



HAL
open science

Orthosystoles and orthokissing numbers

Federica Fanoni, Ara Basmajian

► **To cite this version:**

| Federica Fanoni, Ara Basmajian. Orthosystoles and orthokissing numbers. 2024. hal-04693392

HAL Id: hal-04693392

<https://hal.science/hal-04693392v1>

Preprint submitted on 10 Sep 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ORTHOSSYSTOLES AND ORTHOKISSING NUMBERS

ARA BASMAJIAN AND FEDERICA FANONI

ABSTRACT. For hyperbolic surfaces with geodesic boundary, we study the *orthosystole*, i.e. the length of a shortest essential arc from the boundary to the boundary. We recover and extend work by Bavard completely characterizing the surfaces maximizing the orthosystole in the case of a single boundary component. For multiple boundary components, we construct surfaces with large orthosystole and show that their orthosystole grows, as the genus goes to infinity, at the same rate as Bavard's upper bound.

1. INTRODUCTION

Systoles – shortest closed geodesics – of closed hyperbolic surfaces have been widely studied and have been fundamental in understanding hyperbolic surfaces and their moduli spaces. A simple area argument provides an upper bound on the systole length which is asymptotic to $2 \log g$, as the genus g goes to infinity. Surprisingly, this naive bound provides the correct order of growth, as various authors have constructed sequences of closed hyperbolic surfaces with systole length growing logarithmically in the genus ([6], [9], [13], [21], [22], [16]). The best constructions ([9], [13]) have systole length growing at least as $\frac{4}{3} \log g$. It is a well known open problem to understand the gap between the best construction and the best known upper bound (see also [4] and [11] for improvements on the naive bound).

In analogy with sphere packing problems in Euclidean space, Schmutz Schaller introduced and studied kissing numbers – numbers of systoles – of hyperbolic surfaces. Also for kissing numbers, upper bounds ([20], [10], [11]) and constructions of surfaces with large kissing numbers ([24], [25], [26], [7], [2]) are known. Furthermore, it is known ([23], [24]) that surfaces with large systole must have large kissing number as well.

If we consider compact hyperbolic surfaces with boundary, it is natural to consider another collection of geodesics, instead of closed ones: those starting and ending at the boundary components. Given a hyperbolic surface X with geodesic boundary, we define the *orthosystole* to be the length of a shortest geodesic from boundary to boundary, and we denote it by $\text{osys}(X)$. Moreover, we call *orthokissing number* (and we denote it by $\text{okiss}(X)$) the number of geodesics from boundary to boundary having minimal length.

The first observation is that there are no interesting upper and lower bounds for osys depending only on the topology of the surface: if S has negative Euler characteristic, the orthosystole can be arbitrarily small and arbitrarily large (see Lemma 4.1). As a consequence, we will discuss bounds on the orthosystole for surfaces with restrictions on the boundary lengths. Two natural possibilities are to fix each boundary length or to fix their sum, and it turns out that the two situations are quite different.

If we fix the sum of the boundary lengths, a sharp upper bound has been proven by Bavard in [3]:

Theorem 1.1 (Bavard). *Let X be a surface of signature (g, n) and total boundary length L . Then*

$$\text{osys}(X) \leq 2 \sinh^{-1} \left(\frac{1}{2 \sinh \left(\frac{L}{24g - 24 + 12n} \right)} \right)$$

Date: May 2, 2024.

1991 Mathematics Subject Classification. Primary 20F69, 32G15; Secondary 57K20.

Key words and phrases. hyperbolic surface, moduli space, orthogeodesic, orthokissing number, orthosystole.

with equality if and only if $\text{okiss}(X) = 6g - 6 + 3n$. Moreover, the bound is attained for every L .

Note that Bavard's work is more general and is phrased quite differently. The theorem just stated corresponds to parts (1), (3) and (4) [3, Théorème 1], for $c = \partial X$ and considering only r . In this case, $r = \frac{\text{osys}(X)}{2}$ and one can verify that the condition for equality of part (3) corresponds to $\text{okiss}(X) = 6g - 6 + 3n$. We refer to the Appendix for an explanation of the correspondence between Bavard's formulation and the theorem above.

If, instead of fixing the sum, we fix each boundary length, we cannot always find surfaces with $\text{okiss}(S) = 6g - 6 + 3n$ (as discussed at the beginning of Section 6), so Bavard's result is not always sharp. Of course, if a surface has a single boundary component, fixing its length or the total boundary length is the same, so in this case Bavard's result completely classifies the global maxima for the orthosystole function. Our first result extends Bavard's work, by showing that global and local maxima for the orthosystole function coincide in the case of a single boundary component.

Theorem A. *Let X be a surface of signature $(g, 1)$ and boundary length ℓ . The following are equivalent:*

- (1) X is a global maximum for the orthosystole function;
- (2) X is a local maximum for the orthosystole function;
- (3) $\text{okiss}(X) = 6g - 3$;
- (4) $\text{osys}(X) = 2 \sinh^{-1} \left(\frac{1}{2 \sinh(\ell/(24g - 12))} \right)$.

Moreover, for any $g \geq 1$ the number of local (or global) maxima in each moduli space is exactly

$$\frac{2(6g - 5)!}{12^g g! (3g - 3)!}$$

We remark that our proof is very different from Bavard's — in particular, we do not rely on his work to prove Theorem A.

Note that for the analogous problem for the systole function, global maxima are not known (except in a few low complexity cases) and it is a hard problem even to construct local maxima (see [12]).

For multiple boundary components, while we cannot completely describe global maxima, we are able to show that the orthosystole admits a global maximum and to construct examples of surfaces with large orthosystole.

Theorem B. *Let S be a surface of signature (g, n) and negative Euler characteristic. Fix $0 < \ell_1 \leq \dots \leq \ell_n$.*

- (1) *The function $\text{osys} : \mathcal{M}(S; \ell_1, \dots, \ell_n) \rightarrow \mathbb{R}$ admits a maximum.*
- (2) *Suppose $g \geq n \geq 2$. Then there is a surface $X \in \mathcal{M}(S; \ell_1, \dots, \ell_n)$ with*

$$\text{osys}(X) \geq \cosh^{-1} \left(\frac{\cosh \left(\frac{\ell_n}{6 \lfloor \frac{g}{n} \rfloor} \right)}{\cosh \left(\frac{\ell_n}{6 \lfloor \frac{g}{n} \rfloor} \right) - 1} \right).$$

While the fact osys admits a maximum might seem a triviality, it is more subtle than in the case of the systole. Indeed, continuity of the systole function and Mumford's compactness theorem readily imply the existence of global maxima for the systole function in each moduli space. On the other hand, we cannot replace the systole by the orthosystole in Mumford's criterion (see Lemma 6.3). So to prove (1) we will need to apply a length-expansion result ([27],[19],[18]).

Note furthermore that if n and the ℓ_i are fixed and we let the genus go to infinity, the surfaces we construct have orthosystole length growing as $2 \log g$, which matches the asymptotics of the upper bound given in Theorem 1.1.

On the other hand (see Section 6), the surfaces constructed in Theorem B aren't even *local* maximizers for the orthosystole. While we have some examples of surfaces with larger orthosystole (Lemma 6.2), for multiple boundary components we are not able to exhibit global maximizers for the orthosystole.

The length-expansion result mentioned before allows us also to show that local maxima of the orthosystole function have relatively large orthokissing number, that is:

Proposition C. *Let S be a surface of signature (g, n) and negative Euler characteristic. Fix $0 < \ell_1 \leq \dots \leq \ell_n$. Then if $X \in \mathcal{M}(S; \ell_1, \dots, \ell_n)$ is a local maximum for $osys$, $okiss(S) \geq 2g - 2 + n$.*

This article is structured as follows: after defining the objects we are interested in and recalling or proving some facts we will need (Section 2), in Section 3 we will discuss maximal collections of pairwise disjoint and pairwise non-homotopic arcs, which will play a fundamental role in our work. Section 4 is dedicated to proving some basic facts about orthosystoles and orthokissing numbers and part (1) of Theorem B (see Corollary 4.6). Section 5 is concerned with the case of surfaces with a single boundary component and the proof of Theorem A. The case of surfaces with more than one boundary component and part (2) of Theorem B is proven in Section 6. Finally, in Section 7 we show Proposition C.

ACKNOWLEDGEMENTS

The authors would like to thank Hugo Parlier and Bram Petri for useful discussions. A.B. was partially supported by PSC CUNY Award 65245-00 53 and the Simons Foundation (359956, A.B.).

2. PREREQUISITES

Given a compact surface S of signature (g, n) , we denote by $\text{Teich}(S)$ the Teichmüller space of surfaces with boundary, where the boundary length is allowed to vary. Given $L > 0$, $\text{Teich}(S; L)$ is the subspace of surfaces whose total boundary length is L . For $\ell_1, \dots, \ell_n > 0$, $\text{Teich}(S; \ell_1, \dots, \ell_n)$ is the subspace of surfaces whose i -th boundary components has length ℓ_i , for $i = 1, \dots, n$. We denote similarly the moduli spaces $\mathcal{M}(S)$, $\mathcal{M}(S; L)$ and $\mathcal{M}(S; \ell_1, \dots, \ell_n)$. Moreover, $\mathcal{M}_\varepsilon(S; \ell_1, \dots, \ell_n)$ denotes the subset of $\mathcal{M}(S; \ell_1, \dots, \ell_n)$ given by surfaces all of whose closed geodesics have length at least ε – the so-called (ε) -thick part of moduli space, which is a compact set by Mumford's compactness theorem [17].

Let $X \in \text{Teich}(S)$. Its *systole* $\text{sys}(X)$ is the length of a shortest closed geodesic which is not a boundary component. Its *orthosystole* $osys(X)$ is the length of a shortest orthogeodesic. We will regularly abuse notation and use the terms systole and orthosystole also for a simple closed geodesic (not boundary parallel) or an orthogeodesic of minimal length. Its *kissing number* $\text{kiss}(X)$ is the number of systoles and its *orthokissing number* $okiss(X)$ is the number of its orthosystoles.

Given a homotopically nontrivial simple closed curve γ , we denote by $\ell_\gamma(X)$ the length of the unique shortest closed curve in the free homotopy class of γ , i.e. of the unique simple closed geodesic in the class.

Denote by \mathcal{A} the collection of essential arcs from boundary to boundary, up to homotopy relative to the boundary. For any $\alpha \in \mathcal{A}$, we denote by $\ell_\alpha(X)$ the length of the unique shortest arc in the class, i.e. of the unique orthogeodesic in the class.

By using Fermi coordinates [8, Chapter 1], it is easy to prove:

Lemma 2.1. *Let F be a hyperbolic funnel with boundary curve γ . The area of the collar about γ of width w is $\ell_\gamma(F) \sinh(w)$.*

Moreover:

Lemma 2.2. *Consider a right-angled hexagon of sides (read counterclockwise) $a, \gamma, b, \alpha, c, \beta$ and let h_a be the orthogonal between a and α . Abusing notation we conflate the names of the arcs with their lengths. Let $A = \cosh(a), B = \cosh(b)$ and $C = \cosh(c)$ and set*

$$s(A, B, C) := \cosh^{-1} \left(\frac{A + BC}{\sqrt{(B^2 - 1)(C^2 - 1)}} \right) + \\ + \cosh^{-1} \left(\frac{B + AC}{\sqrt{(A^2 - 1)(C^2 - 1)}} \right) + \cosh^{-1} \left(\frac{C + AB}{\sqrt{(A^2 - 1)(B^2 - 1)}} \right).$$

Then:

- (1) $\alpha = \cosh^{-1} \left(\frac{A + BC}{\sqrt{(B^2 - 1)(C^2 - 1)}} \right)$;
- (2) $\alpha + \beta + \gamma = s(A, B, C)$;
- (3) $\frac{\partial s}{\partial A}(A, B, C) = \frac{A - 1 - B - C}{(A - 1)\sqrt{A^2 + B^2 + C^2 + 2ABC - 1}}$;
- (4) $\lim_{A \rightarrow 1} s(A, B, C) = +\infty = \lim_{A \rightarrow +\infty} s(A, B, C)$;
- (5) $\cosh(h_a) = \frac{\sqrt{A^2 + B^2 + C^2 + 2ABC - 1}}{\sqrt{A^2 - 1}}$; if $b = c$, $\sinh(h_a) = \frac{\cosh(b)}{\sinh(a/2)}$.

Proof. Standard formulas (see for instance [8, Chapter 2]) give us the lengths of α, β , and γ and hence their sum $s(A, B, C)$. The formula for the derivative of s follows by explicit computation.

If $A \rightarrow 1$, then $a \rightarrow 0$, i.e. one side of the hexagon converges to a point in the boundary of the hyperbolic plane, so the length of its adjacent sides goes to infinity. If $A \rightarrow \infty$

$$s(A, B, C) \geq \cosh^{-1} \left(\frac{A + BC}{\sqrt{(B^2 - 1)(C^2 - 1)}} \right) \rightarrow \infty.$$

The expressions for h_a follow again from standard hyperbolic trigonometry ([8, Chapter 2]), by looking at the two right-angled pentagons obtained by cutting the hexagon along the orthogonal from the side of length a to the opposite side. \square

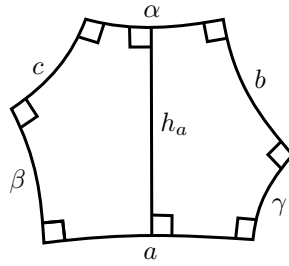


FIGURE 1. A right-angled hexagon

We will also use the Collar Lemma ([14]; see also [8, Chapter 4]):

Lemma 2.3 (Collar Lemma). *A simple closed geodesic α in a hyperbolic surface X has an embedded collar of width*

$$w(\alpha) = \sinh^{-1} \left(\frac{1}{\sinh \left(\frac{\ell_\alpha(X)}{2} \right)} \right).$$

In particular any simple closed geodesic or orthogeodesic intersecting α has length at least $w(\alpha)$ ($2w(\alpha)$ if α is not a boundary component).

The following is due to Wolpert ([29]; see also [15, Theorem 2.1]):

Theorem 2.4. *If $f : X \rightarrow Y$ is a K -quasiconformal homeomorphism of surfaces with boundary and $\gamma \subset X$ is either an essential simple closed curve or an essential arc from boundary to boundary, then*

$$\frac{\ell_{f(\gamma)}(Y)}{\ell_\gamma(X)} \leq K.$$

We will also use a length-expansion result (see [27], [19] and [18]):

Theorem 2.5. *Let X be a surface of signature (g, n) , $n > 0$, and let $\gamma_1, \dots, \gamma_n$ be the boundary geodesics of X . For every $(\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n$ where $\varepsilon_i \geq 0$ for all i , and at least one ε_i is not zero, there exists a surface Y with boundary geodesics of lengths $\ell_{\gamma_1}(X) + \varepsilon_1, \dots, \ell_{\gamma_n}(X) + \varepsilon_n$ such that all corresponding simple closed geodesics in Y are of length strictly greater than those of X .*

Remark 2.6. For surfaces X, Y as in Theorem 2.5, if $\varepsilon_i = \varepsilon_j = 0$ and α is an orthogeodesic from γ_i to γ_j , $\ell_\alpha(Y) > \ell_\alpha(X)$. This follows by the proof of the theorem given in [19].

3. HEXAGON DECOMPOSITIONS

A *hexagon decomposition* H of a topological surface S with boundary is a maximal collection of pairwise non-homotopic and disjoint essential arcs. If $X \in \text{Teich}(S)$, the orthogeodesic representatives of the arcs in H cut X into a union of right-angled hexagons. By Euler characteristic considerations, a hexagon decomposition of a surface of signature (g, n) contains $3|\chi(S)| = 6g - 6 + 3n$ arcs and cuts the surface into $2|\chi(S)| = 4g - 4 + 2n$ (topological) disks.

Given a surface S and a hexagon decomposition $H = \{\alpha_1, \dots, \alpha_{6g-6+3n}\}$, denote by ∂_H the collection of triples $\{i, j, k\}$ (where two indices might coincide) of indices such that $\alpha_i, \alpha_j, \alpha_k \in H$ are three sides of some hexagon defined by H . Define the function¹

$$F_H : \mathbb{R}^{6g-6+3n} \rightarrow \mathbb{R}$$

$$x \mapsto \sum_{\{i,j,k\} \in \partial_H} s(\cosh(x_i), \cosh(x_j), \cosh(x_k)).$$

Ushijima [28] showed:

Theorem 3.1. *Let S be a surface of signature (g, n) and $H = \{\alpha_1, \dots, \alpha_{6g-6+3n}\}$ a hexagon decomposition. Then*

$$\varphi_H : \text{Teich}(S) \rightarrow \mathbb{R}_{>0}^{6g-6+3n}$$

$$X \mapsto (\ell_{\alpha_i}(X))_{i=1}^{6g-6+3n}$$

is a homeomorphism.

Remark 3.2. Note that hexagon decompositions in [28] are called truncated triangles.

For the sake of completeness, we provide here an alternative proof:

Proof. Since right-angled hexagons are determined by the lengths of three alternating sides, and any three positive lengths determine a right-angled hexagon, the map is a bijection. Theorem 2.4 implies that φ_H is continuous. Conversely, Bishop [5] shows that given two right-angled hexagons of alternating side lengths a, b, c and a', b', c' , there is a quasiconformal homeomorphism from one to the other, sending the side of length a (respectively b, c) to the side of length a' (respectively b', c'), linear on the sides, whose quasiconformal constant depends on $a, b, c, \max\{|a - a'|, |b - b'|, |c - c'|\}$ and goes to one as $\max\{|a - a'|, |b - b'|, |c - c'|\}$ goes to zero. By using these maps on each hexagon determined by H , we get a quasiconformal map $\varphi_H^{-1}(x_1, \dots, x_{6g-6+3n}) \rightarrow \varphi_H^{-1}(x'_1, \dots, x'_{6g-6+3n})$, whose quasiconformal constant

¹Here we are slightly abusing notation, since $\{i, j, k\}$ is an unordered triple, so $s(\cosh(x_i), \cosh(x_j), \cosh(x_k))$ is technically not defined, but since s is symmetric, we can choose any ordering for i, j, k to compute s .

depends on $x_1, \dots, x_{6g-6+3n}, \max\{|x_i - x'_i| \mid i = 1, \dots, 6g - 6 + 3n\}$ and goes to one as $\max\{|x_i - x'_i| \mid i = 1, \dots, 6g - 6 + 3n\}$ goes to zero. This proves continuity of φ_H^{-1} . \square

Note that this means in particular that for every arc $\alpha \in \mathcal{A}$, the function

$$\begin{aligned} \text{Teich}(S) &\rightarrow \mathbb{R} \\ X &\mapsto \ell_\alpha(X) \end{aligned}$$

is continuous.

Moreover:

Lemma 3.3. *Let S be a surface of signature (g, n) and $H = \{\alpha_1, \dots, \alpha_{6g-6+3n}\}$ a hexagon decomposition. For every $L > 0$*

$$\varphi_H(\text{Teich}(S; L)) = \left\{ x \in \mathbb{R}_{>0}^{6g-6+3n} \mid F_H(x) = L \right\}.$$

Proof. This is a consequence of Lemma 2.2 and the fact that the boundary of S is the union of all the sides not belonging to H in the hexagons in the decomposition given by H . \square

4. ORTHOSYSTOLES AND ORTHOKISSING NUMBER: GENERAL BOUNDS

This section is dedicated to proving some basic properties of orthosystoles and part (1) of Theorem B. We start by showing that there are no bounds on the orthosystole depending only on the geometry of the surface:

Lemma 4.1. *Let S be a compact surface with nonempty boundary. Then*

$$\inf_{X \in \mathcal{M}(S)} \text{osys}(X) = 0 \quad \text{and} \quad \sup_{X \in \mathcal{M}(S)} \text{osys}(X) = \infty.$$

Proof. Given a pair of pants of boundary lengths a, b, b , the length of the shortest orthogeodesic from and to the boundary component of length a goes to zero if $a \rightarrow \infty$ and $b \rightarrow 0$. By either gluing the two boundary components of length b to each other or attaching a subsurface to them, we get surfaces in each moduli space with arbitrarily short orthosystole. The fact that the supremum is infinite follows from the Collar Lemma, by letting all the boundary lengths go to zero. \square

Just as systoles, orthosystoles are simple. On the other hand, while two systoles can intersect, we prove that orthosystoles are disjoint, which will imply the bound on the orthokissing number mentioned in the introduction.

Lemma 4.2. *Orthosystoles are simple and disjoint. In particular, a surface X of signature (g, n) satisfies $\text{okiss}(X) \leq 6g - 6 + 3n$.*

Proof. Simplicity and disjointness follow from a standard surgery argument: if an orthosystole is not simple or two orthosystoles intersect, we can construct a shorter orthogeodesic using subarcs determined by some (self-)intersection point.

As there are at most $6g - 6 + 3n$ pairwise disjoint orthogeodesics on a surface of signature (g, n) , the bound on $\text{okiss}(X)$ follows immediately. \square

The following criterion to detect orthosystoles will be crucial in the proofs of Theorems A and B.

Lemma 4.3. *Let X be a hyperbolic surface with a hexagon decomposition of orthogeodesics of the same length. Then the orthogeodesics in the hexagon decomposition are precisely the orthosystoles.*

Proof. Denote by a the length of the arcs in the hexagon decomposition and let α be the length of the other sides of a hexagon with alternating sides of length a .

Let γ be an orthogeodesic of X not in the hexagon decomposition. It starts at some point $p \in \partial X$, which needs to be contained in the interior of a side of length α in some hexagon, and it is split into arcs a_1, \dots, a_k by the orthogeodesics in the hexagon decomposition, where $k \geq 2$ (since γ is not in the hexagon decomposition). As there are no hyperbolic triangles with two right angles, a_1 and a_k need to join opposite sides of a hexagon. In particular they both have length at least h , where h is the orthogonal between opposite sides of a hexagon, so $\ell(\gamma) \geq 2h$. Moreover by Lemma 2.2

$$\sinh(h) = \frac{\cosh(a)}{\sinh(a/2)} = \frac{2 \sinh^2(a/2) + 1}{\sinh(a/2)} > \sinh(a/2)$$

i.e. $2h > a$, proving that the orthogeodesics in the hexagon decomposition are the unique ones of minimal length. \square

Next we show that the orthosystole function is locally the minimum of finitely many continuous functions, and thus continuous.

Lemma 4.4. *The orthosystole function*

$$\begin{aligned} \text{osys} : \text{Teich}(S) &\rightarrow \mathbb{R} \\ X &\mapsto \text{osys}(X) \end{aligned}$$

is locally the minimum of the lengths of finitely many orthogeodesics, i.e. for every $X \in \text{Teich}(S)$ there is an open neighborhood U of X and a finite collection $A \subset \mathcal{A}$ such that if $Y \in U$

$$\text{osys}(Y) = \min\{\ell_\alpha(Y) \mid \alpha \in A\}.$$

Up to restricting U , we can assume

$$A = \{\alpha \in \mathcal{A} \mid \ell_\alpha(X) = \text{osys}(X)\}.$$

In particular, osys is a continuous function.

Proof. Let $X \in \text{Teich}(S)$ and $K > 1$. Let d_T be the Teichmüller distance on $\text{Teich}(S)$. Then for every $Y \in \text{Teich}(S)$ such that $d_T(X, Y) \leq \log K$ we have $\text{osys}(Y) \leq K \text{osys}(X)$ (by Theorem 2.4), so

$$\text{osys}(Y) = \min\{\ell_\alpha(Y) \mid \alpha \in \mathcal{A} : \ell_\alpha(Y) \leq K \text{osys}(X)\}.$$

Again by Theorem 2.4

$$\{\alpha \in \mathcal{A} \mid \ell_\alpha(Y) \leq K \text{osys}(X)\} \subset \{\alpha \in \mathcal{A} \mid \ell_\alpha(X) \leq K^2 \text{osys}(X)\} =: A.$$

So for every $Y \in \text{Teich}(S)$ such that $d_T(X, Y) \leq \log K$

$$\text{osys}(Y) = \min\{\ell_\alpha(Y) \mid \alpha \in A\}.$$

Moreover, A is finite by discreteness of the orthospectrum.

Let $M(X) = \{\alpha \in \mathcal{A} \mid \ell_\alpha(X) = \text{osys}(X)\}$; note that $M(X) \subset A$. By discreteness of the orthospectrum and Theorem 2.4 there is $K' \leq K$ such that if $U = \{Y \in \text{Teich}(S) \mid d_T(X, Y) \leq \log K'\}$ and $Y \in U$, for every $\alpha \in A \setminus M(X)$ and every $\beta \in M(X)$

$$\ell_\alpha(Y) \leq K' \ell_\alpha(X) < \frac{1}{K'} \ell_\beta(X) \leq \ell_\beta(Y),$$

so

$$\text{osys}(Y) = \min_{\gamma \in A(X, K)} \ell_\gamma(Y) = \min_{\gamma \in M(X)} \ell_\gamma(Y),$$

as required.

As lengths of orthogeodesics are continuous functions, osys is continuous as well. \square

In the case of simple closed geodesics, it follows directly from Mumford's compactness criterion and continuity of the systole function that there are global systole maximizers in moduli or Teichmüller space. One might wonder if one could show an analogue of Mumford's criterion using orthogeodesic lengths, i.e. proving that for every $\varepsilon > 0$, the set

$$\{X \in \mathcal{M}(S; \ell_1, \dots, \ell_n) \mid \text{osys}(X) \geq \varepsilon\}$$

is a compact subset of moduli space. Unfortunately, this is not the case, as we will show in Section 6. This is the reason why we need a different approach to show that the orthosystole function has a global maximum. To this end, we prove:

Proposition 4.5. *For every S of signature (g, n) and for every $\ell_1, \dots, \ell_n > 0$ there is $\varepsilon > 0$ such that if $X \in \mathcal{M}(S; \ell_1, \dots, \ell_n)$ satisfies $\text{sys}(X) < \varepsilon$, then there is $Y \in \mathcal{M}(S; \ell_1, \dots, \ell_n)$ with $\text{sys}(Y) \geq \varepsilon$ and $\text{osys}(Y) > \text{osys}(X)$.*

Proof. Choose $\varepsilon < \min \left\{ \sinh^{-1}(1), 2 \sinh^{-1} \left(2 \sinh \left(\frac{\ell(\partial X)}{24g-24+12n} \right) \right) \right\}$.

Suppose $\text{sys}(X) < \varepsilon$. Let

$$C_\varepsilon = \{\gamma \in \mathcal{C}(S) \mid \ell_\gamma(X) < \varepsilon\}.$$

Note that if an orthogeodesic crosses a curve in of length at most ε , by Lemma 2.3 it has length at least

$$2 \sinh^{-1} \left(\frac{1}{\sinh(\varepsilon/2)} \right) > 2 \sinh^{-1} \left(\frac{1}{2 \sinh \left(\frac{\ell(\partial X)}{24g-24+12n} \right)} \right).$$

So by Theorem 1.1 any orthosystole is disjoint from C_ε .

Let X' be the surface obtained by cutting X along C_ε . Apply Theorem 2.5 and Remark 2.6 to X' to get a surface Y' where all curves corresponding to curves in C_ε have length ε and such that all curves in Y' are longer than those in X' and all orthogeodesics of Y' joining two curves corresponding to boundary components of X are longer or equal to the corresponding in X' . Glue the boundary components of Y' which corresponded to the same curve in X to get a surface Y . Let $A \subset \mathcal{A}$ be a finite set such that

$$\text{osys}(X) = \min\{\ell_\alpha(X) \mid \alpha \in A\} \quad \text{and} \quad \text{osys}(Y) = \min\{\ell_\alpha(Y) \mid \alpha \in A\}.$$

Since orthosystoles are disjoint from C_ε , we can assume that A contains only arcs disjoint from C_ε . By construction $\text{sys}(Y) = \varepsilon$ and for every $\alpha \in A$,

$$\ell_\alpha(X) < \ell_\alpha(Y),$$

so $\text{osys}(X) < \text{osys}(Y)$. □

An immediate consequence is the following:

Corollary 4.6. *The function osys restricted to $\mathcal{M}(S; \ell_1, \dots, \ell_n)$ admits a maximum.*

Proof. By Proposition 4.5, there is $\varepsilon > 0$ such that

$$\sup\{\text{osys}(X) \mid X \in \mathcal{M}(S; \ell_1, \dots, \ell_n)\} = \sup\{\text{osys}(X) \mid X \in \mathcal{M}_\varepsilon(S; \ell_1, \dots, \ell_n)\}.$$

As $\mathcal{M}_\varepsilon(S; \ell_1, \dots, \ell_n)$ is compact and osys is continuous on moduli space (since it is a mapping class group invariant continuous function on Teichmüller space), osys admits a maximum □

5. SURFACES WITH ONE BOUNDARY COMPONENT

This section is dedicated to the case of surfaces with a single boundary component and the proof of Theorem A. The main reason why having a single boundary component is an advantage is the fact that in this case we can always construct surfaces with a hexagon decomposition of orthosystoles and we can explicitly compute their orthosystole length. The main difficulty in the proof of Theorem A is showing that local maxima for the orthosystole function need to have a hexagon decomposition of orthosystoles.

We start by showing the existence of surfaces with a hexagon decomposition of orthosystoles.

Lemma 5.1. *Let S be a surface of signature $(g, 1)$ and H a hexagon decomposition of S . Let $\ell > 0$. There is a unique $X \in \text{Teich}(S; \ell)$ such that all arcs in H have the same length and*

$$\text{osys}(X) = f(g, \ell) := 2 \sinh^{-1} \left(\frac{1}{2 \sinh(\ell/(24g - 12))} \right).$$

Proof. Uniqueness of the hyperbolic structure follows from Theorem 3.1, so we just need to check existence. Suppose then that all orthogeodesic representatives of arcs in H have length a . All hexagons have three alternating sides of length a ; the other three sides have the same length α given by

$$\cosh(\alpha) = \frac{\cosh(a) + \cosh^2(a)}{\sinh^2(a)} = \frac{\cosh(a) + \cosh^2(a)}{\cosh^2(a) - 1} = \frac{\cosh(a)}{\cosh(a) - 1}.$$

The boundary of X is given by $3(4g - 2)$ α -sides (since there are $4g - 2$ hexagons), i.e.

$$\ell = (12g - 6)\alpha.$$

So

$$\cosh(a) = \frac{\cosh\left(\frac{\ell}{12g-6}\right)}{\cosh\left(\frac{\ell}{12g-6}\right) - 1}$$

and since the right-hand side is bigger than one, there is a (unique) solution a , and standard computations show that

$$a = 2 \sinh^{-1} \left(\frac{1}{2 \sinh(\ell/(24g - 12))} \right).$$

By Lemma 4.3, the arcs in H are precisely the orthosystoles, so the statement about $\text{osys}(X)$ follows. \square

Remark 5.2. The function $f(g, \ell)$ is independent of the hexagon decomposition. In particular non-homeomorphic hexagon decompositions yield non-isometric surfaces with the same orthosystole (indeed, an isometry between surfaces with hexagon decompositions of orthogeodesics of minimal length needs to send the hexagon decomposition of a surface to the hexagon decomposition of the other surface, since the arcs in the hexagon decompositions are the only ones of minimal length).

To show that local maxima for the orthosystole function have a hexagon decomposition of orthosystole, we will need the following technical lemma.

Lemma 5.3. *Let $X \in \text{Teich}(S)$ and $H = \{\alpha_1, \dots, \alpha_{6g-3}\}$ a hexagon decomposition.*

- (1) *If $\ell_{\alpha_i}(X) = \min_j \ell_{\alpha_j}(X)$, then $\frac{\partial F_H}{\partial x_i}(\varphi_H(X)) < 0$.*
- (2) *If $\frac{\partial F_H}{\partial x_j}(\varphi_H(X)) = 0$, then $\ell_{\alpha_j}(X)$ is a global minimum or an inflection point of the function*

$$F_{H, X, i} : \mathbb{R}_+ \rightarrow \mathbb{R}$$

$$y \mapsto F_H(\ell_{\alpha_1}(X), \dots, \ell_{\alpha_{j-1}}(X), y, \ell_{\alpha_{j+1}}(X), \dots, \ell_{\alpha_{6g-6+3n}}(X)).$$

Proof. We have

$$F_H : \mathbb{R}^{6g-6+3n} \rightarrow \mathbb{R}$$

$$x \mapsto \sum_{\{i, j, k\} \in \partial_H} s(\cosh(x_i), \cosh(x_j), \cosh(x_k)).$$

For any i , x_i appears in exactly two terms in the sum, those corresponding to the two hexagons containing the orthogeodesic α_i . Suppose the two terms correspond to triples $\{i, j, k\}$ and $\{i, l, m\}$. Then

$$\frac{\partial F_H}{\partial x_i} = \frac{\partial s(\cosh(x_i), \cosh(x_j), \cosh(x_k))}{\partial x_i} + \frac{\partial s(\cosh(x_i), \cosh(x_l), \cosh(x_m))}{\partial x_i}.$$

Using the expression for the derivative of s (Lemma 2.2), we get

$$\frac{\partial F_H}{\partial x_i} = \sinh(x_i) \left(\frac{\cosh(x_i) - 1 - \cosh(x_j) - \cosh(x_k)}{(\cosh(x_i) - 1) \sqrt{\cosh(x_i)^2 + \cosh(x_j)^2 + \cosh(x_k)^2 + 2 \cosh(x_i) \cosh(x_j) \cosh(x_k) - 1}} + \frac{\cosh(x_i) - 1 - \cosh(x_l) - \cosh(x_m)}{(\cosh(x_i) - 1) \sqrt{\cosh(x_i)^2 + \cosh(x_l)^2 + \cosh(x_m)^2 + 2 \cosh(x_i) \cosh(x_l) \cosh(x_m) - 1}} \right)$$

So if $x_i \leq x_j, x_k, x_l, x_m$ the derivative is negative. In particular, this holds if $\ell_{\alpha_i}(X) = \min_j \ell_X(\alpha_j)$, proving (1).

If $\frac{\partial F_H}{\partial x_j}(\varphi_H(X)) = 0$, then $\ell_{\alpha_i}(X)$ is a zero of the derivative of $F_{H,X,i}$. Denote by c_j, c_k, c_l, c_m the hyperbolic cosines of $\ell_{\alpha_j}(X), \ell_{\alpha_k}(X), \ell_{\alpha_l}(X), \ell_{\alpha_m}(X)$. A point $y \in (0, \infty)$ is a zero of the derivative of $F_{H,X,i}$ if and only if

$$\frac{\cosh(y) - 1 - c_j - c_k}{(\cosh(y) - 1) \sqrt{\cosh(y)^2 + c_j^2 + c_k^2 + 2 \cosh(y) c_j c_k - 1}} + \frac{\cosh(y) - 1 - c_l - c_m}{(\cosh(y) - 1) \sqrt{\cosh(y)^2 + c_l^2 + c_m^2 + 2 \cosh(y) c_l c_m - 1}} = 0$$

which implies

$$(\cosh(y) - 1 - c_j - c_k) \sqrt{\cosh(y)^2 + c_l^2 + c_m^2 + 2 \cosh(y) c_l c_m - 1} = -(\cosh(y) - 1 - c_l - c_m) \sqrt{\cosh(y)^2 + c_j^2 + c_k^2 + 2 \cosh(y) c_j c_k - 1}$$

which in turn implies

$$(\cosh(y) - 1 - c_j - c_k)^2 (\cosh(y)^2 + c_l^2 + c_m^2 + 2 \cosh(y) c_l c_m - 1) = (\cosh(y) - 1 - c_l - c_m)^2 (\cosh(y)^2 + c_j^2 + c_k^2 + 2 \cosh(y) c_j c_k - 1).$$

This is a third degree equation in $z = \cosh(y)$ (the coefficients of $\cosh(y)^4$ cancel out), which has three solutions, $z = 1, z = z_1$ and $z = z_2$. Since $z = 1$ implies that $y = 0$, which is impossible, there are at most two zeroes of the derivative of $F_{H,X,i}$ in the interval $(0, \infty)$. Since by Lemma 2.2

$$\lim_{y \rightarrow 0} F_{H,X,i}(y) = +\infty = \lim_{y \rightarrow +\infty} F_{H,X,i}(y),$$

there should be at least one zero of the derivative, which is a global minimum, and if there are two the second zero can only be an inflexion point. \square

Proposition 5.4. *Let $\ell > 0$ and S be a surface of signature $(g, 1)$. Then if $X \in \text{Teich}(S; \ell)$ is a local maximum for the orthosystole, then $\text{okiss}(X) = 6g - 3$ (i.e. $\text{okiss}(X)$ is maximal).*

Proof. Let $X \in \text{Teich}(S; \ell)$ be a surface with $\text{okiss}(X) = k < 6g - 3$. Let $M = \{\alpha_1, \dots, \alpha_k\}$ be the collection of orthosystoles of X . Complete M to a hexagon decomposition $H = \{\alpha_1, \dots, \alpha_{6g-3}\}$. Our goal is to show that X is not a local maximum.

The idea of the proof is to define surfaces arbitrarily close to X by increasing the lengths of all arcs in M and varying the length of α_{6g-3} in such a way that the boundary still has length ℓ . We want to vary all these lengths by a sufficiently small amount so that, by Lemma 4.4, the orthosystole of the new surfaces is the minimum of the lengths of the curves in M , which in turn implies that the surfaces we construct all have orthosystole strictly longer than $\text{osys}(X)$. For this, we need to know that if we keep the boundary length fixed and we continuously all lengths of arcs in H but one, the length of the last arc varies continuously in terms of the other lengths. Since the boundary length of a surface is defined by a quite explicit function of the lengths of the arcs in H (the function F_H from Section 3), we would want to use the implicit function theorem to show this continuity property. For this, we need the derivative $\frac{\partial F_H}{\partial x_{6g-3}}(\varphi_H(X))$ to

be nonzero, which is unfortunately not always the case. So when this derivative is zero, we will instead increase the lengths of $\alpha_2, \dots, \alpha_k$ and α_{6g-3} by small controlled amounts and prove that to keep the boundary length fixed we need to increase the length of α_1 as well.

Let us formalize the argument. Note first that by Lemma 4.4 and Theorem 3.1 there is $\varepsilon_1 > 0$ such that if $Y \in \text{Teich}(S)$ and

$$\max_{i \in \{1, \dots, 6g-3\}} |\ell_{\alpha_i}(Y) - \ell_{\alpha_i}(X)| < \varepsilon_1,$$

then

$$\text{osys}(Y) = \min_{i \in \{1, \dots, k\}} \ell_{\alpha_i}(Y).$$

We assume that ε_1 is small enough so that for every i , $\ell_{\alpha_i}(X) - \varepsilon_1 > 0$.

Let $a = (a_1, \dots, a_{6g-3}) := \varphi_H(X)$.

Case 1: $\frac{\partial F_H}{\partial x_{6g-3}}(a) \neq 0$.

By the implicit function theorem there is $\varepsilon_2 > 0$ and a continuous function

$$h : U = \left\{ (y_1, \dots, y_{6g-4}) \in \mathbb{R}^{6g-4} \mid \max_{1 \leq i \leq 6g-4} |y_i - a_i| < \varepsilon_2 \right\} \rightarrow \mathbb{R}$$

such that $F_H(y_1, \dots, y_{6g-4}, h(y_1, \dots, y_{6g-4})) = \ell$. We can assume that ε_2 is small enough so that $U \subset (\mathbb{R}_+)^{6g-4}$ and $h(U) \subset (a_{6g-3} - \varepsilon_1, a_{6g-3} + \varepsilon_1)$.

For every $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$ define

$$X_\varepsilon = \varphi_H^{-1}(a_1 + \varepsilon, \dots, a_k + \varepsilon, a_{k+1}, \dots, a_{6g-4}, h(a_1 + \varepsilon, \dots, a_k + \varepsilon, a_{k+1}, \dots, a_{6g-4})).$$

In other words, X_ε is the surface where the lengths of the orthogeodesics in M are all increased by ε and the length of α_{6g-3} is adjusted so that the length of the boundary of X_ε is ℓ . By the choice of $\varepsilon_1, \varepsilon_2$ and ε

$$\text{osys}(X_\varepsilon) = \min_{1 \leq i \leq k} \ell_{\alpha_i}(Y) = \min_{1 \leq i \leq k} a_i + \varepsilon = \text{osys}(X) + \varepsilon.$$

Since X_ε converges to X , X is not a local maximum of osys .

Case 2: $\frac{\partial F_H}{\partial x_{6g-3}}(a) = 0$.

By Lemma 5.3, $\frac{\partial F_H}{\partial x_i}(a) < 0$ for every $1 \leq i \leq k$ and a_{6g-3} is a global minimum or an inflexion point of the function

$$y \mapsto F_H(a_1, \dots, a_{6g-4}, y).$$

In case it's an inflexion point, let us assume that the function is locally monotone increasing around a_{6g-3} ; the case in which the function is locally decreasing is analogous.

By the implicit function theorem, there is $\varepsilon_2, \varepsilon_3 > 0$ and a continuous function

$$h : U = \left\{ (y_2, \dots, y_{6g-3}) \in \mathbb{R}^{6g-4} \mid \max_{2 \leq i \leq 6g-3} |y_i - a_i| < \varepsilon_2 \right\} \rightarrow \mathbb{R}$$

such that $F_H(h(y_2, \dots, y_{6g-3}), y_2, \dots, y_{6g-3}) = \ell$. We can assume that $\varepsilon_2, \varepsilon_3$ are small enough so that $U \subset (\mathbb{R}_+)^{6g-4}$, $h(U) \subset (a_1 - \varepsilon_3, a_1 + \varepsilon_3)$, for some $\varepsilon_3 < \varepsilon_1$, and

$$\frac{\partial F_H}{\partial x_1}(z, y_2, \dots, y_{6g-3}) < 0$$

for all $(y_2, \dots, y_{6g-3}) \in U$ and $z \in (a_1 - \varepsilon_3, a_1 + \varepsilon_3)$.

For every $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$ there is $\delta > 0$, $\delta < \min\{\varepsilon_1, \varepsilon_2\}$, such that the length of the boundary of the surface obtained by increasing² the length of α_{6g-3} by ε and the lengths of $\alpha_2, \dots, \alpha_k$ by δ is more than ℓ . This holds since the boundary length is a continuous function of the lengths of arcs in H and by Lemma 5.3, if we increase the length of α_{6g-3} , the boundary

²If instead the function $y \mapsto F_H(a_1, \dots, a_{6g-4}, y)$ is locally monotone decreasing around a_{6g-3} , we decrease the length of α_{6g-3} by ε .

length increases, while if we increase the length of any arc in M , the boundary length decreases. Now let $X_{\varepsilon, \delta}$ be the surface such that:

$$\begin{cases} \ell_{\alpha_i}(X_{\varepsilon, \delta}) = a_i + \delta & \forall i \in \{2, \dots, k\} \\ \ell_{\alpha_i}(X_{\varepsilon, \delta}) = a_i & \forall i \in \{k+1, \dots, 6g-4\} \\ \ell_{\alpha_{6g-3}}(X_{\varepsilon, \delta}) = a_{6g-3} + \varepsilon \\ \ell_{\alpha_1}(X_{\varepsilon, \delta}) = h(a_2 + \delta, \dots, a_k + \delta, a_{k+1}, \dots, a_{6g-4}, a_{6g-3} + \varepsilon) \end{cases}$$

that is,

$$\varphi_H(X_{\varepsilon, \delta}) = (h(a_2 + \delta, \dots, a_k + \delta, a_{k+1}, \dots, a_{6g-4}, a_{6g-3} + \varepsilon), a_2 + \delta, \dots, a_k + \delta, a_{k+1}, \dots, a_{6g-4}, a_{6g-3} + \varepsilon).$$

Note that $h(a_2 + \delta, \dots, a_k + \delta, a_{k+1}, \dots, a_{6g-4}, a_{6g-3} + \varepsilon) > a_1$ because

$$F_H(a_1, a_2 + \delta, \dots, a_k + \delta, a_{k+1}, \dots, a_{6g-4}, a_{6g-3} + \varepsilon) > \ell$$

and by the assumption on ε_2 (and its consequence on the derivative with respect to x_1) we need to increase a_1 to decrease ℓ . So

$$\text{osys}(X_{\varepsilon, \delta}) = \min_{1 \leq i \leq k} \ell_{\alpha_i}(X_{\varepsilon, \delta}) > \text{osys}(X).$$

As ε tends to zero, so does δ and thus $X_{\varepsilon, \delta}$ converges to X , showing again that X is not a local maximum. \square

We can now prove the equivalence of the statements in Theorem A:

Proof of Theorem A. Recall that the four equivalent conditions are:

- (1) X is a global maximum for the orthosystole function;
- (2) X is a local maximum for the orthosystole function;
- (3) $\text{okiss}(X) = 6g - 3$;
- (4) $\text{osys}(X) = 2 \sinh^{-1} \left(\frac{1}{2 \sinh(\ell/(24g - 12))} \right)$.

It is clear that (1) implies (2). Proposition 5.4 shows that (2) implies (3) and Lemma 5.1 show that (3) implies (4).

To prove that (4) implies (1), we note that all local maxima have the same orthosystole length (since (2) implies (4)) and thus, as we know that osys admits a global maximum (Proposition 5.4), all local maxima are also global maxima. Moreover, this also shows that

$$\sup_{X \in \mathcal{M}(S; \ell_1, \dots, \ell_n)} \text{osys}(X) = 2 \sinh^{-1} \left(\frac{1}{2 \sinh(\ell/(24g - 12))} \right)$$

hence (4) implies (1). \square

Note that the equivalence of (1), (3) and (4) follow from Bavard's work, though we've given here alternative proofs. The following Corollary will finish the proof of Theorem A.

Corollary 5.5. *For every $g \geq 1$, the function $\text{osys} : \mathcal{M}(S; \ell) \rightarrow \mathbb{R}$ has exactly*

$$\frac{2(6g - 5)!}{12^g g! (3g - 3)!}$$

local maxima (which are all global maxima).

Proof. This follows from Theorem A, Remark 5.2, the fact that homeomorphism classes of hexagon decomposition are in correspondence with homeomorphism classes of ideal triangulations of a surface with one puncture, and the computation of the number of homeomorphism classes of such triangulations from [1]. \square

6. SURFACES WITH MULTIPLE BOUNDARY COMPONENTS

Understanding local and global maxima of the orthosystole function in the case of multiple boundary components seems significantly harder. The main reason is that we cannot in general guarantee that there is a surface with a hexagon decomposition of orthosystoles. Indeed, if a surface X of signature (g, n) has a hexagon decomposition of orthosystoles, then each boundary component is a union of arcs of length α , where α is the length of any other side of a right-angled hexagon of alternating sides of length $\text{osys}(X)$. In particular, if the boundary lengths are not rational multiples of each other there won't be any hexagon decomposition of orthosystoles. But even when the boundary lengths are rational multiples of each other it is not clear if it is always possible to have a hexagon decomposition of orthosystoles.

As mentioned before, in [3] it is the sum of the boundary lengths to be fixed, instead of the individual lengths. Note that for every signature and every sum of boundary lengths we can find a hyperbolic surface with a hexagon decomposition of orthosystoles, which means that it attains the bound in Theorem 1.1.

One case in which we can show the existence of such a hexagon decomposition is the case of surfaces with two boundary components of the same length. We will then use this construction to construct surfaces with large orthosystole and prove Theorem B.

Lemma 6.1. *Let $S_{g,2}$ be a surface of signature $(g, 2)$. For any $\ell > 0$ there is $X \in \mathcal{M}(S_{g,2}; \ell, \ell)$ with a hexagon decomposition of orthosystoles and*

$$\text{osys}(X) = \cosh^{-1} \left(\frac{\cosh \left(\frac{\ell}{6g} \right)}{\cosh \left(\frac{\ell}{6g} \right) - 1} \right) = \max_{Y \in \mathcal{M}(S_{g,2}; \ell, \ell)} \text{osys}(Y).$$

Proof. Color one boundary component of $S_{g,2}$ blue and the other red. We will first show, by induction on the genus, that $S_{g,2}$ has a hexagon decomposition such that half of the hexagons have two red sides and one blue side and half of the hexagons have two blue sides and one red side.

In the base case, $g = 1$, an example of such a hexagon decomposition is given in Figure 2.

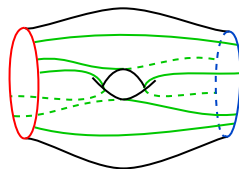
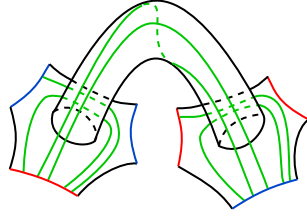


FIGURE 2. A hexagon decomposition on $S_{1,2}$

For the induction step, assume H is a hexagon decomposition with the required properties on $S_{g,2}$. Choose a hexagon H_1 with two red sides and a hexagon H_2 with two blue sides. Remove a disk from the interior of H_1 and one from the interior of H_2 and glue the two new boundary components together. We now have a surface of signature $(g + 1, 2)$ with a collection of disjoint arcs. We add arcs as in Figure 3 to get a hexagon decomposition with the required properties.

So for every $g \geq 1$ we can choose a hexagon decomposition with the properties above; if we choose all arcs in this hexagon decomposition to have the same length a we get a hyperbolic surface X with a hexagon decomposition of orthosystoles (Lemma 4.3), whose boundary components have the same length, equal to $6g\alpha$, where α is the length of the other sides of a hexagon with alternating sides of length a, a, a . So the boundary lengths are ℓ if and only if

$$\ell = 6g \cosh^{-1} \left(\frac{\cosh(a)}{\cosh(a) - 1} \right)$$

FIGURE 3. From genus g to genus $g + 1$

i.e.

$$\cosh(a) = \frac{\cosh\left(\frac{\ell}{6g}\right)}{\cosh\left(\frac{\ell}{6g}\right) - 1}$$

and as the right-hand side is bigger than one, there is a unique solution and by Lemma 4.3

$$\text{osys}(X) = \cosh^{-1}\left(\frac{\cosh\left(\frac{\ell}{6g}\right)}{\cosh\left(\frac{\ell}{6g}\right) - 1}\right).$$

By Theorem [3], X is a global maximum for osys . \square

Using Lemma 6.1, we can prove part (2) of Theorem B:

Proof of Theorem B, part (2). Suppose first $n = 2$ and the genus is odd. Let X_1 and X_2 be hyperbolic surfaces of signature $(\frac{g-1}{2}, 2)$ and boundary components γ_1, δ_1 and γ_2, δ_2 respectively, such that:

- $\ell_{\gamma_i}(X_i) = \ell_{\delta_i}(X_i) = \ell_i$, and
- X_i has a hexagon decomposition of orthogeodesics of minimal length as in Lemma 6.1.

Let Y be a two-holed torus, with boundary components η_1 and η_2 of lengths ℓ_1 and ℓ_2 respectively. Let X be a surface obtained from the X_i and Y by gluing, for every i , δ_i to η_i . Note that any orthogeodesic α in X is either contained in a X_i or it contains a subarc joining two boundary components of a X_i . By construction each X_i contains an orthogeodesic from γ_i to itself of length $\text{osys}(X_i)$, which gives an orthogeodesic on X of the same length. Thus

$$\begin{aligned} \text{osys}(X) &= \min\{\text{osys}(X_1), \text{osys}(X_2)\} = \text{osys}(X_2) = \\ &= \cosh^{-1}\left(\frac{\cosh\left(\frac{\ell_2}{3g-3}\right)}{\cosh\left(\frac{\ell_2}{3g-3}\right) - 1}\right) = \cosh^{-1}\left(\frac{\cosh\left(\frac{\ell_2}{6\lfloor\frac{g}{2}\rfloor}\right)}{\cosh\left(\frac{\ell_2}{6\lfloor\frac{g}{2}\rfloor}\right) - 1}\right), \end{aligned}$$

where the second equality follows from the monotonicity of

$$x \mapsto \cosh^{-1}\left(\frac{\cosh(x)}{\cosh(x) - 1}\right)$$

for $x > 0$.

The case $n = 2$ and even genus is similar; the only difference is that we choose S_1 to have signature $(\frac{g}{2}, 2)$ and S_2 to have signature $(\frac{g}{2} - 1, 2)$. We then get a surface with orthosystole

$$\cosh^{-1}\left(\frac{\cosh\left(\frac{\ell_2}{3g-6}\right)}{\cosh\left(\frac{\ell_2}{3g-6}\right) - 1}\right) \geq \cosh^{-1}\left(\frac{\cosh\left(\frac{\ell_2}{6\lfloor\frac{g}{2}\rfloor}\right)}{\cosh\left(\frac{\ell_2}{6\lfloor\frac{g}{2}\rfloor}\right) - 1}\right),$$

where again we are using the monotonicity of

$$x \mapsto \cosh^{-1} \left(\frac{\cosh(x)}{\cosh(x) - 1} \right)$$

for $x > 0$.

Suppose now $n \geq 3$. Let $q = \lfloor \frac{g}{n} \rfloor$ and let r be such that $g = qn + r$. Consider:

- X_i a hyperbolic surface of signature $q, 2$, with boundary components γ_i, δ_i of length ℓ_i and with a hexagon decomposition as in Lemma 6.1;
- Y a hyperbolic surface of signature q, n and boundary components η_1, \dots, η_n of lengths ℓ_1, \dots, ℓ_n respectively.

Let X be a surface obtained from the X_i to Y by gluing, for every i , δ_i to η_i with any choice of twist. Note that $X \in \text{Teich}(S; \ell_1, \dots, \ell_n)$; furthermore, the same argument as above shows that

$$\text{osys}(X) = \text{osys}(Y_n) = \cosh^{-1} \left(\frac{\cosh \left(\frac{\ell_n}{6 \lfloor \frac{g}{n} \rfloor} \right)}{\cosh \left(\frac{\ell_n}{6 \lfloor \frac{g}{n} \rfloor} \right) - 1} \right).$$

□

With similar techniques to those used in the proof of Proposition 5.4, we can show that the surfaces constructed in the previous proof aren't even local maxima for the orthosystole function. It is actually possible to find sequences of surfaces in different moduli space whose orthosystole is longer than that of the surfaces in Theorem B. We can for example prove the following:

Lemma 6.2. *Fix $n \geq 1$; for any $m \geq 0$ let S_m be a surface of signature $(nm + 1, n)$. Then for every $\ell > 0$ there is $X_m \in \text{Teich}(S_m; \ell, \dots, \ell)$ with a hexagon decomposition of orthogeodesics of minimal length. Moreover*

$$\text{osys}(X_m) = \cosh^{-1} \left(\frac{\cosh \left(\frac{\ell}{12m+6} \right)}{\cosh \left(\frac{\ell}{12m+6} \right) - 1} \right).$$

Proof. Fix $m \geq 0$; the choice of genus allows us to find an order n symmetry φ of S_m , such that a fundamental domain for the action is a subsurface F of signature $(m, 3)$, as in Figure 4.

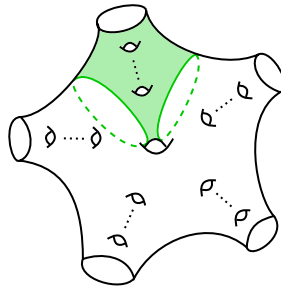


FIGURE 4. The case $n = 5$; the fundamental domain is shaded.

We then construct a hexagon decomposition of S_m as follows.

Look first at F and denote by γ its boundary component which is also a boundary component of S_m . Let \hat{F} be the surface obtained by gluing two disks to the two other boundary components of F . Fix a hexagon decomposition of \hat{F} and homotope the arcs so that they are contained in F and that the two disks in $\hat{F} \setminus F$ are contained in different hexagons. Look now at F ; for

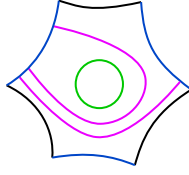


FIGURE 5. The sides in blue are subarcs of γ ; the arcs in purple are to be added.

each hexagon containing a boundary component add two new essential arcs from γ to itself, as in Figure 5.

We now have a collection of $6m - 3 + 4 = 6m + 1$ arcs on F . Take all the arcs in their φ -orbit, so that we have $6mn + n$ arcs on S_m . It is not hard to see that there are n complementary components C_i , for $i = 0, \dots, n - 1$, which are not hexagons, where C_i contains $\varphi^i(\gamma)$ and $\varphi^{i+1}(\gamma)$. For every i , add two arcs in C_i , each joining $\varphi^i(\gamma)$ to $\varphi^{i+1}(\gamma)$. The union of all these arcs is a hexagon decomposition H_m of S_m with the property that each boundary component of S_m is split into the same number of subarcs. This implies that if all arcs in H_m are given the same length, all boundary components of S_m will have the same length. To conclude the proof we just need to show that for every $\ell > 0$ we can choose a length $a = a(\ell, n, m) > 0$ so that if all arcs in H_m have length a , each boundary component of S_m has length ℓ . Since H_m contains $4nm + 2n$ hexagons, by Lemma 2.2 we know that a needs to satisfy

$$n\ell = (4nm + 2n)s(\cosh(a), \cosh(a), \cosh(a))$$

i.e.

$$n\ell = (12nm + 6n) \cosh^{-1} \left(\frac{\cosh(a)}{\cosh(a) - 1} \right)$$

which implies

$$\cosh(a) = \frac{\cosh \left(\frac{\ell}{12m+6} \right)}{\cosh \left(\frac{\ell}{12m+6} \right) - 1}.$$

As the right-hand side is bigger than one, there is a unique solution. Then X_m can be chosen to be the surface where the arcs in H_m have length a and by Lemma 4.3 they are the orthosystoles. \square

On the other hand, we are not able to prove (or disprove) that the surfaces from Lemma 6.2 are local, let alone global, maximizers of the orthosystoles.

We end this section by using Lemma 6.1 to show that there is no version of Mumford's compactness theorem that can be stated using orthosystoles instead of systoles:

Lemma 6.3. *Let $\ell > 0$. For every $\varepsilon > 0$ there is $g_0 \geq 1$ so that for every $g \geq g_0$, if S is a surface of signature $(g, 1)$, the set*

$$\{X \in \mathcal{M}(S; \ell) \mid \text{osys}(X) \geq \varepsilon\}$$

is not compact.

Proof. By Lemma 6.1, there is a surface X_g of signature $(g - 1, 2)$ and boundary lengths ℓ, ℓ with a hexagon decomposition of orthosystoles and

$$\text{osys}(X_g) = \cosh^{-1} \left(\frac{\cosh \left(\frac{\ell}{6(g-1)} \right)}{\cosh \left(\frac{\ell}{6(g-1)} \right) - 1} \right).$$

For any one-holed torus T of boundary length ℓ , we can construct a surface $Y_{g,T}$ by gluing T to a boundary component of X_g with any choice of twist parameter. Note that

$$\text{osys}(Y_{g,T}) \geq \text{osys}(X_g),$$

because any orthogeodesic in $Y_{g,T}$ is either an orthogeodesic of X_g or it contains an arc joining two boundary components of X_g .

Since $\text{osys}(Y_{g,T}) \rightarrow \infty$ as $g \rightarrow \infty$, for every $\varepsilon > 0$ there is g_0 so that $\text{osys}(Y_{g,T}) > \varepsilon$ for every $g \geq g_0$. It is sufficient to choose a sequence T_n of one-holed tori with a curve shrinking to zero to get a sequence $Y_{g,T}$ leaving every compact. \square

7. ORTHOSSYSTOLE MAXIMIZERS AND ORTHOKISSING NUMBER

In this section we prove a bound on the orthokissing number of a surface which is a local maximizer for the orthosystole. The main result we need is the following:

Proposition 7.1. *Let S be a surface of signature (g, n) and let $0 < \ell_1, \dots, \ell_n$. If $X \in \text{Teich}(S; \ell_1, \dots, \ell_n)$ is a local maximum for the orthosystole function, the collection of orthosystoles of X fill the surface, i.e. every essential simple closed geodesic on X intersects at least one orthosystole.*

Proof. By contradiction, suppose not. Denote by $\gamma_1, \dots, \gamma_n$ the boundary components of S . Let α be a simple closed geodesic disjoint from all orthosystoles of X . Cut X along α to get a (possibly disconnected) surface Y with $n + 2$ boundary components, n corresponding to $\gamma_1, \dots, \gamma_n$ and two new ones, which we denote by α_1 and α_2 . By Theorem 2.5, for every $\varepsilon > 0$ there is a surface Y_ε satisfying

$$\ell_{\gamma_i}(Y_\varepsilon) = \ell_{\gamma_i}(Y) \quad \forall i = 1, \dots, n$$

and

$$\ell_{\alpha_i}(Y_\varepsilon) = \ell_{\alpha_i}(Y) + \varepsilon \quad \forall i = 1, 2$$

and such that, by Remark 2.6 and since orthosystoles of X are disjoint from α , for any orthosystole β of X

$$\ell_\beta(Y_\varepsilon) > \ell_\beta(Y).$$

Glue back Y_ε , with the same twist parameter about α as X , to get a surface X_ε such that for any orthosystole β of X

$$\ell_\beta(X_\varepsilon) > \ell_\beta(X).$$

For ε small enough, the set of orthosystoles of X_ε is a subset of the set of orthosystoles of X (by discreteness of the orthogeodesic spectrum), for every ε small enough we get

$$\text{osys}(X_\varepsilon) > \text{osys}(X),$$

contradicting local maximality of X . \square

A consequence of the proposition is that local maximizers of the orthosystole functions lie in a compact subset of moduli space.

Corollary 7.2. *Let S be a surface of signature (g, n) and let $0 \leq \ell_1 \leq \dots \leq \ell_n$. Then there is $\varepsilon > 0$ such that any local maximum for the orthosystole function lies in $\mathcal{M}_\varepsilon(S; \ell_1, \dots, \ell_n)$. We can choose*

$$\varepsilon = 2 \sinh^{-1} \left(2 \sinh \left(\frac{\ell(\partial X)}{24g - 24 + 12n} \right) \right).$$

Proof. If X is a local maximum, its orthosystoles fill. So the systole of X intersects an orthosystole of X and thus

$$\text{osys}(X) \geq 2w(\text{sys}(X)) = 2 \sinh^{-1} \left(\frac{1}{\sinh(\text{sys}(X)/2)} \right).$$

By Theorem 1.1

$$\text{osys}(X) \leq 2 \sinh^{-1} \left(\frac{1}{2 \sinh \left(\frac{\ell(\partial X)}{24g - 24 + 12n} \right)} \right).$$

Combining the two inequalities yields the result. \square

To deduce the lower bound on the orthokissing number of a local maximizer for the orthostole function from Proposition 7.1 we just need to compute how many disjoint arcs are needed to fill:

Lemma 7.3. *Let S be a surface of signature (g, n) . Let \mathcal{A} be a filling collection of disjoint arcs of S . Then*

$$|\mathcal{A}| \geq \begin{cases} 2g - 2 + n & \text{if } n \geq 2 \\ 2g & \text{if } n = 1 \end{cases}$$

and the bound is sharp.

Proof. Suppose $|\mathcal{A}| = k$. As \mathcal{A} is filling, it cuts S into N_1 topological disks and N_2 peripheral annuli. Denote the disks by D_1, \dots, D_{N_1} and the annuli by A_1, \dots, A_{N_2} .

Each disk D_i is a polygon with sides alternating between arcs of \mathcal{A} and arcs of the boundary. Let d_i be the number of arcs of \mathcal{A} , so that D_i is a $2d_i$ -gon.

One boundary component of an annulus A_i is a boundary component of S and the other is a polygon with sides alternating between arcs of \mathcal{A} and arcs of the boundary. Let a_i be the number of arcs on \mathcal{A} in the boundary of A_i . Pick an arc β_i from one boundary component of A_i to the other, so that it intersects the polygonal boundary component of A_i in the middle of an arc of the boundary of S (see Figure 6). Cutting A_i along β_i yields a polygon B_i with $2(a_i + 2)$ sides, half of which are arcs of the boundary of S .

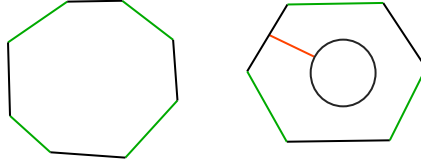


FIGURE 6. A disk and an annulus in $S \setminus \mathcal{A}$. The green sides are arcs of A , the black ones are arcs of the boundary of S and the orange one is a β_i .

We now compute the Euler characteristic using the CW-complex decomposition of S given by the D_i and the B_i . Note that each arc of A appears in exactly two polygons, so

$$\sum_{i=1}^{N_1} d_i + \sum_{i=1}^{N_2} a_i = 2k.$$

Each vertex of a D_i or B_i is identified with another vertex, so the number of 0-cells is

$$\frac{1}{2} \left(\sum_{i=1}^{N_1} (2d_i) + \sum_{i=1}^{N_2} (2(a_i + 2)) \right) = \sum_{i=1}^{N_1} d_i + \sum_{i=1}^{N_2} a_i + \sum_{i=1}^{N_2} 2 = 2k + 2N_2.$$

The edges of D_i or B_i not coming from arcs of the boundary of S are identified in pairs, while the other edges are not identified with any other edge. Thus the number of 1-cells is

$$\frac{1}{2} \left(\sum_{i=1}^{N_1} d_i + \sum_{i=1}^{N_2} (a_i + 2) \right) + \sum_{i=1}^{N_1} d_i + \sum_{i=1}^{N_2} (a_i + 2) = k + N_2 + 2k + 2N_2 = 3k + 3N_2.$$

Finally, the number of 2-cells is $N_1 + N_2$. So

$$2 - 2g - n = 2k + 2N_2 - (3k + 3N_2) + N_1 + N_2 = -k + N_1,$$

which implies that $k \geq 2g - 2 + n + N_1$.

If $n = 1$, all arcs start and end at the unique boundary component, so there cannot be annuli in the complement of \mathcal{A} , i.e. $N_2 = 0$ and $N_1 \geq 1$. So $k \geq 2g$, with equality if and only if $N_1 = 1$.

If $n \geq 2$, $k \geq 2g - 2 + n$, with equality if and only if $N_1 = 0$.

To prove that the two bounds are sharp, we just need to exhibit:

- if $n = 1$, a collection of arcs whose complement is a single polygon;
- if $n \geq 2$, a collection of arcs whose complement is a union of annuli.

These are given in Figure 7.

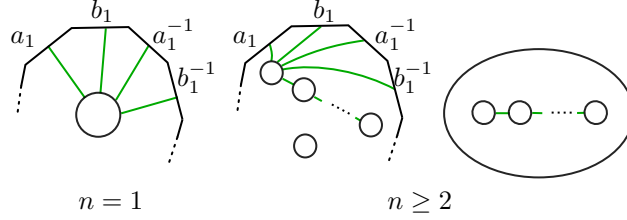


FIGURE 7. Filling collections of arcs of minimal size

□

Proposition C is now an easy corollary of the results of this section:

Corollary 7.4. *Let S be a surface of signature (g, n) , for $n \geq 2$, and let $0 < \ell_1 \leq \dots \leq \ell_n$. Then if $X \in \text{Teich}(S; \ell_1, \dots, \ell_n)$ is a local maximum for the orthosystole function, $\text{okiss}(S) \geq 2g - 2 + n$.*

Proof. This follows from Proposition 7.1, Lemma 4.2 and Lemma 7.3. □

APPENDIX A. ORTHOSSYSTOLE AND INJECTIVITY RADIUS OF THE BOUNDARY

The goal of this appendix is to explain why Bavard’s result in [3] can be stated in terms of orthosystoles and orthokissing numbers.

Let us start with the setup by Bavard. He considers a compact hyperbolic surface X and a collection c of pairwise disjoint simple closed geodesics, which is assumed to contain the boundary components of X , if there are any. For such a collection, he defines $\Sigma \subset X$ to be the union of all points $x \in X$ such that there are at least two (geodesic) paths from x to c realizing the distance of x from c . He proves ([3, Lemme 1]) that Σ is a graph whose edges are geodesic segments. The *injectivity radius* of c is

$$\text{inj}(c) := \min_{x \in \Sigma} d(x, c).$$

We recall here the statements from (1), (3) and (4) of [3, Théorème 1] for the injectivity radius of $c = \partial X$.

Theorem A.1. *Let X be a compact hyperbolic surface with non-empty boundary of signature g, b and let L be the sum of its boundary lengths. If r is the injectivity radius of ∂X , then*

$$\sinh(r) \sinh\left(\frac{L}{12(6g - 6 + 3n)}\right) \leq \frac{1}{2},$$

with equality if and only if all edges of Σ have the same length and meet at $\frac{2\pi}{3}$ angles. Moreover, for every L , there is a hyperbolic surface attaining the bound.

The equivalence of this theorem and Theorem 1.1 follows from the next two lemmas:

Lemma A.2. *If $c = \partial X$, $\text{inj}(c) = \frac{\text{osys}(X)}{2}$.*

Proof. If α is an orthosystole, the midpoint x of α belongs to Σ and thus $\frac{\text{osys}(X)}{2} \geq \text{inj}(c)$. Conversely, if $x \in \Sigma$ realizes the injectivity radius of c , look at two distance-realizing paths a_1 and a_2 from x to c . The concatenation $a_2 * a_1^{-1}$ is an essential arc from the boundary to the boundary, so if α is the orthogeodesic in the homotopy class, we have

$$\text{osys}(X) \leq \ell_\alpha(X) \leq \ell_{a_1}(X) + \ell_{a_2}(X) = 2d(x, c) = 2\text{inj}(c).$$

□

Lemma A.3. *Suppose $c = \partial X$; then Σ has edges of the same length which meet at $\frac{2\pi}{3}$ angles if and only if $\text{okiss}(X) = 6g - 6 + 3n$.*

Proof. Suppose first that Σ has edges of the same length which meet at $\frac{2\pi}{3}$ angles. Drop the perpendiculars from the vertices of Σ to c ; by [3, Lemme 1], we get a decomposition of X into isometric quadrilateral with angles $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}$, where the side between the right angles is a segment of a boundary component and the opposite one is an edge of Σ . In particular, for every edge of Σ , there is an orthogeodesic which intersects it once orthogonally at its midpoint. By construction of Σ , this orthogeodesic has length $2 \text{inj}(c)$, so by Lemma A.2 it is an orthosystole. As observed after the statement of [3, Théorème 1], since X deformation retracts onto Σ , $2 - 2g - n = v - e$, where v is the number of vertices of Σ and e the number of edges. As Σ is trivalent, $v = \frac{2}{3}e$. So there are at least

$$e = 6g - 6 + 3n$$

orthosystoles, and by Lemma 4.2 this means that $\text{okiss}(X) = 6g - 6 + 3n$.

Conversely, $\text{okiss}(X) = 6g - 6 + 3n$, X admits a hexagon decomposition of orthosystoles, and it is not hard to show that Σ is the graph obtained by gluing one tripod per hexagon, and the tripod is given by the barycenter and the three perpendiculars from the barycenter to the three orthosystoles. As all hexagons are isometric, Σ satisfies the conditions in the statement. \square

REFERENCES

- [1] Roland Bacher and Alina Vdovina. Counting 1-vertex triangulations of oriented surfaces. *Discrete Math.*, 246(1-3):13–27, 2002.
- [2] Florent Balacheff, Eran Makover, and Hugo Parlier. Systole growth for finite area hyperbolic surfaces. *Ann. Fac. Sci. Toulouse, Math. (6)*, 23(1):175–180, 2014.
- [3] C. Bavard. Anneaux extrémaux dans les surfaces de Riemann. *Manuscripta Math.*, 117(3):265–271, 2005.
- [4] Christophe Bavard. Disques extrémaux et surfaces modulaires. *Ann. Fac. Sci. Toulouse Math. (6)*, 5(2):191–202, 1996.
- [5] Christopher J. Bishop. Quasiconformal mappings of Y -pieces. *Rev. Mat. Iberoamericana*, 18(3):627–652, 2002.
- [6] Robert Brooks. Injectivity radius and low eigenvalues of hyperbolic manifolds. *J. Reine Angew. Math.*, 390:117–129, 1988.
- [7] Robert Brooks. Platonic surfaces. *Comment. Math. Helv.*, 74(1):156–170, 1999.
- [8] Peter Buser. *Geometry and spectra of compact Riemann surfaces*. Modern Birkhäuser Classics. Birkhäuser Boston, Ltd., Boston, MA, 2010. Reprint of the 1992 edition.
- [9] Peter Buser and Peter Sarnak. On the period matrix of a Riemann surface of large genus. *Invent. Math.*, 117(1):27–56, 1994. With an appendix by J. H. Conway and N. J. A. Sloane.
- [10] Federica Fanoni and Hugo Parlier. Systoles and kissing numbers of finite area hyperbolic surfaces. *Algebr. Geom. Topol.*, 15(6):3409–3433, 2015.
- [11] Maxime Fortier Bourque and Bram Petri. Linear programming bounds for hyperbolic surfaces. *arXiv e-prints*, page arXiv:2302.02540, February 2023.
- [12] Maxime Fortier Bourque and Kasra Rafi. Local maxima of the systole function. *J. Eur. Math. Soc. (JEMS)*, 24(2):623–668, 2022.
- [13] Mikhail G. Katz, Mary Schaps, and Uzi Vishne. Logarithmic growth of systole of arithmetic Riemann surfaces along congruence subgroups. *J. Differential Geom.*, 76(3):399–422, 2007.
- [14] Linda Keen. Collars on Riemann surfaces. In *Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)*, pages 263–268. Ann. of Math. Studies, No. 79, 1974.
- [15] Lixin Liu, Athanase Papadopoulos, Weixu Su, and Guillaume Théret. Length spectra and the Teichmüller metric for surfaces with boundary. *Monatsh. Math.*, 161(3):295–311, 2010.
- [16] Mingkun Liu and Bram Petri. Random surfaces with large systoles. *arXiv e-prints*, page arXiv:2312.11428, December 2023.
- [17] David Mumford. A remark on Mahler’s compactness theorem. *Proc. Amer. Math. Soc.*, 28:289–294, 1971.
- [18] Athanase Papadopoulos and Guillaume Théret. Shortening all the simple closed geodesics on surfaces with boundary. *Proc. Amer. Math. Soc.*, 138(5):1775–1784, 2010.
- [19] Hugo Parlier. Lengths of geodesics on Riemann surfaces with boundary. *Ann. Acad. Sci. Fenn. Math.*, 30(2):227–236, 2005.
- [20] Hugo Parlier. Kissing numbers for surfaces. *J. Topol.*, 6(3):777–791, 2013.
- [21] Bram Petri. Hyperbolic surfaces with long systoles that form a pants decomposition. *Proc. Am. Math. Soc.*, 146(3):1069–1081, 2018.
- [22] Bram Petri and Alexander Walker. Graphs of large girth and surfaces of large systole. *Math. Res. Lett.*, 25(6):1937–1956, 2018.

- [23] Paul Schmutz. Congruence subgroups and maximal Riemann surfaces. *J. Geom. Anal.*, 4(2):207–218, 1994.
- [24] Paul Schmutz. Arithmetic Fuchsian groups and the number of systoles. *Math. Z.*, 223(1):13–25, 1996.
- [25] Paul Schmutz. Compact Riemann surfaces with many systoles. *Duke Math. J.*, 84(1):191–198, 1996.
- [26] Paul Schmutz Schaller. Extremal Riemann surfaces with a large number of systoles. In *Extremal Riemann surfaces. From the proceedings of the AMS special session with related papers, January 4–5, 1995, San Francisco, CA, USA*, pages 9–19. Providence, RI: American Mathematical Society, 1997.
- [27] William P. Thurston. A spine for Teichmüller space. *Preprint*.
- [28] Akira Ushijima. A canonical cellular decomposition of the Teichmüller space of compact surfaces with boundary. *Comm. Math. Phys.*, 201(2):305–326, 1999.
- [29] Scott Wolpert. The length spectra as moduli for compact Riemann surfaces. *Ann. of Math. (2)*, 109(2):323–351, 1979.

(ARA BASMAJIAN) THE GRADUATE CENTER, CUNY, 365 FIFTH AVE., N.Y.,N.Y., 10016, USA AND HUNTER COLLEGE, CUNY, 695 PARK AVE., N.Y.,N.Y., 10065, USA

E-mail address: abasmajian@gc.cuny.edu

(FEDERICA FANONI) CNRS, UNIV PARIS EST CRETEIL, UNIV GUSTAVE EIFFEL, LAMA, UMR8050, F-94010 CRETEIL, FRANCE

E-mail address: federica.fanoni@u-pec.fr