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Decision problems on geometric tilings

Wang potatoes and Finite Local Complexity

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Abstract

We study decision problems on geometric tilings. First, we study a variant of the Domino problem where square tiles are replaced by geometric tiles of arbitrary shape. We show that, under some weak assumptions, this variant is undecidable regardless of the shapes, extending the results of [11] on rhombus tiles. This result holds even when the geometric tiling is forced to belong to a fixed set. Second, we consider the problem of deciding whether a geometric subshift has finite local complexity, which is a common assumption when studying geometric tilings. We show that it is undecidable even in a simple setting (square shapes with small modifications).

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Disclaimer

The figures in this article are somewhat optimised for being viewed on a computer.

1 Introduction

There are two main settings when talking about tilings: geometric tilings and symbolic tilings. Geometric tilings are coverings of the plane by arbitrary shapes, while symbolic tilings are tilings of \mathbb{Z}^2 by coloured unit squares where some forbidden patterns are not allowed to appear. One main fundamental difference between the two settings is that geometric tilings have no rules outside of the geometric constraints given by the shapes themselves, while in the symbolic case the corners of the unit squares must be on integer coordinates.

In this article, we study two decisions problems concerning geometric tilings: the domino problem in a geometric tiling, and the problem of finite local complexity; we prove that both problems are undecidable.

The undecidability of the domino problem [5] is a founding result of the theory of symbolic tilings: given a set of forbidden patterns, is it possible to tile the plane while avoiding this set of forbidden pattern? The domino problem has subsequently been studied on many structures: many groups [2], the hyperbolic plane [17, 8], on tilings by rhombuses by the two first authors [11], inside other symbolic tilings [3]. In this article we focus on a fusion of the two models, *symbolic geometric tilings*: they are geometric tilings with added decorations and a finite number of forbidden patterns on these decorations (precise definitions are in Subsection 2.4).

Our first problem of interest is the domino problem, as studied in [11] for rhombus tilings: fixing a nonempty set of geometric tilings X , given as inputs a set of decorations on the shapes of X and a set of forbidden patterns, can we tile the plane with shapes of X decorated in such a way that they avoid all forbidden patterns? We show that this problem is always undecidable under weak assumptions on the shapes and X .

In geometric tilings there are potentially infinitely many ways to put two shapes side by side inside a valid tiling. This is the notion of *local complexity*: how many patterns may appear in a ball of some radius up to translation. In symbolic tilings, this complexity is always finite. Geometric tilings sharing this property are said to have *finite local complexity* or *FLC* for short. FLC is a fundamental property in the study of geometric tilings and has been introduced by Kenyon [14]¹.

Most results about geometric tilings are under the assumption of FLC and most well known examples such as Penrose tilings or the hat [20] have FLC, although tilings without FLC have also been studied (see e.g. [7]). FLC makes geometric tilings behave similarly to symbolic tilings, in the sense that locally valid patterns may be enumerated by an algorithm.

The second problem we study is the following: given a geometric tileset, is it decidable whether it has FLC?

The paper is organised as follows. After providing the relevant definitions in Section 2, we prove the undecidability of the domino problem for symbolic geometric tilings in Section 3 and the undecidability of FLC for geometric tilings in Section 4.

2 Definitions and settings

2.1 Computability

A decision problem $A : \mathbb{N} \subseteq \{0, 1\}$ is said to be *computable* when there exists a Turing machine M that halts on all inputs $x \in \mathbb{N}$, and which outputs 1 iff $A(x) = 1$. The canonical undecidable problem is the *halting problem*, denoted **HP**: given (the number corresponding to) a Turing machine as an input, does it halt on an empty input tape?

The halting problem is *recursively enumerable*: there is a Turing machine which never halts and outputs every input that map to 1 (in this case, every Turing machine that halts). Moreover, it is complete for the class of recursively enumerable problems under *many-one reduction*, defined as follows: we say that $A \leq_m B$ if there exists a total computable function f such that $B(x) = 1 \Leftrightarrow A(x) = 1$. The completeness of the halting problem means that for any recursively enumerable problem A , $A \leq_m \mathbf{HP}$.

We denote the class of r.e. and co-r.e. problems as Σ_1^0 and Π_1^0 , respectively, since they are the first levels of the arithmetical hierarchy. This means that **HP** is Σ_1^0 -complete and its complement $\overline{\mathbf{HP}}$ is Π_1^0 -complete.

2.2 Symbolic tilings

Given a finite alphabet Σ , a *configuration* c is an assignment of letters to each coordinate of the plane \mathbb{Z}^2 : $c : \mathbb{Z}^2 \rightarrow \Sigma$. A *pattern* $p : D \rightarrow \Sigma$ is an assignment of letters to a finite portion of the plane $D \subseteq \mathbb{Z}^2$. The shift of a configuration c by a vector $v \in \mathbb{Z}^2$ is denoted $\sigma_v(c)$: $\sigma_v(c)(z) = c(z + v)$. A pattern $p : D \rightarrow \Sigma$ *appears* in a configuration c , denoted $p \sqsubseteq c$, if there exists $z \in \mathbb{Z}^2$ such that $\sigma_z(c)|_D = p$. Conversely, we denote $p \not\sqsubseteq c$ when a pattern does not appear in a configuration.

¹ Under the terminology *finite number of local pictures*.

We define a distance between configurations, two configurations x and y being near each other when they coincide on a big central ball:

$$d(x, y) = 2^{-\min\{\|z\| \mid z \in \mathbb{Z}^2, x(z) \neq y(z)\}}.$$

This defines a topology on the set of all configurations $\Sigma^{\mathbb{Z}^2}$. A *subshift* is a closed, shift-invariant subset of $\Sigma^{\mathbb{Z}^2}$. Subshifts are exactly the sets of configurations which can be defined by forbidding sets of patterns. That is to say, for each subshift $X \subseteq \Sigma^{\mathbb{Z}^2}$, there exists a family of forbidden patterns F such that

$$X = \left\{ x \in \Sigma^{\mathbb{Z}^2} \mid \forall p \in F, p \not\sqsubseteq x \right\}$$

A *subshift of finite type* or *SFT* is a subshift for which there exists a finite family of forbidden patterns defining it.

► **Definition 1** (Classical domino problem [13]). *The domino problem denoted DOMINO is the following problem: given as input be an SFT X given by an alphabet A and a family of forbidden patterns F , is X non-empty?*

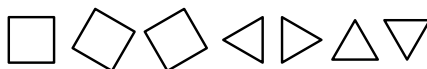
► **Theorem 2** ([5]). *The classical domino problem is undecidable and Π_1^0 -complete.*

It is well-known that every SFT X is conjugate to another SFT $\phi(X)$ defined by nearest-neighbour forbidden patterns (i.e. connected support of size 2), and $\phi(X)$ is computable from X . As a direct corollary, the classical nearest-neighbour domino problem is also undecidable and Π_1^0 -complete.

2.3 Geometric tilings

► **Definition 3** (Geometric tilings). *A shape t is a closed topological disk (compact and simply connected). A shapese set \mathcal{T} is a finite set of shapes. A tiling T is a covering of the plane by countably many shapes with disjoint interior. A \mathcal{T} -tiling is a tiling where all shapes are in \mathcal{T} up to translation.*

As a running example we consider the square-triangle shapese set \mathcal{T}_Δ containing the square (in 3 orientations) and the equilateral triangle (in 4 orientations), illustrated on Fig 1. For technical reasons detailed in Section 3 we consider all sides equal to $\frac{5}{2}$.



■ **Figure 1** The square-triangle shapese set \mathcal{T}_Δ .

► **Remark 4** (Shapes and tiles). The usual terminology is tiles rather than shapes. However we chose to differentiate between shapes (purely geometric) and tiles (both geometric and symbolic) that we define below.

► **Definition 5** (Pattern and support). *We call pattern \mathbf{P} a finite and simply connected set of non-overlapping shapes. The support of \mathbf{P} , denoted $\text{supp}(\mathbf{P})$ is the union of its shapes. We usually consider patterns up to translation, and the context should make it clear.*

We write $\mathbf{P} \in T$ when pattern \mathbf{P} is a subset of a tiling T , and $\mathbf{P} \sqsubseteq T$ (resp. $\mathbf{P} \sqsubseteq X$) when a pattern appears up to translation in T (resp. in some tiling $T \in X$).

We choose a distance between geometric tilings that generalises the usual distance on \mathbb{Z}^2 configurations. Two tilings T_1 and T_2 are ε -close if they are identical on a disk of radius ε^{-1} up to a translation of norm at most ε . See [18, 16] for full details.

With the topology induced by this distance we generalise the concept of subshift to geometric subshifts as follows.

► **Definition 6** (Geometric subshifts). *Given a finite set of shapes \mathcal{T} , a geometric subshift X on \mathcal{T} is a set of \mathcal{T} -tilings that is invariant under \mathbb{R}^2 -translations and closed for the geometric tiling topology.*

In particular, a geometric subshift is purely geometric if it contains exactly all possible \mathcal{T} -tilings for some shapeset \mathcal{T} ; in other words, there are no other constraints than those imposed by the geometry of \mathcal{T} . Any geometric subshift is a subset of the corresponding purely geometric subshift that avoids some set of forbidden patterns:

► **Definition 7** (Valid tiling). *Given a finite set of patterns G , we say that a tiling T is valid for G or avoids G when for any $\mathbf{P} \in G$, $\mathbf{P} \not\sqsubseteq T$.*

We introduce restriction on the geometric tilings on which our main result applies:

► **Definition 8** (Potatoes). *A shape is a potato if its boundary contains at most countably many maximal horizontal and vertical segments (that are not included in a larger segment).*

Our results hold if this assumption applies for any two fixed non-colinear directions, but this makes the proofs more tedious.

► **Remark 9** (Smooth shapes, polygons). As argued in [14], any geometric tiling by topological disks is combinatorially equivalent to a polygonal tiling (where all shapes are polygons), though some properties of the tilings such as self-similarity might be lost in this equivalence. We use the very weak potato assumption for technical reasons.

► **Definition 10** (FLC). *A set of geometric tilings X has finite local complexity (FLC) when for any compact $K \subset \mathbb{R}^2$, up to translation, there exist finitely many patterns $\mathbf{P} \sqsubseteq X$ such that $\text{supp}(\mathbf{P}) \subset K$. Equivalently, there are finitely many 2-tile patterns in X .*

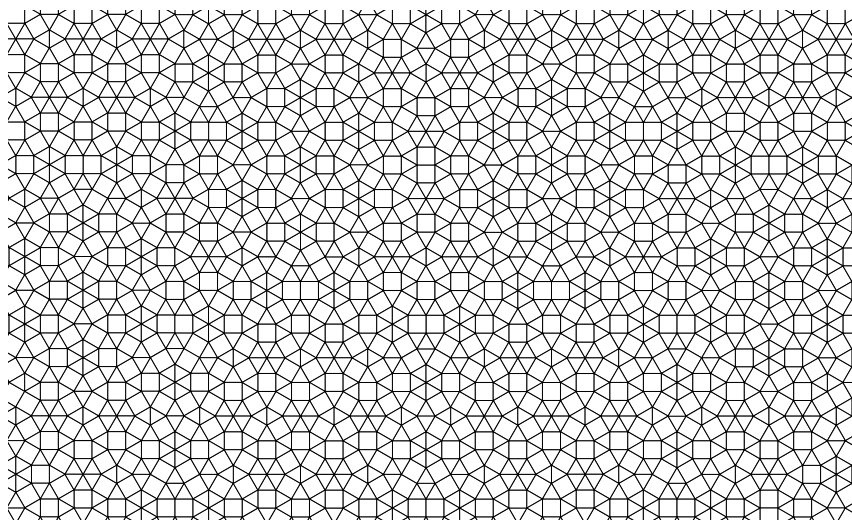
Typical examples of geometric subshifts with FLC are the polygonal edge-to-edge tilings. For example, the purely geometric subshift given by \mathcal{T}_Δ does not have FLC (even a tiling using only a single square tile can have lines or columns shifted by arbitrary offsets), but restricting to edge-to-edge tilings gives it FLC. Continuing, the subshift X_Δ of edge-to-edge tilings containing no bars (3 consecutive squares) and no tribars (4 unit triangles aligned as in the triangular grid) also has FLC; see Fig. 2.



■ **Figure 2** The two forbidden patterns of X_Δ up to rotation and reflexion. There are in total 12 forbidden patterns up to translation.

2.4 Symbolic-geometric tiling and the domino problem

► **Definition 11** (Symbolic-geometric tilings). *Given a finite alphabet A and a finite shapeset \mathcal{T} , a symbolic-geometric tile \mathbf{t} is a shape t with a label a , i.e., $\mathbf{t} = (t, a) \in \mathcal{T} \times A$. We define the label eraser or projection π as $\pi((t, a)) := t$. We call symbolic-geometric tileset a subset \mathbf{T} of $\mathcal{T} \times A$. We call symbolic-geometric tiling T a covering of \mathbb{R}^2 by non-overlapping symbolic-geometric tiles, that is $\pi(T) := \{\pi(\mathbf{t}), \mathbf{t} \in T\}$ is a geometric tiling.*



■ **Figure 3** A typical fragment of a tiling in X_Δ .

We generalise as expected the notions of symbolic-geometric patterns \mathbf{P} and \mathbf{T} -tiling.

► **Definition 12** (Domino on X). *Given a finite shapset \mathcal{T} and a set of \mathcal{T} -tilings X , the problem DOMINO_X is defined as:*

Input: a finite alphabet A , a tileset $\mathbf{T} \subset \mathcal{T} \times A$ and a finite set G of forbidden \mathbf{T} -patterns.

Output: does there exist a \mathbf{T} -tiling T that is valid for G and such that $\pi(T) \in X$?

The case where \mathcal{T} contains only a unit square and X is the subshift of edge-to-edge square tilings (containing a single point up to translation) corresponds to the classical domino problem. This definition, as well as an equivalent of the following result, appeared in [11] in the particular case of rhombus shapes.

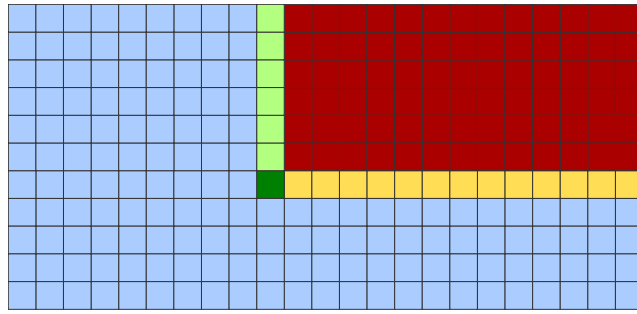
3 The domino problem is undecidable on every set of tilings

► **Theorem 13** (Main result). *Let X be a non-empty set of tilings on potato shapes with finite local complexity. The domino problem on X is Π_1^0 -hard and therefore undecidable.*

This generalises the classical domino problem (when X is a single edge-to-edge tiling using a single square shape) as well as Theorem 1 from [11] for geometric subshifts on rhombus shapes. However, the proof in *op. cit.* made heavy use of the particular structure of rhombus tilings, while the constructions in the present paper are much more generic.

We proceed by reduction to the classical nearest neighbour domino problem. Let (A, F) be an input, that is, A is a finite alphabet and F is a finite set of nearest-neighbour forbidden \mathbb{Z}^2 patterns. We build a tileset \mathbf{T} on shapes \mathcal{T} and a set of forbidden patterns G such that there exists a \mathbf{T} -tiling T valid for G with $\pi(T) \in X$ if, and only if, the SFT (A, F) is not empty.

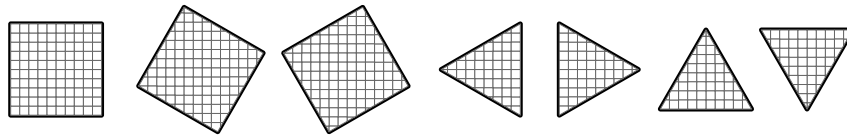
We illustrate our construction throughout on the classical quarter-plane subshift (see Fig. 4) and the geometric subshift X_Δ , where the edge-to-edge condition implicitly induces the list \mathcal{L} of allowed 2-shapes patterns.



■ **Figure 4** A fragment of a typical tiling of the nearest-neighbour quarter-plane subshift on blue and red tiles (with green and yellow boundary tiles). Any two-tiles pattern not appearing in this fragment is forbidden.

3.1 Coloured tiles

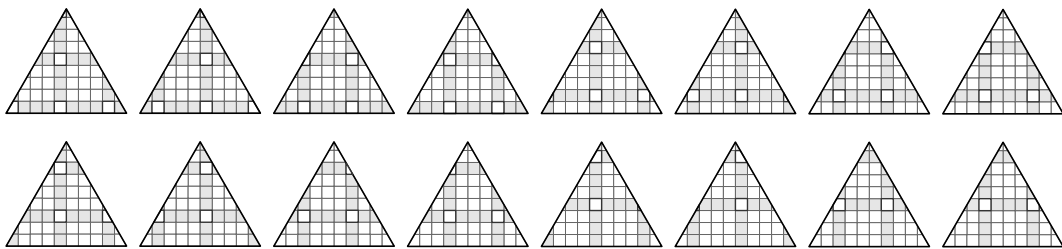
We assume (by rescaling if necessary) that each shape in \mathcal{T} strictly contains a unit square and do not have in its boundary two vertical or horizontal segments at integer distance from each other, using the potato assumption. For each shape, we pick an arbitrary origin point from which we draw a square grid of width $\frac{1}{4}$, as illustrated in Fig. 5 (we pick a vertex as origin point in the figures). We call *checkered tiles* these shapes with the superimposed $\frac{1}{4}$ -grid.



■ **Figure 5** The $\frac{1}{4}$ square grid superimposed on the square and triangle shape of X_Δ forming the checkered square-triangle tiles.

For every $t \in \mathcal{T}$, we define 16 placeholder tiles, one for each $0 \leq i, j \leq 3$, as shown in Fig. 6. In the (i, j) -placeholder tile, the cell (i', j') is a *coding cell* if $i' = i \pmod 4$ and $j' = j \pmod 4$ or a *linking cell* if either $i' = i \pmod 4$ or $j' = j \pmod 4$. A cell such that $i' = i \pmod 4$ is also a row cell, and a cell such that $j' = j \pmod 4$ is also a column cell; note that a coding cell is both row and column.

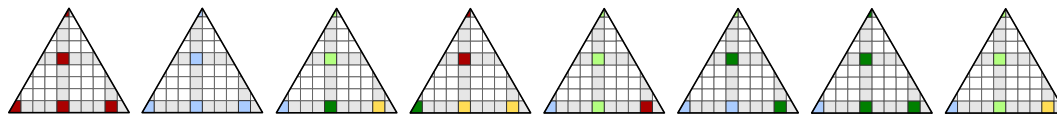
We note \mathbf{T}_0 the set of placeholder tiles.



■ **Figure 6** The 16 placeholder tiles for a triangle shape. The coding cells are shown with a bold boundary and the linking cells in gray.

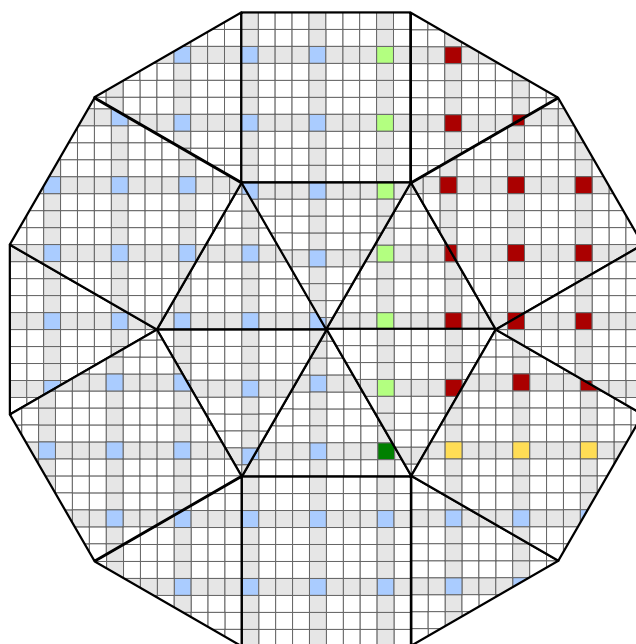
For each placeholder tile \mathbf{t}_0 in \mathbf{T}_0 , we define a coloured tile \mathbf{t} for each Σ -labelling of the coding cells. We note \mathbf{T} the set of coloured tiles obtained from this process. As the shapset is finite, the size of the shapes is bounded; assuming each tile in \mathbf{T}_0 contains at most n

coding cells, it defines at most $|\Sigma|^n$ coloured tiles, so \mathbf{T} is finite.



■ **Figure 7** 8 colourings of the first placeholder tile of Fig. 6.

As there are 5 coding cells and 5 colours, the total number of coloured tiles for this placeholder is 5^5 , most of which are later forbidden by G_F . The five rightmost tiles are allowed and the three leftmost tiles are later forbidden as they contain a forbidden pattern or force a forbidden pattern.



■ **Figure 8** A typical allowed pattern of coloured tiles.

3.2 Forbidden patterns

We define a set of forbidden patterns $G = G_{grid} \cup G_{coding} \cup G_F$ that enforces a global grid structure among coding cells of all coloured tiles. We forbid all coloured patterns that break the grid structure (G_{grid}), do not encode a \mathbb{Z}^2 tiling in the coding cells (G_{coding}), or induce a forbidden pattern for F in the coding cells (G_F). Figure 8 illustrates a typical allowed pattern.

As the set X has FLC, there is a finite list \mathcal{L} of allowed 2-shapes patterns. That is, any pattern \mathbf{P} of two adjacent tiles appearing in any tiling $T \in X$ is in \mathcal{L} up to translation. Notice that the set X (or its tileset) is not a part of the input, so we do not require the list \mathcal{L} to be computable from X .

3.2.1 Support of the forbidden patterns

► **Definition 14** (*r*-patterns). For $r > 0$, a *r*-pattern P is a geometric \mathcal{T} -pattern locally allowed by \mathcal{L} , such that there is a center $x \in \text{supp}(P)$ such that the (square) ball $B(x, r)$ is

included in $\text{supp}(P)$ and P is minimal for x in the sense that removing a tile from P breaks simple connectedness or the condition on $\text{supp}(P)$.

By analogy, a labelled r -pattern is a labelled \mathbf{T} -pattern whose projection is a r -pattern.

Notice that the minimality condition is relative to the choice of center x , so a r -pattern needs not be minimal for inclusion, in the sense that a r -pattern may be strictly included in another r -pattern. However, different choices of x may be possible for the same r -pattern.

Let D_{max} be the maximal diameter of the shapes in \mathcal{T} . Given a r -pattern c , we may assume that its center is 0 up to translating the pattern. Since a tile in c either intersects $B(0, r)$ or fills a hole left by tiles that intersect $B(0, r)$, it follows that c is contained in $B(0, r + D_{max})$. By finite local complexity, the set of labelled r -patterns is finite.

The forbidden patterns defined below are all D_{max} -patterns.

3.2.2 Grid structure

► **Definition 15** (Grid-like patterns). *A labelled (or placeholder) pattern P is grid-like if there is a grid $g_{x_0, y_0} := \{(x, y) \in \mathbb{R}^2 \mid (x - x_0) \in \mathbb{Z} \vee (y - y_0) \in \mathbb{Z}\}$ such that:*

1. *the intersection of g_{x_0, y_0} with the tile boundaries and the placeholder grid boundaries does not contain any segment;*
2. *g_{x_0, y_0} contain no tile triple point, that is, a position that belong to the boundary of three tiles or more;*
3. *if a horizontal (resp. vertical) line in g_{x_0, y_0} crosses the interior of a cell c , then c is a row cell (resp. column cell).*

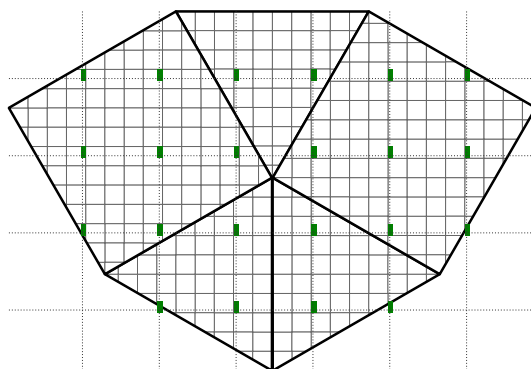
This means that a cell whose interior intersects both a horizontal and vertical line of the grid g_{x_0, y_0} is a coding cell. If the coding cell is a full cell (not cut by the boundary of a tile), it contains a grid vertex.

We define G_{grid} as the set of all non grid-like D_{max} -patterns, which is a finite set.

Recall that by hypothesis, D_{max} is not an integer and $D_{max} \geq \sqrt{2}$ as the interior of any tile contains a unit square. Note that, given a labelled pointed D_{max} -pattern P and a choice of center x , P contains the tile containing x and all of its neighbours.

Grid-like labelled patterns exist on any geometric pattern. Indeed, take a pattern P_0 of unlabelled shapes and the induced pattern P_1 of checkered tiles. Given a cell c of a checkered tile, the set of positions (x_0, y_0) in the interior of c such that the grid g_{x_0, y_0} contains a tile triple point or a segment intersection with the tile boundaries or the placeholder grid boundary is countable (since shapes are potatoes, they have at most countably many maximal horizontal and vertical segments). By a simple counting argument, there are uncountably many positions (x_0, y_0) in the interior of c whose grid satisfies conditions 1 and 2, and for each such position it is straightforward to check that there is a unique choice of placeholder tile for each checkered tile that satisfies condition 3: since each tile contains a vertex of the grid, this forces the position of a coding cell.

In a coloured pattern, the set of valid choices for (x_0, y_0) (modulo 1) for condition 3 is contained in a rectangle drawn in green in Fig. 9. This stems from the fact that each placeholder tile contains a full coding cell which yield a square of possible grid positions of side length $1/4$, so the set of valid choices is a finite intersection of squares. The pattern is grid-like iff this rectangle is not empty, as conditions 1 and 2 remove only countably many positions, and so deciding if a pattern is grid-like is a computable process.

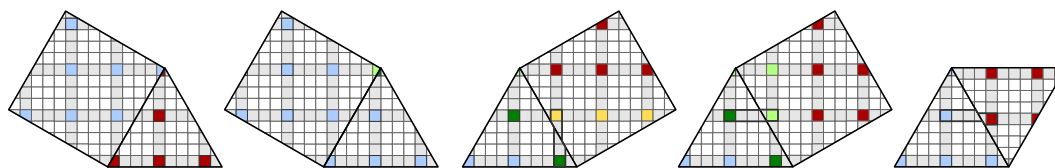


■ **Figure 9** Finding a valid grid placement in a checkered pattern. In green: the set of valid choices for the center of the grid.

3.2.3 Coding forbidden patterns

► **Definition 16** (Coherent patterns). *A grid-like pattern is coherent if two coding cells at distance at most $\frac{1}{4}$ have the same colour.*

The set of forbidden patterns G_{coding} contains every (coloured) grid-like D_{max} -patterns that are not coherent. This ensures a global coding consistency in allowed patterns and tilings.



■ **Figure 10** To the left: two non-coherent patterns. Any D_{max} -pattern containing one of them is forbidden in G_{coding} . To the right: three patterns that induce a forbidden patterns from F . Any D_{max} -pattern containing one of them is forbidden in G_F .

3.2.4 Induced forbidden patterns

To each coherent grid-like pattern P , we canonically associate an induced \mathbb{Z}^2 -pattern P' as follows. Pick some valid grid placement g_{x_0, y_0} . If n and m are such that $(x_0 + n, y_0 + m) \in \text{supp}(P)$, define $P'_{n, m}$ as the colour of the coding cell containing $(x_0 + n, y_0 + m)$ (by the grid-like hypothesis). All valid placements differ by at most $1/4$, so by coding coherence, every valid placement yields the same symbolic pattern P' .

We now define G_F as the set of coherent grid-like D_{max} -patterns P whose induced \mathbb{Z}^2 pattern P' contains a forbidden pattern from F . Notice that, for a D_{max} -pattern P , the support of P' contains a 2×2 square since $D_{max} \geq \sqrt{2}$.

3.3 Equivalence with the original domino problem

We prove Theorem 13 by reduction to the classical nearest-neighbour domino problem.

Given a geometric tileset \mathbf{T} and an instance (A, F) of the nearest-neighbour \mathbb{Z}^2 -domino problem, define X' as the set of \mathbf{T} -tilings T that avoid the forbidden patterns $G = G_{grid} \cup$

$G_{coding} \cup G_F$ defined above and such that $\pi(T) \in X$, that is, removing the colours gives a tiling in X .

We prove that there is a valid configuration c for (A, F) if, and only if, X' is not empty.

(\Rightarrow). Let T be a geometric tiling in X (which is nonempty by assumption), and let T_0 be the corresponding checkered tiling (see Section 3.1).

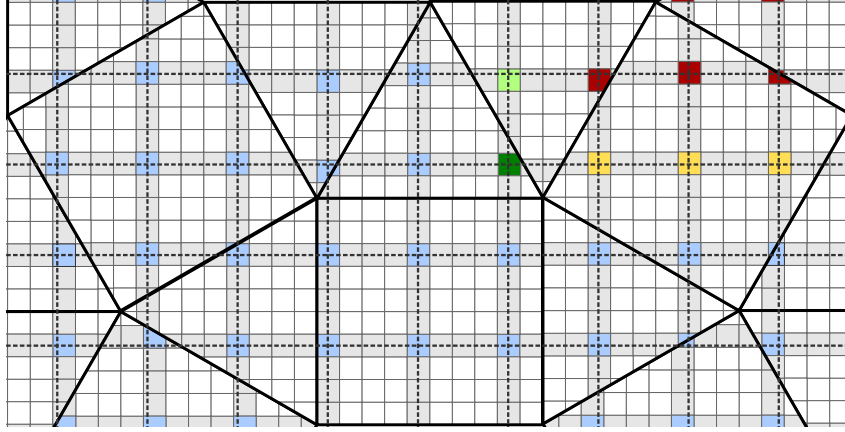
For $x_0, y_0 \in \mathbb{R}^2$, define the unit grid $g_{x_0, y_0} = \{(x, y) : (x - x_0) \in \mathbb{Z} \vee (y - y_0) \in \mathbb{Z}\}$.

There are countably many triple points and horizontal or vertical segments in the boundaries of the tiles and of the cells, so there is a choice of x_0 and y_0 such that g_{x_0, y_0} intersects none of them (these are the conditions for being grid-like as in Definition 15).

Now define the placeholder tiling T_1 as follows: for every tile of T_0 , we choose the only placeholder tile that corresponds to g_{x_0, y_0} in the sense that a cell that intersects a row, resp. a column, of g_{x_0, y_0} is a row cell, resp. a column cell (it is a coding cell if it intersects both).

By construction, we did not create a pattern from G_{grid} , since every D_{max} -pattern is grid-like using the grid g_{x_0, y_0} .

We define a coloured tiling T' as follows: for each coding cell, if $(x_0 + n, y_0 + m)$ is the closest vertex of the grid g_{x_0, y_0} , then the cell gets assigned the colour $c_{n, m}$.



■ **Figure 11** Construction of the coloured tiling T' from a geometric tiling T and a configuration c .

Since each tile contains at least one vertex of g_{x_0, y_0} , the cell containing this vertex is a coding cell, so every coding cell in this tile is at distance at most $1/4$ from a vertex of g_{x_0, y_0} . It follows that two different coding cells that are close get assigned the same vertex and no pattern from G_{coding} may appear.

By construction, T' contains no pattern in G_F since c contains no pattern in F .

We have proved that T' does not contain any pattern from G . Since $\pi(T') = T \in X$, $T' \in X'$. ◀

Notice that the construction of T' from T may not be computable (especially relative to the choice of x_0 and y_0) but this is not a part of reduction.

(\Leftarrow). Let T' be a tiling in X' . We build a function $f : \mathbb{Z}^2 \rightarrow \mathbb{R}^2$ such that for any (n, m) , $f(n, m)$ is in the interior of a coding cell. This defines a configuration c where $c_{n, m}$ has the colour of the cell containing $f(n, m)$.

Up to translating the whole tiling, assume that $(0, 0)$ is in the interior of a coding cell of T' . As the D_{max} -pattern P around $(0, 0)$ is grid-like, choose a position $f(0, 0)$ at distance at most $\frac{1}{4}$ from $(0, 0)$ such that $g_{f(0, 0)}$ is a valid grid position for P .

We repeat this process inductively around positions already defined, by increasing distance from $(0, 0)$. Let $(n, m) \in \mathbb{Z}^2$ such that $f(n, m)$ is defined but not $f(n+1, m)$ (other directions are similar). As the pattern $P_{n,m}$ around $f(n, m)$ is grid-like with $g_{f(n,m)}$ as a valid grid placement, the position $f(n, m) + (1, 0)$ is vertex of $g_{f(n,m)}$ so it is inside a coding cell. As the pattern $P_{n+1,m}$ around $f(n, m) + (1, 0)$ is grid-like, pick a position $f(n+1, m)$ at distance at most $\frac{1}{4}$ from $f(n, m) + (1, 0)$ such that $g_{f(n+1,m)}$ is a valid grid position of $P_{n+1,m}$.

Even though this process is not deterministic, it is possible to prove that the configuration c does not depend on the order in which the values of f are defined or on the choices made in the construction. For simplicity, we prove the weaker statement that any forbidden pattern in c yields a pattern in G_F .

Suppose that the pattern $c_{(0,1),(1,1)}$ is forbidden; the same proof holds for other positions and directions. If $f(1, 1)$ was defined from $f(0, 1)$, this would create a pattern from G_F by construction. Assume that $f(1, 1)$ was defined from $f(1, 0)$. Since $f(0, 1)$ and $f(1, 0)$ were both defined from $f(0, 0)$, it follows that $p = f(1, 0) + (0, 1)$ is at distance at most $\frac{1}{2}$ from $p' = f(0, 1) + (1, 0)$ (by triangular inequality), and they are both inside some coding cell (in possibly different tiles t and t').

Consider the D_{max} -pattern P around $f(1, 1)$: it contains p' and so it contains the whole tile t' . By assumption on the size of the tiles, t' contains some vertex of the grid centered on $f(1, 1)$ which is therefore in a coding cell. p' is in a coding cell of t' so, by definition of the placeholder tile, it is at distance at most $\frac{1}{4}$ from a vertex v of the grid. It follows that:

$$\begin{aligned} d(f(1, 1), v) &\leq d(f(1, 1), f(0, 1) + (1, 0)) \\ &\quad + d(f(0, 1) + (1, 0), f(1, 0) + (0, 1)) + d(f(1, 0) + (0, 1), v) \\ &< \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1, \end{aligned}$$

which implies that $f(1, 1) = v$ (they are vertices on a unit grid). Since p' is in a coding cell and is at distance $< \frac{1}{4}$ from a coding cell with colour $c_{1,1}$, the two cells have the same colour by G_{coding} . It follows that the grid centered on $f(0, 1)$ "sees" the pattern $c_{(0,1),(1,1)}$, so the D_{max} -pattern around $f(0, 1)$ belongs to G_F .

We now prove that c is valid for F . The previous argument shows that any pattern $P' \sqsubset c$ of size 2×2 appears in a grid-like D_{max} -pattern. The tiling T' avoids G_F , so P' does not contain any forbidden pattern in F . \blacktriangleleft

► **Remark 17.** Since tilings in X can be arbitrarily complex, there is no reason for DOMINO_X to be Π_1 -complete. However, if there is an algorithm that, given r , enumerates all central r -patterns of all $t \in X$, we have a semi-algorithm for DOMINO_X by trying all possible colourings of these patterns and rejecting if we found no valid colouring. When X is a subshift, it is enough to have such an algorithm for enumerating locally allowed patterns by a compactness argument; this corresponds to effective subshifts. In this case DOMINO_X is Π_1 -complete.

3.4 Hidden quasi-isometries

We notice that our reduction to the domino problem relies on the existence of a quasi-isometry between the adjacency graph of the tiling and \mathbb{Z}^2 ; since quasi-isometries have appeared in other contexts as an invariant between spaces that preserve the difficulty of the domino problem, we discuss this phenomenon in this section.

► **Definition 18** (Adjacency graphs). *Fix a set of geometric tiles, a geometric tiling x with FLC and an enumeration of all possible 2-tile patterns in x . The adjacency graph $G(x)$ is the graph whose vertices correspond to tiles, edges link tiles that are adjacent in x , and each edge is labelled according to the corresponding 2-tile pattern.*

The problem $\text{DOMINO}_{\{x\}}$ can be seen as the domino problem on the graph $G(x)$. Most existing results regarding the domino problem take place on a single fixed Cayley graph. A specificity of the DOMINO_X problem is that it corresponds to a domino problem on a family of labelled graphs $G(X) = \{G(x) : x \in X\}^2$. Still, our result has some similarities with existing results on fixed graphs.

► **Definition 19** (Quasi-isometry). *A map $\phi : X \rightarrow Y$ between metric spaces (X, d_X) and (Y, d_Y) is a quasi-isometry if there exists $k > 0$ such that*

$$\frac{1}{k} \cdot d_X(x, x') - k \leq d_Y(\phi(x), \phi(x')) \leq k \cdot d_X(x, x') + k \quad \text{and} \quad Y = \bigcup_{x \in X} B(\phi(x), k).$$

► **Remark 20.** The adjacency graph of a geometric tiling with finite local complexity is planar and quasi-isometric to \mathbb{Z}^2 .

In fact, our reduction yields an explicit quasi-isometry $G(x) \rightarrow \mathbb{Z}^2$ for every $x \in X$: draw the grid on x , fix an origin $(0, 0)$ for the grid arbitrarily, and define $\phi : G(x) \rightarrow \mathbb{Z}^2$ such that $\phi(v)$ is any element of \mathbb{Z}^2 in the grid inside v .

In the case of finitely presented groups it is known that the undecidability of the domino problem is a quasi-isometry invariant [6]. In particular, the domino problem on any Cayley graph that is quasi-isometric to \mathbb{Z}^2 is undecidable. However the proof uses heavily the group structure and we do not claim that the undecidability of the domino problem is a quasi-isometry invariant on general (infinite) graphs. There are other quasi-isometry invariant group properties related with the existence of strongly aperiodic subshifts [4].

Adjacency graphs are very far from being Cayley graphs: say two vertices are equivalent if they have the same neighbourhoods for all radii. Cayley graphs are transitive (all vertices are equivalent) while adjacency graphs are not even quasi-transitive (there may be infinitely many equivalence classes). Still, our proofs seem to hint that quasi-isometries play a role even in this context.

4 Finite local complexity is undecidable for geometric tilings

Given the important role played by the FLC hypothesis in carrying symbolic results to the geometric setting, we study the decidability of this property.

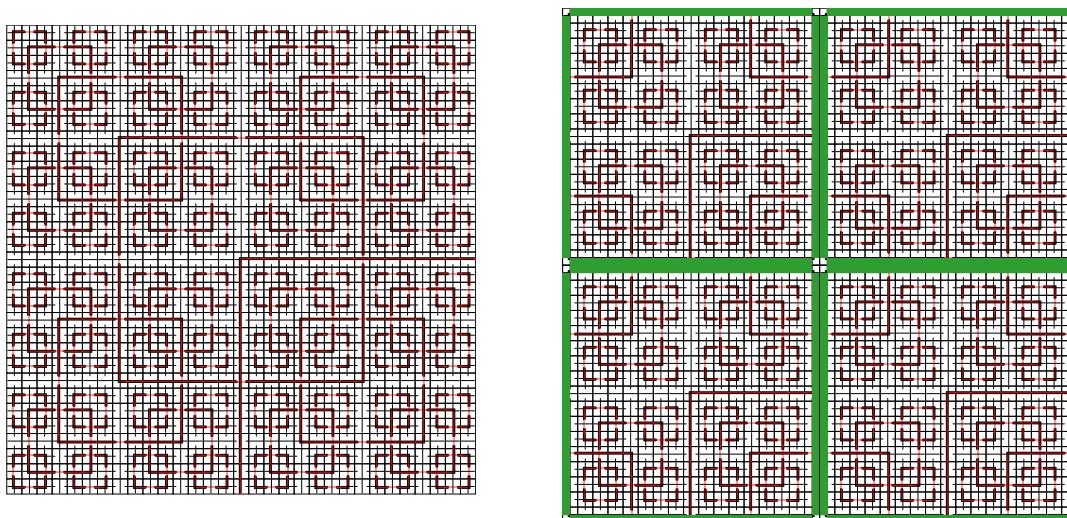
► **Definition 21.** *The FLC problem for purely geometric subshifts:*

Input: a shapset \mathcal{T} .

Output: does the purely geometric subshift on \mathcal{T} have finite local complexity?

► **Theorem 22.** *The FLC problem for purely geometric subshifts is Π_1^0 -hard and therefore undecidable.*

² This family can be described by forbidding subgraphs that correspond to forbidden patterns in X , which means that it is a graph subshift in the sense of [1].

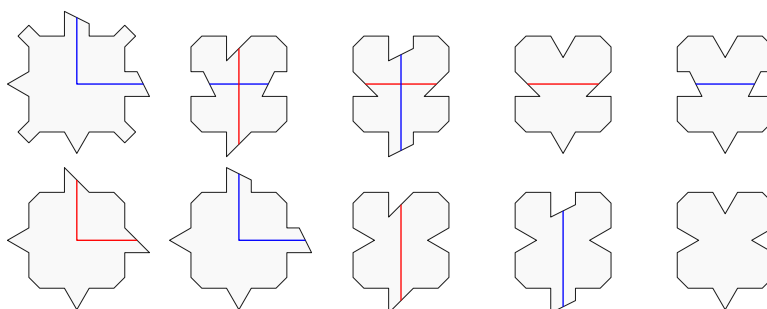


■ **Figure 12** Two tilings by the Gurevich-Koryakov tileset. On the left: when the embedded Turing machine does not halt, it gives aperiodic tilings. On the right: when it does halt, the smallest square that witnesses the halting state gets a special colour on the border that forces a periodic tiling.

Proof. We proceed by reduction to the halting problem for Turing machines. For each Turing machine M , we build a shapeset \mathcal{T} such that the purely geometric subshift is not empty and has finite local complexity iff M does not halt on the empty tape.

We start from 1×1 square tiles decorated with the Robinson [19] decorations as modified by Gurevich and Koryakov [10]: this tileset tiles the plane periodically iff the Turing machine encoded in it halts and aperiodically otherwise, as shown in Figure 12. Robinson’s tileset can be defined either by purely geometric or symbolic tilings: see Fig. 13. Gurevich and Koryakov’s tileset can also be made purely geometric: see Figs. 14 and 15.

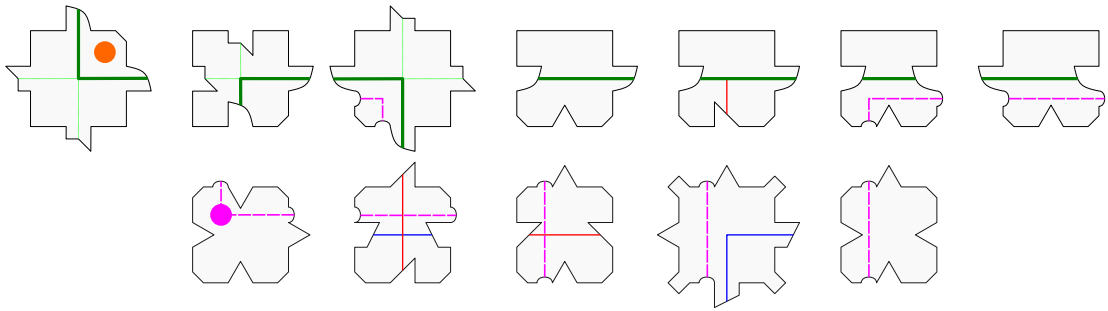
In the periodic case, the only tilings consist of the smallest Robinson squares containing the whole computation repeated periodically, as shown in Fig. 17. The aperiodic case consists either of only Robinson structures, or of two half-planes or four quarter-planes of Robinson structures separated by a shear line of Gurevich-Koriakov border, illustrated on Fig. 18.



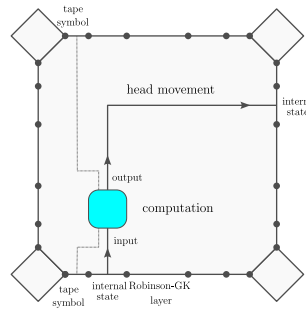
■ **Figure 13** A purely geometric presentation of Robinson tiling. Decorations are shown for clarity only, and the computing layer is not represented.

Let us adapt this construction to our setting. For a given Turing machine, we define the associated geometric subshift using Gurevich and Koryakov’s tileset with one modification: the outer borders of the green squares do not have any markings/indentations except at the

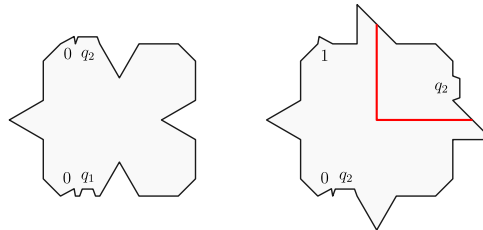




■ **Figure 14** New tiles (in addition to Fig. 13) giving the Gurevich-Koryakov variant. If a tile with the violet signal appears (bottom row), then the next square in the hierarchy will be green (top row) which forces a periodic tiling by green squares of the same size. Green squares can not appear otherwise. Every tile where the halting state appears on the computing layer must start a violet signal (only one such tile is represented here). Decorations are shown for clarity only, and the computing layer is not represented.



(a) Schematics of the embedding of computation in a Robinson or Gurevich-Koryakov tile.

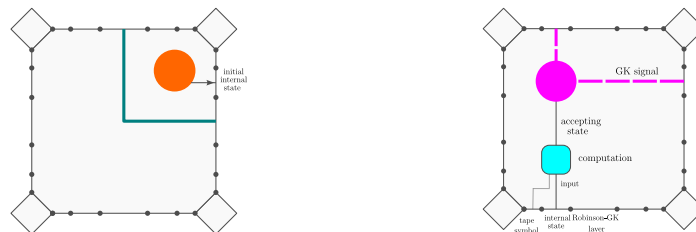


(b) Two examples of computing tiles.

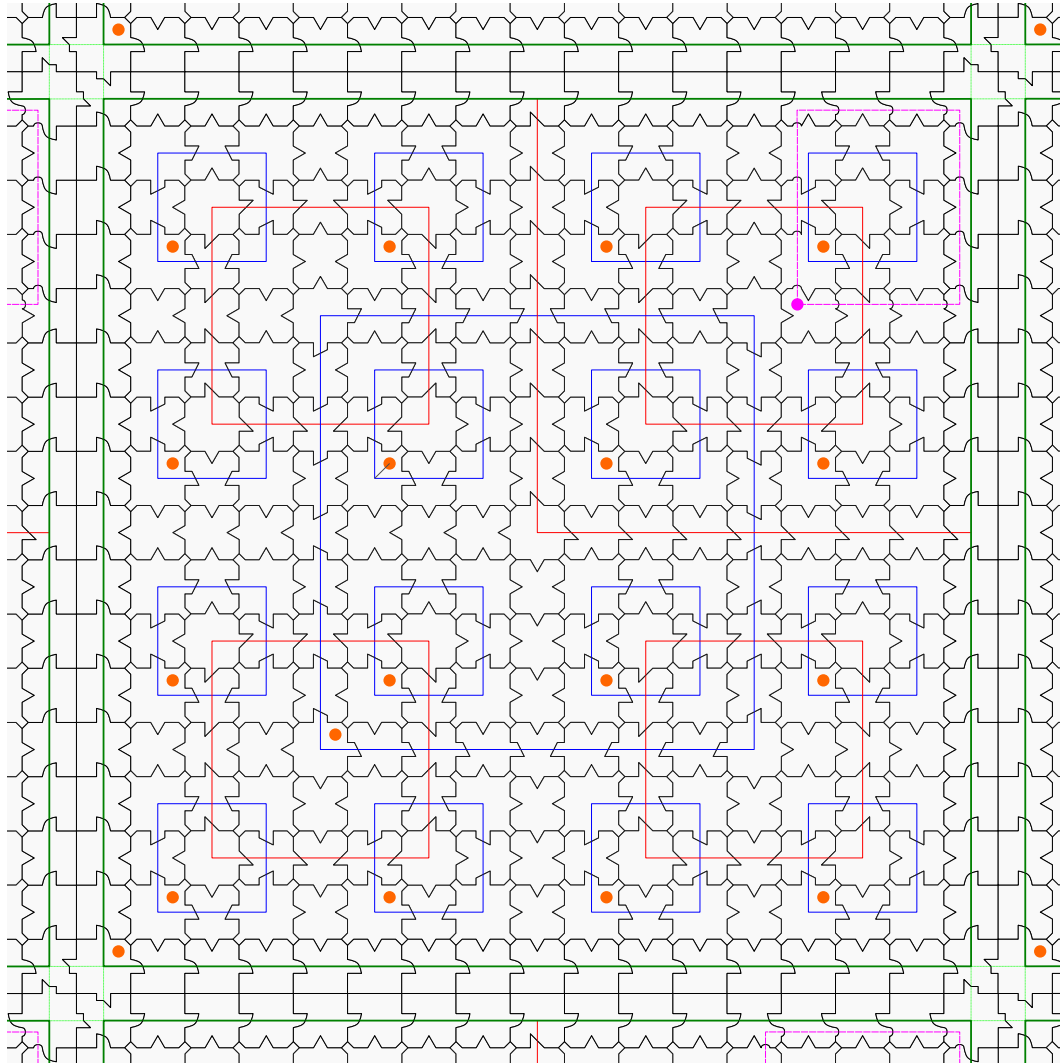
Left: from state q_1 , reading a 0, the machine writes a 0, goes in state q_2 and does not move.

Right: from state q_2 , reading a 0, the machine writes a 1, stays in state q_2 and moves to the right.

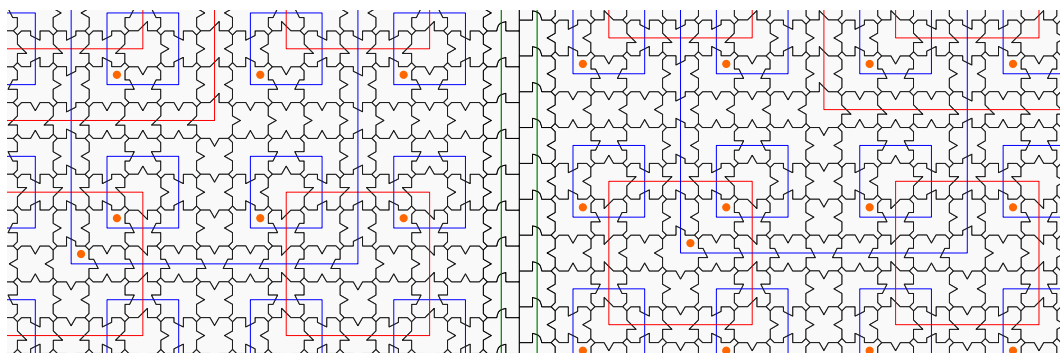
■ **Figure 15** Embedding computation in tiles. Additional *auxiliary tiles* are used to initialise the tape and connect the computation areas.



■ **Figure 16** Right: the seed tile starts the computation (the square can be blue or green). Left: any computation tile that reaches the halting state triggers a violet signal.



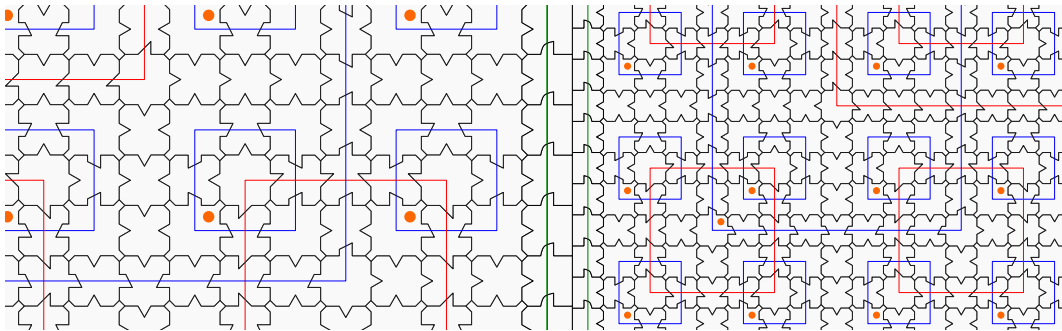
■ **Figure 17** A period of the Gurevich-Koryakov tiling. The computation layer is not represented.



■ **Figure 18** If the simulated machine does not halt, some tilings consist of two "infinite green squares", that is, independent half-planes. Such half-planes may be shifted continuously relative to each other as only the corners of the green squares synchronise.

corners (to enforce periodicity when the machine halts and the green squares are finite; see Fig. 17). In the case where the machine does not halt, there exist tilings consisting of two half-planes of Robinson structures separated by a green border line called a *shear line*. Any vertical shift of a half-plane relative to the other is possible as shown on Fig. 18. This means that the subshift X does not have finite local complexity when the machine does not halt, so X has finite local complexity if and only if the machine halts. ◀

► **Remark 23 (FLC for subshifts).** If M does not halt, any individual tiling in the subshift X has finite local complexity (even though X doesn't), since it contains at most one shear line with a given offset which yields only finitely many 2-shapes patterns. The construction can be tweaked so that individual tilings do not have FLC in this case by duplicating our shapset into squares of size 1 and squares of size $\varphi = (1 + \sqrt{5})/2$ (or any irrational number). Tiles inside a finite or infinite green square synchronise so that only one size is used. However, when there is a shear line (in the case where the machine does not halt), the two half-planes may use different shapsets, as illustrated on Fig. 19. The offset between adjacent tiles across the shear lines are the multiples of φ modulo 1 (plus an initial offset) which are dense in the interval $[0, 1]$. So such a tiling does not have finite local complexity.



■ **Figure 19** A tiling with infinite local complexity. Tiles in the left half-plane are scaled by φ .

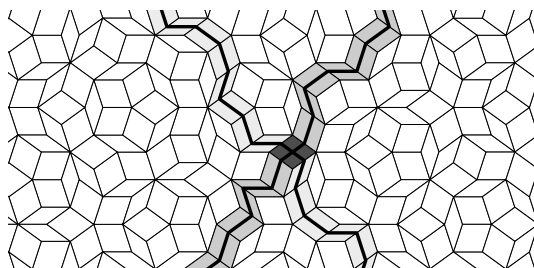
A Aperiodic Wang tilesets from aperiodic geometric tilesets

In this section, we present a folklore construction to define a Wang tileset from a purely geometric subshift in some cases. This is relatively orthogonal to Section 3 where we show how to simulate Wang tilesets in symbolic-geometric tilings on arbitrary shapes.

This construction was first used in [9, §11.1] for Kite-and-Darts Penrose tilings to produce an aperiodic Wang tileset of 32 tiles, and left as exercise (11.1.2) for rhombus Penrose tilings to produce an aperiodic Wang tileset of 24 tiles; full details on this latter case can be found in [12, §5]. We give an overview of this construction, in particular to shed light on geometric and combinatorial conditions that make it work.

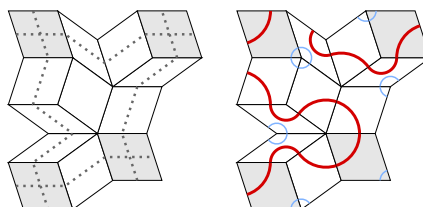
We present the construction for rhombus tilings though it can also work for e.g. the Kites-and-Darts. Let \mathcal{T} be a finite set of rhombus shapes and \mathbf{T} a finite labelled tileset on \mathcal{T} . Let X be an edge-to-edge \mathbf{T} nearest-neighbour subshift of finite type, that is, defined by finitely many forbidden patterns made of two edge-adjacent tiles.

In edge-to-edge rhombus tilings, the shapes form *chains* or *ribbons* of rhombuses sharing an edge-direction [15], so that each rhombus of edges \vec{u} and \vec{v} is the intersection of two chains of direction \vec{u} and \vec{v} (and conversely). This is illustrated in Fig. 20.



■ **Figure 20** Two chains intersecting in an edge-to-edge rhombus tiling.

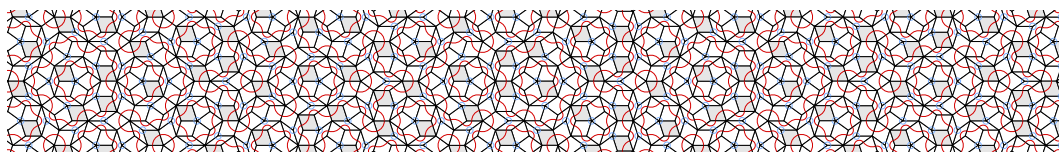
Let t_0 be a rhombus with edge directions \vec{u}_0 and \vec{v}_0 . Call a t_0 -chain-square a pattern delimited by two pairs of chains of same edge direction \vec{u}_0 and \vec{v}_0 , respectively, and intersecting no other chains from these directions, see Fig. 21. The four *corners* of such a pattern are four tiles of shape t_0 .



■ **Figure 21** Left: a chain-square. Right: the same shape with the labels.

Assume that $t_0 \in \mathcal{T}$ is uniformly recurrent in X , that is, there exists $R > 0$ such that any disk of radius R in any tiling $T \in X$ contains a tile t such that $\pi(t) = t_0$. Under this assumption,

- any tile t in any tiling $T \in X$ belongs to at least one t_0 -chain-square either as a corner (if t is a copy of t_0), in a border (if it shares one edge direction) or in the interior.
- the size of t_0 -chain-squares is bounded, so there are finitely many different t_0 -chain-squares in X . A formal proof of a very similar statement is given in e.g. [11, Lemma 21].

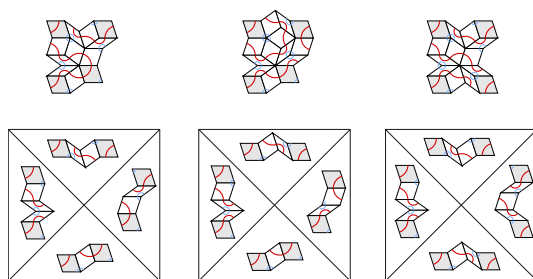


■ **Figure 22** A fragment of labelled Penrose rhombus tiling with occurrences of a tile highlighted.

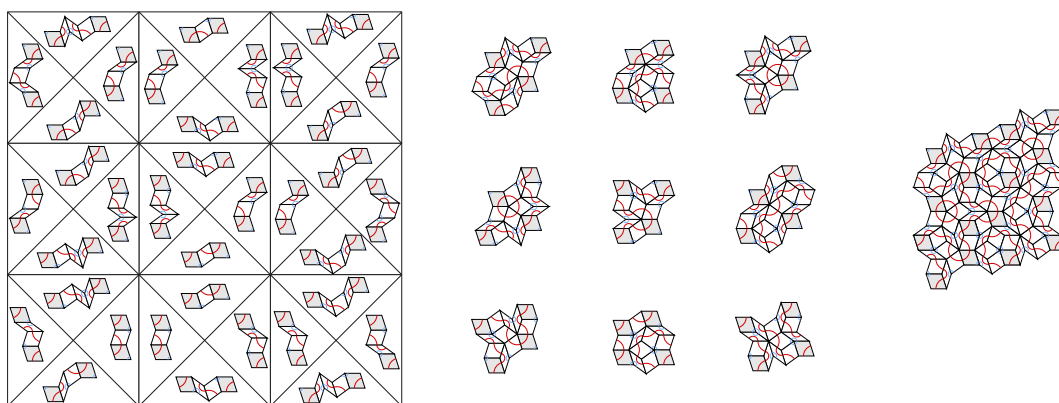
We define a set \mathbf{T}_{X,t_0} of Wang tiles as in Fig. 23: for each t_0 -chain-square P , $\mathbf{t}(P)$ is the Wang tile whose labels are the sides of P . Horizontal, resp. vertical labels are segments of \vec{u}_0 -chains, resp. \vec{v}_0 -chains between two occurrences of t_0 .

Each locally allowed pattern of \mathbf{T}_{X,t_0} Wang tiles can be converted to a geometric \mathbf{T} -pattern that is locally allowed in X , as is Fig. 24, because the subshift X only has nearest-neighbour forbidden patterns. It follows that each tiling in X in which t_0 is uniformly recurrent can be converted to a \mathbf{T}_{X,t_0} tiling, and conversely.

In particular, if X contains no periodic tiling, then neither does \mathbf{T}_{X,t_0} . Note however that the symmetries of the geometric tiling are not necessarily preserved. Therefore, if X

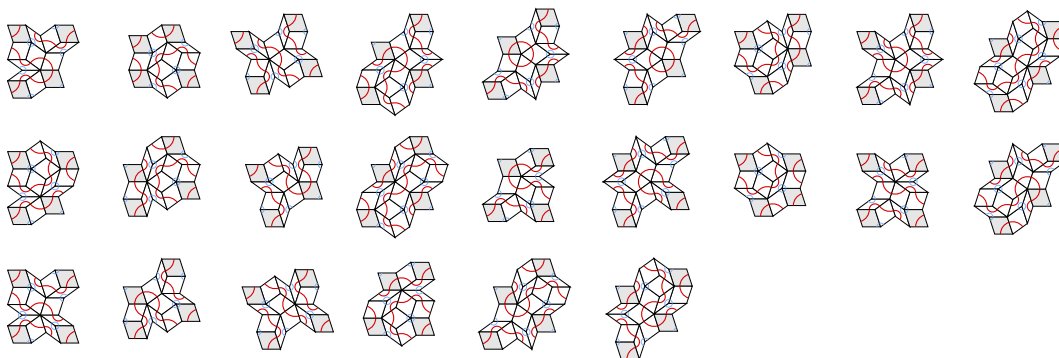


■ **Figure 23** Converting chain-squares to Wang tiles.



■ **Figure 24** Converting a patch of Wang tiles to a rhombus tiling.

contains no periodic configuration and assuming that one can enumerate the complete set of t_0 -chain-squares that are locally allowed in X , we obtain in this way a corresponding aperiodic Wang tileset.



■ **Figure 25** The 24 chain-squares that appear in Penrose rhombus tilings, with labels.

The key ingredients for this construction are the chains of tiles, and the presence of a uniformly recurrent tile that ensures that there are finitely many corresponding chain-squares. Some non-parallellogram tiles, such as the Penrose Kite-and-Darts, have a combinatorial structure that satisfy the first condition.

References

- 1 P. Arrighi, A. Durbec, and P. Guillon. Graph Subshifts. In *Unity of Logic and Computation: 19th Conference on Computability in Europe, CiE 2023*, page 261–274. Springer-Verlag, 2023. doi:10.1007/978-3-031-36978-0_21.
- 2 N. Aubrun, S. Barbieri, and E. Jeandel. About the domino problem for subshifts on groups. In *Sequences, Groups, and Number Theory*, pages 331–389. Springer International Publishing, 2018.
- 3 N. Aubrun, J. Esnay, and M. Sablik. Domino Problem Under Horizontal Constraints. In Christophe Paul and Markus Bläser, editors, *37th International Symposium on Theoretical Aspects of Computer Science (STACS 2020)*, volume 154 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 26:1–26:15, Dagstuhl, Germany, 2020. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPICs.STACS.2020.26.
- 4 S. Barbieri, M. Sablik, and V. Salo. Groups with self-simulable zero-dimensional dynamics, 2022. arXiv:2104.05141.
- 5 R. Berger. *The Undecidability of the Domino Problem*. American Mathematical Society, 1966.
- 6 David Bruce Cohen. The large scale geometry of strongly aperiodic subshifts of finite type. *Advances in Mathematics*, 308:599–626, 2017.
- 7 N. P. Frank. *Tilings with Infinite Local Complexity*, pages 223–257. Springer Basel, 2015. doi:10.1007/978-3-0348-0903-0_7.
- 8 C. Goodman-Strauss. A hierarchical strongly aperiodic set of tiles in the hyperbolic plane. *Theoretical Computer Science*, February 2010. doi:10.1016/j.tcs.2009.11.018.
- 9 B. Grünbaum and G. C. Shephard. *Tilings and Patterns*. Dover Publications, 1987.
- 10 Yu. Sh. Gurevich and I. O. Koryakov. Remarks on Berger’s paper on the domino problem. *Siberian Mathematical Journal*, 1972. doi:10.1007/BF00971620.
- 11 B. Hellouin de Menibus, V. H. Lutfalla, and C. Noûs. The Domino problem is undecidable on every rhombus subshift. In *Developments in Language Theory*. Springer Nature Switzerland, 2023. doi:10.1007/978-3-031-33264-7_9.
- 12 H. Jang. *Directional Expansiveness*. PhD thesis, George Washington University, 2021.
- 13 A. S. Kahr, E. F. Moore, and H. Wang. Entscheidungsproblem reduced to the aea case. *Proceedings of the National Academy of Sciences*, 48(3):365–377, 1962. doi:10.1073/pnas.48.3.365.
- 14 R. Kenyon. Rigidity of planar tilings. *Inventiones mathematicae*, 107:637–651, 1992. doi:10.1007/BF01231905.
- 15 R. Kenyon. Tiling a polygon with parallelograms. *Algorithmica*, 9:382–397, 1993. doi:10.1007/BF01228510.
- 16 V. H. Lutfalla. Geometrical tilings : Distance, topology, compactness and completeness, 2022. URL: <https://arxiv.org/abs/2208.14922>, arXiv:2208.14922.
- 17 M. Margenstern. The domino problem of the hyperbolic plane is undecidable. *Theoretical Computer Science*, 407(1-3):29–84, 2008. doi:10.1016/j.tcs.2008.04.038.
- 18 E. A. Robinson. Symbolic dynamics and tilings of \mathbb{R}^d . In *Proceedings of Symposia in Applied Mathematics*, volume 60, pages 81–120, 2004. doi:10.1090/psapm/060.
- 19 R. M. Robinson. Undecidability and nonperiodicity for tilings of the plane. *Inventiones mathematicae*, 12:177–209, 1971. doi:10.1007/BF01418780.
- 20 D. Smith, J. S. Myers, C. S. Kaplan, and C. Goodman-Strauss. An aperiodic monotile. *Combinatorial theory*, 4(1), 2024. doi:10.5070/C64163843.