundary point:

A point  $\xi \in \mathcal{NC}^\dagger(S)$  is *visible in  $\mathcal{NC}(S \setminus P)$* , for some finite set  $P \subset S$ , if, for any sequence of curves  $\{\alpha_i \mid i \in \mathbb{N}\} \subset \mathcal{NC}^\dagger(S)$  converging to  $\xi$ , the projection of  $\{\alpha_i \mid i \in \mathbb{N}\}$  to  $\mathcal{NC}(S \setminus P)$  is unbounded.

Lemma 3.2 shows that boundary points corresponding to the attracting fixed point of a pseudo-Anosov homeomorphism  $f$  (see [BHW24, Proposition 4.10]) are not visible in any finite approximation, if we use this definition of visibility. On the other hand, we would want them to be, as we can “see” such points at infinity by just looking at the mapping class  $[f]$  and any orbit of a homotopy class of a curve in  $\mathcal{NC}(S)$ . Note that the sequence of curves in  $\mathcal{NC}^\dagger(S)$  constructed in the proof of Lemma 3.2 is not a quasi-geodesic.

Taking this into account leads us to consider Definition 1.1. With respect to this definition, boundary points corresponding to attracting fixed points of a pseudo-Anosov homeomorphism are visible in some finite approximation.

As Theorem A shows that not every boundary point is visible in some finite approximation, a natural question is whether every boundary point is visible in some curve graph approximation, if we allow the set of punctures to be infinite. Concretely, we ask:

**Question 1.2.** Let  $\xi \in \partial_\infty \mathcal{NC}^\dagger(S)$ ; is there a Cantor set  $C \subset S$  such that for any quasi-geodesic ray  $\{\alpha_i \mid i \in \mathbb{N}\} \subset \mathcal{NC}^\dagger(S)$  converging to  $\xi$ , the projection of the quasi-geodesic to  $\mathcal{NC}(S \setminus C)$  is unbounded?

We suspect the answer to this question to be positive, but the problem is more subtle than one might think at first. We refer to Section 3 for more details about this question.

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#### 2. PRELIMINARIES AND NOTATION

Unless otherwise stated, by *surface* we mean a connected orientable two-manifold without boundary. We denote by  $S_g$  the closed connected orientable surface of genus  $g$ . A *curve* on a surface will be assumed to be simple, closed and non-contractible. A curve is *nonseparating* if it does not disconnect the surface.

The *nonseparating fine curve graph*  $\mathcal{NC}^\dagger(S)$  of a surface of positive genus  $S$  has nonseparating curves as vertices and edges correspond to disjointness. The *nonseparating curve graph*  $\mathcal{NC}(S)$  has homotopy classes of nonseparating curves as vertices and two classes are adjacent if they admit disjoint representatives. **If  $S$  is a torus, edges instead correspond to curves intersecting at most once.** We will denote by  $d^\dagger$  (respectively,  $d$ ) the distance in  $\mathcal{NC}^\dagger(S)$  (respectively,  $\mathcal{NC}(S)$ ).

Given a surface  $S$  and a finite collection of points  $P$  on  $S$ , there is a 1-Lipschitz map

$$\pi_P : \mathcal{NC}^\dagger(S) \rightarrow \mathcal{NC}(S \setminus P)$$

(see [BHW22]), which to a curve  $\alpha$  associates the homotopy class of  $\alpha$ , or of a small deformation of it if  $\alpha \cap P \neq \emptyset$ .

All the graphs mentioned above are Gromov hyperbolic (by work of Rasmussen [Ras20] and Bowden, Hensel and Webb [BHW22]), which allows us to talk about their boundary and to classify their isometries. We recall here the definitions and results related to Gromov hyperbolic spaces that we will need in the article. For more details, we refer to [BH99].

**2.1. Gromov hyperbolic spaces.** Let  $X$  be a metric space. We say that it is  $\delta$ -*hyperbolic* if any side of any geodesic triangle is contained in the  $\delta$ -neighborhood of the union of the other two sides. We say that  $X$  is *Gromov hyperbolic* if it is  $\delta$ -hyperbolic for some  $\delta \geq 0$ . For the remainder of this section, we assume that  $X$  is a Gromov hyperbolic space.

Pick a basepoint  $x_0 \in X$ . We define the *Gromov product* of  $x, y \in X$  to be

$$(x, y)_{x_0} := \frac{1}{2} d(x_0, x) + d(x_0, y) - d(x, y).$$

A sequence of points  $\{y_i \mid i \in \mathbb{N}\}$  is *admissible* if

$$(y_i, y_j)_{x_0} \rightarrow \infty \quad \text{as } i, j \rightarrow \infty$$

and two admissible sequences  $\{y_i \mid i \in \mathbb{N}\}$  and  $\{z_i \mid i \in \mathbb{N}\}$  are equivalent if, as  $i, j \rightarrow \infty$ ,

$$(y_i, z_j) \rightarrow \infty.$$

The *Gromov boundary*  $\partial_\infty X$  of  $X$  is the set of equivalence classes of admissible sequences, and the Gromov product can be extended to pairs of boundary points by setting **this needs to be a sup, that's a typo in Bridson-Haefliger as well...**

$$(\xi, \eta)_{x_0} := \inf \left\{ \liminf_{i,j \rightarrow \infty} (y_i, z_j)_{x_0} \mid \{y_i \mid i\} \in \xi, \{z_i \mid i\} \in \eta \right\}.$$

Moreover  $X \cup \partial_\infty X$  has a topology generated by the open sets of  $X$  together with, for every boundary point  $\xi$ , the collection  $\{U(\xi, K) \mid K \geq 0\}$ , where

$$U(\xi, K) := \{\eta \in X \cup \partial_\infty X \mid (\xi, \eta)_{x_0} > K\}.$$

As the notation suggests,  $\partial_\infty X$  and the topology on  $X \cup \partial_\infty X$  do not depend on the choice of basepoint.

For any isometry  $f$  of  $X$  and any point  $x \in X$  the limit

$$\lim_{n \rightarrow \infty} \frac{d(x, f^n(x))}{n}$$

exists and is independent of  $x$ . We call this limit the *asymptotic translation length* of  $f$ .

An isometry  $f$  of  $X$  is of one of three types:

- *elliptic*, if it has bounded orbits;
- *parabolic*, if it has no bounded orbits, but its asymptotic translation length is zero;
- *hyperbolic*, if its asymptotic translation length is positive.

**2.2. Rotation sets.** Let  $F$  be a homotopically trivial homeomorphism of the torus  $T \simeq \mathbb{R}^2 / \mathbb{Z}^2$ . Choose a lift  $\tilde{F}$  of  $F$  to the universal cover  $\mathbb{R}^2$ . The *rotation set*  $\rho(\tilde{F})$  of  $\tilde{F}$  is the set of all limits of converging sequences of the form

$$\frac{\tilde{F}^{n_k}(x_k) - x_k}{n_k},$$

where  $\{x_k\}_k$  is a sequence of points in the plane and  $\{n_k\}_k$  is a sequence of integers going to infinity. By a result of Misiurewicz and Ziemian [MZ89],  $\rho(\tilde{F})$  coincides with the convex hull of the *pointwise rotation set* of  $\tilde{F}$ , which is the collection of all limit points of converging subsequences of the form

$$\frac{\tilde{F}^{n_k}(x) - x}{n_k},$$

where  $\{n_k\}_k$  is a sequence of integers going to infinity.

If we choose another lift  $\tilde{F}'$  of  $F$ , the rotation sets of  $\tilde{F}$  and  $\tilde{F}'$  differ only by an integer translation. We can therefore define the *rotation set*  $\rho(F)$  of  $F$  to be  $\rho(\tilde{F})$  modulo  $\mathbb{Z}^2$ , for any lift  $\tilde{F}$  of  $F$ . In particular, properties such as having empty interior or being one-dimensional are well-defined for rotation sets of homotopically trivial homeomorphisms of  $T$ .

### 3. BOUNDARY POINTS AND APPROXIMATIONS

The goal of this section is to look at boundary points of the fine nonseparating curve graph of surfaces  $S$ , and to understand whether they are visible in some finite approximation, namely in  $\mathcal{NC}(S \setminus P)$ , for some finite collection of points  $P$ .

As mentioned in the introduction, one might think of different ways of making the concept of *being visible in some approximation* precise. The following lemma shows that if we simply ask for the existence of a sequence of curves converging to a boundary point, whose projection in some nonseparating curve graph is unbounded, every boundary point

$\xi \in \partial_\infty \mathcal{NC}^\dagger(S)$  is visible in  $\mathcal{NC}(S)$ , but the curves in  $\mathcal{NC}(S)$  might converge to any boundary point  $\zeta \in \partial_\infty \mathcal{NC}(S)$ :

**Lemma 3.1.** *Let  $S = S_g$ , for  $g \geq 2$ , and  $\xi \in \partial_\infty \mathcal{NC}^\dagger(S)$ . Then for every  $\zeta \in \partial_\infty \mathcal{NC}(S)$ , there is a sequence of curves  $\alpha_i \in \mathcal{NC}^\dagger(S)$  converging to  $\xi$  and such that  $[\alpha_i] \in \mathcal{NC}(S)$  converges to  $\zeta$ .*

*Proof.* Fix a hyperbolic structure on  $S$  and  $\zeta \in \partial_\infty \mathcal{NC}(S)$ . Then  $\zeta$  corresponds to a geodesic lamination  $\lambda$  (see [Kla22] and [Ham14]). Fix a sequence of simple closed geodesics  $\gamma_i$  which converge, in the Hausdorff distance, to  $\lambda$ . Denote by  $\xi_\lambda$  the point in  $\partial_\infty \mathcal{NC}^\dagger(S)$  towards which the  $\gamma_i$  converge. As  $\text{Homeo}_0(S)$  acts minimally on  $\partial_\infty \mathcal{NC}^\dagger(S)$  (see [BHW24]), we can find a sequence  $f_i \in \text{Homeo}_0(S)$  such that  $f_i(\xi_\lambda) \rightarrow \xi$ . Therefore there is a subsequence  $n_i$  so that  $\alpha_i = f_i(\gamma_{n_i}) \rightarrow \xi$ . Since  $f_i \in \text{Homeo}_0(S)$ ,  $[\alpha_i] = [\gamma_{n_i}] \in \mathcal{NC}(S)$ , and by construction the sequence of  $[\gamma_i]$  is unbounded in  $\mathcal{NC}(S)$ .  $\square$

If we strengthen the requirement for visibility to asking that *any* sequence converging to a boundary point has unbounded image in the finite approximation, we end up having “clearly visible” boundary points which are not visible according to our definition:

**Lemma 3.2.** *Let  $\lambda$  the attracting fixed point of a pseudo-Anosov homeomorphism  $f$  of a surface  $S = S_g$ , for  $g \geq 2$ , and let  $\xi_\lambda$  be the associated point  $\partial_\infty \mathcal{NC}^\dagger(S)$ . Then there is a sequence of curves  $\{\alpha_i \mid i \in \mathbb{N}\} \subset \mathcal{NC}^\dagger(S)$  converging to  $\xi_\lambda$  such that the projection to  $\mathcal{NC}(S \setminus P)$ , for any finite set of points  $P$ , is finite.*

*Proof.* We consider a homeomorphism  $F$  given by the first construction in the proof of [BHM<sup>+</sup>22, Theorem 1.1], which pushes into a cusp of a complementary triangle of the lamination. Pick a curve  $\alpha_0$  intersecting  $\lambda$  transversely. As shown in [BHM<sup>+</sup>22], the sequence  $\alpha_i = f^i(\alpha_0)$  converges to  $\xi_\lambda$ . On the other hand, for any finite set of points  $P$ , the sequence  $[\alpha_i] \in \mathcal{NC}(S \setminus P)$  is eventually constant.  $\square$

We therefore consider Definition 1.1, where we restrict to looking at quasi-geodesics converging to boundary points. As mentioned, we can show that not every boundary point is visible in some finite approximation in this sense either:

**Theorem 3.3** (Theorem A). *Let  $S = S_g$ , for  $g \geq 2$ . There are  $\xi \in \partial_\infty \mathcal{NC}^\dagger(S)$  and a quasi-geodesic  $\{\alpha_i \mid i \in \mathbb{N}\} \subset \mathcal{NC}^\dagger(S)$  converging to  $\xi$  such that the projection of the quasi-geodesic to  $\mathcal{NC}(S \setminus P)$ , for every finite  $P$ , is finite.*

In the proof we will use the following result, which is a simplified version of [Min05, Lemma 4.2]:

**Lemma 3.4.** *Let  $X$  be a  $\delta$ -hyperbolic (geodesic) space. Then for every  $C \geq 14\delta$  there are constants  $L = L(C, \delta)$  and  $k = k(C, \delta)$  so that the following holds: if  $x_0, x_1, \dots, x_k$  is a sequence of points so that for every  $i$*

- $d(x_i, x_{i+1}) \geq L$ , and
- $(x_{i-1}, x_{i+1})_{x_i} \leq C$

*then for every choice of geodesics  $g_i$  from  $x_{i-1}$  to  $x_i$ , their concatenation is a  $k$ -quasi-geodesic.*

We will also need:

**Lemma 3.5.** *Let  $\mathcal{F}$  be a foliation of  $S$ ,  $\alpha$  a curve in  $\mathcal{NC}^\dagger(S)$  and  $\beta$  a curve on a  $K$ -quasi-geodesic from  $\alpha$  to the point at infinity corresponding to  $\mathcal{F}$ . Let  $\delta$  be the hyperbolicity constant of  $\mathcal{NC}^\dagger(S)$ . Then there is a constant  $R = R(\delta, K)$  such that the following holds. Let  $\ell_i$  be a sequence of leaf segments of  $\mathcal{F}$  of unbounded length. If  $\gamma_i$  is the push along  $\ell_i$  of  $\alpha$ , we have*

$$(\alpha, \gamma_i)_\beta \leq R$$

for every  $i$  sufficiently large.

*Proof.* By [BHW24],  $\gamma_i \rightarrow \xi$ , where  $\xi$  is the point at infinity corresponding to  $\mathcal{F}$ .

Let  $\{\beta_n \mid n \in \mathbb{N}\}$  be the  $K$ -quasi-geodesic in the statement, with  $\alpha = \beta_0$  and  $\beta = \beta_N$ . For every  $n \geq N$ , since quasi-geodesics fellow-travel in Gromov hyperbolic spaces (see [BH99, Theorem 1.7]), there are a constant  $R' = R'(K, \delta)$  and a curve  $\eta_n$  on a geodesic between  $\alpha$  and  $\beta_n$  such that  $d^\dagger(\beta, \eta_n) \leq R'$ . Therefore we have

$$\begin{aligned} (\alpha, \beta_n)_\beta &= \frac{1}{2}(d^\dagger(\alpha, \beta) + d^\dagger(\beta_n, \beta) - d^\dagger(\alpha, \beta_n)) = \\ &\leq \frac{1}{2}(d^\dagger(\alpha, \eta_n) + d^\dagger(\eta_n, \beta_n) - d^\dagger(\alpha, \beta_n) + 2R') = R'. \end{aligned}$$

Since  $\beta_n \rightarrow \xi$ , and by properties of the Gromov product (see e.g. [Väi05, Section 5]) we deduce that

$$(\alpha, \xi)_\beta \leq R' + 2\delta.$$

Moreover,  $\gamma_i \rightarrow \xi$  implies that  $(\alpha, \gamma_i)_\beta$  is eventually less than  $(\alpha, \xi)_\beta + 2\delta$ , so if  $i$  is large enough,

$$(\alpha, \gamma_i)_\beta \leq R' + 4\delta.$$

□

*Proof of Theorem A.* Let  $\delta$  be the hyperbolicity constant of  $\mathcal{NC}^\dagger(S)$ . Fix two distinct foliations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  sharing a foliated box  $B$ , that is:  $B \simeq [0, 1] \times [0, 1]$  and the restrictions of  $\mathcal{F}_1$  and  $\mathcal{F}_2$  to  $B$  coincide with the horizontal foliation of  $[0, 1] \times [0, 1]$ . Choose also a curve  $\gamma \in \mathcal{NC}^\dagger(S)$  disjoint from  $B$ . Let  $\xi_i \in \partial_\infty \mathcal{NC}^\dagger(S)$  be the point corresponding to  $\mathcal{F}_i$ , for  $i = 1, 2$ . Fix two  $K$ -quasi-geodesics  $r_1$  and  $r_2$  from  $\gamma$  to  $\xi_1$  and  $\xi_2$ , respectively.

By [AM21, Lemma 2.10], there is a constant  $c = c(\delta, K)$  such that for every  $\alpha_i$  on a  $K$ -quasi-geodesic from  $\gamma$  to  $\xi_i$  we have

$$(\alpha_1, \alpha_2)_\gamma \leq (\xi_1, \xi_2)_\gamma + c.$$

Let  $R = R(K, \delta)$  be given by Lemma 3.5 and fix  $C \geq \max\{15\delta, (\xi_1, \xi_2)_\gamma + c + \delta, R + \delta\}$  and let  $L$  and  $k$  be given by Lemma 3.4 for the space  $\mathcal{NC}^\dagger(S)$  and the constant  $C$ .

Choose moreover, using Lemma 3.5:

- curves  $\gamma_i \in \mathcal{NC}^\dagger(S)$  on  $r_i$  and at distance at least  $L$  from  $\gamma$ ;
- countably many horizontal segments  $\sigma_j \subset B$  with pairwise disjoint regular neighborhoods  $N(\sigma) \subset B$ ;
- sufficiently long leaf segments  $\ell_i$  of  $\mathcal{F}_i$ , with regular neighborhoods  $N(\ell_i) \subset S \setminus B$ , so that the curves  $\eta_i$  obtained by pushing  $\gamma$  along  $\ell_i$  satisfy:
  - $d^\dagger(\gamma_i, \eta_i) \geq L$ , and
  - $(\eta_i, \gamma)_{\gamma_i} \leq C - \delta$ ;



Note that

$$(\gamma_1, \gamma_2)_\gamma \leq (\xi_1, \xi_2)_\gamma + c \leq C - \delta,$$

where the first inequality follows from [AM21] and the second from our choice of  $C$ .

For  $j = 1, 2$ , let  $L_j = d^\dagger(\gamma, \gamma_j) + d^\dagger(\gamma_j, \eta_j)$ . Let  $m = \min\{L_1, L_2\}$  and  $M = \max\{L_1, L_2\}$ .

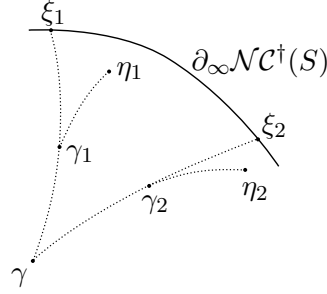


FIGURE 1. The curves  $\gamma, \gamma_i$  and  $\eta_i$  in the proof of Proposition A

We now iteratively define:

- two sequences of curves  $\beta_0, \beta_1, \dots$  and  $\alpha_0, \alpha_1, \dots$ ,
- collection of arcs  $\Omega_1 \subset \Omega_2, \dots$ , with pairwise disjoint regular neighborhoods (denote by  $N(\Omega_i)$  the union of the regular neighborhoods of the arcs in  $\Omega_i$ ),

such that

- (1) for every  $i \geq 1$ ,  $d^\dagger(\beta_{i-1}, \alpha_i) \geq L$ ,
- (2) for every  $i \geq 1$ ,  $(\alpha_{i-1}, \alpha_i)_{\beta_i} \leq C$  and  $(\beta_i, \beta_{i+1})_{\alpha_i} \leq C$ , and
- (3) for every  $i \geq 1$ ,  $\alpha_i$  is the push of  $\alpha_0$  along  $\Omega_i$ .

Set  $\alpha_0 = \beta_0 = \gamma$ ,  $\beta_1 = \gamma_1$ ,  $\alpha_1 = \eta_1$  and  $\Omega_1 = \ell_1$ .

Suppose we have constructed curves and collections of arcs for every  $i \leq n$ . Note that there is a homeomorphism  $F$  of the surface such that:

- $F(\alpha_n) = \gamma$ ,
- $F(\Omega_n) \subset \bigcup_j (\sigma_j)$  and  $F(N(\Omega_n)) \subset \bigcup_j (N(\sigma_j))$ .

By  $\delta$ -hyperbolicity,

$$\min\{(\gamma_1, F(\beta_n))_\gamma, (\gamma_2, F(\beta_n))_\gamma\} \leq (\gamma_1, \gamma_2)_\gamma + \delta \leq C$$

so we can choose  $j$  so that  $(\gamma_j, F(\beta_n))_\gamma \leq C$ . Then we set:

- $\beta_{n+1} = F^{-1}(\gamma_j)$  and  $\alpha_{n+1} = F^{-1}(\eta_j)$ ,
- $\Omega_{n+1} = \Omega_n \cup F^{-1}(\ell_j)$  and  $N(\Omega_{n+1}) = N(\Omega_n) \cup F^{-1}(N(\ell_j))$ .

One can check that the sequence we construct satisfies the three conditions (1), (2) and (3). In particular, for any choice of geodesic segments  $a_i$  between  $\alpha_{i-1}$  and  $\beta_i$  and  $b_i$  between  $\beta_i$  and  $\alpha_i$ , the concatenation

$$\dots b_n * a_n \dots b_2 * a_2 * b_1 * a_1$$

is a  $k$ -quasi-geodesic by Lemma 3.4 and it has a well-defined point at infinity, which we denote by  $\xi$ . Note moreover that for every  $i$ , the length of the quasi-geodesic between  $\alpha_i$  and  $\alpha_{i+1}$  is either  $m$  or  $M$ . Therefore, the sequence of the  $\alpha_i$ 's is a quasi-geodesic, converging to  $\xi$ . Given any finite collection of points  $P$ , they can be contained in only



finitely many neighborhoods of the arcs in the  $\Omega_n$ . In particular, there is some  $N \geq 1$  so that, in  $\mathcal{NC}(S \setminus P)$ ,

$$[\alpha_n] = [\alpha_N] \quad \forall n \geq N,$$

since pushing along an arc which doesn't contain a point of  $P$  doesn't change the homotopy class of a curve in  $S \setminus P$ .

□

Since boundary points are not always visible via finite approximations, the next question (see Question 1.2) is whether every boundary point is visible via some possibly infinite-type approximation, i.e. via  $\mathcal{NC}(S \setminus E)$ , where  $E$  is an endset. By *endset* we mean a totally disconnected second-countable compact subset<sup>1</sup> of  $S$ , so that  $S \setminus E$  is a possibly infinite-type surface whose space of ends is homeomorphic to  $E$ . Note that the question makes sense because, as shown by Aramayona and Valdez [AV18] and Rasmussen [Ras20], the nonseparating curve graph of an infinite-type surface of finite positive genus is connected, has infinite diameter and is Gromov hyperbolic.

Answering this question seems to be harder than expected. To explain why, let us discuss a couple of naive strategies to construct such an endset and see why they fail.

Suppose we have a quasi-geodesic  $\{\alpha_i \mid i \in \mathbb{N}\} \subset \mathcal{NC}^\dagger(S)$  converging to a boundary point. Let us assume that the curves are transverse to each other. What we would like to do is to show that there is some endset  $E$  so that (up to possibly passing to subsequences) the distance in  $\mathcal{NC}(S \setminus E)$  between  $\pi_E(\alpha_1)$  and  $\pi_E(\alpha_i)$  goes to infinity.

The first idea to construct such an endset is to pick a point in each bigon that  $\alpha_i$  forms with  $\alpha_1$ , for every  $i$ , and to define  $E$  to be the closure of this set of points. Then the  $\alpha_i$  would be in minimal position with respect to  $\alpha_1$  on  $S \setminus E$  and we would be able to conclude. Bigons, on the other hand, can accumulate badly in the surface: if we just follow this procedure,  $E$  could for instance contain nontrivial segments.

Of course, this first idea doesn't use at all the fact that the sequence is a quasi-geodesic. By using the fact that the  $\alpha_i$  form a quasi-geodesic, one can show that it is enough to fix an annular neighborhood  $A$  of  $\alpha_0$  and puncture only bigons which are not contained in  $A$ . So another strategy is to reduce to proving that any countable collections of arcs from  $\alpha_0$  to the boundary of  $A$  can be punctured by an endset.

The problem is that such collections can be very wild. For instance, suppose we identify  $A$  with  $S^1 \times [0, 1]$ . Let  $\mathcal{A}$  be the countable collection given by all simple piecewise-linear arcs from bottom to top, starting at a rational point of  $S^1 = \mathbb{R}/\mathbb{Z}$ , such that all slopes are rational and the non-smooth points have rational coordinates. As  $\mathcal{A}$  is countable, if our strategy is to work, there should be an endset  $E$  which intersects all arcs in  $\mathcal{A}$ . Assume such an endset exists. Note that there is an arc  $b$  joining opposite sides of the annulus and disjoint from  $E$ . Then  $b$  and  $E$  have some positive distance  $\varepsilon > 0$ . As arcs in  $\mathcal{A}$  can approximate  $b$  arbitrarily well, there is some  $a \in \mathcal{A}$  in the  $\varepsilon/2$ -neighborhood of  $b$ , and thus  $a$  is disjoint from  $E$ , a contradiction.

As a consequence, if Question 1.2 has a positive answer, it doesn't seem to be possible by construct an endset using only "abstract considerations": we will instead need to get a more in-depth understanding of the topological properties of curves belonging to a quasi-geodesic in  $\mathcal{NC}^\dagger(S)$ .

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<sup>1</sup>This is equivalent to  $E$  being homeomorphic to a closed subset of a Cantor set, which is the reason Question 1.2 is stated in terms of Cantor sets

Instead of asking whether boundary points are visible via endsets, we can ask if homeomorphisms are visible. In this case, we know that the answer is negative:

**Proposition 3.6.** *Let  $S$  be a closed surface of genus at least two. There are homeomorphisms  $f$  of  $S$  acting parabolically on  $\mathcal{C}^\dagger(S)$  so that, if  $E \subset S$  is an  $f$ -invariant endset, the induced map on  $\mathcal{C}(S \setminus E)$  is elliptic.*

Note that this contrasts with the hyperbolic case: indeed, it was shown by Guihéneuf and Militon [GM23a] that a homeomorphism of a surface acts as a hyperbolic element on  $\mathcal{NC}^\dagger(S)$  if and only if it acts as a hyperbolic element on  $\mathcal{NC}(S \setminus P)$ , for some invariant set of points  $P$ .

*Proof of Proposition 3.6.* Fix a hyperbolic structure on  $S$ . We will consider the first construction for [BHM<sup>+</sup>22, Theorem 1.1], the time-one map of a flow pushing into the cusps of a minimal filling geodesic lamination  $\lambda$  on  $S$ . More precisely, identify a triangle  $\Delta$  in the complement of  $\lambda$  with the ideal triangle in  $\mathbb{H}^2$  with vertices  $0, 1$  and  $\infty$ . Fix  $0 < a_1 < b_1 < b_2 < a_2 < 1$  and define the strips  $A = [a_1, a_2] \times [1, \infty)$  and  $B = [b_1, b_2] \times [2, \infty)$ . Let  $f : \Delta \rightarrow \Delta$  be a homeomorphism supported on  $A$ , such that

$$f(x, y) = \begin{cases} \left( x, y + \frac{x-a_1}{b_1-a_1} \right) & \text{if } (x, y) \in [a_1, b_1] \times [2, \infty) \\ (x, y + 1) & \text{if } (x, y) \in B \\ \left( x, y + \frac{a_2-x}{a_2-b_2} \right) & \text{if } (x, y) \in [b_2, a_2] \times [2, \infty) \end{cases}$$

Let  $F$  be the map obtained by extending  $f$  to the identity outside of the triangle.

As shown in [BHM<sup>+</sup>22],  $F$  acts as a parabolic element on  $\mathcal{C}^\dagger(S)$ . Fix now a simple closed geodesic  $\alpha$  transverse to  $\lambda$ . Suppose by contradiction that  $E$  is an  $F$ -invariant endset such that  $[F]$  does not act as an elliptic element on  $\mathcal{C}(S \setminus E)$ . The iterates of  $\alpha$  under the action of  $F$  coincide with  $\alpha$  outside of  $A$ , so for the  $[F]$ -orbit of  $[\alpha]$  to be infinite on  $\mathcal{C}(S \setminus E)$ , we need  $E$  to intersect the interior of  $A$ . Moreover, if  $E \cap A$  contains only points with uniformly bounded height (in  $\Delta$ ), eventually all  $F^n(\alpha)$  are homotopic to each other. So there is a point  $p = (x_0, y_0) \in [a_1, a_2] \times [2, \infty)$  which belongs to  $E$ . By invariance, and since  $E$  is closed, the closure of the  $F$ -orbit of  $p$  is contained in  $E$ . The hyperbolic distance in  $\Delta$  between successive iterates of  $p$  tends to zero by construction, but the iterates go arbitrarily far in the cusp. Therefore, for every  $\varepsilon > 0$  there is some  $h > 0$  such that the orbit of  $p$  is  $\varepsilon$ -dense in  $r_h = \{x_0\} \times [h, \infty)$  and  $r_h$  is within an  $\varepsilon$ -neighborhood of a ray of  $\Delta$ . Since every half-leaf of  $\lambda$  is dense in  $\lambda$ , we deduce that the closure of the orbit of  $p$  contains  $\lambda$ . In particular,  $E$  cannot be an endset.  $\square$

#### 4. A PARABOLIC ELEMENT OF A BIG GRAPH OF CURVES

The goal of this section is to construct a parabolic isometry of a graph of curves associated to an infinite-type surface. Concretely, the surface will be the torus with a Cantor set and an isolated point removed, and the graph will be the nonseparating curve graph.

Let  $f : S^1 \rightarrow S^1$  be a Denjoy map, arising from a rotation of some angle  $\alpha \notin \mathbb{Q}$  by blowing up the orbit of a point. Denote by  $K$  the invariant Cantor set. Let  $T = S^1 \times S^1$  be the mapping torus of  $f$  and consider the vertical vector field  $V$  and the associated flow  $\varphi_t$ . The Cantor set  $C := K \times \{0\}$  is invariant under  $\varphi_1$ .

Pick a point  $p = (x_0, 0) \notin C$  and  $\varepsilon > 0$  sufficiently small so that the  $\varepsilon$ -neighborhood  $U$  of  $\{\varphi_t(p) \mid t \in [0, 1/4]\}$  is contained in the complement of  $\{\varphi_t(C) \mid t \in \mathbb{R}\}$ , and disjoint from  $S^1 \times \{\frac{1}{2}\}$ . Consider a smooth function  $\rho : T \rightarrow \mathbb{R}_{\geq 0}$  such that:

- $\rho$  is positive outside  $\{\varphi_t(p) \mid t \in [0, 1/4]\}$ ;
- $\rho$  is zero on  $\{\varphi_t(p) \mid t \in [0, 1/4]\}$ ;
- $\rho$  is one outside  $U$ .

Then the time-one map  $F$  of the dampened vector field  $\rho V$  is an orientation-preserving homeomorphism of  $S := T \setminus C \setminus \{p\}$  and it is homotopic to the identity. To prove Theorem B our goal is to show:

**Theorem 4.1.** *The map  $F$  acts parabolically on  $\mathcal{NC}^\dagger(T)$  and induces a mapping class acting parabolically on  $\mathcal{NC}(S)$ .*

The theorem follows from two claims:

**Claim 4.2.** *The rotation set of  $F$  acting on  $\mathcal{NC}^\dagger(T)$  is contained in a line.*

**Claim 4.3.** *There are no bounded orbits of  $[F]$  acting on  $\mathcal{NC}(S)$ .*

Indeed, Claim 4.2, together with [BHM<sup>+</sup>22, Theorem 1.3], implies that  $F$  does not act hyperbolically on  $\mathcal{NC}^\dagger(T)$  and therefore (since the projection  $\mathcal{NC}^\dagger(T) \rightarrow \mathcal{NC}(S)$  is distance non-increasing),  $[F]$  does not act hyperbolically on  $\mathcal{NC}(S)$ . By Claim 4.3,  $[F]$  is not an elliptic element, so it is parabolic. In particular,  $F$  is not elliptic either, and therefore it is parabolic as well.

Let us prove the claims.

*Proof of Claim 4.2.* The intuition is that  $F$  preserves the flow-lines of  $V$ , so the rotation set must be one-dimensional.

More precisely, we can pick a lift  $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$  of  $f$  such that  $\tilde{f}(0) \geq 0$  and is as small as possible. Using this, we can construct a lift  $\tilde{F}$  of  $F$  such that  $\tilde{F}(x, y) = (\tilde{f}(x), y + 1)$  for every  $x \in \tilde{C}$ , where  $\tilde{C}$  is the lift of the Cantor set  $C$ .

Given any  $(x, y) \in [0, 1]^2$ , its sequence of iterates is contained in a broken line of the form

$$\{\dots, (\tilde{f}^{-m}(z), -m), \dots, (\tilde{f}^{-1}(z), -1), (z, 0), (\tilde{f}(z), 1), \dots, (\tilde{f}^m(z), m), \dots\}.$$

By construction of the Denjoy map,  $|\tilde{f}^m(z) - (z + m\alpha)| < 1$  for every  $z$  and  $m$ , so the average slope of the broken line is  $\frac{1}{\alpha}$ . This implies that

$$\lim_{n \rightarrow \infty} \frac{\tilde{F}^n(x, y) - (x, y)}{n}$$

is proportional to  $(\alpha, 1)$ . As a consequence, the rotation set of  $F$  is contained in the line  $y = \frac{x}{\alpha}$ .  $\square$

*Proof of Claim 4.3.* Let  $\gamma$  be the curve  $S^1 \times \{\frac{1}{2}\}$  and  $\gamma'$  the curve  $S^1 \times \{\frac{3}{4}\}$ . We will show that the orbit in  $\mathcal{NC}(S)$  of  $[\gamma]$  under the action of  $[F]$  is unbounded, which implies that every  $[F]$ -orbit is unbounded.

To do so, we first study the intersection pattern between  $\gamma'$  and  $F^n(\gamma)$  for large  $n$ . This is easier to do in the universal cover  $\mathbb{R}^2$  of the torus. Let  $\tilde{V}$  be a lift of the vector field  $V$  used in the construction of  $F$ , and let  $\tilde{F}$  be a lift of  $F$ . This is time-one map of the dampened vector field  $\rho \circ \pi \tilde{V}$  where  $\pi$  is the covering projection. Hence,  $\tilde{F}^n$  is the

time- $n$ -map of that vector field. Intuitively, this map has the effect of “flowing along  $\tilde{V}$ , but getting caught at each lift of  $p$ ” — see Figure 2.

From this description we can see that every bigon (in  $S$ ) that  $F^n(\gamma)$  forms a bigon with the curve  $\gamma'$  contains the marked point  $p$  or a point of  $C$ . In other words,  $F^n(\gamma)$  and  $\gamma'$  are in minimal position on  $T \setminus (\{p\} \cup C)$ . We then have

$$d([\gamma], [F^n(\gamma)]) = d([\gamma'], [F^n(\gamma)]) \geq d^\dagger(\gamma', F^n(\gamma)) \geq d^\dagger(\gamma, F^n(\gamma)) - 1.$$

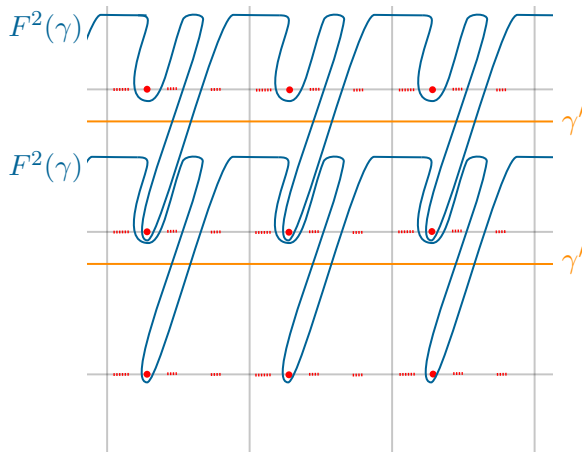


FIGURE 2. Lifts of  $\gamma'$  (in orange) and of  $F^2(\gamma)$  (in blue) to the universal cover of  $T$ . The lifts of  $p$  and  $C$  are in red.

If by contradiction the  $[F]$ -orbit of  $[\gamma]$  is bounded, there is some  $K \geq 0$  so that for every  $n$

$$d^\dagger(\gamma, F^n(\gamma)) \leq d([\gamma], [F^n(\gamma)]) + 1 \leq K.$$

Fix a genus-two branched cover  $S_2 \rightarrow T$ , branched over a point in the Cantor set; then the elevations of  $\gamma$  and of  $F^n(\gamma)$  also have bounded distance if  $\mathcal{NC}^\dagger(S_2)$ . By [BHM<sup>+</sup>22], there is a finite-sheeted cover  $X \rightarrow S_2$  and elevations of the elevations of  $\gamma$  and  $F^n(\gamma)$  which are disjoint.

Let  $\tilde{\gamma}_1, \dots, \tilde{\gamma}_N \subset X$  be the elevations of  $\gamma$ . Let  $\lambda \subset T$  be the flow lines of all points in  $C$ :

$$\lambda := \{\varphi_t(C) \mid t \in \mathbb{R}\}.$$

Denote by  $\lambda_2$  the lift of  $\lambda$  in  $S_2$  and  $\lambda_X$  the lift to  $X$ . Fix a metric on  $X$ . We will show that any sufficiently long segment  $L$  of  $\lambda_X$  is so that, for every  $i$ ,  $\tilde{\gamma}_i \cup L$  intersects all nonseparating curves of  $X$ .

As a consequence, if  $\{\beta_n\}_n \subset \mathcal{NC}^\dagger(T)$  is a sequence of curves which are disjoint from leaf segments of  $\lambda$  of length going to infinity,  $d^\dagger(\beta_n, \gamma) \rightarrow \infty$ . As the  $F^n(\gamma)$  satisfy the condition of being disjoint from such leaf segments, this concludes the proof of Claim 4.3.

To prove our statement, consider the map  $c : T \rightarrow S^1 \times S^1 =: \hat{T}$  which collapses the bi-infinite strips obtained by flowing the complementary segments of  $C$ . The covering maps  $X \rightarrow S_2 \rightarrow T$  induce covering maps  $\hat{X} \rightarrow \hat{S}_2 \rightarrow \hat{T}$ , which commute with analogous

collapsing maps  $c_2 : S_2 \rightarrow \hat{S}_2$  and  $c_X : X \rightarrow \hat{X}$ :

$$\begin{array}{ccc} \lambda_X, \tilde{\gamma}_1, \dots, \tilde{\gamma}_N \subset X & \xrightarrow{c_X} & \hat{X} \supset \hat{\lambda}_X \\ \downarrow & & \downarrow \\ \lambda_2 \subset S_2 & \xrightarrow{c_2} & \hat{S}_2 \supset \hat{\lambda}_2 \\ \downarrow & & \downarrow \\ \lambda, \gamma \subset T & \xrightarrow{c} & \hat{T} \supset \hat{\lambda} \end{array}$$

We think of the surfaces  $\hat{S}_2$  and  $\hat{X}$  as square-tiled surfaces with the tiling coming from the projections to  $\hat{T}$ . Define  $\hat{\lambda} := c(\lambda)$ ,  $\hat{\lambda}_2 := c_2(\lambda_2)$  and  $\hat{\lambda}_X := c_X(\lambda_X)$ . Then  $\hat{\lambda}$  is an irrational slope foliation of the torus, and  $\hat{\lambda}_2$  and  $\hat{\lambda}_X$  are irrational slope foliations of square-tiled surfaces.

Set  $\hat{\gamma}_i := c_X(\tilde{\gamma}_i)$ . Since  $\hat{\lambda}_X$  is an irrational slope foliation, there is some length  $\hat{\ell}$  so that for every segment  $\hat{L}$  of  $\hat{\lambda}_X$  of length at least  $\hat{\ell}$ , the complementary components are  $\hat{L} \cup \hat{\gamma}_i$  rectangles horizontal sides and sides with irrational slope, possibly with one or two irrational slope slits (if  $\hat{L}$  does not start or end on the curve). Let  $\ell$  be such that every segment of  $\lambda_X$  of length at least  $\ell$  projects onto a segment of  $\hat{\lambda}_X$  of length at least  $\hat{\ell}$ . Fix a segment  $L \subset \lambda_X$  of length at least  $\ell$ . Suppose by contradiction that  $\beta$  is a nonseparating curve disjoint from  $L \cup \tilde{\gamma}_i$ , i.e.  $\beta$  is contained in a connected component of  $X \setminus (L \cup \tilde{\gamma}_i)$ . Such a component comes from a component of  $\hat{X} \setminus (\hat{L} \cup \hat{\gamma}_i)$ , possibly blown-up along some irrational slope segment. In particular, any such component is contractible and cannot contain a nonseparating curve, a contradiction.  $\square$

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