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Sterile Insect Technique in a Patch System: Influence of Migration Rates on Optimal Single-Patch Releases Strategies

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Abstract

The Sterile Insect Technique (SIT) is a biological control method used to reduce or eliminate pest populations or disease vectors. This technique involves releasing sterilized insects that, upon mating with the wild population, produce no offspring, leading to a decline or eventual eradication of the target species. We incorporate a spatial dimension by modeling the pest/vector population as being distributed across multiple patches, with both wild and released sterile insects migrating between these patches at predetermined rates.

This study has two primary objectives: first, within an n -patch model, sufficient conditions are derived for achieving the elimination of the wild population through SIT, whether releases occur in a subset of patches or across all patches. Second, we focus on the two-patch scenario, showing that optimal SIT control within one patch can successfully reduce the wild population in that patch to a desired level within a finite time frame, provided that the migration rates between patches are sufficiently low. Numerical simulations are employed to illustrate these results and further analyze the outcomes.

Keywords: *Sterile Insect Technique; patch model; migration; elimination; population reduction; control; optimal control; optimality conditions*

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1 Introduction

Pests and disease vectors have become a critical global concern, impacting both food security and public health. To address these challenges, a range of chemical and non-chemical control methods have been developed, with varying degrees of success. However, there is now a strong emphasis on sustainable and biological control strategies, which are designed to target specific pests or vectors while minimizing environmental impact and safeguarding human health. Among these strategies, the Sterile Insect Technique (SIT) is recognized as a particularly promising approach [25]. For SIT to be effectively implemented in the field, preliminary studies, including modeling and simulations, are essential to assess its viability. SIT involves the mass rearing of insects, followed by the release of sterilized males (typically achieved through ionizing radiation). These sterilized males mate with wild females, transferring sterile sperm and leading to no viable offspring. Over time, this results in a gradual decline in the population of the targeted pest or vector. Despite its potential, the success of SIT is heavily dependent on the ability to release only sterile males, which poses a significant challenge due to the need for an efficient sex-separation method.

To address this challenge, genetic-sexing strains have been developed for species such as the Mediterranean fruit fly (*Ceratitis capitata*) [10] and the oriental fruit fly (*Bactrocera dorsalis*) [36]. These strains enable sex separation based on traits like temperature sensitivity or puparium coloration.

In cases where genetic-sexing strains are unavailable, sex-separation is performed mechanically. This process can be time-consuming and labor-intensive, with the added risk of inadvertently releasing sterile females. This is particularly concerning for disease vectors like mosquitoes, where female contamination can undermine control efforts [24, 23]. However, for certain pests, such as fruit flies, the release of both sterile males and females has been shown to have potential benefits [30]. On Réunion Island, where a genetic-sexing strain for *Bactrocera dorsalis* is not available, both sterile males and females are released.

Since the 1950s, the SIT has been applied with varying degrees of success against numerous pests, such as the Mediterranean fruit fly, as well as disease vectors, including *Anopheles* and *Aedes* mosquitoes [25]. Additionally, alternative sterilization methods have been developed, including genetic approaches like the RIDL method [45] (which stands for "Release of Insects carrying Dominant Lethals") and techniques involving the *Wolbachia* bacteria [42]. Since the pioneering model by E. Knippling [32], a range of mathematical and computational models – simple, sex-structured, and/or stage-structured – have been developed and extensively studied to enhance the effectiveness of SIT (see for instance [13, 38, 12, 47, 35, 29, 44, 11, 15, 8]). To date, although numerous field programs are underway, only a few incorporate a mathematical modeling and simulation component.

While conceptually straightforward, the SIT is challenging to implement in the field and even more so on an industrial scale. Even with expertise in mass rearing and access to ionization facilities, several factors can lead to failures. To mitigate these risks, mathematical modeling can play a crucial role in formalizing SIT strategies [13], as well as addressing specific challenges associated with SIT deployment. These challenges include residual fertility, where not all sterile males are entirely infertile; re-mating, where females of the targeted species may mate multiple times; and sterile female contamination, which occurs when sex separation is less than 100% efficient.

Some of these issues have been explored through mathematical modeling [8, 24, 23, 22], yielding valuable theoretical results and threshold values based on SIT and vector/pest parameters, which are essential to ensure the success of SIT prior to mass production and release.

When the area targeted for SIT is not naturally isolated, the International Atomic Energy Agency (IAEA) and other SIT experts recommend establishing a “buffer zone” to minimize the migration of wild individuals from untreated areas into the treated zone. This scenario can be represented simply as a model with two interconnected areas – one treated and one untreated – linked by migration rates, which depend on the effectiveness of the buffer zone. Therefore, it is essential to estimate the migration rates under which releases in the targeted area alone remain effective, and possibly optimal. This scenario will be modeled using a two-patch metapopulation model.

Regarding release strategies for SIT, numerous models have been developed and analyzed [46, 2, 21, 16]. These strategies often leverage the strong Allee effect induced by SIT, which suggests that local elimination of pests or vectors is feasible once the system reaches the basin of attraction of the zero equilibrium. For instance, [3] demonstrated that both massive and small release strategies can be effective, but this is only true if the targeted area is isolated from surrounding areas—an ideal condition that is not always practically achievable.

In [17], the incorporation of immigration terms into the model highlighted that local elimination is only possible if migration is effectively controlled, meaning that immigration must be halted. However, the migration model used in [17] was limited, as it did not account for the fact that some released sterile males might migrate out of the targeted area, potentially reducing the effectiveness of the SIT. This underscores the importance of incorporating spatial components into SIT models. Spatial dynamics have been explored through partial differential equations (PDEs). Initial studies, such as [34], investigated wave extinction phenomena due to SIT releases, and subsequent research, like [31], continued to build on this approach. PDEs have also been used to include environmental factors, such as wind and landscape features [20].

Another approach is the metapopulation model, which implicitly considers spatial components. This model, widely used in ecology and conservation biology since the early 1980s [27, 40, 7], has been applied to mathematical epidemiology [9, 6, 18]. However, it has only been explored in the context of a two-patch SIT control in two papers only: first, in a very recent paper [14], where the authors studied the conditions on constant or impulsive periodic releases to reach elimination in one patch using a complex mosquito model; and also in [49], where the authors examined the release of sterile males in one patch only and performed standard qualitative analysis to determine conditions for the existence of positive (stable) equilibria. Our work aims to extend these last results by deriving additional insights specific to this scenario.

We aim to advance the analysis by applying insights from the metapopulation/patches approach [26], utilizing the monotone dynamical systems theory [37, 43, 28]. We will establish sufficient conditions to demonstrate that complete elimination across all patches is achievable by releasing an adequate number of sterile insects, ideally across as many patches as possible. Focusing on a two-patch system, our analysis reveals that if diffusion rates are sufficiently low, it is feasible to reduce the wild population to below a given threshold in patch 1 using an optimal SIT control strategy, even if releases occur only in patch 1. If diffusion rates cannot be reduced effectively, it becomes preferable to treat both patches simultaneously.

The paper is organized as follows. In Section 2 we present a n -patch model for a single population of pest or (vector-borne disease) vector. In Subsection 2.2, we derive the n -patch model for the sterile insects releases with constant releases. For the 2-patches case, we recall some analytical results for the single population model. Then, in Subsection 2.5, we present the 2-patches SIT model and describe the optimal control problem of interest, specifically focusing on scenarios where sterile insects are released only in patch 1. In a nutshell, we demonstrate that if migration is sufficiently low, the optimal strategy is the same as in the absence of migration. The precise results are stated in Section 2.6. Section 3 examines the SIT model with constant releases, providing conditions to ensure the elimination of the wild population in both patches, and extends these findings to the n -patch scenario. These results are instrumental in understanding the

underlying dynamics of controlled systems and in establishing the existence of an optimal control strategy for the two-patch case. Sections 4 and 5 are devoted to proving the main theoretical results regarding small migration rates and characterizing the optimal control, respectively. In Section 6 we present numerical simulations to both illustrate and extend our theoretical results. Additionally, we explore the practical implications and applications of these findings. We conclude the paper with several perspectives for future research in Section 7.

2 The patchy model - Main results

Before addressing the two-patch system, we first define the general n -patch problem for $n \geq 2$, considering both scenarios with and without SIT control.

2.1 The n -patch pest/vector population model

We consider a system of n interconnected patches, each occupied by a single species:

$$\frac{dP_i}{dt} = P_i f_i(P_i) + \sum_{j=1, j \neq i}^n d_{ij} P_j - \sum_{j=1, j \neq i}^n d_{ji} P_i, \quad i = 1, \dots, n.$$

In this equation, P_i denotes the population density of the species in patch i (where $i = 1, \dots, n$), and $f_i(P_i)$ represents the specific dynamics of the wild population in that patch. The dispersion rate d_{ij} (for $i \neq j$) is the non-negative coefficient governing the species' movement from patch j to patch i . Specifically, $d_{ii} = -\sum_{j=1, j \neq i}^n d_{ji}$. Therefore, we define the matrix $D = (D_{ij})_{1 \leq i, j \leq n}$ as follows:

$$D_{ij} = \begin{cases} d_{ij} & \text{for } i \neq j, \\ -\sum_{k=1, k \neq i}^n d_{ki}, & \text{for } i = j. \end{cases}$$

This yields a differential system that can be expressed in the concise form:

$$\frac{dP}{dt} = P \cdot F(P) + DP, \quad (1)$$

with $(F(P))_i = f_i(P_i)$ for $i = 1, \dots, n$ and

$$f_i(x) = b_i - \mu_{i,1} - \mu_{i,2}x, \quad i = 1, \dots, n,$$

where $b_i > 0$ represents the daily birth-rate per individual, $\mu_{1,i} > 0$ represents the daily death-rate, and $\mu_{2,i} > 0$ represents the daily density death-rate in patch i . Thus, setting $r_i = b_i - \mu_{i,1}$ and $K_i = r_i/\mu_{i,2}$, we rewrite

$$f_i(x) = r_i \left(1 - \frac{x}{K_i} \right), \quad i = 1, \dots, n.$$

This type of system has been studied in the context of species persistence and extinction, beginning with the two-patch system analyzed in [1] and later extended in [5, 26]. Notably, [37] enhanced and generalized the results of [1] by applying the theory of cooperative monotone systems [28]. Additionally, these systems have been explored as 'source-source' systems [41, 5, 26], where all $r_i > 0$, and as 'source-sink' systems [48, 7], where some $r_i < 0$. In this manuscript, we focus on the 'source-source' case. Finally, we assume that the matrix D is irreducible, implying that the species can migrate between any patches.

2.2 The n -patch sterile insects model

We assume that sterile insects are released in each patch at a rate of u_i for $i = 1, \dots, n$, with a mortality rate of μ_S . The population of sterile individuals in patch i , denoted as $M_{s,i}$ for $i = 1, \dots, n$, is modeled by:

$$\frac{dM_{s,i}}{dt} = u_i(t) - \mu_S M_{s,i} + \omega \sum_{j=1, j \neq i}^n d_{ij} M_{s,j} - \omega \sum_{j=1, j \neq i}^n d_{ji} M_{s,i}, \quad (2)$$

where ω is a parameter representing the impact of radiation on the dispersal ability of the sterile insects. It is important to note that system (2) can be reformulated as:

$$\frac{dM_s}{dt} = U(t) + \mathcal{A}M_s, \quad (3)$$

with $M_s = (M_{s,i})_{i=1,\dots,n}$, $U = (u_i)_{i=1,\dots,n}$, two vectors, and $\mathcal{A} = \omega D - \mu_S I_n$, where I_n is the identity matrix of size n .

Remark 2.1 (Properties of the matrix \mathcal{A}). *It is notable that \mathcal{A} enjoys several properties:*

- (i) *It is a Metzler matrix, meaning that $\mathcal{A}_{ij} \geq 0$ for all $i, j = 1, \dots, n$ with $i \neq j$;*
- (ii) *Additionally, $\mathcal{A}_{ii} < 0$ for all $i = 1, \dots, n$;*
- (iii) *Since $\sum_{j=1}^n \mathcal{A}_{ji} = -\mu_S < 0$ for all $i = 1, \dots, n$, it follows that $\sum_{j=1, j \neq i}^n \mathcal{A}_{ji} < -\mathcal{A}_{ii}$ for all $i = 1, \dots, n$. In other words, \mathcal{A} is strictly diagonally dominant.*

These properties imply that \mathcal{A} is nonsingular and hence invertible. Moreover, \mathcal{A} is Hurwitz, which means that every eigenvalue of \mathcal{A} has a strictly negative real part.

We also assume that $M_s(0) = \mathbf{0}$, meaning there are no sterile insects at time $t = 0$. It is straightforward to show that the solution of system (3) under this initial condition is given by

$$M_s(t) = \int_0^t e^{\mathcal{A}(t-s)} U(s) ds.$$

Now, assuming U is constant and summing all the equations in (2) yields

$$\frac{dM_{tot}}{dt} = \sum_{i=1}^n u_i - \mu_S M_{tot},$$

where $M_{tot} = \sum_{i=1}^n M_{s,i}$ represents the total number of sterile males across all patches. According to Remark 2.1, it follows that M_{tot} converges to

$$M_{tot}^* := \frac{1}{\mu_S} \sum_{i=1}^n u_i$$

as $t \rightarrow +\infty$. It is noteworthy that the equilibrium number of sterile insects is independent of the diffusion rates d_{ij} for $i, j = 1, \dots, n$ with $i \neq j$.

Remark 2.2 (Case of constant releases). *Assuming further that the release rates u_i are constant, and given that \mathcal{A} is Hurwitz, the linear stability theory allows us to conclude that $M_s^* := -\mathcal{A}^{-1}U \geq 0$ is globally asymptotically stable (GAS) for System (3).*

2.3 Conclusion: the n -patch SIT model of a single vector species

When sterile insects are released, they primarily disrupt the mating process. Consequently, in each patch i , the birth rate is modified to $b_i \frac{P_i}{P_i + \gamma M_{s,i}}$, where $\frac{P_i}{P_i + \gamma M_{s,i}}$ represents the probability of a fertile mating occurring in patch i , and γ is a parameter reflecting the average competitiveness of the sterile males.

Thus, by combining (1) and (3) as discussed in Sections 2.1 and 2.2, we can derive the n -patch SIT model:

$$\begin{cases} \frac{dP}{dt} = P \cdot F_s(P, M_s) + DP, & t > 0 \\ \frac{dM_s}{dt} = U(t) + \mathcal{A}M_s, \end{cases} \quad (4)$$

with

$$(F_s)_i = f_{s,i}(x, y) = \frac{b_i}{x + \gamma y} - \mu_{i,1} - \mu_{i,2}x, \quad i = 1, \dots, n.$$

It is important to emphasize that releasing sterile insects introduces a strong Allee effect into model (4). Consequently, the steady state $\mathbf{0}$ becomes Locally Asymptotically Stable (LAS), with the size of its basin of attraction potentially depending on the release rates.

Next, we will focus on a two-patch model. We will specify the model and review some well-known results related to the asymptotic behavior of its solutions.

2.4 The two-patch model of a single vector species

Let $T > 0$ be the fixed time horizon over which the control strategy is applied. We consider a two-patch model comprising two interconnected patches with population sizes P_1 and P_2 , respectively. These patches may represent different geographic regions or habitats, as illustrated in Fig. 1.

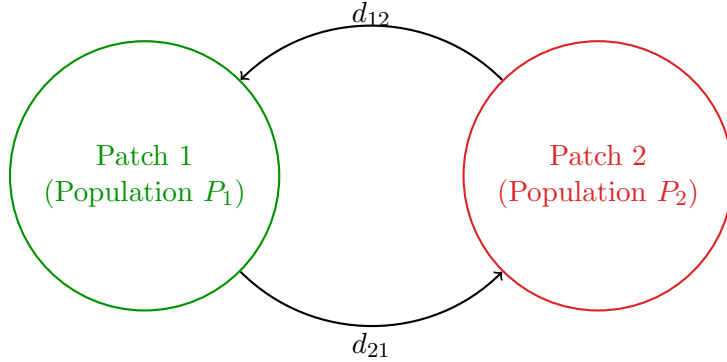


Figure 1: The two-patch model

As detailed in Section 2.1, we examine the 2-patch logistic model given by:

$$\begin{cases} \frac{dP_1}{dt} = (b_1P_1 - \mu_1 - \mu_2P_1)P_1 + d_{12}P_2 - d_{21}P_1 & \text{in } (0, T), \\ \frac{dP_2}{dt} = (b_2P_2 - \mu_1 - \mu_2P_2)P_2 + d_{21}P_1 - d_{12}P_2 & \text{in } (0, T). \end{cases} \quad (5)$$

The persistence and elimination properties of this two-patch system are summarized in the following result:

Theorem 2.3 (see [1, Theorem 1]). *Let us denote by $r_i = b_i - \mu_{1,i}$ and $K_i = \frac{r_i}{\mu_{2,i}}$ so that System (5) can be recast as*

$$\begin{cases} \frac{dP_1}{dt} = r_1 \left(1 - \frac{P_1}{K_1}\right) P_1 + d_{12}P_2 - d_{21}P_1 & \text{in } (0, T), \\ \frac{dP_2}{dt} = r_2 \left(1 - \frac{P_2}{K_2}\right) P_2 + d_{21}P_1 - d_{12}P_2 & \text{in } (0, T). \end{cases}$$

The extinction behavior, $\lim_{t \rightarrow +\infty} P_i(t) = 0$ for $i = 1, 2$, occurs if, and only if, the following two conditions are satisfied:

- (i) $r_1 - d_{12} < 0$ and $r_2 - d_{21} < 0$,
- (ii) $d_{12}d_{21} - (r_1 - d_{12})(r_2 - d_{21}) \leq 0$.

On the other hand, if either (i) or (ii) do not hold, then $\lim_{t \rightarrow +\infty} P_i(t) = P_i^* > 0$, for $i = 1, 2$.

Assuming that System (5) is permanent, and defining $d_{12} = \beta/\gamma_1$ and $d_{21} = \beta/\gamma_2$, where β , γ_1 , and γ_2 are positive real numbers with β representing the migration intensity and γ_2/γ_1 quantifying the migration asymmetry, we obtain the following results:

Proposition 2.4 (see [5, Propositions 1 and 2]). *Assuming that System (5) has a positive equilibria (P_1^*, P_2^*) . Let $P^* = P_1^* + P_2^*$. We have*

- When $r_1 = r_2$, migration is always unfavorable to P^* , i.e. $P^* < K_1 + K_2$.
- Assume $r_1 < r_2$. Then

– If γ_1, γ_2 satisfy $\frac{\gamma_1}{\gamma_2} < \frac{r_1 K_2}{r_2 K_1}$, then there exists β_C such that:

$$0 \leq \beta \leq \beta_C \implies P^* > K_1 + K_2, \quad \text{and} \quad \beta_C \leq \beta \implies P^* < K_1 + K_2.$$

- If γ_1, γ_2 satisfy $\frac{r_1 K_2}{r_2 K_1} < \frac{\gamma_1}{\gamma_2} < \frac{K_1}{K_2}$, then $P^* > K_1 + K_2$ for every β .
- If γ_1, γ_2 satisfy $\frac{K_2}{K_1} < \frac{\gamma_1}{\gamma_2}$, then $P^* < K_1 + K_2$ for every β .

Remark 2.5. *The behavior of the system depends on whether migration occurs between the patches. In the absence of migration (i.e., when $d_{12} = d_{21} = 0$ in (5)), the solution (P_1, P_2) to System (6) converges pointwise to the equilibrium (K_1, K_2) as $t \rightarrow +\infty$. However, when migration is present, and if $r_1 \neq r_2$ and $K_1 \neq K_2$, the sum of the equilibrium states may not equal the sum of the carrying capacities of the individual patches. Specifically, $P_1 + P_2$ may converge to a value different from $K_1 + K_2$, either less than or greater than $K_1 + K_2$. This result highlights that, in the presence of migration, the total equilibrium population across patches can deviate from the sum of the carrying capacities. This deviation underscores the inherently nonlinear dynamics of patch systems.*

2.5 The two-patch SIT model

Next, we consider a two-patch SIT-control model, where u_i represents the amount of sterile insects released in patch i . The time evolution of the wild population is then modeled by the following SIT system:

$$\begin{cases} \frac{d}{dt} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} f_1(P_1, M_{s,1}) \\ f_2(P_2, M_{s,2}) \end{pmatrix} + D \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} & \text{in } [0, T], \\ \frac{d}{dt} \begin{pmatrix} M_{s,1} \\ M_{s,2} \end{pmatrix} = -\mu_s \begin{pmatrix} M_{s,1} \\ M_{s,2} \end{pmatrix} + D_S \begin{pmatrix} M_{s,1} \\ M_{s,2} \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} & \text{in } [0, T], \\ (P_1(0), P_2(0)) = (P_1^*, P_2^*), \\ M_{s,1}(0) = M_{s,2}(0) = 0, \end{cases} \quad (6)$$

where (P_1^*, P_2^*) is the positive equilibrium of System (5) and

$$D = \begin{pmatrix} -d_{21} & d_{12} \\ d_{21} & -d_{12} \end{pmatrix}$$

stands for the migration matrix modeling the interactions between the two patches. Assuming that the migration of sterile insects does not exceed that of the wild population, the matrix $D_S = \omega D$ (with $0 \leq \omega \leq 1$) represents the migration matrix for the sterile insects.

Building on previous work [3] and the discussions in Sections 2.4, we consider

$$f_i(P_i, M_{s,i}) = g_i(P_i, M_{s,i})P_i \quad \text{with} \quad g_i(P_i, M_{s,i}) = b_i \frac{P_i}{P_i + \gamma M_{s,i}} - \mu_{1,i} - \mu_{2,i}P_i,$$

where $b_i > 0$, $\mu_{1,i} > 0$ and $\mu_{2,i} > 0$, $i = 1, 2$ are given.

In the context of SIT control, we address the following two key biological issues:

Question 1:

Is it possible to find a release threshold to reduce or insure the elimination of the pest/vector population in all patches?

Question 2:

To reduce the population P_1 under a given value $\varepsilon > 0$, is it optimal to act only with u_1 ($u_2 = 0$) when migration is weak?

We concentrate on the control problem in patch 1, aiming to reduce the vector population in this patch below a specified threshold $\varepsilon > 0$ using the minimal amount of control effort u_i . Our goal is to explore how the migration parameters d_{12} and d_{21} influence the structure of optimal controls, especially when these parameters are small. The optimal control problem we will consider hence reads

$$\boxed{\inf_{u \in \mathcal{U}_{\varepsilon, M}} J(u_1, u_2),} \quad (\mathcal{P}_{\varepsilon, M})$$

where $M = (M_1, M_2) \in (\mathbb{R}_+^*)^2$, where M_1 and M_2 are the maximal daily release rates, and $\varepsilon > 0$ are given,

$$J(u_1, u_2) = \int_0^T (u_1(t) + u_2(t)) dt$$

and

$$\mathcal{U}_{\varepsilon, M} = \{u = (u_1, u_2) \in L^\infty(0, T; \mathbb{R}^2) \mid 0 \leq u_i \leq M_i \text{ a.e. in } (0, T), i = 1, 2 \text{ and } P_1(T) \leq \varepsilon\}.$$

2.6 Main results: the 2-patches SIT model ($\mathcal{P}_{\varepsilon, M}$)

We now analyze problem ($\mathcal{P}_{\varepsilon, M}$) under the assumption that the migration parameters are small. It is important to note that sterile insects can be released in both patches simultaneously, meaning that controls u_1 and u_2 can be activated concurrently.

As a preliminary step, we present a result on the use of constant controls. This approach is not only biologically significant but also essential for ensuring that the admissible set of controls in problem ($\mathcal{P}_{\varepsilon, M}$) is non-empty. In simple terms, with a sufficient number of sterilized males, it is possible to achieve the constraint $P_1(T) \leq \varepsilon$. Roughly speaking, with enough sterilized males it is possible to reach the constraint $P_1(T) \leq \varepsilon$.

Proposition 2.6. *There exist $\bar{T} > 0$ and $\bar{M} > 0$ such that for each $T > \bar{T}$ and $M_i > \bar{M}$, $i = 1, 2$, Problem ($\mathcal{P}_{\varepsilon, M}$) has a solution (u_1, u_2) .*

This result is derived by carefully analyzing the system (6) under constant controls. The impact of these constant controls on the system's equilibrium across different regimes (with and without migration) is thoroughly examined in Section 3. The proof of Proposition 2.6 is presented in section 3.3. We now present one of the main results of this article, which addresses the qualitative behavior of the optimal controls.

Theorem 2.7. *There exist $\bar{T} > 0$ and $\bar{M} > 0$ such that for each $T > \bar{T}$ and $M_i > \bar{M}$, $i = 1, 2$, Problem ($\mathcal{P}_{\varepsilon, M}$) admits a solution (u_1, u_2) , and one has necessarily*

$$P_1(T) = \varepsilon.$$

Furthermore, there exists $\bar{\delta} > 0$, depending on the parameters of the problem except d_{12} and d_{21} , such that

$$d_{12} + d_{21} \leq \bar{\delta} \Rightarrow u_2(\cdot) = 0.$$

Remark 2.8 (Biological interpretation of this result.). *The preceding theorem establishes that an optimal control strategy exists to reduce (and potentially eliminate) the pest or vector population in patch 1 within a finite time by releasing sterile males exclusively in patch 1. However, this is only feasible if the migration rates between the two patches are sufficiently low.*

Theorem 2.9 (Characterization of the control). *Assume that $\mu_s \geq \mu_{1,1}$. There exist $\bar{T} > 0$, $\bar{M} > 0$ and $\bar{\delta} > 0$ such that, for each $T \geq \bar{T}$, $M_i > \bar{M}$ and $\delta = d_{12} + d_{21} \in [0, \bar{\delta}]$, every solution (u_1, u_2) of the optimal control problem ($\mathcal{P}_{\varepsilon, M}$) satisfies:*

- $u_2 = 0$ on $[0, T]$;
- there exists $0 < t_0 < t_1 < T$ such that $u_1 = 0$ on $[0, t_0]$ and on $(t_1, T]$.

Furthermore, in the case where $\delta = 0$ (no migration), every control u_1 satisfies

$$\begin{cases} \frac{d}{dt} P_1 = f_1(P_1, M_{s1}) & \text{in } [0, T], \\ \frac{d}{dt} M_{s1} = -\mu_s M_{s1} + u_1 & \text{in } [0, T], \\ u(t) = \left(\frac{\partial^2 f}{\partial M_s^2} \right)^{-1} \left(\frac{\partial f_1}{\partial M_s} \frac{\partial f_1}{\partial P_1} + \mu_{1,s} M_s \frac{\partial^2 f}{\partial M_s^2} - f_1 \frac{\partial^2 f_1}{\partial M_s \partial P_1} \right) \Big|_{(P_1(t), M_{s1}(t))} & \text{in } [t_0, t_1], \\ u(t) = 0 & \text{in } [0, t_0] \cup [t_1, T]. \end{cases}$$

Remark 2.10. *With the notations of Theorem 2.9 above, it is notable that, when the parameter δ decreases to 0, any family of solution (u_1^δ, u_2^δ) to Problem ($\mathcal{P}_{\varepsilon, M}$) with migration coefficients d_{12} and d_{21} such that $\delta \leq \bar{\delta}$ converges towards a solution (u_1^0, u_2^0) to Problem ($\mathcal{P}_{\varepsilon, M}$) with no migration, in the sense of the weak-star topology of $L^\infty(0, T)$. This results immediately from the Γ -convergence argument, in the first step of the proof of Theorem 2.7, in Section 4.5.*

3 Analysis of constant controls (proof of Proposition 2.6)

In this section, we assume that the controls u_1 and u_2 are constant over the interval $[0, T]$. System (6) can be divided into two components. The second component pertains to the dynamics of the sterile males, which is modeled by system (3) with $n = 2$. As discussed in Section 2.2, page 4, we know that the solution $(M_{s,1}, M_{s,2})$ converges exponentially to $(M_{s,1}^*, M_{s,2}^*)$ as $t \rightarrow +\infty$, where

$$\begin{pmatrix} M_{s,1}^* \\ M_{s,2}^* \end{pmatrix} = -\mathcal{A}^{-1}U = \begin{pmatrix} \frac{u_1(\mu_S + \omega d_{12}) + \omega d_{12}u_2}{(\mu_S + \omega d_{21} + \omega d_{12})\mu_S} \\ \frac{u_2(\mu_S + \omega d_{21}) + \omega d_{21}u_1}{(\mu_S + \omega d_{12} + \omega d_{21})\mu_S} \end{pmatrix}. \quad (7)$$

Therefore, the long-term behavior of System (6) is equivalent to that of the following system:

$$\begin{cases} \frac{dP_1}{dt} = \left(b_1 \frac{P_1}{P_1 + \gamma M_{s,1}^*} - (\mu_{1,1} + d_{21}) - \mu_{2,1}P_1 \right) P_1 + d_{12}P_2, \\ \frac{dP_2}{dt} = \left(b_2 \frac{P_2}{P_2 + \gamma M_{s,2}^*} - (\mu_{1,2} + d_{12}) - \mu_{2,2}P_2 \right) P_2 + d_{21}P_1. \end{cases} \quad (\text{SIT})$$

System (SIT) defines a monotone cooperative system (see [28]) on $\mathcal{D} = \mathbb{R}_+^2$ for any values of $t \in (0, +\infty)$. Let us consider two cases :

3.1 Case without migration

Assuming first that $d_{12} = d_{21} = 0$, then the steady states are $\mathbf{0} = (0, 0)$, $(P_1^*, 0)$, $(0, P_2^*)$ and (P_1^*, P_2^*) where P_i^* are solutions of

$$b_i \frac{P_i}{P_i + \gamma M_{s,i}^*} - \mu_{1,i} - \mu_{2,i}P_i = 0.$$

Setting $\mathcal{N}_i = b_i/\mu_{1,i}$, the basic offspring number related to population P_i , and assuming $\mathcal{N}_i > 1$, long but straightforward computations (see for instance [3]), show that there exists a threshold value $M_{s,i}^{crit}$, defined as follows

$$M_{s,i}^{crit} = \frac{\mu_{1,i}}{\mu_{2,i}} \left(\sqrt{\mathcal{N}_i} - 1 \right)^2 = \frac{1}{Q_i} \left(\sqrt{\mathcal{N}_i} - 1 \right)^2, \quad (8)$$

where $Q_i = \frac{\mu_{2,i}}{\mu_{1,i}}$, such that we can deduce the following result

Proposition 3.1. *Assume $\mathcal{N}_i > 1$, for $i = 1, 2$, then the following results hold:*

- When $\gamma M_{s,i}^* > M_{s,i}^{crit}$, $i = 1, 2$, then $\mathbf{0}$ is the only equilibrium.
- When $\gamma M_{s,i}^* = M_{s,i}^{crit}$, $i = 1, 2$, then only one positive equilibrium, P_i^\dagger , exists where

$$P_i^\dagger = \frac{\mu_{1,i}}{\mu_{2,i}} \sqrt{\mathcal{N}_i}, \quad i = 1, 2.$$

- When $0 < \gamma M_{s,i}^* < M_{s,i}^{crit}$, $i = 1, 2$, then there exist two positive equilibria P_1^* and P_2^* :

$$P_{1,i}^* = \frac{1}{2Q_i} \left(\mathcal{N}_i - 1 - Q_i \gamma M_{s,i}^* - \sqrt{(\mathcal{N}_i - 1 - Q_i \gamma M_{s,i}^*)^2 - 4Q_i \gamma M_{s,i}^*} \right), \quad i = 1, 2,$$

and

$$P_{2,i}^* = \frac{1}{2Q_i} \left(\mathcal{N}_i - 1 - Q_i \gamma M_{s,i}^* + \sqrt{(\mathcal{N}_i - 1 - Q_i \gamma M_{s,i}^*)^2 - 4Q_i \gamma M_{s,i}^*} \right), \quad i = 1, 2.$$

With both patches being isolated, and employing the approach used in [4, 3], we obtain the following stability results:

Proposition 3.2. *Assume $\mathcal{N}_i > 1$, for $i = 1, 2$. The following results then hold:*

- When $\gamma M_{s,i}^* > M_{s,i}^{crit}$, then $\mathbf{0}$ is GAS.
- When $0 < \gamma M_{s,1}^* \leq M_{s,1}^{crit}$ and $\gamma M_{s,2}^* > M_{s,2}^{crit}$ then $(P_{1,2}^*, 0)$ and $\mathbf{0}$ are LAS
- When $\gamma M_{s,1}^* > M_{s,1}^{crit}$ and $0 < \gamma M_{s,2}^* \leq M_{s,2}^{crit}$ then $(0, P_{2,2}^*)$ and $\mathbf{0}$ are LAS
- When $0 < \gamma M_{s,i}^* < M_{s,i}^{crit}$, $i = 1, 2$, then $(P_{1,2}^*, P_{2,2}^*)$, $(P_{1,2}^*, 0)$, $(0, P_{2,2}^*)$ and $\mathbf{0}$ are LAS.

It is crucial to understand that SIT introduces a significant Allee effect, ensuring that $\mathbf{0}$ is always Locally Asymptotically Stable (LAS) once sterile males are released, i.e., $M_{s,i}^* > 0$ for $i = 1, 2$. The size of the basin of attraction for $\mathbf{0}$ primarily depends on the magnitude of the releases.

Remark 3.3. This reasoning extends to n isolated patches. For each patch, the critical release value specified in formula (8) remains valid for $i = 1, 2, \dots, n$. Consequently, as shown in Proposition 3.2, $\mathbf{0}$ is Globally Asymptotically Stable (GAS) if $\gamma M_{s,i}^* > M_{s,i}^{crit}$ for $i = 1, 2, \dots, n$. In other words, the release rates u_i must satisfy $u_i > \mu_s M_{s,i}^{crit} / \gamma$ for all $i = 1, 2, \dots, n$.

3.2 Case with migration

We consider either $d_{12} > 0$ and $d_{21} \geq 0$ or $d_{12} \geq 0$ and $d_{21} > 0$. We will use a result from [37] to show that $\mathbf{0}$ can be GAS, provided that enough sterile males are released.

Once sterile insects are released, with conditions such as $u_1 > 0$ and $u_2 \geq 0$ or $u_1 \geq 0$ and $u_2 > 0$, the dynamics can become quite intricate. For example, [49] examined the scenario with $u_1 > 0$ and $u_2 = 0$, revealing up to eight positive equilibria depending on the release rate u_1 . Only some of these equilibria are Locally Asymptotically Stable (LAS). The following phase diagrams (see Figs. 2 and 3, with parameter values from Table 1 and migration rates $d_{12} = d_{21} = 0.002$) illustrate the complexity of the dynamics for varying $u_1 > 0$ and $u_2 > 0$. They show that multiple equilibria can coexist, but only a few are LAS (indicated by green dots). As the values of u_1 and u_2 vary, the number of possible steady states can transition between 7, 5, 3, or 1. Figure 3 (Right) demonstrates that for sufficiently large u_1 and u_2 , only the equilibrium $\mathbf{0}$ remains, which is Globally Asymptotically Stable (GAS). We will also show how to determine the minimal release rates required in each patch to ensure elimination in both patches.

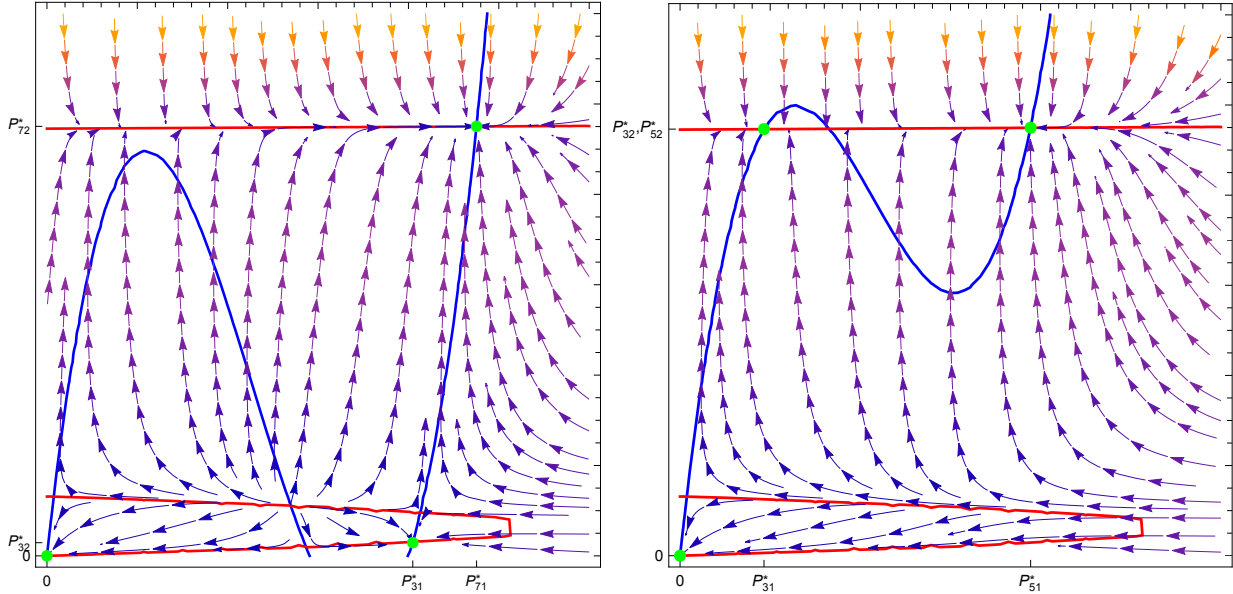


Figure 2: (Left) Phase Portrait for $u_1 = 1090$ and $u_2 = 1090$: 7 equilibria case. (Right) Phase Portrait for $u_1 = 1100$ and $u_2 = 1090$: 5 equilibria case. Each green dots represents a LAS equilibrium.

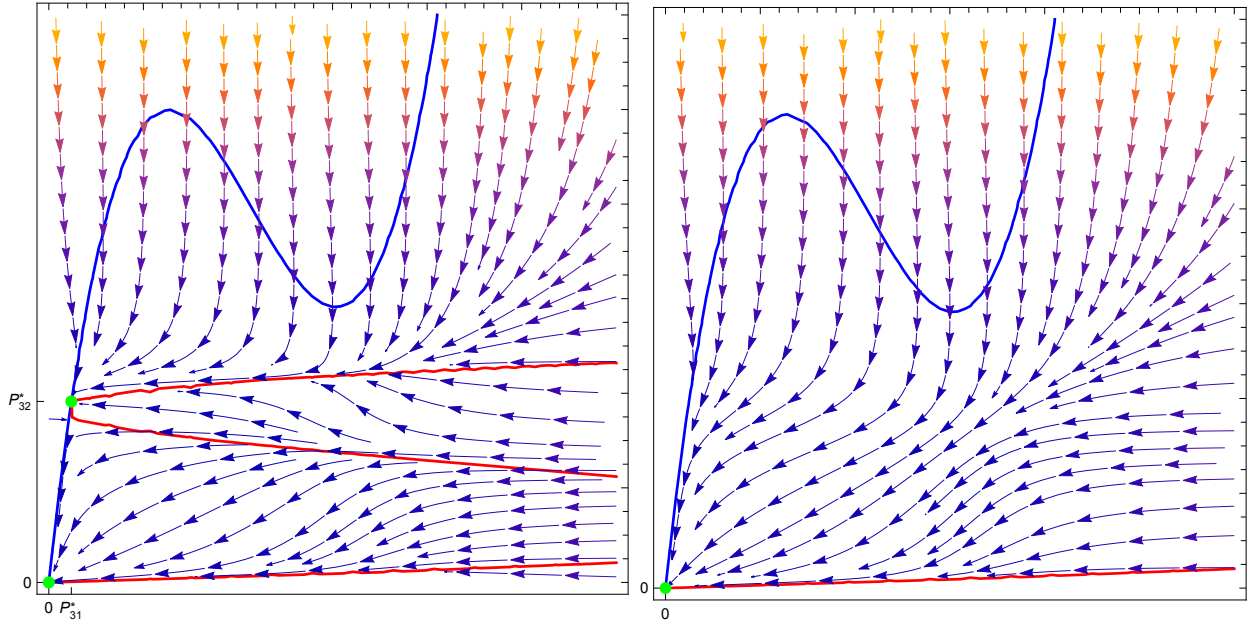


Figure 3: (Left) Phase Portrait for $u_1 = 1100$ and $u_2 = 1170$: : 3 equilibria case (Right) Phase Portrait for $u_1 = 1100$ and $u_2 = 1080$: : 1 equilibrium case. Each green dots represent a LAS equilibrium.

First of all, once $M_{s,i}^* > 0$, we easily check that $\mathbf{0}$ is always Locally Asymptotically Stable (LAS), even in the presence of migration. Indeed, computing the Jacobian matrix of system (SIT) at $\mathbf{0}$ leads to

$$J(\mathbf{0}) = \begin{pmatrix} -(\mu_{1,1} + d_{21}) & d_{12} \\ d_{21} & -(\mu_{1,2} + d_{12}) \end{pmatrix},$$

that is Hurwitz since $\text{Tr}(J(\mathbf{0})) < \mathbf{0}$ and $\det(J(\mathbf{0})) > \mathbf{0}$.

Let us set $R_i = \sup_{x \geq 0} g_i(x)$ for $i = 1, 2$ and $A_R = D + \text{diag}([R_1, R_2])$. According to [37, Theorem 1(i)], extinction occurs (i.e. $\mathbf{0}$ is GAS) if the spectral abscissa $s(A_R)$ of A_R is negative.

One computes

$$g'_i(x) = b_i \frac{\gamma M_{s,i}^*}{(x + \gamma M_{s,i}^*)^2} - \mu_{2,i} = \frac{1}{(x + \gamma M_{s,i}^*)^2} \left(b_i \gamma M_{s,i}^* - \mu_{2,i} (x + \gamma M_{s,i}^*)^2 \right),$$

such that $g'_i(x) \geq 0$, when $x \in [x_1^*, x_2^*]$, where

$$x_1^* = -\sqrt{\frac{b_i \gamma M_{s,i}^*}{\mu_{2,i}} - \gamma M_{s,i}^*} \leq 0 \quad \text{and} \quad x_2^* = \sqrt{\frac{b_i \gamma M_{s,i}^*}{\mu_{2,i}} - \gamma M_{s,i}^*},$$

and $g'_i(x) < 0$, when $x \notin [x_1^*, x_2^*]$. If $x_2^* \geq 0$, then $g_i(x) \leq g_i(x_2^*)$ for all $x \geq 0$. We have $x_2^* \geq 0$ if, and only if, $\gamma M_{s,i}^* \leq b_i / \mu_{2,i}$. Thus we deduce that, for $i = 1, 2$, we have

$$R_i = \begin{cases} \left(\sqrt{b_i} - \sqrt{\mu_{2,i} \gamma M_{s,i}^*} \right)^2 - \mu_{1,i} & \text{when } 0 \leq \gamma M_{s,i}^* \leq \frac{b_i}{\mu_{2,i}}, \\ -\mu_{1,i} & \text{when } \gamma M_{s,i}^* > \frac{b_i}{\mu_{2,i}}. \end{cases}$$

First case: $\gamma M_{s,i}^* > \frac{b_i}{\mu_{2,i}}$ for $i = 1, 2$. Then,

$$A_R = \begin{pmatrix} -(\mu_{1,1} + d_{21}) & d_{12} \\ d_{21} & -(\mu_{1,2} + d_{12}) \end{pmatrix}$$

is such that $s(A_R) < 0$. Thus, $\mathbf{0}$ is GAS.

Remark 3.4 (Generalization). *Note that this reasoning extends to the general case with n patches, where $i = 1, 2, \dots, n$. Specifically, if $\gamma M_{s,i}^* > \frac{b_i}{\mu_{2,i}}$ for $i = 1, 2, \dots, n$, then the matrix A_R shares the same properties as the matrix A outlined in Remark 2.1. In particular, A_R is Hurwitz, which implies that $\mathbf{0}$ is Globally Asymptotically Stable (GAS).*

Second case: $\gamma M_{s,i}^* \leq \frac{b_i}{\mu_{2,i}}$ for $i = 1, 2$. Then,

$$A_R = \begin{pmatrix} (\sqrt{b_1} - \sqrt{\mu_{2,1}\gamma M_{s,1}^*})^2 - \mu_{1,1} - d_{21} & d_{12} \\ d_{21} & (\sqrt{b_2} - \sqrt{\mu_{2,2}\gamma M_{s,2}^*})^2 - \mu_{1,2} - d_{12} \end{pmatrix}.$$

The matrix A_R is again a Metzler matrix. To ensure that A_R is (strictly) column diagonally dominant, we need to find values of $\gamma M_{s,i}^*$ such that, for $i = 1, 2$,

$$\sum_{j=1}^2 (A_R)_{ji} = \left(\sqrt{b_i} - \sqrt{\mu_{2,i}\gamma M_{s,i}^*} \right)^2 - \mu_{1,i} < 0. \quad (9)$$

Since $\gamma M_{s,i}^* \leq \frac{b_i}{\mu_{2,i}}$, straightforward calculations show that (9) holds when $M_{s,i}^{\text{crit}} < \gamma M_{s,i}^*$ for $i = 1, 2$, where $M_{s,i}^{\text{crit}}$ is defined in (8). Moreover, if $M_{s,i}^{\text{crit}} < \gamma M_{s,i}^*$ for $i = 1, 2$, it automatically ensures that $(A_R)_{ii} < 0$, because

$$\begin{aligned} (A_R)_{ii} < 0 &= \left(\sqrt{b_i} - \sqrt{\mu_{2,i}\gamma M_{s,i}^*} \right)^2 - \mu_{1,i} - \sum_{\substack{j=1 \\ j \neq i}}^2 d_{j,i} \\ &< \left(\sqrt{b_i} - \sqrt{\mu_{2,i}\gamma M_{s,i}^*} \right)^2 - \mu_{1,i} < 0, \end{aligned}$$

for $i = 1, 2$, thanks to (9). Therefore, when $M_{s,i}^{\text{crit}} < \gamma M_{s,i}^* \leq \frac{b_i}{\mu_{2,i}}$ for $i = 1, 2$, we can conclude that A_R is Hurwitz. Then, we derive the following result for a 2-patches SIT-system

Theorem 3.5. *Assume $\mathcal{N}_i > 1$ and $d_{ij} \geq 0$ for $1 \leq i \neq j \leq 2$. Then, the following results holds*

- (a) *When $M_{s,i}^* > 0$, $i = 1, 2$, then $\mathbf{0}$ is always LAS.*
- (b) *When $\gamma M_{s,1}^* > M_{s,1}^{\text{crit}}$ and $\gamma M_{s,2}^* > M_{s,2}^{\text{crit}}$, then $\mathbf{0}$ is GAS.*

In fact, the reasoning above extends to n connected patches, allowing the previous theorem to be generalized to

Theorem 3.6. *For all $i = 1, \dots, n$, assume $\mathcal{N}_i > 1$, and $d_{i,j} \geq 0$ for $1 \leq i \neq j \leq n$. Then, the following results holds*

- (a) *When $M_{s,i}^* > 0$, $i = 1, 2, \dots, n$, then $\mathbf{0}$ is always LAS.*
- (b) *When $\gamma M_{s,i}^* > M_{s,i}^{\text{crit}}$, $i = 1, 2, \dots, n$, then $\mathbf{0}$ is GAS.*

Remark 3.7. *This result, while not optimal regarding critical release rates, provides a way to estimate the minimum number of sterile insects to release needed to ensure long-term elimination, even without knowing the migration rates between patches.*

Remark 3.8 (The particular case $u_2 = 0$). *Using Theorem 3.5, we can analyze the specific case where $u_2 = 0$, meaning that releases occur only in Patch 1, with $d_{21} > 0$. Applying (7), we obtain*

$$(M_{s,1}^*, M_{s,2}^*) = \left(\frac{\mu_S + \omega d_{12}}{\mu_S + \omega d_{12} + \omega d_{21}}, \frac{\omega d_{21}}{\mu_S + \omega d_{12} + \omega d_{21}} \right) \frac{u_1}{\mu_S}.$$

According to Theorem 3.5, we have to check that

$$\gamma \frac{\mu_S + \omega d_{12}}{(\mu_S + \omega d_{12} + \omega d_{21})} \frac{u_1}{\mu_S} > M_{s,1}^{\text{crit}} \quad \text{and} \quad \gamma \frac{\omega d_{21}}{(\mu_S + \omega d_{12} + \omega d_{21})} \frac{u_1}{\mu_S} > M_{s,2}^{\text{crit}},$$

that is, the release rate in Patch 1 must satisfy the following constraint to achieve elimination in both patches: $u_1 > \max\{u_1^{crit}, u_2^{crit}\}$ where

$$u_1^{crit} = \frac{\mu_S (\mu_S + \omega d_{12} + \omega d_{21})}{\gamma (\mu_S + \omega d_{12})} M_{s,1}^{crit} \quad \text{and} \quad u_2^{crit} = \frac{\mu_S (\mu_S + \omega d_{12} + \omega d_{21})}{\gamma \omega d_{21}} M_{s,2}^{crit}. \quad (10)$$

Thanks to (10), we can highlight the following points:

- If the dispersal of the sterile insects is significantly reduced by the sterilization process, i.e., $\omega \ll 1$, then, according to the second condition, u_2^{crit} can be very large. This indicates that the release rate u_1 in Patch 1 must also be very large.
- Even if $\omega = 1$, meaning the dispersal ability of the sterile insects is unaffected by the sterilization process, a problem arises if d_{21} , the migration rate from Patch 1 to Patch 2, is low, i.e., $d_{21} \ll 1$. This is particularly noteworthy because it shows that reducing the displacement between patches, for instance by using traps, can negatively impact the control when sterile insects are only released in Patch 1.
- When $d_{12} = 0$, meaning no insects are migrating from Patch 2 to Patch 1, elimination in Patch 1 is possible only if $\gamma M_{s,1}^* > M_{s,1}^{crit}$. Thus, if the targeted area is Patch 1, it seems feasible to significantly reduce the wild population by releasing sterile insects only in Patch 1.

In the general case, where $u_i > 0$ for $i = 1, 2$, using (7), we need to select u_1 and u_2 such that

$$\gamma \frac{u_1 (\mu_S + \omega d_{21}) + \omega d_{21} u_2}{(\mu_S + \omega d_{12} + \omega d_{21}) \mu_S} > M_{s,1}^{crit} \quad \text{and} \quad \gamma \frac{u_2 (\mu_S + \omega d_{12}) + \omega d_{12} u_1}{(\mu_S + \omega d_{12} + \omega d_{21}) \mu_S} > M_{s,2}^{crit}, \quad (11)$$

that is

$$\frac{u_1 (\mu_S + \omega d_{21}) + \omega d_{21} u_2}{(\mu_S + \omega d_{12} + \omega d_{21})} > \frac{\mu_S}{\gamma} M_{s,1}^{crit} =: u_{s,1}^{crit} \quad \text{and} \quad \frac{u_2 (\mu_S + \omega d_{12}) + \omega d_{12} u_1}{(\mu_S + \omega d_{12} + \omega d_{21})} > \frac{\mu_S}{\gamma} M_{s,2}^{crit} =: u_{s,2}^{crit}.$$

Remark 3.9. According to (11), the parameters γ and μ_S can influence the release rates. Specifically, a larger value of γ (indicating better competitiveness) and/or a smaller value of μ_S (indicating a longer lifespan) will reduce the necessary release rates. This underscores the importance of accurately estimating both parameters. These parameters depend on the radiation dose used for sterilization as well as any additional treatments, such as Ginger oil root, which has been shown to increase the competitiveness parameter γ (see [22] and references therein).

To determine the minimal total release rate $u_1 + u_2$, where $u = (u_1, u_2)^T \geq 0$, we need to solve the inequality (11). Specifically, we formulate the following linear optimization program: find $u = (u_1, u_2)^T$ such that

$$\inf\{u_1 + u_2, -\mathcal{A}_2^{-1}u \geq b_2 \text{ and } u \geq 0\}, \quad (12)$$

where

$$-\mathcal{A}_2^{-1} = \frac{\gamma}{(\mu_S + \omega d_{12} + \omega d_{21}) \mu_S} \begin{pmatrix} \mu_S + \omega d_{12} & \omega d_{12} \\ \omega d_{21} & \mu_S + \omega d_{21} \end{pmatrix}, \quad b_2 = \begin{pmatrix} M_{s,1}^{crit} \\ M_{s,2}^{crit} \end{pmatrix}.$$

This approach will help determine both the minimal total release rate required to achieve elimination in both patches and the optimal distribution of this rate between the two patches. In essence, this result will specify the minimum daily production of sterile insects needed.

Remark 3.10 (Generalization). Similarly, the previous reasoning can be extended to n connected patches. In this case, we need to determine the vector $u = (u_1, \dots, u_n)^T$ that satisfies the following linear programming problem:

$$\inf \left\{ \sum_{i=1}^n u_i, -\mathcal{A}^{-1}u \geq b_n \text{ and } u \geq 0 \right\},$$

where the matrix \mathcal{A} has been introduced in Section 2.2, and $b_n = (M_{s,i}^{crit})_{1 \leq i \leq n} \in \mathbb{R}^{n \times 1}$.

Note that some optimal release rates may eventually be zero. In Section 6, we will present simulations for the two-patch case and discuss the results.

3.3 Proof of Proposition 2.6

According to Theorem 3.5, the set of admissible controls $\mathcal{U}_{\varepsilon, M}$ is non-empty provided that T and M are large enough. We now investigate the existence property. For the sake of clarity, we temporarily denote by $(P_1^u, P_2^u, M_{s,1}^u, M_{s,2}^u)$ the solution of System (6) associated with u .

Let $(u_n)_{n \in \mathbb{N}}$ be a minimizing sequence. According to the Banach-Alaoglu-Bourbaki theorem, the set $L^\infty(0, T; [0, M])$ is compact for the weak- $*$ topology of $L^\infty(0, T)$ and, up to a subsequence, $(u_n)_{n \in \mathbb{N}}$ converges to $u \in L^\infty(0, T; [0, M])$. The functional J is obviously continuous with respect to the weak- $*$ topology of $L^\infty(0, T)$, so it remains to prove that the set $\{u \in L^\infty(0, T) \mid P_1^u(T) \leq \varepsilon\}$ is closed for this topology, in other words, that $P_1^u(T) \leq \varepsilon$. First, we have

$$M_s^{u_n}(t) = \int_0^t e^{-(\mu_s I_2 + D_S)(t-s)} u_n(s) ds, \quad \text{for all } t \geq 0.$$

It follows that $(M_s^{u_n})_{n \in \mathbb{N}}$ is bounded in $W^{1, \infty}(0, T)$ and therefore converges, up to a subsequence, to M_s^u strongly in $C^0([0, T])$ by the Ascoli theorem.

Furthermore, using the fact that $f(0, M) = 0$ for all $M \geq 0$, we infer from the Cauchy-Lipschitz theorem that $P_i(\cdot) \geq 0$ for $i = 1, 2$. It follows that

$$\frac{dP_i^{u_n}}{dt}(t) \leq b_i P_i^{u_n}(t) + \|D\| \|P^{u_n}\|, \quad t \geq 0, \quad i = 1, 2,$$

where $\|D\|$ denotes the spectral norm of D . Therefore, $\left\| \frac{dP^{u_n}}{dt} \right\|(t) \leq (\|b_1, b_2\| + \|D\|) \|P^{u_n}(t)\|$ for all $t \geq 0$, and it follows from Gronwall's lemma that $\|P^{u_n}\|$ is uniformly bounded. By applying the Ascoli theorem as above, it follows that, up to a subsequence, $(P^{u_n})_{n \in \mathbb{N}}$ converges in $C^0([0, T])$ towards some element P . We continue to denote this subsequence by $(P^{u_n})_{n \in \mathbb{N}}$ with a slight abuse of notation. Now, passing to the limit in the relation

$$P^{u_n}(t) = P^{u_n}(0) + \int_0^t \left(\begin{pmatrix} f_1(P_1^{u_n}, M_{s,1}^{u_n}) \\ f_2(P_2^{u_n}, M_{s,2}^{u_n}) \end{pmatrix} + DP(s) \right) ds, \quad t \geq 0,$$

yields that the pair (P^u, M_s^u) solves the system (6). Finally, from the inequality $P_1^{u_n}(T) \leq \varepsilon$, we obtain $P_1^u(T) \leq \varepsilon$ by C^0 convergence. This concludes the proof.

4 Analysis of the optimal control strategy (proof of Theorem 2.7)

4.1 First order optimality conditions

Let (u_1, u_2) be an optimal control for Problem $(\mathcal{P}_{\varepsilon, M})$ and (P, M_s) be the corresponding trajectories, solutions of (6) for u . To write the first-order optimality conditions, we will use the Pontryagin Maximum Principle (PMP). To take into account the integral constraint on u_i ($i = 1, 2$), it is convenient to introduce an artificial state variable y_i solving the o.d.e.

$$y'(t) = u_1(t) + u_2(t) \quad \text{on } [0, T] \quad \text{and} \quad y(0) = 0$$

in such a way that the constraint $\int_0^T u + u_2 dt \leq C$ rewrites as the terminal condition $y(T) \leq C$.

Let us introduce the Hamiltonian of Problem $(\mathcal{P}_{\varepsilon, M})$, given by

$$\begin{aligned} \mathcal{H} : \mathbb{R}^4 \times \mathbb{R}^6 \times \mathbb{R}^2 \times \{-1, 0\} &\rightarrow \mathbb{R} \\ ((P, M_s, y), (q, \lambda), u, p^0) &\mapsto \left\langle \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \begin{pmatrix} f_1(P_1, M_{s,1}) \\ f_2(P_2, M_{s,2}) \end{pmatrix} + AP \right\rangle_{\mathbb{R}^2} + \left\langle \begin{pmatrix} q_3 \\ q_4 \end{pmatrix}, (D_S - \mu_s I_2) M_s + u \right\rangle_{\mathbb{R}^2} \\ &\quad + \lambda(u_1 + u_2), \end{aligned}$$

where, here and the rest, $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ will denote the standard inner product in \mathbb{R}^n . $\|\cdot\|_{\mathbb{R}^n}$ will denote the standard euclidean norm in \mathbb{R}^n . When there is no ambiguity, we will also write $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to denote those quantities.

According to the Maximum Principle (see, e.g. [33]), there exists an absolutely continuous mapping $q : [0, T] \rightarrow \mathbb{R}^4$ called adjoint vector such that the so-called extremal $((P, M_s, y), (p, \lambda), u)$ satisfies a.e. in $[0, T]$:

- *Adjoint equations:*

$$-\frac{d}{dt} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \left(\begin{pmatrix} \frac{\partial f_1}{\partial P_1}(P_1, M_{s1}) & 0 \\ 0 & \frac{\partial f_2}{\partial P_2}(P_2, M_{s2}) \end{pmatrix} + A^\top \right) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad (13)$$

$$-\frac{d}{dt} \begin{pmatrix} q_3 \\ q_4 \end{pmatrix} = (D_S^\top - \mu_s I_2) \begin{pmatrix} q_3 \\ q_4 \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial M_{s1}}(P_1, M_{s1}) & 0 \\ 0 & \frac{\partial f_2}{\partial M_{s2}}(P_2, M_{s2}) \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad (14)$$

and in addition, $d\lambda/dt = 0$ which implies that λ is a constant vector (still denoted λ with a slight abuse of notation).

- *Maximality condition:*

$$\text{for a.e. } t \in [0, T], \quad u^*(t) \text{ solves the problem } \max_{\substack{0 \leq v_1 \leq M_1 \\ 0 \leq v_2 \leq M_2}} \left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} q_3 \\ q_4 \end{pmatrix} \right\rangle_{\mathbb{R}^2} + \lambda(v_1 + v_2).$$

and therefore, one has

$$\begin{aligned} q_3 + \lambda \leq 0 \text{ on } \{u_1 = 0\}, \quad q_3 + \lambda = 0 \text{ on } \{0 < u_1 < M_1\}, \quad q_3 + \lambda \geq 0 \text{ on } \{u_1 = M_1\} \\ q_4 + \lambda \leq 0 \text{ on } \{u_2 = 0\}, \quad q_4 + \lambda = 0 \text{ on } \{0 < u_2 < M_2\}, \quad q_4 + \lambda \geq 0 \text{ on } \{u_2 = M_2\} \end{aligned} \quad (15)$$

- *Transversality conditions:* we impose the terminal conditions

$$q_1(T) = p^0, \quad q_2(T) = 0, \quad q_3(T) = q_4(T) = 0, \quad \lambda(T) = \lambda_T \quad (16)$$

on the adjoint state, where $\lambda_T \in \mathbb{R}_+$ satisfies moreover the complementary condition $\lambda_T(P_1(T) - C) = 0$.

We infer from (16) that $\lambda \geq 0$.

Furthermore, one has $p^0 = -1$. Indeed, in the converse case, one has $p^0 = 0$ and q solves a homogeneous linear system with $0_{\mathbb{R}^4}$ as final condition. It follows that $q = 0$ which is in contradiction with the fact that the pair (q, p^0) is nontrivial. The expected result follows.

Let us sum up hereafter the optimality conditions we have obtained.

$$\boxed{\begin{aligned} -\frac{d}{dt} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \left(\begin{pmatrix} \frac{\partial f_1}{\partial P_1}(P_1, M_{s1}) & 0 \\ 0 & \frac{\partial f_2}{\partial P_2}(P_2, M_{s2}) \end{pmatrix} + A^\top \right) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ -\frac{d}{dt} \begin{pmatrix} q_3 \\ q_4 \end{pmatrix} &= (D_S^\top - \mu_s I_2) \begin{pmatrix} q_3 \\ q_4 \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial M_{s1}}(P_1, M_{s1}) & 0 \\ 0 & \frac{\partial f_2}{\partial M_{s2}}(P_2, M_{s2}) \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ q_1(T) &= -1, \quad q_2(T) = 0, \quad q_3(T) = q_4(T) = 0 \\ q_3 - \lambda &\leq 0 \text{ on } \{u_1 = 0\}, \quad q_3 - \lambda = 0 \text{ on } \{0 < u_1 < M_1\}, \quad q_3 - \lambda \geq 0 \text{ on } \{u_1 = M_1\} \\ q_4 - \lambda &\leq 0 \text{ on } \{u_2 = 0\}, \quad q_4 - \lambda = 0 \text{ on } \{0 < u_2 < M_2\}, \quad q_4 - \lambda \geq 0 \text{ on } \{u_2 = M_2\} \\ \lambda &\geq 0 \end{aligned}} \quad (17)$$

Notice that for all $t \in [0, T]$, one has

$$\begin{pmatrix} q_3(t) \\ q_4(t) \end{pmatrix} = \int_t^T e^{-D_S^\top(t-s)} e^{\mu_s(t-s)} \begin{pmatrix} \frac{\partial f_1}{\partial M_{s1}}(P_1, M_{s1}) & 0 \\ 0 & \frac{\partial f_2}{\partial M_{s2}}(P_2, M_{s2}) \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}(s) ds.$$

The expression of the differential of J_1 in the next result is an immediate byproduct of the Pontryagin Maximum Principle above.

Lemma 4.1. *For $i = 1, 2$, let $u_i \in \mathcal{U}_{\varepsilon_i, M_i}$ and let h_i denote an admissible perturbation¹ at u_i . Consider $G : u \mapsto P_1(T)$. The functional G is differentiable at $u = (u_1, u_2)$ and one has*

$$DG(u)h = - \int_0^T (q_3 h_1 + q_4 h_2) dt.$$

¹More precisely, we call ‘‘admissible perturbation’’ any element of the tangent cone $\mathcal{T}_{u_i, \mathcal{U}_{\varepsilon_i, M_i}}$ to the set \mathcal{U}_{C_i, M_i} at u_i . Recall that the cone $\mathcal{T}_{u_i, \mathcal{U}_{\varepsilon_i, M_i}}$ is the set of functions $h_i \in L^\infty(0, T)$ such that, for any sequence of positive real numbers ε_n decreasing to 0, there exists a sequence of functions $h_{i,n} \in L^\infty(0, T)$ converging to h_i as $n \rightarrow +\infty$, and $u_i + \varepsilon_n h_{i,n} \in \mathcal{U}_{\varepsilon_i, M_i}$ for every $n \in \mathbb{N}$

4.2 A series of intermediate results

Let us recall or state several auxiliary results.

Lemma 4.2. *Let α and β denote two positive real numbers and set*

$$A = \begin{pmatrix} -\alpha & \beta \\ \alpha & -\beta \end{pmatrix}.$$

Then, for every $t \geq 0$, one has

$$\exp(tA) = \frac{1}{\alpha + \beta} \begin{pmatrix} \beta + \alpha e^{-t(\alpha+\beta)} & \beta(1 - e^{-t(\alpha+\beta)}) \\ \alpha(1 - e^{-t(\alpha+\beta)}) & \alpha + \beta e^{-t(\alpha+\beta)} \end{pmatrix}.$$

In particular, all the coefficients of $\exp(tA)$ are positive whenever $t \geq 0$.

Proof. This computation is straightforward, observing that

$$A = PDP^{-1} \quad \text{with } P = \begin{pmatrix} \beta & -1 \\ \alpha & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{\alpha + \beta} \begin{pmatrix} 1 & 1 \\ -\alpha & \beta \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & -(\alpha + \beta) \end{pmatrix}.$$

The result follows by using that $e^{tA} = Pe^{tD}P^{-1}$. □

Lemma 4.3. *Let $T > 0$ and let z denote the solution of the Cauchy system*

$$\begin{cases} z'' = Az + B & \text{in } [0, T], \\ z(0) = z'(0) = 0, \end{cases}$$

where A is positively continuous and B is a nonnegative measurable function. Then, z is nondecreasing and nonnegative.

Proof. Let z_ε denote the solution to the Cauchy system

$$\begin{cases} z_\varepsilon'' = Az_\varepsilon + B & \text{in } [0, T], \\ z_\varepsilon(0) = \varepsilon, \quad z_\varepsilon'(0) = 0. \end{cases}$$

We claim that z_ε is increasing (and therefore positive). Indeed, assume by contradiction the existence of $t_\varepsilon \in (0, T)$ such that $z'(t_\varepsilon) = 0$. Without loss of generality, we assume that t_ε is the first positive solution t of the equation $z'_\varepsilon(t) = 0$. Then, z_ε is positive and increasing on $[0, t_\varepsilon]$, and one has

$$0 = z'_\varepsilon(t_\varepsilon) = \int_0^{t_\varepsilon} (Az_\varepsilon + B) > \varepsilon \int_0^{t_\varepsilon} A > 0,$$

whence a contradiction. We conclude the proof of this lemma by using the continuity property of z_ε with respect to the parameter ε . From $z_\varepsilon(\cdot) > 0$ on $[0, T]$, we infer that $z_0 = z(\cdot) \geq 0$ on $[0, T]$. Finally, since

$$z'(t) = \int_0^{t_\varepsilon} (Az + B) \geq 0,$$

the expected conclusion follows. □

Lemma 4.4. *Let $n \in \mathbb{N}^*$ and K denote a compact set of \mathbb{R}^n . Let $T > 0$, $\varepsilon_0 > 0$, $t_0 \in [0, T]$, $\bar{y} \in \mathbb{R}^n$ and let y_ε denote the maximal solution to*

$$\begin{cases} y'_\varepsilon(t) = f(t, y_\varepsilon(t) + \eta_\varepsilon(t)) + U_\varepsilon(t)y_\varepsilon(t), \\ y_\varepsilon(t_0) = \bar{y}, \end{cases}$$

for $\varepsilon \in [0, \varepsilon_0]$, where f belongs to $C^1([0, T], K)$, η_ε belongs to $C^0([0, T]; \mathbb{R}^n)$ and U_ε belongs to $C^0([0, T], \mathcal{M}_n(\mathbb{R}))$. Let us assume that y_ε is defined on $[0, T]$ and that there exists $L_T > 0$ such that $\|U_\varepsilon\|_{C^0} \leq L_T\varepsilon$. There exists $k_T > 0$ such that

$$\|y_\varepsilon - y_0\|_{C^0([0, T])} \leq k_T(\varepsilon + \|\eta_\varepsilon - \eta_0\|_{C^0([0, T])}).$$

Proof. According to the assumptions, f is in particular Lipschitz with respect to its second variable. Let us hence introduce $K_T > 0$ such that

$$\forall(t, y_1, y_2) \in [0, T] \times K^2, \quad \|f(t, y_1) - f(t, y_2)\| \leq K_T \|y_1 - y_2\|.$$

Let us set $\delta y = y_\varepsilon - y_0$, where $y_0 = y_{\varepsilon=0}$. Thus, δy satisfies

$$\begin{cases} \delta y'(t) = f(t, y_\varepsilon(t) + \eta_\varepsilon(t)) - f(t, y_0(t) + \eta_0(t)) + U_\varepsilon(t)y_\varepsilon(t), \\ \delta y(t_0) = 0, \end{cases}$$

so that, by setting $r(t) = \|\delta y(t)\|_{\mathbb{R}^2}$, r satisfies $r(0) = 0$ and

$$\begin{aligned} r'(t) &= \frac{\langle y_\varepsilon(t) - y_0(t), y'_\varepsilon(t) - y'_0(t) \rangle}{r(t)} \\ &= \frac{\langle y_\varepsilon(t) - y_0(t), f(t, y_\varepsilon(t) + \eta_\varepsilon(t)) - f(t, y_0(t) + \eta_0(t)) \rangle}{r(t)} + \frac{\langle U_\varepsilon(t)y_\varepsilon(t), y_\varepsilon(t) - y_0(t) \rangle}{r(t)} \\ &\leq K_T r(t) + K_T \|\eta_\varepsilon(t) - \eta_0(t)\| + \|U_\varepsilon(t)\|_{\mathcal{M}_n(\mathbb{R})} \|y_\varepsilon(t)\| \\ &\leq (K_T + \|U_\varepsilon(t)\|_{\mathcal{M}_n(\mathbb{R})})r(t) + \|U_\varepsilon(t)\|_{\mathcal{M}_n(\mathbb{R})} \|y_0\|_{C^0([0, T])} + K_T \|\eta_\varepsilon(t) - \eta_0(t)\| \end{aligned}$$

by using the Cauchy-Schwarz and the triangle inequalities, where the notation $\|\cdot\|_{\mathcal{M}_n(\mathbb{R})}$ stands for the operator norm associated with the Euclidean one. Then, by applying a standard Gronwall lemma, and since $r(0) = 0$, one has

$$r(t) \leq \frac{L_T \|y_0\|_{C^0([0, T])} \varepsilon + K_T \|\eta_\varepsilon - \eta_0\|_{C^0([0, T])}}{K_T + L_T \varepsilon} \left(e^{(K_T + \varepsilon L_T)(t-t_0)} - 1 \right)$$

for all $t \geq t_0$. The desired result follows. \square

4.3 Saturation of the constraint

We investigate hereafter whether the integral constraint is activated.

Proposition 4.5. *Let $M > \bar{M}$, $T > \bar{T}$ and let us assume that $\omega > 0$. Let (u_1, u_2) denote a solution of Problem $(\mathcal{P}_{\varepsilon, M})$. Then, one has*

$$P_1(T) = \varepsilon.$$

Proof. Let us argue by contradiction, assuming that $P_1(T) < \varepsilon$. There exists $\xi \in L^\infty(0, T; \mathbb{R}_+)$ a measurable set $I \subset (0, T)$ such that $\xi > 0$ a.e. on I and $u_1 + \nu\xi$ is admissible whenever $\nu < 0$ is small enough.

For $i = 1, 2$, let us denote temporarily by $P_i[u]$ and $M_{s_i}[u]$ the solutions to System (6) to underline the dependency with respect to the control function u . According to this system, by linearity of M_{s_1} and M_{s_2} with respect to u , one has obviously

$$M_{s_i}[(u_1 + \nu\xi, u_2)](t) - M_{s_i}[(u_1, u_2)](t) = \nu\delta M_{s_i}, \quad i = 1, 2$$

for all $t \in [0, T]$, where $\delta M_{s_i}(t)$ is the i -th coordinate ($i = 1, 2$) of

$$\int_0^t e^{D_S(t-s)} e^{-\mu_S(t-s)} \xi(s) ds.$$

According to Lemma 4.2 and since $\xi > 0$ a.e. on I , we get that for $i = 1, 2$, the right-hand side δM_{s_i} does not vanish identically and is nonnegative on $[0, T]$. In other words, there exists $J \subset (0, T)$ of positive measure such that $\delta M_{s_i} > 0$ on J .

From now on, we will rather use the notation $P_i[M_{s_1}]$, $i = 1, 2$ to underline the dependency of P_i with respect to M_{s_1} . Let us now compute the differential of $P_i[M_{s_1}]$ with respect to M_{s_1} . We will not prove that the mapping $M_{s_1} \in C^0([0, T]) \mapsto P_i[u] \in H^1(0, T)$, $i = 1, 2$, is differentiable, since this follows from a standard argument based on the implicit function theorem.

Let us introduce the notations

$$\dot{P}_i = \lim_{\nu \searrow 0} \frac{P_i[M_{s_1} + \nu\delta M_{s_1}] - P_i[M_{s_1}]}{\nu}, \quad i = 1, 2.$$

An easy computation shows that the pair (\dot{P}_1, \dot{P}_2) solves the system

$$\begin{cases} \frac{d}{dt} \begin{pmatrix} \dot{P}_1 \\ \dot{P}_2 \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial f_1}{\partial P_1} - d_{21}\right) \dot{P}_1 + d_{12} \dot{P}_2 + \frac{\partial f_1}{\partial M_{s1}} \delta M_{s1} \\ \left(\frac{\partial f_2}{\partial P_2} - d_{12}\right) \dot{P}_2 + d_{21} \dot{P}_1 \end{pmatrix} & \text{in } [0, T], \\ (\dot{P}_1(0), \dot{P}_2(0)) = (0, 0), \end{cases}$$

where the partial derivatives of f_1 and f_2 are respectively evaluated at the trajectory (P_1, M_{s1}) and (P_2, M_{s2}) . We do not write it explicitly in these computations for the sake of notational simplicity.

Let us introduce the auxiliary functions γ_1 and γ_2 defined on $[0, T]$ by

$$\gamma_1(t) = d_{21}t - \int_0^t \frac{\partial f_1}{\partial P_1} \quad \text{and} \quad \gamma_2(t) = d_{12}t - \int_0^t \frac{\partial f_2}{\partial P_2}.$$

The system above can be recast as

$$\begin{cases} \frac{d}{dt} \begin{pmatrix} e^{\gamma_1} \dot{P}_1 \\ e^{\gamma_2} \dot{P}_2 \end{pmatrix} = \begin{pmatrix} e^{\gamma_1} \left(d_{12} \dot{P}_2 + \frac{\partial f_1}{\partial M_{s1}} \delta M_{s1} \right) \\ d_{21} e^{\gamma_2} \dot{P}_1 \end{pmatrix} & \text{in } [0, T], \\ (\dot{P}_1(0), \dot{P}_2(0)) = (0, 0). \end{cases} \quad (18)$$

Setting $w_i = e^{\gamma_i} \dot{P}_i$ for $i = 1, 2$, one gets

$$\begin{cases} \begin{pmatrix} \frac{dw_1}{dt} \\ e^{\gamma_1 - \gamma_2} \frac{dw_2}{dt} \end{pmatrix} = \begin{pmatrix} e^{\gamma_1} \left(d_{12} e^{-\gamma_2} w_2 + \frac{\partial f_1}{\partial M_{s1}} \delta M_{s1} \right) \\ d_{21} w_1 \end{pmatrix} & \text{in } [0, T], \\ (w_1(0), w_2(0)) = (0, 0), \end{cases}$$

and we infer that w_2 solves the ODE

$$\begin{cases} \frac{d}{dt} \left(e^{\gamma_1 - \gamma_2} \frac{dw_2}{dt} \right) = d_{21} d_{12} e^{\gamma_1 - \gamma_2} w_2 + d_{21} e^{\gamma_1} \frac{\partial f_1}{\partial M_{s1}} \delta M_{s1} & \text{in } [0, T], \\ (w_2(0), \frac{dw_2}{dt}(0)) = (0, 0). \end{cases}$$

Let us consider the standard change of variable for Sturm-Liouville type systems,

$$v = \int_0^t e^{\gamma_2(s) - \gamma_1(s)} ds$$

and let us set $z_2(v) := w_2(t)$ for all $t \in [0, T]$. Direct computations yield that

$$z_2''(v) = e^{\gamma_1(t) - \gamma_2(t)} \frac{d}{dt} \left(e^{\gamma_1(t) - \gamma_2(t)} \frac{dw_2}{dt}(t) \right),$$

and therefore,

$$\begin{cases} z_2''(v) = d_{21} d_{12} e^{2(\gamma_1(t) - \gamma_2(t))} z_2(v) + d_{21} e^{2\gamma_1(t) - \gamma_2(t)} \frac{\partial f_1}{\partial M_{s1}}(t) \delta M_{s1}(t) & \text{in } [0, U], \\ (z_2(0), \frac{dz_2}{dv}(0)) = (0, 0), \end{cases}$$

where t has now to be seen as a function of u and

$$V = \int_0^T e^{\gamma_2(s) - \gamma_1(s)} ds.$$

According to Lemma 4.3 applied to $z = -z_2$, since $\partial f / \partial M_{s1}(\cdot) < 0$, the function $z_2(\cdot)$ is non-positive on $[0, T]$. By undoing all the change of variables above, it follows that $\dot{P}_2(\cdot) \leq 0$ on $[0, T]$ and $\dot{P}_2(T) < 0$. Now, according to (18), $\frac{d}{dt}(e^{\gamma_1} \dot{P}_1)$ writes as the sum of two non-positive terms and does not vanish identically. Since moreover $\dot{P}_1(0) = 0$, we infer that $\dot{P}_1(T) < 0$.

Coming back to the beginning of this reasoning, one has

$$P_1[u_1 + \nu \xi, u_2](T) - P_1[u_1, u_2](T) = D_{M_{s1}} P_1(T) \cdot \delta M_{s1} + D_{M_{s2}} P_1(T) \cdot \delta M_{s2} + o(\nu),$$

where $D_{M_{si}} P_1(T)$ stands for the differential of $P_1(T)$ with respect to M_{si} , $i = 1, 2$. The quantity $\dot{P}_1(T)$ above coincides precisely with $D_{M_{s1}} P_1(T) \cdot \delta M_{s1}$. We claim that, mimicking all the computations above, one shows similarly that $D_{M_{s2}} P_1(T) \cdot \delta M_{s2} < 0$.

We have reached a contradiction. The desired result follows. \square

4.4 Estimates on P_1 and P_2

In this section, we highlight that the populations P_1 and P_2 are bounded.

Proposition 4.6. *Let $(P_1^0, P_2^0) \in (\mathbb{R}_+^*)^2$ and $(u_1, u_2) \in \mathcal{U}_{\varepsilon, M}$, then the functions $M_{s_1}, M_{s_2}, P_1, P_2$ are all positive on $[0, T]$. Moreover, one has the estimate*

$$P_1(t)^2 + P_2(t)^2 \leq \left(e^{\rho_m(d_{21}, d_{12})t} \sqrt{(P_1^0)^2 + (P_2^0)^2} + \frac{1}{\rho_m(d_{21}, d_{12})} \|b\| (e^{\lambda_m(d_{21}, d_{12})t} - 1) \right)^2$$

for every $t \in [0, T]$, where

$$\rho_m(d_{21}, d_{12}) = \frac{1}{2}(d_{21} + d_{12}) + \frac{1}{\sqrt{2}} \sqrt{d_{21}^2 + d_{12}^2} > 0.$$

Proof. By exploiting (6), we get

$$\begin{pmatrix} M_{s_1}(t) \\ M_{s_2}(t) \end{pmatrix} = \int_0^t e^{-\mu_s(t-s)} e^{(t-s)D_s} \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} ds.$$

Since $u = (u_1, u_2) \in \mathcal{U}_{\varepsilon, M}$, they are in particular non-negative. According to Lemma 4.2, all the coefficients of $e^{(t-s)D_s}$ are positive on $(0, T]$ and it follows that M_{s_1} and M_{s_2} are positive on $(0, T]$.

One has $(P_1(0), P_2(0)) = (P_1^0, P_2^0) \in (\mathbb{R}_+^*)^2$. To prove the positivity of P_1 and P_2 , let us argue by contradiction, assuming that either P_1 or P_2 vanishes on $(0, T]$. Let t_0 denote the first such time and assume without loss of generality that $P_1(t_0) = 0$. In particular, one has necessarily $dP_1/dt(t_0) \leq 0$. We have already seen that $P_2(t) \geq 0$ on $(0, t_0)$ and we claim moreover that $P_2(t_0) > 0$. Indeed, in the converse case, one would have $(P_1(t_0), P_2(t_0)) = (0, 0)$ which would yield, according to the Cauchy-Lipschitz theorem, that (P_1, P_2) would vanish identically on $[0, T]$. Thus, one has

$$\begin{pmatrix} f_1(P_1(t_0), M_{s_1}(t_0)) \\ f_2(P_2(t_0), M_{s_2}(t_0)) \end{pmatrix} + A \begin{pmatrix} P_1(t_0) \\ P_2(t_0) \end{pmatrix} = \begin{pmatrix} d_{12}P_2(t_0) \\ f_2(P_2(t_0), M_{s_2}(t_0)) - d_{12}P_2(t_0) \end{pmatrix}$$

and therefore, $dP_1/dt(t_0) > 0$, whence a contradiction.

By mimicking this reasoning, one sees similarly that if $t'_0 \in (0, T)$ denotes the first time such that $P_2(t'_0) = 0$ (whenever it exists), then, one has necessarily $P_1(t'_0) > 0$ and $dP_2/dt(t'_0) > 0$. Since we have moreover $dP_2/dt(t'_0) \leq 0$, we obtain a contradiction. The first claim follows.

Regarding the last claim of this lemma, let us introduce the function L given by

$$L(t) = \frac{1}{2}(P_1(t)^2 + P_2(t)^2).$$

Then, one has

$$\frac{dL}{dt} = P_1 f_1(P_1, M_{s_1}) + P_2 f_2(P_2, M_{s_2}) + \left\langle \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, D \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \right\rangle$$

for all $t \in [0, T]$.

Since $P_i > 0$ and $M_{s_i} \geq 0$ on $[0, T]$ for $i = 1, 2$, we infer that $P_i^2/(P_i + \gamma M_{s_i}) \leq P_i$ and therefore,

$$\frac{dL}{dt} \leq b_1 P_1 + b_2 P_2 + \left\langle \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, D \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \right\rangle.$$

Furthermore, set $D_s = \frac{1}{2}(D + D^\top)$ and $D_a = \frac{1}{2}(D - D^\top)$, so that $D = D_s + D_a$. One has

$$\langle DX, X \rangle = \langle D_s X, X \rangle \leq \rho_m(d_{12}, d_{21}) \|X\|^2$$

for all $X \in \mathbb{R}^2$. Indeed, the eigenvalues of D_s are $-\frac{1}{2}(d_{12} + d_{21}) \pm \frac{1}{\sqrt{2}} \sqrt{d_{12}^2 + d_{21}^2}$, whence the claim. Setting $b = [b_1, b_2]^\top$, it follows from the Cauchy-Schwarz inequality that

$$\frac{dL}{dt}(t) \leq 2\rho_m(d_{21}, d_{12})L(t) + \sqrt{2}\|b\|\sqrt{L(t)},$$

which rewrites

$$\frac{d}{dt} \left(L(t) e^{-2\rho_m(d_{21}, d_{12})t} \right) \leq \sqrt{2} \|b\| e^{-\rho_m(d_{21}, d_{12})t} \sqrt{e^{-2\rho_m(d_{21}, d_{12})t} L(t)}.$$

Integrating this inequality thus yields

$$\sqrt{e^{-2\rho_m(d_{21}, d_{12})t} L(t)} - \sqrt{L(0)} \leq \frac{1}{\sqrt{2}\rho_m(d_{21}, d_{12})} \|b\| (1 - e^{-\rho_m(d_{21}, d_{12})t})$$

and therefore

$$\sqrt{L(t)} \leq e^{\rho_m(d_{21}, d_{12})t} \sqrt{L(0)} + \frac{1}{\sqrt{2}\rho_m(d_{21}, d_{12})} \|b\| (e^{\rho_m(d_{21}, d_{12})t} - 1)$$

whence the claim. \square

4.5 Conclusion

Let us now assume that D and D_S are nonzero matrices. We respectively replace D and D_S by νD and νD_S . In that case, we respectively denote by $(P_1^\nu, P_2^\nu, M_{s_1}^\nu, M_{s_2}^\nu)$ and $(q_i^\nu)_{i \in \llbracket 1, 4 \rrbracket}$ the corresponding solutions to the state system (6) and its adjoint (13). In what follows, we will also denote by $u_\nu = (u_1^\nu, u_2^\nu)$ a solution to Problem $(\mathcal{P}_{\varepsilon, M})$.

For the sake of clarity, we split this proof into several steps.

Step 1. A Γ -convergence property. Let us temporarily denote by J_ν the quantity given by $J_\nu = \int_0^T (u_1^\nu + u_2^\nu) dt$ and by $\mathcal{U}_{\varepsilon, M}^\nu$, the set

$$\mathcal{U}_{\varepsilon, M}^\nu = \{u = (u_1, u_2) \in L^\infty(0, T; \mathbb{R}^2) \mid 0 \leq u_i \leq M_i \text{ a.e. in } (0, T), i = 1, 2 \text{ and } P_1^\nu(T) \leq \varepsilon\}.$$

We investigate the behaviour of a family of minimizers u_ν as $\nu \searrow 0$.

Lemma 4.7. *Let $(u_\nu)_{\nu > 0}$ denotes a family of minimizers over $\mathcal{U}_{\varepsilon, M}$ (i.e. for Problem $(\mathcal{P}_{\varepsilon, M})$), $(u_\nu)_{\nu > 0}$ converges, up to a subsequence, weakly-* in $L^\infty(0, T; \mathbb{R}^2)$ to a minimizer of J_0 over $\mathcal{U}_{\varepsilon, M}$ and*

$$\lim_{\nu \searrow 0} J_\nu = \lim_{\nu \searrow 0} \min_{u \in \mathcal{U}_{\varepsilon, M}^\nu} J(u) = \min_{u \in \mathcal{U}_{\varepsilon, M}^0} J(u) = J_0.$$

Proof. Recall that, according to Proposition 4.5, one has $P_1^\nu(T) = \varepsilon$. For simplicity, let us write $u^\nu = (u_1^\nu, u_2^\nu)$. Let us start from the inequality

$$J(u^\nu) \leq J(v) \tag{19}$$

for all $v = (v_1, v_2)$ in $L^\infty(0, T; [0, M_1] \times [0, M_2])$ such that $P_1^\nu[v](T) \leq \varepsilon$, where we write $P_1^\nu(T)[v]$ instead of $P_1^\nu(T)$ to underline the dependency in v . Observe first that, by compactness of $L^\infty(0, T; [0, M_1] \times [0, M_2])$ for the weak-* topology, the sequence (u_1^ν, u_2^ν) converges up to a subsequence for this topology towards some element \tilde{u}_0 . We aim at letting ν tend to 0 in (19). Since J is obviously continuous for this topology, the only point to check is the continuity of the mapping $\nu \mapsto P_1^\nu[u^\nu](T)$.

Note first that, identifying $u_\nu(s)$ with a column vector, one has

$$\begin{pmatrix} M_{s_1}^\nu(t) \\ M_{s_2}^\nu(t) \end{pmatrix} = \int_0^t e^{-\mu_s(t-s)} e^{(t-s)\nu D_S} u_\nu(s) ds \xrightarrow{\nu \searrow 0} \int_0^t e^{-\mu_s(t-s)} \tilde{u}_0(s) ds = \begin{pmatrix} \tilde{M}_{s_1}^0(t) \\ \tilde{M}_{s_2}^0(t) \end{pmatrix},$$

by weak convergence, where M_s^0 is associated to the control \tilde{u}_0 . We infer that M_s^ν converge pointwise (and even continuously) towards \tilde{M}_s^0 in $[0, T]$. Recall moreover that, according to Proposition 4.6, (P_1^ν, P_2^ν) is bounded in $C^0([0, T])$. Since (P_1^ν, P_2^ν) solves the ODE system (6), we infer that $(dP_1^\nu/dt, dP_2^\nu/dt)$ is also bounded in $C^0([0, T])$. According to the Ascoli theorem, it follows that (P_1^ν, P_2^ν) converges up to a subsequence towards some limit in $C^0([0, T])$. Note that this limit is necessarily the solution $(\tilde{P}_1^0, \tilde{P}_2^0)$ associated to \tilde{u}_0 . This can be easily seen by rewriting the ODE system as the fixed-point system

$$\begin{pmatrix} P_1^\nu(t) \\ P_2^\nu(t) \end{pmatrix} = \begin{pmatrix} P_1^0 \\ P_2^0 \end{pmatrix} + \int_0^t \left[\begin{pmatrix} f_1(P_1^\nu, M_{s_1}^\nu) \\ f_2(P_2^\nu, M_{s_2}^\nu) \end{pmatrix} + D \begin{pmatrix} P_1^\nu \\ P_2^\nu \end{pmatrix} \right],$$

passing to the limit, and concluding by the uniqueness of solutions of the resulting Cauchy-Lipschitz system. We thus infer that

$$\varepsilon = \lim_{\nu \searrow 0} P_1^\nu(T) = \tilde{P}_1^0(T).$$

It remains to prove that \tilde{u}_0 solves Problem $(\mathcal{P}_{\varepsilon, M})$ for $\nu = 0$. Let u_0 denote a solution of Problem $(\mathcal{P}_{\varepsilon, M})$ without migration ($\nu = 0$). We define $v_\nu = u_0 + h_\nu$ with $h_\nu = (\sqrt{\nu}(M - u_0^1), 0)$. By construction, v_ν belongs to $L^\infty(0, T; [0, M]^2)$ whenever ν is small enough. Introduce the mapping $G : (\nu, u) \mapsto P_1^\nu[u](T)$. It is well-known that, for a fixed u , the mapping $\nu \mapsto G(\nu, u)$ is differentiable due to the continuous dependence of the solution of an ODE system on its parameters. Using Lemma 4.1, we get that

$$\begin{aligned} P_1^\nu[v_\nu](T) &= P_1^0[u_0](T) + D_u G(0, u_0)h_\nu + \nu \left. \frac{\partial G(0, u_0)}{\partial \nu} \right|_{\substack{u=u_0 \\ \nu=0}} + o(\nu, \|h_\nu\|_{L^\infty}) \\ &= P_1^0[u_0](T) - \sqrt{\nu} \int_0^T (M - u_0)q_3^0 + o(\sqrt{\nu}). \end{aligned}$$

Since there exists $C > 0$ such that $q_3^0(\cdot) > C > 0$, then, for ν small enough, $P_1^\nu[v_\nu](T) \leq \varepsilon$. It allows us to take $v = v_\nu$ in (19) and letting ν tend to $+\infty$, we obtain

$$J(\tilde{u}_0) \leq J(u_0).$$

Thus \tilde{u}_0 is a minimiser and the conclusion follows. \square

Step 2. Estimating the adjoint states for small migration rates. Let us provide several estimates of the solutions to (6) associated to a minimizer $u_\nu = (u_1^\nu, u_2^\nu)$. As previously, we will use either the superscript ν or the subscript ν to underline the dependency on ν . Observe first that the state system (6) in that case can be recast under the form

$$Y_\nu' = F(Y_\nu) + \nu M Y_\nu + U_\nu,$$

where

$$Y_\nu = \begin{pmatrix} P_1^\nu \\ P_2^\nu \\ M_{s1}^\nu \\ M_{s2}^\nu \end{pmatrix}, \quad F \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_3) \\ f_2(y_2, y_4) \\ -\mu_S y_3 \\ -\mu_S y_4 \end{pmatrix}, \quad M = \begin{pmatrix} [c|c]D & 0 \\ 0 & D_S \end{pmatrix}, \quad U_\nu = \begin{pmatrix} 0 \\ 0 \\ u_1^\nu \\ u_2^\nu \end{pmatrix}.$$

We aim at applying Lemma 4.4. Nevertheless, as such, it cannot be directly used because of the source term U_ν . To overcome this difficulty, let us introduce Z_ν given by

$$Z_\nu(t) = \int_0^t U_\nu(s) ds$$

and recast the system above as

$$W_\nu' = F(W_\nu + Z_\nu) + \nu M(W_\nu + Z_\nu),$$

where we have set $W_\nu = Y_\nu - Z_\nu$. Under this form, one can now apply Lemma 4.4, showing the existence of $C_T^1 > 0$ such that

$$\|W_\nu - W_0\|_{C^0([0, T])} \leq C_T^1 \left(\nu + \sup_{t \in [0, T]} \left\| \int_0^t (u_\nu(s) - u_0(s)) ds \right\| \right)$$

and therefore there exists $C_T^2 > 0$ such that

$$\|Y_\nu - Y_0\|_{C^0([0, T])} \leq C_T^2 \left(\nu + \sup_{t \in [0, T]} \left\| \int_0^t (u_\nu(s) - u_0(s)) ds \right\| \right). \quad (20)$$

Let us now provide estimates of the adjoint state Q_ν . The adjoint state equations (17) can be recast as

$$-Q_\nu'(t) = \hat{H}(t)Q_\nu(t) + H_\nu(t)Q_\nu(t),$$

where

$$Q_\nu = \begin{pmatrix} q_1^\nu \\ q_2^\nu \\ q_3^\nu \\ q_4^\nu \end{pmatrix}, \quad \hat{H} = \begin{pmatrix} \frac{\partial f_1}{\partial P_1}(P_1^0, M_{s1}^0) & 0 & 0 & 0 \\ 0 & \frac{\partial f_2}{\partial P_2}(P_2^0, M_{s2}^0) & 0 & 0 \\ \frac{\partial f_1}{\partial M_{s1}}(P_1^0, M_{s1}^0) & 0 & -\mu_s & 0 \\ 0 & \frac{\partial f_2}{\partial M_{s2}}(P_2^0, M_{s2}^0) & 0 & -\mu_s \end{pmatrix}$$

and

$$H_\nu = \nu \begin{pmatrix} [c|c]D^\top & 0 \\ 0 & D_S^\top \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial P_1}(P_1^\nu, M_{s1}^\nu) - \frac{\partial f_1}{\partial P_1}(P_1^0, M_{s1}^0) & 0 & 0 & 0 \\ 0 & \frac{\partial f_2}{\partial P_2}(P_2^\nu, M_{s2}^\nu) - \frac{\partial f_2}{\partial P_2}(P_2^0, M_{s2}^0) & 0 & 0 \\ \frac{\partial f_1}{\partial M_{s1}}(P_1^\nu, M_{s1}^\nu) - \frac{\partial f_1}{\partial M_{s1}}(P_1^0, M_{s1}^0) & 0 & 0 & 0 \\ 0 & \frac{\partial f_2}{\partial M_{s2}}(P_2^\nu, M_{s2}^\nu) - \frac{\partial f_2}{\partial M_{s2}}(P_2^0, M_{s2}^0) & 0 & 0 \end{pmatrix}.$$

Using (20), it is easy to see that

$$\|H_\nu\|_{C^0([0,T];\mathcal{M}_4(\mathbb{R}))} = O\left(\nu + \sup_{t \in [0,T]} \left\| \int_0^t (u_\nu(s) - u_0(s)) ds \right\|\right) = o(1) \quad \text{as } \nu \searrow 0,$$

the last equality resulting from Lemma 4.7, and it follows hence directly from Lemma 4.4 and from (20) that there exists $C_T^3 > 0$ such that

$$\|Q_\nu - Q_0\|_{C^0([0,T])} \leq C_T^3 \left(\nu + \sup_{t \in [0,T]} \left\| \int_0^t (u_\nu(s) - u_0(s)) ds \right\| \right). \quad (21)$$

Step 3: Conclusion. Let us choose $\bar{M} > 0$ such that

$$\int_0^T (u_1^0 + u_2^0) = \int_0^T u_1^0 \leq \frac{T\bar{M}}{2}$$

and let us assume ν small enough so that

$$\int_0^T (u_1^\nu + u_2^\nu) \leq \frac{3T\bar{M}}{4}. \quad (22)$$

which is possible according to the Γ -convergence property stated in Lemma 4.7.

For the sake of notational simplicity, let us denote by R_ν the remainder term

$$R_\nu := \nu + \sup_{t \in [0,T]} \left\| \int_0^t (u_\nu(s) - u_0(s)) ds \right\|$$

and observe that $(R_\nu)_{\nu > 0}$ tends to 0 as $\nu \searrow 0$ by virtue of Lemma 4.7.

According to (21), we infer the existence of $C_T^4 > 0$ such that

$$q_3^\nu(t) \geq -C_T^4 R_\nu + q_3^0(t) \quad \text{and} \quad q_4^\nu(t) \leq C_T^4 R_\nu \quad (23)$$

for all $t \in [0, T]$.

By applying Proposition 4.6 in the special case where $A = 0$ and using

$$q_1^0(t) = -\exp\left(\int_0^t \frac{\partial f_1}{\partial P_1}(P_1(s), M_{s1}(s)) ds\right),$$

we get the existence of $C_T^5 > 0$ such that $q_1^0(\cdot) \leq -C_T^5$ on $[0, T]$, and using that

$$q_3^0(t) = -\int_t^T e^{\mu_s(t-s)} \frac{\partial f_1}{\partial M_{s1}}(P_1, M_{s1}) \exp\left(-\int_0^s \frac{\partial f_1}{\partial P_1}(P_1(\xi), M_{s1}(\xi)) d\xi\right) ds,$$

we infer the existence of $C_T^6 > 0$ such that

$$q_3^0(t) \geq C_T^6 \int_t^T e^{\mu_s(t-s)} ds = \frac{C_T^6}{\mu_s} (1 - e^{\mu_s(t-T)}),$$

so that

$$q_3^\nu(t) \geq -C_T^4 R_\nu + \frac{C_T^6}{\mu_s} (1 - e^{\mu_s(t-T)})$$

for all $t \in [0, T]$.

Let $\lambda_\nu \geq 0$ denote the Lagrange multiplier introduced in the optimality system (17). Now, we have to distinguish between two cases: if $\lambda_\nu > C_T^4 R_\nu$, then $q_4^\nu(\cdot) < \lambda_\nu$ a.e. in $[0, T]$ according to (23), and $u_2^\nu(\cdot) = 0$. In the converse case, namely $\lambda_\nu \leq C_T^4 R_\nu$, an elementary computation shows that

$$t < T_\nu \quad \Rightarrow \quad q_3^\nu(t) \geq -C_T^4 R_\nu + \frac{C_T^6}{\mu_s} (1 - e^{\mu_s(t-T)}) > C_T^4 R_\nu \geq \lambda_\nu,$$

where

$$T_\nu := T + \frac{1}{\mu_s} \ln \left(1 - \frac{2\mu_s C_T^4 R_\nu}{C_T^6} \right) = T + O(\nu).$$

It follows that one can assume without loss of generality that ν has been chosen small enough so that

$$T_\nu \geq \frac{9}{10}T.$$

According to (17), one has $u_1^\nu = M_1$ on $[0, T_\nu]$, and therefore

$$\int_0^T u_1^\nu \geq T_\nu M_1 \geq \frac{9TM_1}{10} \geq \frac{9T\bar{M}}{10}.$$

Combining this estimate with (22), we obtain

$$\frac{9T\bar{M}}{10} \leq \int_0^T u_1^\nu \leq \int_0^T (u_1^\nu + u_2^\nu) \leq \frac{3T\bar{M}}{4},$$

a contradiction. It follows that the case where $\lambda_\nu \leq C_T^4 R_\nu$ cannot arise whenever ν is small enough and one necessarily has $u_2^\nu(\cdot) = 0$ in $[0, T]$.

5 Characterisation of the optimal control (proof of Theorem 2.9)

First, let us consider the scenario with no migration, i.e., $D = D_S = 0_{\mathcal{M}_2(\mathbb{R})}$. In this case, System (6) becomes uncoupled. For this reason, it is straightforward that any pair (u_1, u_2) solving problem $(\mathcal{P}_{\varepsilon, M})$ has to satisfy $u_2(\cdot) = 0$. It follows that the optimal control problem $(\mathcal{P}_{\varepsilon, M})$ can be reformulated as

$$\inf_{u_1 \in \mathcal{U}_{\varepsilon, M_1}^0} J_1^0(u_1), \quad (24)$$

where $\varepsilon > 0$, $M_1 > 0$ are given,

$$J_1^0(u_1) = \int_0^T u_1$$

and

$$\mathcal{U}_{\varepsilon, M_1}^0 = \{u_1 \in L^\infty(0, T; \mathbb{R}) \mid 0 \leq u_1 \leq M_1 \text{ a.e. in } (0, T) \text{ and } P_1(T) \leq \varepsilon\},$$

and P_1 is solution to the time evolution

$$\begin{cases} \frac{d}{dt} P_1 = f_1(P_1, M_{s,1}) & \text{in } [0, T], \\ \frac{d}{dt} M_{s,1} = -\mu_s M_{s,1} + u_1 & \text{in } [0, T], \\ P_1(0) = P_1^* \in \mathbb{R}_+^* \end{cases} \quad (25)$$

with

$$f_1(P_1, M_{s,1}) = \left(b_1 \frac{P_1}{P_1 + \gamma M_{s,1}} - \mu_{1,1} - \mu_{2,1} P_1 \right) P_1,$$

where $b_1 > 0$, $\mu_{1,1} > 0$ and $\mu_{2,1} > 0$ are given.

Proposition 5.1 (Characterisation of the control without migration). *Assume that $D = D_S = 0$ and $\mu_s \geq \mu_1$. There exists $\bar{T} > 0$ and \bar{M} such that for each $T > \bar{T}$ and $M_1 > \bar{M}$, there exists $t_0, t_1 \in [0, T]$ such that*

- $u_1 = 0$ on $[0, t_0] \cup [t_1, T]$,
- $u_1 \in (0, M_1)$ on (t_0, t_1) with

$$u_1(t) = \left(\frac{\partial^2 f_1}{\partial M_{s,1}^2} \right)^{-1} \left(\frac{\partial f_1}{\partial M_{s,1}} \frac{\partial f_1}{\partial P_1} + \mu_{1,s} M_{s,1} \frac{\partial^2 f_1}{\partial M_{s,1}^2} - f_1 \frac{\partial^2 f_1}{\partial M_{s,1} \partial P_1} \right) \Bigg|_{(P_1(t), M_{s,1}(t))}. \quad (26)$$

Moreover, if \bar{U} is large enough, then $u_1 = 0$ on $(0, t_0)$.

Before presenting the proof of this proposition, we will establish a series of useful lemmas. The dual system (13) can be reduced to

$$\begin{cases} -q'_1 = \frac{\partial f_1}{\partial P_1} q_1 & \text{in } [0, T], \\ -q'_3 = \frac{\partial f_1}{\partial M_{s,1}} q_1 - \mu_s q_3 & \text{in } [0, T], \\ q_1(T) = 1, \quad q_3(T) = 0 \end{cases} \quad (27)$$

and the first optimality conditions become

$$\begin{aligned} q_3 - \lambda &\leq 0 \text{ on } \{u_1 = 0\}, \\ q_3 - \lambda &= 0 \text{ on } \{0 < u_1 < M_1\}, \\ q_3 - \lambda &\geq 0 \text{ on } \{u_1 = M_1\}, \\ \lambda &\geq 0. \end{aligned}$$

Indeed, in (13), if $q_2(T) = q_4(T) = 0$, we have $q_2 = q_4 = 0$.

Lemma 5.2. *Assume that $D = D_S = 0$. There exists $\bar{T} > 0$ and \bar{M} such that for each $T > \bar{T}$ and $M_1 > \bar{M}$ such that the solution to (24) satisfies $P_1(T) = \varepsilon$.*

For the proof, we will make a minor adjustment to the proof of Proposition 4.5.

Lemma 5.3. *Assume $D = D_S = 0$. There exist constants $\bar{T} > 0$ and \bar{M} such that for every $T > \bar{T}$ and $M_1 > \bar{M}$, if $u = u_1^*$ is a solution to the optimal control problem (24), and $(P_1, M_{s,1})$ is the corresponding solution to (25), then $P'_1 \leq 0$ on the interval $(0, T)$.*

We refer to [2, Lemma 4.10] for a proof.

Lemma 5.4. *Assume that $D = D_S = 0$. There exists $\bar{T} > 0$ and \bar{M} such that for each $T > \bar{T}$ and $M_1 > \bar{M}$, there exist three positive constant C_0, C_1 and C_2 , that only depend on \bar{T} and \bar{M} , such that*

$$|M_{1,s}| < C_0 \quad \text{and} \quad 0 < C_1 < q_1, q_3, P_1 < C_2 \quad \text{on } [0, T].$$

We refer to [2, Lemma 4.13] for a proof. The following lemma is derived through straightforward computations, the details of which are omitted for brevity.

Lemma 5.5. *We have*

$$\frac{\partial f_1}{\partial M_{s,1}} = \frac{-b_1 \gamma P_1^2}{(P_1 + \gamma M_{s,1})^2} < 0 \quad \text{and} \quad \frac{\partial^2 f_1}{\partial M_{s,1}^2} = \frac{b_1 \gamma^2 P_1^2}{(P_1 + \gamma M_{s,1})^3} > 0.$$

Building on these intermediate results, we are now ready to prove the Proposition 5.1.

Proof of Proposition 5.1. For a complete proof, we refer to Theorem 3.3 in [2], as the behavior of the function f_1 with respect to $M_{s,1}$ is analogous to our case. Indeed, as shown by Lemma 5.5.

$$\frac{\partial f_1}{\partial M_{s,1}} < 0 \quad \text{and} \quad \frac{\partial^2 f_1}{\partial M_{s,1}^2} > 0.$$

The main difference lies in the expression of the function f_1 , which leads to variations in the final computation. Additionally, in the present proof, we require explicit bounds that remain independent of ε , especially to handle the case with small migration. Note that

$$-q_3'' = \mathcal{G}(P_1, M_{s,1}, u_1)q_1 - \mu_s q_3',$$

where

$$\mathcal{G}(P_1, M_{s,1}, u_1) = \frac{\partial^2 f_1}{\partial M_{s,1}^2}(u_1 - \mu_s M_{1,s}) + \frac{\partial^2 f_1}{\partial M_{s,1} \partial P_1} f_1 - \frac{\partial f_1}{\partial M_{s,1}} \frac{\partial f_1}{\partial P_1}.$$

Let I be an open interval of (the open set) $\{q_3 > \lambda\}$. The main idea, inspired by [2, Proof of Theorem 3.3], is as follows: if q_3'' maintains a constant sign on I , then I can be extended to an interval $(0, t_0)$ or (t_1, T) . In other words, the set where $q_3 > \lambda$ consists of either a single interval that reaches one boundary of $[0, T]$, or the union of two intervals, each touching an opposite extreme point of $[0, T]$.

Consider $t^* \in I$ a time at which q_3 admits a local extremum. Then, at t^* , it holds

$$-q_3''(t^*) = \mathcal{G}(P_1(t^*), M_{s,1}(t^*), M_1)q_1(t^*).$$

According to Lemma 5.2, 5.3 and 5.4, one has $P' \leq 0$, $P_1(T) = \varepsilon$ and $M_{s,1}$ is uniformly bounded with respect to T and M_1 by a constant C_0 . Hence, we have

$$\frac{\partial^2 f_1}{\partial M_{s,1}^2} \geq \frac{b_1 \gamma^2 (P_{1,0}^1)^2}{(\varepsilon + \gamma C_0)^3} > C > 0. \quad (28)$$

Using the fact that $M_{s,1}$ and P_1 are uniformly bounded with respect to T and M_1 , we deduce that q_3'' does not vanish which is in contradiction with the fact that q_3 admits a local extremum at t^* . Thus, $\{q_3 > \lambda\}$ can only be $(0, t_0)$ or (t_1, T) or $(0, t_0) \cup (t_1, T)$.

Let now I be an open interval of $\{q_3 < \lambda\}$. Consider $t^* \in I$ a time at which q_3 admits a local extremum. Then, at t^* , it holds

$$-q_3''(t^*) = \mathcal{G}(P_1(t^*), M_{s,1}(t^*), 0)q_1(t^*)$$

with

$$\mathcal{G}(P_1, M_{s,1}, 0) = \frac{\partial^2 f_1}{\partial M_{s,1}^2}(-\mu_s M_{s,1}) + \frac{\partial^2 f_1}{\partial M_{s,1} \partial P_1} f_1 - \frac{\partial f_1}{\partial M_{s,1}} \frac{\partial f_1}{\partial P_1}.$$

Note that

$$\frac{\partial^2 f_1}{\partial M_{s,1} \partial P_1} = \frac{-2b_1 \gamma P_1 (P_1 + \gamma M_{s,1}) + 2b_1 \gamma P_1^2}{(P_1 + \gamma M_{s,1})^3} = \frac{-2b_1 \gamma^2 P_1 M_{s,1}}{(P_1 + \gamma M_{s,1})^3}$$

and

$$\frac{\partial f_1}{\partial P_1} = \frac{f_1}{P_1} + \left(b_1 \frac{\gamma M_{s,1}}{(P_1 + \gamma M_{s,1})^2} - \mu_{2,1} \right) P_1.$$

By combining this observation with Lemma 5.5, we can deduce that

$$\begin{aligned} \mathcal{G}(P_1, M_{s,1}, 0) &= \frac{b_1 \gamma^2 P_1^2}{(P_1 + \gamma M_{s,1})^3} (-\mu_s M_{s,1}) + \frac{-2b_1 \gamma^2 P M_{s,1}}{(P_1 + \gamma M_{s,1})^3} f_1 \\ &\quad + \frac{b_1 \gamma P_1^2}{(P_1 + \gamma M_{s,1})^2} \left(\frac{f_1}{P_1} + \left(b_1 \frac{\gamma M_{s,1}}{(P_1 + \gamma M_{s,1})^2} - \mu_{2,1} \right) P_1 \right). \end{aligned}$$

or equivalently,

$$\begin{aligned} \mathcal{G}(P_1, M_{s,1}, 0) &= \frac{b_1 \gamma^2 P_1^2}{(P_1 + \gamma M_{s,1})^3} \left(-\mu_s M_{s,1} - \frac{2M_{s,1}}{P_1} f_1 + \frac{f_1}{P_1 \gamma} (P_1 + \gamma M_{s,1}) \right. \\ &\quad \left. + \left(b_1 \frac{\gamma M_{s,1}}{(P_1 + \gamma M_{s,1})^2} - \mu_{2,1} \right) \frac{P_1 (P_1 + \gamma M_{s,1})}{\gamma} \right). \end{aligned}$$

Using Lemma 5.3, we infer

$$f_1 = \left(b_1 \frac{P_1}{P_1 + \gamma M_{s,1}} - \mu_{1,1} - \mu_{2,1} P_1 \right) P_1 < 0.$$

it follows that

$$\mathcal{G}(P_1, M_{s,1}, 0) < \frac{b_1 \gamma^2 P_1^2}{(P + \gamma M_{s,1})^3} (-\mu_s M_{s,1} - \frac{f_1}{P_1 \gamma} \gamma M_{s,1} + b_1 \frac{P_1 M_{s,1}}{(P_1 + \gamma M_{s,1})} - \mu_{2,1} \frac{P_1 (P_1 + \gamma M_{s,1})}{\gamma}).$$

Using the expression of f_1 ,

$$\begin{aligned} \mathcal{G}(P_1, M_{s,1}, 0) < \frac{b_1 \gamma^2 P_1^2}{(P + \gamma M_{s,1})^3} (-\mu_s M_{s,1} - \left(b_1 \frac{P_1}{P_1 + \gamma M_{s,1}} - \mu_{1,1} - \mu_{2,1} P_1 \right) M_{s,1} \\ + b_1 \frac{P_1 M_{s,1}}{(P_1 + \gamma M_{s,1})} - \mu_{2,1} \frac{P_1 (P_1 + \gamma M_{s,1})}{\gamma}). \end{aligned}$$

Therefore, one has

$$\mathcal{G}(P_1, M_{s,1}, 0) < \frac{b_1 \gamma^2 P_1^2}{(P + \gamma M_{s,1})^3} (-\mu_s M_{s,1} + \mu_{1,1} M_{s,1} - \mu_{2,1} \frac{P_1^2}{\gamma}).$$

If $\mu_s > \mu_1$, then by applying Lemma 5.4,

$$\mathcal{G}(P_1, M_{s,1}, 0) < -\frac{\mu_{2,1} b_1 \gamma P_1^4}{(P_1 + \gamma M_{s,1})^3} < -\frac{\mu_{2,1} b_1 \gamma \varepsilon^4}{(P_1^* + \gamma C_0)^3}. \quad (29)$$

The conclusion of the proof follows a similar approach to that of [2, Theorem 3.3]. We outline the key steps below. Since $q_1 > 0$, it follows that q_3'' cannot vanish, which contradicts the assumption that q_3 has a local extremum at t^* . Therefore, the set $\{q_3 < \lambda\}$ must coincide with either $(0, t_0)$, (t_1, T) , or $(0, t_0) \cup (t_1, T)$. Consequently, q_3 must equal λ on (t_0, t_1) , and thus (26) holds on this interval.

We remark that $q_3 < 0$ on an interval (t^*, T) , since $\lambda \geq 0$, it follows that $u_1 = 0$ on (t^*, T) , then also on (t_1, T) . Now, assume that for all $T > 0$, we have $u_1 = M_1$ on $(0, t_0)$. By using (29), there exists a constant $C > 0$ independent of T such that $u_1 > C$ on $(0, t_1)$. Given that $\int_0^T u_1$ is uniformly bounded with respect to T , the same must hold for t_1 . However, this contradicts the fact that $T - t_1$ is uniformly bounded as T increases. Therefore, there exists a sufficiently large T^* such that $u_1 = 0$ on $(0, t_0)$. Consequently, $u_1 = 0$ on $(0, t_0)$ for all $T \geq T^*$. □

Proof of Theorem 2.9. Let $d_{21}, d_{12} \geq 0$. Consider $P = (P_1, P_2)$, $M_s = (M_{s,1}, M_{s,2})$, and $q = (q_1, q_2)$ as the solution to $(\mathcal{P}_{\varepsilon, M})$, (6), and (13). Let $P^0 = (P_1^0, P_2^0)$, $M_s^0 = (M_{s,1}^0, M_{s,2}^0)$, and $q^0 = (q_i^0)_i$ be the corresponding solution to (24), (25), (27). Let I be an open interval of the open set $\{q_3 > \lambda\}$. Consider $t^* \in I$ a time at which q_3 admits a local extremum. Then, at t^* , it holds

$$\begin{aligned} -q_3''(t^*) &= \mathcal{G}(P_1, M_{s,1}, M_1) q_1 - \omega d_{21} q_3' + \omega d_{12} q_4' \Big|_{t=t^*} \\ &= \mathcal{G}(P_1^0, M_{s,1}^0, M_1) q_1^0 + O(\|(P, M_s, q) - (P^0, M_s^0, q^0)\|) \Big|_{t=t^*}. \end{aligned}$$

Thanks to (28) and Lemma 5.4, we deduce that q_3'' does not vanish, allowing us to conclude as previously. Similarly, let I be an open interval of the open set $\{q_3 < \lambda\}$. Consider $t^* \in I$ a time at which q_3 admits a local extremum. Then, at t^* , it holds

$$-q_3''(t^*) = \mathcal{G}(P_1^0, M_{s,1}^0, 0) q_1^0 + O(\|(P, M_s, q) - (P^0, M_s^0, q^0)\|) \Big|_{t=t^*}.$$

Using the fact that q_1^0 is bounded below by a positive constant and that (P, M_s, q) converges uniformly to (P^0, M_s^0, q^0) as $d_{21} = d_{12}$ approaches zero, we can deduce that q_3'' does not vanish. Consequently, we can draw the same conclusion as before. □

	b_i	$\mu_{1,i}$	$\mu_{2,i}$	ω	μ_s	γ	\mathcal{N}_i	$u_{s,i}^{crit}$
Patch 1	8.15	0.035	0.0015	1.0	0.231	1.0	232.85	1095.79
Patch 2			0.0014					1174.06

Table 1: Parameters values for the 2-patches system

6 Numerical illustrations and simulations

Based on [22], we use the numerical values provided in Table 1, which are appropriate for populations such as fruit flies (e.g., *B. dorsalis*) or potentially mosquitoes in a favorable tropical environment.

Assuming that 50% of the sterile insects die within 3 days, we obtain $\mu_s = \frac{\log 2}{3} \approx 0.231$ [22]. By selecting appropriate death rates $\mu_{1,i}$ and $\mu_{2,i}$, we ensure that without control, the wild population would reach 5410 individuals in the first patch and 5797 in the second patch. These values will serve as the initial conditions for the upcoming simulations. For a pest like *B. dorsalis*, the basic offspring number \mathcal{N} , as provided in Table 1, is notably high, ranging between 100 and 500, depending on the fruit host. Using (11), we estimate the critical release rates $u_{s,i}^{crit}$ for both patches, assuming no migration. Specifically, the total critical release rate is $u_{s,1}^{crit} + u_{s,2}^{crit} = 2271$ individuals per unit of time.

We will first present and discuss simulations with constant controls and varying values for the dispersal/migration parameters d_{12} , d_{21} , and ω to illustrate the theoretical results.

6.1 Minimal release rates estimate to reach elimination

We use the results of section 3 and solve the linear optimization problem using Matlab [39].

6.1.1 Case $d_{12} = d_{21} = 0.01$

Solving the linear optimization problem (12) with the parameters provided in Table 1 yields a critical release rate of $u^{crit} = (1093, 1178)$, resulting in a total release rate of 2271 individuals per unit of time. As anticipated, with the populations in Patch 1 and Patch 2 being nearly equal, the control efforts required for each patch are almost identical. This release rate represents the minimum required for each patch to achieve control. In practice, increasing these values can enhance control effectiveness and accelerate the reduction of the wild population. The key is to determine the minimal daily release rate of sterile males and the optimal distribution between patches to ensure effective control.

Assuming $\omega = 0.5$, indicating low dispersal of sterile insects, we find $u^{crit} = (1095, 1176)$ for a total release rate of 2271 individuals. This result is nearly identical to that obtained with $\omega = 1$. Since both patches are very similar in terms of carrying capacity and diffusion rates, the dispersal rate of sterile insects has a minimal impact. However, it will take longer to achieve elimination compared to the scenario where $\omega = 1$.

When we set $u_2 = 0$, meaning that releases are only made in Patch 1, solving (12) yields $u^{crit} = (29, 486, 0)$. It is evident that if releases in Patch 2 are not permitted for any reason, the control effort required is significantly higher compared to a scenario where both patches are treated. Clearly, to achieve elimination in both patches, the most effective strategy is to release sterile insects in both patches.

In the worst-case scenario where $u_2 = 0$ and $\omega = 0.5$, we find $u^{crit} = (56, 624, 0)$. This result is consistent with expectations: when dispersal is low, the number of sterile insects migrating from Patch 1 to Patch 2 is minimal, necessitating a substantially higher release rate in Patch 1 to achieve elimination in both patches. Naturally, increasing the release rates will expedite the decline of the wild population in each patch to $\mathbf{0}$.

6.1.2 Case $d_{12} = 0.05$ and $d_{21} = 0.01$

Solving system (12) with the parameters provided in Table 1 yields $u^{crit} = (889, 1381)$, resulting in a total release rate of 2271 individuals per unit of time. As anticipated, the release rate is higher in Patch 2 compared to Patch 1. This is because the diffusion rate from Patch 2 to Patch 1 is greater than that from Patch 1 to Patch 2, meaning that sterile insects released in Patch 2 move more quickly to Patch 1. It is feasible to increase the total release rate beyond 2271 individuals per unit of time, provided that u_1 and u_2

satisfy the constraint given in (11). A higher release rate will accelerate the decline of the wild population in both patches towards elimination. However, the practical feasibility of this approach depends on the daily production capacity of sterile insects.

Assuming $\omega = 0.5$, indicating low dispersal of sterile insects, we find $u^{\text{crit}} = (993, 1278)$ with the same total release rate. This result is logical: to compensate for the reduced mobility of sterile insects, a higher release rate is required in Patch 1, resulting in a lower release rate for Patch 2.

When $u_2 = 0$, meaning we only release sterile insects in Patch 1, solving (12) yields $u^{\text{crit}} = (34, 184, 0)$. This represents a significantly higher release rate compared to scenarios where both patches are treated. It is evident that excluding Patch 2 from treatment necessitates a much larger control effort to achieve elimination in both patches.

In the worst-case scenario, where $u_2 = 0$ and $\omega = 0.5$, we find $u^{\text{crit}} = (61, 322, 0)$. It is important to note that extensive releases are typically needed only for a finite period, T . According to the phase diagram in Fig. 3(a), once the population falls within the basin of attraction of $\mathbf{0}$, meaning the population size is sufficiently reduced, it becomes feasible to switch to smaller release rates—though they should not be too small—similar to the approach suggested in [3]. This strategy is practical and aligns with real-world applications.

6.1.3 Case $d_{21} = 0.01$ and $d_{12} = 0.0002$ or $d_{12} = 0$

In scenarios where the diffusion rate from Patch 2 to Patch 1 is very low or negligible, we observe the following results:

- When $d_{12} = 0.0001$, solving (12) yields $u^{\text{crit}} = (1144, 1127)$ for a total release rate of 2271 individuals per unit of time. If $u_2 = 0$, then $u_1^{\text{crit}} = 28,314$.
- When $d_{12} = 0$, the results are consistent: $u^{\text{crit}} = (1144, 1127)$ for both patches, and $u_1^{\text{crit}} = 28,312$ when releases are restricted to Patch 1.
- If both diffusion rates are very low, such as $d_{21} = d_{12} = 0.0002$, we find $u^{\text{crit}} = (1096, 1175)$. In the worst-case scenario where $u_2 = 0$, the required release rate in Patch 1 is exceedingly high, $u_1^{\text{crit}} = 1,359,214$.

These results highlight that extremely low diffusion rates limit the movement of sterile insects into Patch 2, necessitating very large release rates to achieve elimination in both patches. For population reduction strategies, various results and approaches are feasible, depending on the specific parameters and goals.

Remark 6.1. *The previous simulations underscore the necessity of releasing sterile insects in both patches to achieve elimination. Specifically, a positive release rate in each patch is required, with a minimal total release rate of 2271 individuals per unit of time. In the two-patch scenario, the dispersal rates primarily influence how the release is distributed between the patches. Thus, while dispersal rates affect the distribution strategy, they do not change the total release rate required for elimination. Naturally, increasing the production rate will lead to faster population reduction.*

We now turn to a numerical investigation of similar scenarios, aiming to determine an optimal release strategy that reduces the population in Patch 1 by 99% within a specified finite time T .

6.2 Solving the optimal control problem

Based on the parameter values provided in Table 1 and Theorem 3.5, we will explore various scenarios with different migration rates d_{12} and d_{21} , specifically considering values such as 0.0, 0.0002, and 0.01. As established, when $M_{s,1}^*$ and $M_{s,2}^*$ exceed their critical thresholds, for any $\varepsilon > 0$, there exists a time T such that Problem $(\mathcal{P}_{\varepsilon, M})$ has a solution.

We set $T = 50$, $M_{1,s}^* = M_{2,s}^* = 20,000$, and $\varepsilon = 0.01 \times P_1^*$. In Figures 4, 5, 6, and 7, we analyze scenarios with migration rates d_{12} and d_{21} taking various values such as 0.0, 0.0002, 0.002, 0.005, and 0.01.

6.2.1 Case $d_{21} = d_{12} = 0$ and $d_{21} = d_{12} = 0.0002$

We observe the theoretical result from Theorem 2.7: ε is reached within the finite time T . Given that the migration rates are sufficiently small, it is unnecessary to intervene in the second patch, as demonstrated

in Figures 4 and 5. Notably, the release rate profiles in Figures 4₂ and 5₂ exhibit remarkable similarity: a very large initial release is followed by progressively smaller releases until a certain time t_1 , at which point the releases cease, consistent with the expectations in Proposition 5.1.

The codes are available here :

<https://github.com/michelduprez/Sterile-Insect-Technique-in-a-Two-Patch-Model>

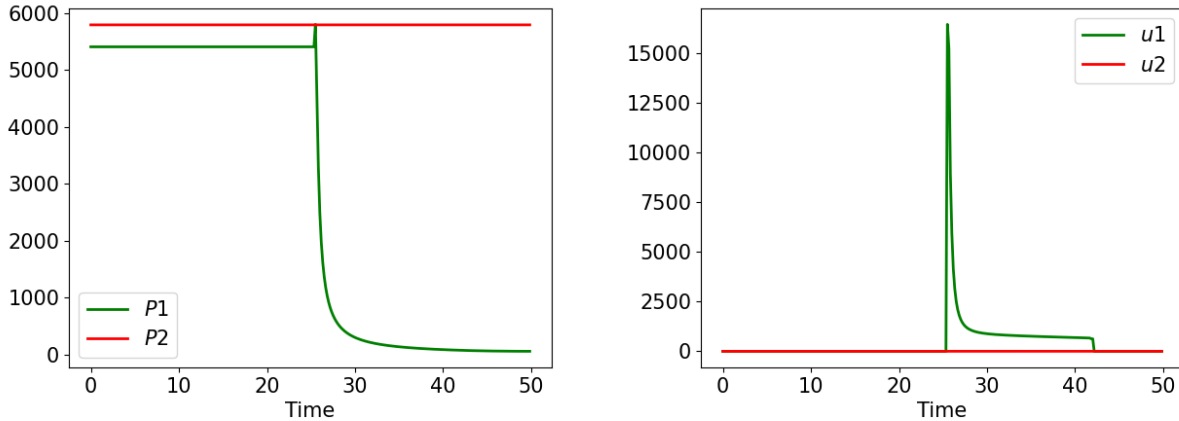


Figure 4: Numerical solution of the optimal control problem $(\mathcal{P}_{\varepsilon, M})$ with the parameter values given in Table 1, $T = 50$, $M_{1,s}^* = M_{2,s}^* = 20000$, $\varepsilon = 0.01 \times P_1^*$ and $d_{12} = d_{21} = 0$.

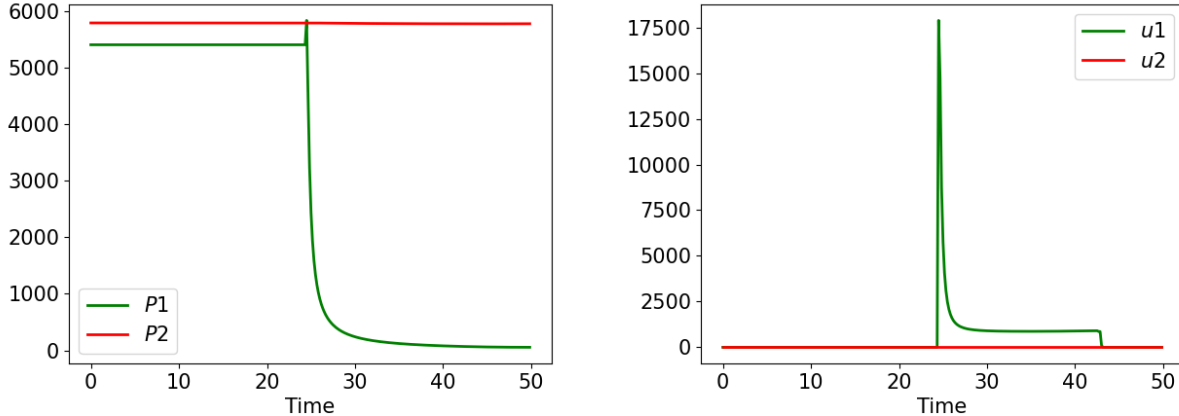


Figure 5: Numerical solution of the optimal control problem $(\mathcal{P}_{\varepsilon, M})$ with the parameter values given in Table 1, $T = 50$, $M_{1,s}^* = M_{2,s}^* = 20000$, $\varepsilon = 0.01 \times P_1^*$, $d_{12} = d_{21} = 0.0002$.

6.2.2 Case $d_{12} = 0.0002$ and $d_{21} = 0.01$

In Figure 6, we show that the same outcome is achieved when the diffusion from Patch 2 to Patch 1 is sufficiently small only. This finding is intuitive, as the primary objective is to reduce the wild insect population in Patch 1. Indeed, having a large diffusion from patch 1 to patch 2 is even more advantageous while it is crucial to reduce the diffusion from patch 2 to patch 1. However, in practice, reducing the migration in both way, rather than in one way, is done using massive trapping, in a buffer zone between both patches. Thus, reducing $d_{12} + d_{21}$ makes sense, from the practical point of view.

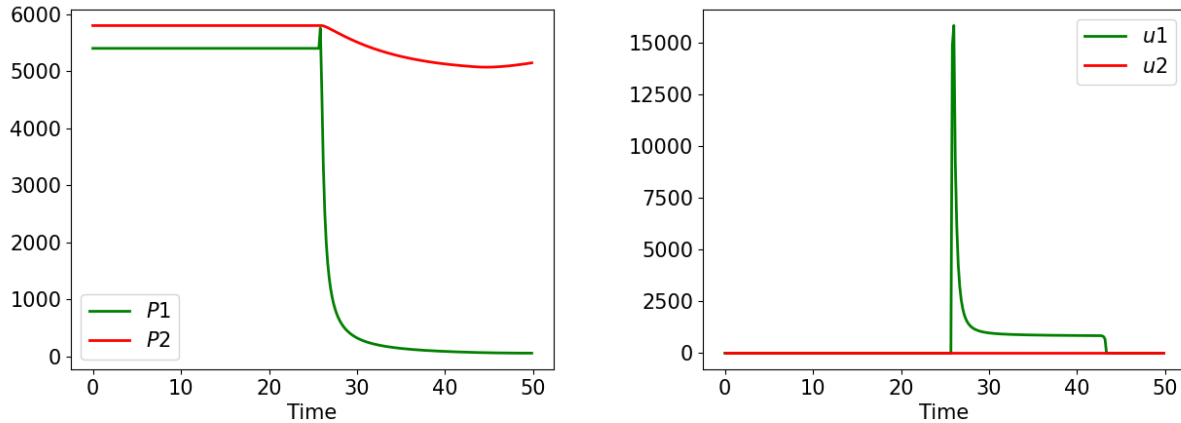


Figure 6: Numerical solution of the optimal control problem $(\mathcal{P}_{\varepsilon, M})$ with the parameter values given in Table 1, $T = 50$, $M_{1,s}^* = M_{2,s}^* = 20000$, $\varepsilon = 0.01 \times P_1^*$, $d_{12} = 0.0002$ and $d_{21} = 0.01$

6.2.3 Case $d_{12} = d_{21} = 0.005$

Finally, in Figure 7, we illustrate that when diffusion rates are high, the optimal solution requires releases in both patches to achieve a 99% population reduction in Patch 1 by time $T = 50$. This result is crucial from a practical standpoint. It indicates that if it is not feasible to lower the diffusion rates between Patch 1 and Patch 2 through mass trapping or other control measures in the buffer zone, then releases in Patch 2 become necessary. Consequently, a sterile insect production facility must be capable of generating a substantially larger quantity of sterile insects—well beyond the amount needed solely for Patch 1. The choice of strategy, therefore, can significantly influence the success of field trials and, naturally, the associated costs.

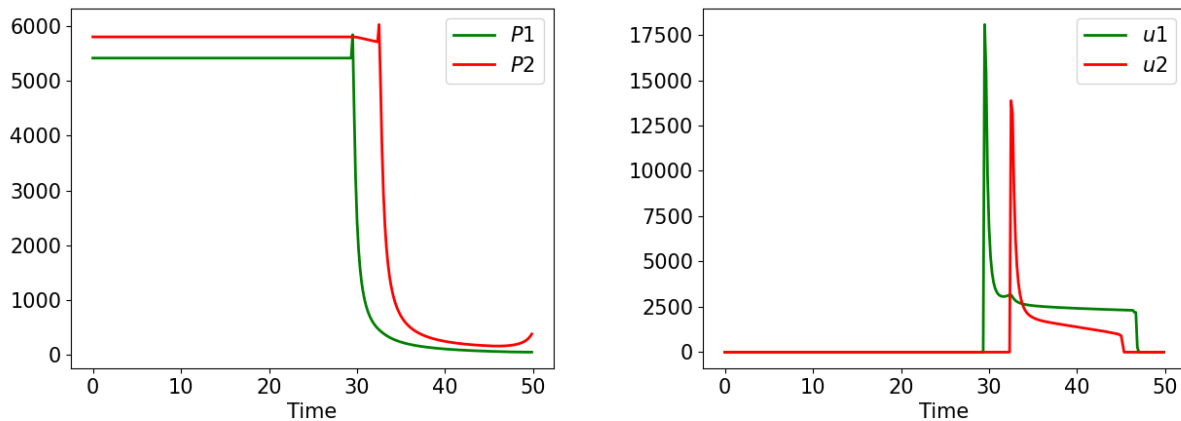


Figure 7: Numerical solution of the optimal control problem $(\mathcal{P}_{\varepsilon, M})$ with the parameter values given in Table 1, $T = 50$, $M_{1,s}^* = M_{2,s}^* = 20000$, $\varepsilon = 0.01 \times P_1^*$, $d_{12} = d_{21} = 0.005$.

6.2.4 Case $d_{12} = d_{21} = 0.01$

It is also insightful to examine the optimal control solution when $d_{12} = d_{21} = 0.01$: see Figure 8. In this scenario, the optimal strategy involves releases in both patches. As expected, the control starts with massive releases, followed by small or no releases. However when μ_S increases, i.e. $\mu_S = 0.1$, the numerical optimal solution can be false: see Fig. 9. Indeed, from a practical standpoint, this approach is neither

feasible nor desirable, as it begins with minimal releases at $t = 0$ and concludes with a surge of massive releases. This "suboptimal" result from a biological perspective can be corrected by reducing, for instance, the final time T to 30 days, as shown in Figure 10. In this adjusted strategy, the releases begin with a very large number, as expected, and then gradually decrease. Since the releases are conducted in both patches, the wild populations in both areas will decline accordingly.

These examples highlight that optimal release strategies must be evaluated through a biological lens. In other words, while the optimal solution may be mathematically sound, it does not always align with the best practices from a biological standpoint. Last but not least, our simulations underscore the critical importance of isolation between patches. If Patch 1 is not sufficiently isolated from Patch 2, then releasing sterile insects in both patches becomes necessary, even if the goal is to reduce the population in Patch 1 only. From an experimental perspective, it is therefore essential to accurately estimate all diffusion rates and their variations throughout the release process, enabling adjustments to the release strategy as needed.

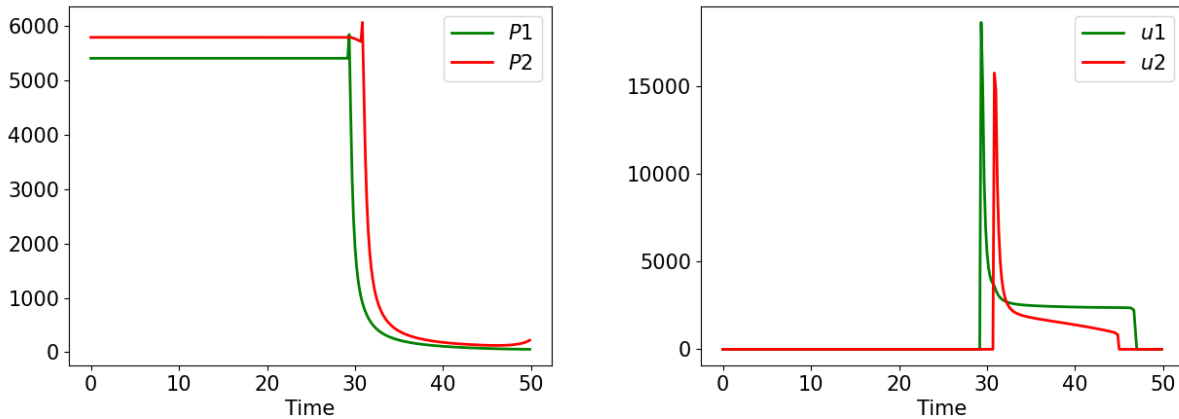


Figure 8: Numerical solution of the optimal control problem $(\mathcal{P}_{\varepsilon, M})$ with the parameter values given in Table 1, $T = 50$, $M_{1,s}^* = M_{2,s}^* = 20000$, $\varepsilon = 0.01 \times P_1^*$, $d_{12} = d_{21} = 0.01$.

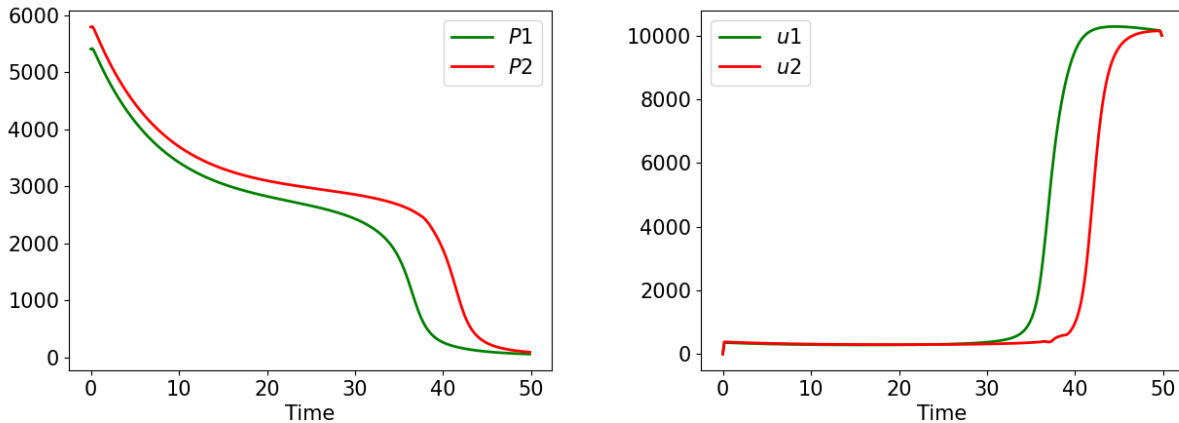


Figure 9: Numerical solution of the optimal control problem $(\mathcal{P}_{\varepsilon, M})$ with the parameter values given in Table 1, $T = 50$, $M_{1,s}^* = M_{2,s}^* = 20000$, $\varepsilon = 0.01 \times P_1^*$, $d_{12} = d_{21} = 0.01$, and $\mu_S = 0.1$

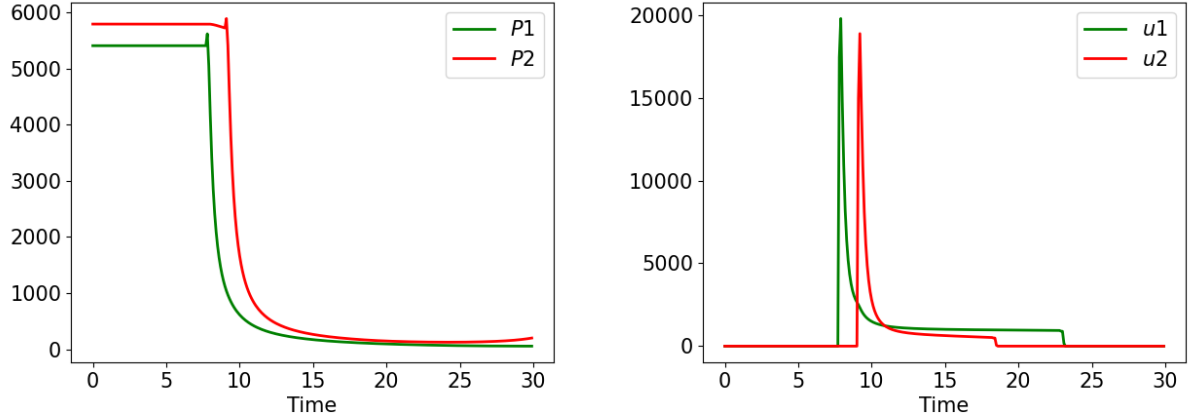


Figure 10: Numerical solution of the optimal control problem $(\mathcal{P}_{\varepsilon, M})$ with the parameter values given in Table 1, $T = 30$, $M_{1,s}^* = M_{2,s}^* = 20000$, $\varepsilon = 0.01 \times P_1^*$, $d_{12} = d_{21} = 0.01$, and $\mu_S = 0.1$

7 Conclusion

In this work, we have demonstrated two key results. First, by ensuring that the release rates, u_i , satisfy a specific vector inequality, $U \geq \frac{\mu_S}{\gamma} \mathcal{A}^{-1} M_S^{\text{crit}}$, we guarantee the possibility of pest or vector elimination across all patches. Second, using optimal control techniques, we have shown in the two-patch scenario that an optimal control strategy exists to reduce the wild population below a specified threshold in Patch 1, by releasing sterile insects only in Patch 1, provided that the migration rates are sufficiently small. This scenario is frequently encountered in practice in SIT programs. Additionally, numerical simulations confirms that this result also holds as long as the migration rate from Patch 2 to Patch 1 remains sufficiently low.

Some perspectives for future work include:

- **Impact of Time Control.** We have fixed the control time T , but it could be treated as a variable. Allowing T to vary might lead to different optimal strategies and insights into the control dynamics.
- **Migration Rates from Patch 2 to Patch 1.** In Theorem 2.7, we established that the sum of migration rates $d_{12} + d_{21}$ must be small to achieve $P_1 = \varepsilon$ in finite time. However, simulations, such as those in Fig. 6, suggest that it is sufficient for d_{12} alone to be small. It would be valuable to derive this result theoretically.
- **Elimination in n Patches with Control in $m < n$ Patches.** This scenario arises when different stakeholders do not permit the release of sterile insects in their areas. It is crucial to determine whether the non-controlled patches negatively impact the overall control strategy. Understanding the significance of the network connecting the patches is essential. Additionally, evaluating the extra costs or the additional amount of sterile insects required when an optimal strategy cannot be applied is important for production considerations.
- **Optimal Control Model for n Patches.** Extending our optimal control model to n patches raises the question: "Is it optimal to omit action in some patches when there is low dispersal or migration between them?" While directly studying the n -patch optimal control problem is challenging, it might be possible to address this by considering that if migration corresponds to a discretized Laplacian, the n -patch model could converge to a parabolic equation. It is well known that for such equations, and with a cost function involving the L^1 norm of the solution, the optimal control can be sparse (see [19] for a review). Thus, exploring the convergence of the n -patch control to the control of the corresponding parabolic equation could be insightful.
- **Impulsive Control Strategy.** In practice, sterile insect releases are conducted periodically, typically every τ day. Considering a model with periodic pulsed releases would be more realistic. The results obtained from continuous release scenarios could provide useful estimates for the amount of sterile insects needed per release.

- **Incorporating seasonality in the search for optimal strategies.** Last but not least, knowing the relationship between the model’s parameters and environmental parameters, like temperature and rainfall, it could be possible to derive, numerically, the best strategies along the year, like in [21].

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