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Dominance Criteria on Grids for Measuring Competitive Balance in Sports Leagues

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Abstract

The paper proposes a dominance criterion that assesses whether a seasonal outcome of a sports league is more balanced than another. This criterion is based on a novel third-order stochastic dominance defined on finite sets of evenly spaced seasonal points (seasonal grids), called downward seasonal balance (DSB). The DSB criterion makes the same assessments as the well-known Lorenz criterion. However, the converse is not true: The DSB criterion makes assessments even in cases where the Lorenz criterion cannot. The former is then less incomplete than the latter. The assessments of the DSB criterion reflect the unanimity of a class of competitive balance indices. A seasonal outcome is more balanced than another according to the DSB criterion if and only if every index of the class agrees. Such a class is axiomatically characterized so that the indices place at least as much emphasis on the balance between leading competitors as on the balance occurring among the nonleading competitors. An empirical application provides comparisons of seasonal outcomes of the five most competitive soccer leagues in Europe from 2014-2015 to 2018-2019.

JEL Classification: D6 ; Z2. *Keywords:* Competitive balance ; Grids ; Stochastic dominance ; Upside transfer sensitivity.

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1 Introduction

Introduced by Rottenberg (1956) and Neale (1964), the so-called *uncertainty-of-outcome hypothesis* is commonplace in the economics of sport. It requires that demand for stadium attendance or broadcast increases with respect to competitive balance, that is balance among competitors. Cairns *et al.* (1986) decompose competitive balance into three lengths: short-, medium-, and long-term competitive balance. *Short-term competitive balance* amounts to the balance of strength between two opponents in a game. *Medium-term competitive balance* refers to the balance among participants in a contest during a season. Long-term competitive balance deals with an interval of several seasons. According to Szymanski (2003), the championship race is a distinctive attribute of medium-term competitive balance. This race is determinant in providing empirical support for the uncertainty-of-outcome hypothesis (Jennett, 1984; Pawlowski and Anders, 2012; Scelles *et al.*, 2013; Andreff and Scelles, 2015; Pawlowski *et al.*, 2018; among others). In this sense, this paper aims to measure medium-term competitive balance, emphasizing the championship race. Such a measurement is performed by means of pairwise comparisons of distributions of seasonal points.

As this work focuses on distributional aspects, the concepts and results of the measurement of income inequality can be translated into concepts and results of the measurement of competitive (im)balance. One of the most-used methods for measuring inequality is the Lorenz, or second-order stochastic, dominance criterion.¹ Distribution \mathbf{y} is said to *Lorenz dominate* distribution \mathbf{x} when no part of the Lorenz curve of distribution \mathbf{y} lies below that of distribution \mathbf{x} . In sports terms, this means that \mathbf{y} is at least as balanced as \mathbf{x} . Quirk and Fort (1992) and Szymanski and Kuypers (1999) employ the Lorenz criterion to measure long-term competitive balance. Additionally, Michie and Oughton (2004) employ the Lorenz criterion for analyzing *medium-term* competitive balance.

The normative content of the Lorenz criterion is derived from the higher of two non-intersecting curves obtaining from the lower by a sequence of *progressive transfers*, that is, transfers of points from a stronger to a weaker competitor. For instance, assume a contest has three competitors: B , C , and D . A first hypothetical distribution of points is $\mathbf{x} = (B, 8; C, 2; D, 2)$, where B , the strongest, is awarded 8 points, while the two weakest— C and D —obtain 2 points. A second hypothetical distribution is $\mathbf{y} = (B, 6; C, 4; D, 2)$. Distribution \mathbf{y} can be obtained from \mathbf{x} through a progressive transfer of two points from B to C . According to the Lorenz criterion, \mathbf{y} is at least as balanced as \mathbf{x} . This judgment is consistent with the so-called *principle of transfers* (Pigou, 1912; Dalton, 1920). In sports terms, this principle requires competitive balance to increase as the result of a progressive transfer. According to

¹Using the results of Rothschild and Stiglitz (1970), Atkinson (1970) demonstrated equivalence between Lorenz dominance and second-order stochastic dominance employed to rank uncertain prospects.

the principle of transfer, \mathbf{y} is more balanced than \mathbf{x} . Dasgupta, Sen, and Starrett (1973) show that a first-distribution Lorenz dominates a second distribution if and only if *all* inequality indices that satisfy the principle of transfers state that the first is at least as balanced as the second. This equivalence only stands for pairs of distributions with equal means.

Horowitz (1997), Schmidt and Berri (2001, 2002), Utt and Fort (2002), Borooah and Mangan (2012) and Gayant and Le Pape (2017a), among others, propose several inequality indices to measure competitive balance. Moreover, Triguero Ruiz and Avila-Cano (2019) introduce a cardinal index, which is based on the concept of distance. Since these indices satisfy the principle of transfers, their judgment is unanimous when one distribution Lorenz dominates another. For instance, all indices state that \mathbf{y} is at least as balanced as \mathbf{x} . However, these indices render conflicting judgments when both progressive and *regressive* (weaker-to-stronger) transfers are needed to convert one distribution into another. Building on the previous example, consider a third hypothetical distribution $\mathbf{z} = (B, 7; C, 4; D, 1)$. The well-known Gini coefficient indicates that \mathbf{z} is as balanced as \mathbf{x} . Atkinson indices (Atkinson, 1970) state that distribution \mathbf{x} is more balanced than \mathbf{z} . Alternatively, some generalized entropy indices (with parameters higher than 2) state that \mathbf{z} is more balanced than \mathbf{x} . When the indices that satisfy the principle of transfers are not unanimous, Lorenz curves intersect, and the criterion is inconclusive. With intersecting Lorenz curves, the criterion cannot be used to judge whether \mathbf{z} is at least as balanced as \mathbf{x} . Moyes (1999) and Aaberge (2009), among others, point out that Lorenz curves often intersect in practice.

To overcome this limitation, the paper proposes three equivalent aspects of the measurement of competitive balance. First, the notion of an increase in balance is associated with a progressive transfer and a *balancing composite transfer* (BCT). A composition of two transfers, a BCT is a regressive transfer between weaker competitors and a progressive transfer between stronger competitors. [It is equivalent to the UNFACT introduced by Gayant and Le Pape \(2017a\)](#). A BCT should increase competitive balance because [“competition for top positions is considerably more important than competition in the “dull” midfield” \(Bundzinski and Pawlowski, 2017, pp. 116-117\)](#). [Applying concepts of behavioral economics, Pawlowski and Bundzinski \(2014\) show that soccer fans care more about a balanced championship race than about any other balanced subpart of the competition](#). Second, all indices that satisfy this requirement and the principle of transfers may be used to make a unanimous judgment on two distributions if and only if a distribution can be obtained from the other through a sequence of progressive transfers and/or BCTs. The set for which this unanimity is reached includes the generalized entropy indices (with parameters higher than 2) proposed by Gayant and Le Pape (2017a) and the index of Triguero Ruiz and Avila-Cano (2019).² In

²[The index of Triguero Ruiz and Avila-Cano \(2019\) is ordinally equivalent to the generalized entropy](#)

practice, it seems impossible to check whether one of the two above equivalent conditions is fulfilled. For instance, an infinite number of comparisons should be performed to check whether these indices make a unanimous judgment.

The third aspect of the measurement of competitive balance regards an operational method to verify the validity of these conditions. It is shown that a first-distribution downward seasonal balance (DSB) dominates a second distribution if and only if a sequence of progressive transfers and/or BCTs is needed to convert the second into the first distribution.³ The DSB criterion is a novel third-order stochastic dominance quasiorder. It is weaker, and so less incomplete, than the Lorenz criterion. When the Lorenz criterion can be employed to make a judgment, the DSB criterion allows the same judgment, whereas the converse is not true. According to the previous example, \mathbf{y} DSB dominates \mathbf{x} because \mathbf{y} Lorenz dominates \mathbf{x} . Moreover, \mathbf{z} DSB dominates \mathbf{x} because \mathbf{z} can be obtained from \mathbf{x} through a BCT of one point among B , C , and D .

Fishburn and Willig (1984), Gayant and Le Pape (2017b), and Dubois and Mussard (2019), among others, explore third-, fourth- and higher-order dominance criteria to rank income distributions. Unlike these approaches, this paper proposes a reverse third-order stochastic dominance. Applied to distributions of seasonal points, the classical third-order stochastic dominance is based on the aggregation from lower point levels to larger ones, whereas DSB is defined on the aggregation from larger point levels to lower ones.

In contrast to measuring income inequality, measuring medium-term competitive balance generally includes populations with fewer than 33 competitors, each with an integer-valued number of points. Thus, both Lorenz and DSB criteria are designed to compare distributions of finite numbers of integers (grids). They are easily implementable, as the required data are lists of wins-ties-losses of each competitor. Such data are freely available and do not suffer from bias or measurement errors.

This work performs intranational judgments of seasonal distributions from 2014-2015 to 2018-2019 of the five most competitive soccer leagues in Europe. The Lorenz criterion was conclusive in only 34% of the 50 pairwise comparisons that could to be made, whereas the DSB criterion was conclusive for 66% of these cases. According to these criteria, the competitive balance of the English *Premier League*, the German *Bundesliga*, and the Italian *Serie A* globally decreased during the period. Only the competitive balance of the Spanish *LaLiga* globally increased from 2014-2015 to 2018-2019. More precisely, the competitive balance of the French *Ligue 1*, the Bundesliga, and Serie A first decreased generally from

index with the parameter equal to 2. I thank a referee who points out that the former is an increasing transformation of the latter.

³The name of the DSB criterion is inspired by Michie and Oughton (2004), who called the Lorenz criterion the “Lorenz Seasonal Balance” criterion.

2014-2015 to 2016-2017 and later generally increased from 2017-2018 to 2018-2019.

Section 2 introduces the framework. Section 3 presents the second-order stochastic dominance criterion on grids (equivalently, the Lorenz criterion). Section 4 examines the DSB criterion. Section 5 discusses an empirical application, whose results are summarized in Table 5.1. Section 6 concludes the article.

2 Setup

A population of $n \geq 2$ competitors (teams) is investigated. Teams compete in a “once home-once away” championship without playoffs so that each team plays $2(n - 1)$ games, and the total number of games played is $n(n - 1)$. The degree of competitive balance of a league is measured according to the distribution of points at the end of the contest. Assume a *point award system* where, for every game it plays, a team is awarded q_v points for a victory, q_t for a tie, and q_d for a defeat such that $q_v, q_t, q_d \in \mathbb{Z}_+$ and $q_v > q_t > q_d$.⁴ Denote $p_0 \in \mathbb{Z}_+$ as the number of points of every team at the outset of the contest. A team has the smallest possible number of points, $p_0 + 2(n - 1)q_d$, if it loses the $2(n - 1)$ games. The second smallest, $p_0 + (2n - 3)q_d + q_t$, is the number of points of a team if it loses all the games it plays but one for which it makes a tie. This calculation continues up to the greatest possible number of points, $p_0 + 2(n - 1)q_v$, if a team wins the $2(n - 1)$ games it plays.⁵ The seasonal number of points of a team lies in the following finite set:

$$\mathcal{P} = \{p_0 + j : j = 2(n - 1)q_d, (2n - 3)q_d + q_t, \dots, (2n - 3)q_v + q_t, 2(n - 1)q_v\}.$$

The *distribution* (function) of points $f : \mathcal{P} \rightarrow [0, 1] \cap \mathbb{Q}$ lies in the following set:⁶

$$\Omega = \left\{ f : f(p_0 + j) \in [0, 1] \cap \mathbb{Q}, \forall p_0 + j \in \mathcal{P} \text{ and } \sum_{j=2(n-1)q_d}^{2(n-1)q_v} f(p_0 + j) = 1 \right\}$$

where $f(p_0 + j) \times 100$ is the percentage of teams that hold $p_0 + j$ points at the end of the championship. The cumulative distribution function $F : \mathcal{P} \rightarrow [0, 1] \cap \mathbb{Q}$ is such that

$$F(p_0 + j) = \sum_{\substack{i \in \mathcal{P}: \\ i \leq p_0 + j}} f(i)$$

where $F(p_0 + j) \times 100$ is the percentage of teams awarded by at most $p_0 + j$ points at the end of the contest.

⁴The set \mathbb{Z}_+ is the set of nonnegative integers.

⁵This statement follows from the so-called principle of saturation; see Avila-Cano *et al.* (2021).

⁶The set \mathbb{Q} is the set of rational numbers.

Following Kolm (1969), Atkinson (1970), and Sen (1973), additively separable competitive (im)balance (inequality) indices may be derived from the following *aggregate value of points*:

$$A_f = \sum_{j=2(n-1)q_d}^{2(n-1)q_v} v(p_0 + j)f(p_0 + j)$$

where $v(p_0 + j)$ is the *value* of points $p_0 + j$ and $v : \mathcal{P} \rightarrow \mathbb{R}$ is assumed to be strictly increasing. The aggregate value of points is formally analogous to a standard additively separable social welfare function. Therefore, each competitive (im)balance index I_f can be considered a function of A_f with a particular value function v so that for two distributions $f, g \in \Omega$ with equal means,

$$A_f \geq A_g \iff I_f \leq I_g. \tag{1}$$

The aggregate value of points states that *f is at least as balanced as g*, which is equivalent to the assessment that *f* is at most as imbalanced as *g* according to the competitive (im)balance index. The clause in italics of the previous sentence is commonly used in sports economics and is employed throughout the paper.⁷

Inequality indices suitably describe a totally equal distribution, say **b**, as the most balanced one. Mainly because of the bilaterality of the relationship among competitors, the identification of the least balanced distribution is the subject of debate. According to the indices, a distribution is the *least balanced* if it is less balanced than any other possible distribution.⁸ On the one hand, for the European soccer point award system: $p_0 = 0$, $q_v = 3$, $q_t = 1$, and $q_d = 0$, Avila-Cano *et al.* (2021) characterize such a distribution according to the so-called Herfindahl-Hirschman index. In this framework, the least balanced is a *truncated-cascade distribution*, which is made up of a cascade among some of the strongest teams, and an equal distribution by ties among the remainder of the competitors. The *cascade* can be objectified as the strongest team winning its $2(n-1)$ games, the second strongest team losing 2 games and winning $2(n-2)$ games, the third strongest team losing 4 games and winning $2(n-3)$ games, *etc.* The authors show that the number of teams in the cascade of the least balanced distribution depends on the total number of competitors.

On the other hand, Gayant and Le Pape (2015a, 2017a) argue that every index should state the complete-cascade distribution, say **i**, as the least balanced.⁹ In the *complete-cascade distribution*, all n participants are in the cascade. It is formalized by Avila-Cano *et al.* (2021).

⁷In this sense, competitive balance is simply the opposite concept of competitive imbalance. The latter notion is evoked to clarify the analogy with income inequality.

⁸The set of possible distributions of points is a subset of Ω defined according to the rules of the competition under consideration.

⁹Distribution **i** is characterized by Borooah and Mangan (2012), Fort and Quirk (1995), Horowitz (1997), Utt and Fort (2002), Owen *et al.* (2007), Avila-Cano *et al.* (2021), and Gayant and Le Pape (2015a, 2015b, 2017a).

Gayant and Le Pape (2015a) demonstrate that this requirement is respected only if the point award system is *stable*, that is $2q_t = q_v + q_d$. Stability implies that all distributions in Ω have the same mean, and the converse is not true.

The next two sections propose (incomplete) quasiorders, which theoretically cannot make assessments for all pairs of distributions in Ω . Therefore, the quasiorders do not require stability as they do not identify the least balanced distribution. However, assessments provided by the quasiorders may be consistent with some statements of the indices only if they involve distributions with equal means. This necessary condition comes from the equivalence stated in (1).

It is worth mentioning that distributions of shares from any unstable system have the same mean for a given n and could consistently be compared by the quasiorders. However, this would require characterizing the set of seasonal shares. This complex task would complicate the notation. For the sake of simplicity, it is henceforth assumed that $p_0 = 0$, $q_v = 2$, $q_t = 1$, and $q_d = 0$. Each team is awarded a number of points, which lies in the following grid:

$$\mathcal{P}_* = \{p \in \mathbb{Z}_+ : p \leq 4n - 4\}.$$

3 Second-order stochastic dominance for points

As seen above, an essential definition shared in the measurement of both income inequality and competitive balance is progressive transfer. According to the following definition, a progressive transfer is a “stronger-to-weaker” transfer. The stronger team is awarded $p + \epsilon + \delta$ points before the transfer and $p + \epsilon$ after the transfer. The weaker team is awarded with p points before the transfer and $p + \delta$ afterward.

Definition 3.1. *Let f and g belong to Ω . Distribution f is obtained from g by means of a progressive transfer if four point levels exist: $p, p + \delta, p + \epsilon, p + \epsilon + \delta \in \mathcal{P}_*$ with $\delta > 0$, and $\epsilon \geq \delta$, such that:*

$$\begin{aligned} f(p) &= g(p) - \frac{1}{n} ; & f(p + \delta) &= g(p + \delta) + \frac{1}{n} ; \\ f(p + \epsilon) &= g(p + \epsilon) + \frac{1}{n} ; & f(p + \epsilon + \delta) &= g(p + \epsilon + \delta) - \frac{1}{n} ; \end{aligned}$$

f and g are identical everywhere else. Equivalently, g is obtained from f by means of a regressive transfer.

“ $f(p) = g(p) - \frac{1}{n}$ ” means that at point level p , the mass of teams in distribution f is lower by $\frac{1}{n} \times 100$ percent than in distribution g . Moreover, “ $f(p + \delta) = g(p + \delta) + \frac{1}{n}$ ” means that at point level $p + \delta$, the mass of teams in distribution f is higher by $\frac{1}{n} \times 100$ percent than in

distribution g . Aggregating both interpretations yields that $\frac{1}{n} \times 100$ percent of the teams—in fact, one team—with points p in distribution g receives δ additional points in distribution f . Definition 3.1 means that distribution f is obtained from distribution g through a progressive transfer of points δ . The restriction $\epsilon \geq \delta$ implies that the stronger team before a transfer is still stronger than the weaker team after a transfer.

Axiom 3.1. *Principle of transfers: A principle requiring that competitive balance does not decrease as the result of a progressive transfer.*

The principle of transfers holds if and only if the value of points increases at a (non-strictly) decreasing rate over \mathcal{P}_* . That is, the marginal gain in competitiveness due to an increase in points received by a weaker team at least compensates for the marginal loss due to a decrease in points suffered by a stronger team. Formally, for

$$\Delta_1 v(p) = v(p+1) - v(p) \text{ and } \Delta_2 v(p) = \Delta_1 v(p+1) - \Delta_1 v(p),$$

the value function should lie in V_2 defined with respect to V_1 as follows:

$$\begin{aligned} V_1 &= \{v : \mathcal{P}_* \rightarrow \mathbb{R} : \Delta_1 v(p) > 0, \forall p = 0, \dots, 4n-5\}; \\ V_2 &= \{v \in V_1 : \Delta_2 v(p) \leq 0, \forall p = 0, \dots, 4n-6\}. \end{aligned}$$

Functions v in V_2 are usually referred to as Jensen concave of order 1.

To align with the principle of transfers, it is possible to perform comparisons of distributions without specifying the form of v insofar as the function belongs to V_2 . However, this way of proceeding raises the practical question of its implementation, as without specifying $v \in V_2$, it is impossible to directly verify whether a distribution is more balanced than another. Consider two alternative distributions, f and g in Ω , and denote the difference by $\Delta f = f - g$; then, the difference in the aggregate value of points is:

$$\Delta A_f = \sum_{p=0}^{4n-4} v(p) \Delta f(p).$$

Applying Abel's lemma, we then have:¹⁰

$$\Delta A_f = \sum_{p=0}^{4n-5} \Delta F(p) [v(p) - v(p+1)] + \Delta F(4n-4) v(4n-4)$$

¹⁰Abel's lemma is a summation formula that shows the following: Let $b_0, \dots, b_n, c_0, \dots, c_n$ be real numbers. Set $B_j = \sum_{k=0}^j b_k$. Then:

$$\sum_{j=0}^n b_j c_j = \sum_{j=0}^{n-1} B_j (c_j - c_{j+1}) + B_n c_n.$$

where $\Delta F \equiv F - G$. Making use of the fact that $\Delta F(4n - 4) = 0$, we obtain:

$$\Delta A_f = \sum_{p=0}^{4n-5} \Delta F(p)[v(p) - v(p+1)] = \sum_{p=0}^{4n-5} \Delta D^1(p) \Delta_1 v(p) \quad (2)$$

where $\Delta D^1(p) = -\Delta F(p)$. Once again applying Abel's lemma:

$$\Delta A_f = - \sum_{p=0}^{4n-6} \Delta D^2(p) \Delta_2 v(p) + \Delta D^2(4n-5) \Delta_1 v(4n-5) \quad (3)$$

where $\Delta D^2(p) = \sum_{q=0}^p \Delta D^1(q) = -\sum_{q=0}^p \Delta F(q)$ for $p = 0, \dots, 4n-5$.

The following remark recalls the well-known equivalence between the criterion expressed by $\Delta D^2(p)$, which is formally the second-order stochastic dominance relation (Rothschild and Stiglitz, 1970) and the Lorenz criterion.¹¹

Remark 3.1. *For f and g in Ω , the two following assertions are equivalent:*

- (a) f Lorenz dominates g .
- (b) $\Delta D^2(p) \geq 0$ for all $p = 0, \dots, 4n-5$.

Proof. The proof follows from two equivalences. First, statement (b) is equivalent to *traditional* second-order stochastic dominance, that is, dominance of distributions defined on a continuum (Fishburn and Lavalley 1995). Second, traditional second-order stochastic dominance is equivalent to Lorenz dominance (Atkinson 1970). \square

The following theorem proposes a simple procedure to decide whether f is at least as balanced as g when attention is restricted to unspecified value functions being first-order Jensen concave.

Theorem 3.1. *For f and g in Ω , the following three assertions are equivalent:*

- (i) For all $v \in V_2$, $\Delta A_f \geq 0$.
- (ii) f can be reached from g through a finite sequence of progressive transfers.
- (iii) $\Delta D^2(p) \geq 0$ for all $p = 0, \dots, 4n-5$.

Proof. Fishburn and Lavalley (1995) show that statement (iii) is equivalent to traditional second-order stochastic dominance. Therefore, the theorem is a direct adaptation of the result of Hardy, Littlewood, and Polya (1934). A proof of this theorem is also provided by Marshall, Olkin, and Arnold (2011). Moreover, (iii) \Rightarrow (i) is straightforward from (3). \square

¹¹A distribution f Lorenz dominates another distribution g if the $\frac{h}{n} \times 100\%$ weakest competitors in f hold at least as much points (scaled down by the total amount) as the $\frac{h}{n} \times 100\%$ weakest competitors in g , for all $h \in \{1, 2, \dots, n\}$.

Statement (i) makes a judgment: Distribution f is at least as balanced as g . Moreover, the value function v should be Jensen concave of order 1 for the functions A to be unanimous on this judgment. This means that all functions A , equivalently for all indices, that satisfy the principle of transfers unanimously state that f at least is as balanced as g . Statement (ii) stresses that progressive transfers are the required transformations to convert a distribution into another for this unanimity to be possible. Statement (iii) exhibits how to perform a pairwise comparison between f and g in practice. Distribution f Lorenz dominates g if and only if the verdict implied in (i) and stated by (ii) holds. In contrast, if the Lorenz curves of two distributions cross, then two indices exist that make opposite conclusions about these two distributions.

4 Downward third-order stochastic dominance for points

Gayant and Le Pape (2017a) base their work on the notion of [balancing composite transfer \(BCT\)](#) to provide a formal characterization of what an increase in competitive balance of the league could be.¹² As mentioned above, the BCT is a combination of a regressive transfer between weaker teams (e.g., the two arrows on the left in Figure 4.1) and a progressive transfer between stronger teams (e.g., the two arrows on the right in Figure 4.1), involving the same number of points and the same amplitude.

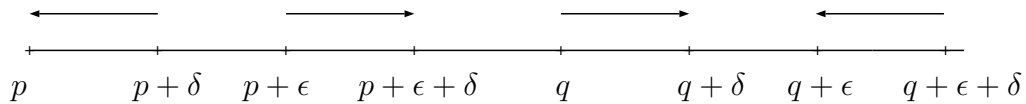


Figure 4.1. An example of BCT

The number of points δ is transferred from a team with $p + \delta$ to a stronger team with $p + \epsilon$ and also transferred from a team with $q + \epsilon + \delta$ to a weaker team with q . The integer ϵ ensures that the pretransfer difference in points between teams involved in the regressive transfer is the same as the posttransfer difference in points between teams involved in the progressive transfer.

The following definition sets a general presentation of the BCT.

Definition 4.1. *Let f and g belong to Ω . Distribution f is obtained from g by means of a BCT if eight point levels exist, $p, p + \epsilon, p + \epsilon + \delta, q, q + \delta, q + \epsilon, q + \epsilon + \delta \in \mathcal{P}_*$ with $p < q$,*

¹²Precisely, Gayant and Le Pape (2017a) denote the BCT by UNFACT, *i.e.* UNFAvorable Composite Transfer. This name comes from the widespread view that an UNFACT increases inequality of income (Shorrocks and Forster, 1987). Although a BCT (or UNFACT) is generally undesirable when dealing with comparisons of income distributions, Gayant and Le Pape (2017a) claim that the competitive balance of a league increases as the result of a BCT.

$0 < \delta \leq q - p$, and $\epsilon \geq \delta$ such that:

$$\begin{aligned} \Delta f(p) &= \frac{1}{n} ; \Delta f(p + \delta) = -\frac{1}{n} ; \Delta f(p + \epsilon) = -\frac{1}{n} ; \Delta f(p + \epsilon + \delta) = \frac{1}{n} ; \\ \Delta f(q) &= -\frac{1}{n} ; \Delta f(q + \delta) = \frac{1}{n} ; \Delta f(q + \epsilon) = \frac{1}{n} ; \Delta f(q + \epsilon + \delta) = -\frac{1}{n} ; \end{aligned}$$

f and g are identical everywhere else. Equivalently, g is obtained from f by means of an unbalancing composite transfer.

In Definition 4.1, both “ $\Delta f(p) = \frac{1}{n}$ ” and “ $\Delta f(p + \delta) = -\frac{1}{n}$ ” formally correspond to the left arrow on the left side of Figure 4.1. That is, $\frac{1}{n} \times 100$ percent of the teams move from point level $p + \delta$ to point level p . Hence, the weakest team affected by the BCT “loses” points at the expense of a stronger team. The other pairs of equations in the definition correspond to the other arrows of Figure 4.1.

As an illustration, consider $\mathbf{x} = (B, 8; C, 2; D, 2)$ and $\mathbf{y} = (B, 6; C, 6; D, 0)$. Distribution \mathbf{y} may be obtained from \mathbf{x} through a BCT. In sports terms, \mathbf{x} may be viewed as a distribution, where B wins all its games, C loses its games against B and ties D twice (so D loses its games against B and ties C twice). Moreover, \mathbf{y} may be viewed as a distribution where B wins its games against D and ties C twice; C wins its games against D and ties B twice, and D loses all its games. While \mathbf{x} is locally more balanced at the bottom, \mathbf{y} is more balanced at the top of the distribution. The following principle states that \mathbf{y} is at least as balanced as \mathbf{x} .

Axiom 4.1. *Upside transfer sensitivity (Gayant and Le Pape, 2017a). The principle requires that competitive balance does not decrease as the result of a BCT.*

Upside transfer sensitivity requires that at least as much emphasis be placed on the balance among stronger teams as among weaker teams. [This is in line with the empirical findings reported by Budzinski and Pawlowski \(2017\) insofar as the championship race is more important than any other sub-part of the contest.](#) The aggregate function of points A satisfies this principle and the principle of transfers if and only if its value function v has a first-order forward difference that is decreasing and concave over \mathcal{P}_* . In brief, the marginal valuation of points decreases at a nondecreasing rate over \mathcal{P}_* . That is, v belongs to the following class:

$$V_3 = \{v \in V_2 : \Delta_3 v(p) = \Delta_1(\Delta_2 v(p)) \leq 0, \forall p = 0, \dots, 4n - 7\}.$$

Functions in V_3 are Jensen concave of order 2.

Applying the lemma of downward summation formula to (3), it follows:¹³

$$\Delta A_f = - \sum_{p=1}^{4n-6} \Delta D^3(p) \Delta_3 v(p-1) - \Delta D^3(0) \Delta_2 v(0) + \Delta D^2(4n-5) \Delta_1 v(4n-5) \quad (4)$$

¹³The lemma of downward summation formula is presented and demonstrated in the appendix.

where

$$\Delta D^3(p) = \sum_{q=p}^{4n-6} \Delta D^2(q) \text{ for } p = 0, \dots, 4n - 6.$$

From (4), distribution f is at least as balanced as g , that is, $\Delta A_f \geq 0$ if

$$\Delta D^3(p) \geq 0 \text{ for all } p = 0, \dots, 4n - 6, \text{ and } \Delta D^2(4n - 5) \geq 0.$$

This sufficient condition is the formal means of stating that f DSB dominates g . The DSB dominance criterion greatly depends on the function D^3 , which aggregates the D^2 values from the third-largest point level, $4n - 6$, to the lowest one in \mathcal{P}_* . In a continuous setting, Aaberge (2009) names this kind of reversal condition downward dominance. It is part of the following theorem.

Theorem 4.1. *For f and g in Ω , the three following assertions are equivalent:*

- (i) *For all $v \in V_3$, $\Delta A_f \geq 0$.*
- (ii) *f can be reached from g through a finite sequence of progressive transfers and/or BCTs.*
- (iii) *$\Delta D^3(p) \geq 0$ for all $p = 0, \dots, 4n - 6$ and $\Delta D^2(4n - 5) \geq 0$.*

Proof. See the appendix. □

Statement (i) displays a judgment: Distribution f is at least as balanced as g . Moreover, the value function v should be Jensen concave of order 2 for the functions A to be unanimous on this judgment. This means that all functions A (equivalently, all indices), which satisfy the principle of transfers and upside transfer sensitivity, unanimously state that f is at least as balanced as g .¹⁴ Statement (ii) stresses that progressive transfers and/or BCTs are the necessary transformations to convert one distribution into another for this unanimity to be possible. Statement (iii) exhibits how to perform the pairwise comparison between f and g in practice. Distribution f DSB dominates g if and only if the verdict implied in (i) and stated by (ii) holds. Assertions (i–iii) in Theorem 3.1 imply assertions (i–iii) in Theorem 4.1, demonstrating that the DSB criterion is weaker than the Lorenz criterion.

Graphically, a distribution f is at least as balanced as a distribution g if two conditions are fulfilled. First, the DSB point of f does not lie below the DSB point of g at every abscissa

¹⁴Chateauneuf, Gajdos, and Wilthien (2002) show that a principle that involves a BCT tacitly rules out all rank-dependent methods to measure income inequality. Consequently, upside transfer sensitivity implies excluding all these methods to measure competitive balance. The Gini coefficient is one of the rank-dependent methods.

value from the third largest point level up to the smallest in \mathcal{P}_* . Second, $\Delta D^2(4n-5) \geq 0$.¹⁵ Consider a simple example. For a population of 3 teams, $\mathcal{P}_* = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, let $(0, 6, 6)$ be represented by $f \in \Omega$, and $(2, 2, 8)$ be represented by $g \in \Omega$:

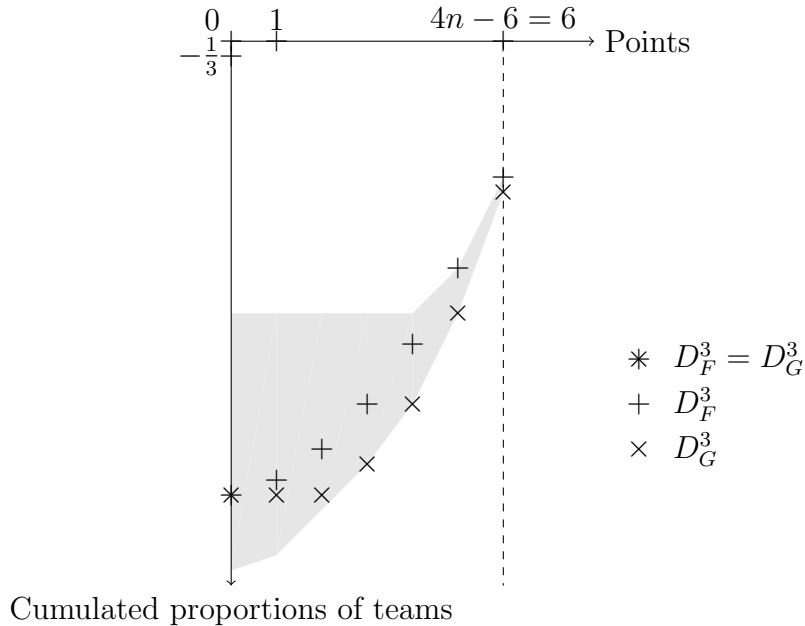
$$f = \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ 4 & 0 \\ 5 & 0 \\ 6 & \frac{2}{3} \\ 7 & 0 \\ 8 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 2 & \frac{2}{3} \\ 3 & 0 \\ 4 & 0 \\ 5 & 0 \\ 6 & 0 \\ 7 & 0 \\ 8 & \frac{1}{3} \end{bmatrix}.$$

Denote $D_F^2(p) = \sum_{q=0}^p (-F(q))$; hence, $D_F^3(p) = \sum_{q=p}^6 D_F^2(q)$. Moreover, $D_F^2(4n-5) = D_G^2(4n-5) = -4$.

p_i	0	1	2	3	4	5	6
$D_F^3(p)$	-10	-9.67	-9	-8	-6.67	-5	-3
$D_G^3(p)$	-10	-10	-10	-9.33	-8	-6	-3.33

Table 4.1. Coordinates of the DSB points of the 3-team example.

The following figure illustrates that distribution f is at least as balanced as g according to the DSB criterion.



¹⁵For variables defined on grids, Fishburn and Lavalley (1995, Corollary 3) remark that a third-order stochastic dominance condition is stronger than its counterpart defined on continuous variables. It turns out that, at least for this subcase, assertion (iii) in Theorem 4.1 is stronger than its counterpart defined on continuous variables.

Figure 4.2. An example of DSB dominance.

The totally equal distribution, **b**, is (roughly) represented by the upper contour of the gray area. Moreover, the lower contour stands for the perfect competitive imbalance distribution, **i**.¹⁶

5 Competitive balance in European soccer leagues

This section briefly illustrates the results from the data covering features of the five top-ranked national soccer leagues of the Union of European Football Associations (UEFA).¹⁷ The five soccer leagues comprise the German Bundesliga, the English Premier League, the Spanish LaLiga, the French Ligue 1, and the Italian Serie A. The period analyzed stretches from the 2014-2015 season to the 2018-2019 season.¹⁸

The relevant data include numbers of victories, ties, and defeats experienced by each team at the end of each season. The data are freely available on *skysports.com*, *google.com*, *wikipedia.org* and *espndeportes.espn.com*, among others. Distributions of points can be generated from these data in accordance with the point award system proposed in the setup. For instance, at the end of the Ligue 1 season 2018-2019, the weakest team won 5 games, tied 12 times, and lost 21 games, so the team was awarded 22 points. The second-weakest team won 7 games, had 12 ties, and lost 19 games, so it was awarded 26 points; and so on up to the champion team, which won 29 games, had 4 ties, and lost 5 games, so that it was awarded 62 points.

Only intranational comparisons were examined to check whether competitive balance decreased during the covered period. Obviously, international comparisons could be performed to check which league was the most balanced and which was the least balanced. The Lorenz criterion was able to be used to assert whether a season was more or less balanced than another season in the same league in only **17 cases out of 50** pairwise comparisons. In such cases, all the indices that satisfy the principle of transfers unanimously made the same conclusion as the Lorenz criterion. The DSB criterion was conclusive in **33 cases out of the 50** pairwise comparisons. All the indices that satisfy the principle of transfers and upside

¹⁶Rigorously, distributions **b** and **i** should be represented by 18 points. However, these additional points would worsen the clarity of the figure.

¹⁷The statement relies on the UEFA clubs' coefficient rankings from the 2018-2019 season, which itself depends on the four seasons considered in the application. More information is available at <https://www.uefa.com/memberassociations/uefarankings/country/#/yr/2019>

¹⁸The analysis of pandemic effects on the competitive balance in soccer is studied by Cueva (2020), Fisher and Haucap (2020), and Bryson et al. (2021), among others. As stressed by a referee, this issue is beyond the scope of this article. For this reason, data from the 2019-2020 season are not taken into consideration in this section.

transfer sensitivity were unanimously in line with the DSB criterion in these cases. The generalized entropy index with a parameter equal to two and the distance to competitive balance are two indices that satisfy these two axioms. Triguero Ruiz and Avila-Cano (2019) measured the competitive balance of the five leagues under consideration from 1997-1998 to 2016-2017 by means of these two indices. Their conclusions were ordinally consistent with the DSB criterion from 2014-2015 to 2016-2017, which is the period shared by their study and by that in this section.

All comparisons are presented in Table 5.1. The table should be read as follows: Below the mention of “Premier League,” the row named 2015-2016 “ \succ_D ” (DSB dominates) the column named 2014-2015. This means that the Premier League season 2015-2016 was at least as balanced as the Premier League season of 2014-2015, according to the DSB criterion. Nevertheless, below the mention of the “Premier League,” the row named 2016-2017 “ \preceq_L ” (is Lorenz dominated by) the column named 2014-2015. This means that the Premier League season 2016-2017 was at most as balanced as the Premier League season of 2014-2015, according to both the Lorenz and DSB criteria. The symbols “ \preceq_D ” and “ \succ_D ” allow for making a conclusion according to the DSB criterion while the Lorenz criterion was inconclusive. In Table 5.1, “NC” indicates that both the Lorenz and DSB criteria were unable to be used to draw a conclusive comparison between the two distributions under consideration.

The competitive balance of the Premier League tended to decrease from 2014-2015. Indeed, the 2018-2019 season was at most as balanced as the 2016-2017, 2015-2016, and 2014-2015 seasons. Additionally, the 2017-2018 season was at most as balanced as the 2015-2016 and 2014-2015 seasons. The most balanced season of the period was 2015-2016.

The DSB criterion provided a small number of judgments considering Ligue 1. In the first part of the period, competitive balance did not increase insofar as seasons 2017-2018 and 2016-2017 were at most as balanced as season 2014-2015. Ultimately, competitive balance did not decrease: Season 2018-2019 was at least as balanced as seasons 2016-2017 and 2017-2018.

The competitive balance of the Bundesliga tended to decrease over the observed period: Season 2018-2019 was at most as balanced as season 2014-2015. Beyond this trend, the evolution of the competitive balance of the Bundesliga was similar to that of Ligue 1. In the first part of the period, competitive balance did not increase insofar as seasons 2016-2017 and 2015-2016 were at most as balanced as season 2014-2015. Later, competitive balance did not decrease: Season 2017-2018 was at least as balanced as 2016-2017, which was at least as balanced as 2015-2016.

The competitive balance of Serie A tended to decrease over the period under consideration. Indeed, the 2018-2019 season was at most as balanced as the 2014-2015 season. In the first part of the period, competitive balance did not increase, reaching a minimum level in 2017-

2018. The 2017-2018 season was at most as balanced as the 2016-2017 season, which was at most as balanced as season 2015-2016, which, in turn, was at most as balanced as season 2014-2015. At the end of the period, competitive balance did not decrease, since the 2018-2019 season was at least as balanced as the 2017-2018 and 2016-2017 seasons.

The DSB criterion provided a small number of judgments for LaLiga. The competitive balance tended to increase from 2014-2015 to 2018-2019. Moreover, the 2018-2019 season was the most balanced season of the period.

	2014-2015	2015-2016	2016-2017	2017-2018
Premier League				
2015-2016	\succ_D			
2016-2017	\preceq_L	\preceq_D		
2017-2018	\preceq_D	\preceq_D	NC	
2018-2019	\preceq_L	\preceq_L	\preceq_D	NC
Ligue 1				
2015-2016	NC			
2016-2017	\preceq_D	NC		
2017-2018	\preceq_D	NC	NC	
2018-2019	NC	NC	\succ_D	\preceq_L
Bundesliga				
2015-2016	\preceq_L			
2016-2017	\preceq_L	\succ_D		
2017-2018	NC	\succ_D	\succ_D	
2018-2019	\preceq_L	NC	NC	NC
Serie A				
2015-2016	\preceq_D			
2016-2017	\preceq_L	\preceq_L		
2017-2018	\preceq_L	\preceq_L	\preceq_D	
2018-2019	\preceq_L	NC	\preceq_L	\preceq_L
LaLiga				
2015-2016	NC			
2016-2017	NC	NC		
2017-2018	NC	NC	NC	
2018-2019	\preceq_L	\preceq_D	\preceq_L	\preceq_L

Table 5.1. Competitive balance comparisons in five soccer leagues in Europe.

6 Concluding remarks

The DSB criterion is proposed as a means to measure competitive balance when the Lorenz criterion is inconclusive. It makes a conclusive comparison of any pair of distributions of points when a sequence of progressive transfers and/or BCTs is necessary to convert one

distribution into another. The normative view of competitive balance supported by this criterion is characterized by two axioms: The principle of transfers and upside transfer sensitivity. Moreover, the DSB criterion draws a conclusion if and only if all competitive balance indices that satisfy these axioms unanimously make the same conclusion as this criterion. The set for which unanimity is reached includes the indices proposed by Gayant and Le Pape (2017a) and Triguero Ruiz and Avila-Cano (2019).

Depken (1999), Pawlowski *et al.* (2010), Gayant and Le Pape (2012), and Triguero Ruiz and Avila-Cano (2019) propose methodologies to make comparisons between distributions of different sizes for measuring competitive balance. Further axioms than those introduced in this paper are needed for the DSB criterion to make such comparisons. For instance, it is commonplace to impose the Dalton (population) principle when comparing income distributions of different sizes. This principle requires that the level of inequality remain the same if a distribution is ℓ -fold replicated, for ℓ a positive integer. However this axiom might be not so suitable for measuring competitive balance. Should $\mathbf{x} = (B, 8; C, 2; D, 2)$ be as balanced as $\mathbf{x}' = (B, 8; B', 8; C, 2; C', 2; D, 2; D', 2)$? Placing emphasis on the championship race, \mathbf{x}' may be considered more balanced than \mathbf{x} . Indeed, in \mathbf{x}' , there are two competitors— B and B' —fighting for the championship whereas in \mathbf{x} , one team is merely but clearly stronger than the others. Although the generalized entropy indices satisfy the Dalton principle, not all indices consistent with the DSB criterion fulfill it. For instance, the distance to competitive balance does not.

The DSB criterion is a quasiorder; hence there are cases where it is inconclusive. To overcome this limitation, two strategies might be used: (A) an additional criterion that places even more emphasis on transfers that occur at the top of the distribution than the DSB criterion does; or (B) an additional criterion that satisfies the same principles as the DSB criterion and places more emphasis on the bottom of the distribution of points than the “dull midfield”. The choice of strategy (B) rather than (A) may be motivated by the possibility of relegation and promotion in some leagues, such as European soccer leagues. This point is stressed by Gayant and Le Pape (2017a).

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Appendix

Downward summation formula

Lemma. Downward summation formula. Let $b_0, \dots, b_n, c_0, \dots, c_n$ be real numbers. Set $C_j = \sum_{k=j}^n c_k$. Then:

$$\sum_{j=0}^n b_j c_j = \sum_{j=1}^n C_j (b_j - b_{j-1}) + C_0 b_0.$$

Proof. Let $b_0, \dots, b_n, c_0, \dots, c_n$ be real numbers. Set $C_j = \sum_{k=j}^n c_k$. Then for every $j > 0$,

$$c_j = \sum_{k=j}^n c_k - \sum_{k=j+1}^n c_k = C_j - C_{j+1}.$$

It turns out that:

$$\begin{aligned} \sum_{j=0}^n b_j c_j &= \sum_{j=0}^{n-1} b_j (C_j - C_{j+1}) + b_n c_n \\ &= \sum_{j=0}^{n-1} b_j C_j - \sum_{j=0}^{n-1} b_j C_{j+1} + b_n c_n = \sum_{j=1}^{n-1} b_j C_j + b_0 C_0 - \sum_{j=0}^{n-2} b_j C_{j+1} - b_{n-1} C_n + b_n c_n \\ &= \sum_{j=1}^{n-1} (b_j - b_{j-1}) C_j + b_0 C_0 - b_{n-1} C_n + b_n c_n \end{aligned}$$

By definition $C_n = c_n$, thus:

$$\sum_{j=0}^n b_j c_j = \sum_{j=1}^n (b_j - b_{j-1}) C_j + b_0 C_0$$

which concludes the proof. □

Proof of Theorem 4.1

Proof. [(i) \Rightarrow (ii)] Building on Theorem 3.1, for all v lying in V_2

$$\sum_{p=0}^{4n-4} v(p) f(p) \geq \sum_{p=0}^{4n-4} v(p) g(p)$$

implies that f can be reached from g by a finite sequence of progressive transfers. Since $V_3 \subseteq V_2$, then for v lying in V_3 the same implication remains valid.

Moreover, let us demonstrate that if (i) holds, then f can be obtained from g by a finite sequence of BCTs (first step) or by a finite sequence of BCTs and progressive transfers (second step).

First step: According to the definition of V_3 , for any $p \in [0, 4n - 7] \cap \mathcal{P}$, and for all $v \in V_3$,

$$\begin{aligned} \Delta_3 v(p) &\leq 0 \\ \Leftrightarrow v(p+3) - v(p+2) - v(p+2) + v(p+1) \\ &\quad - v(p+2) + v(p+1) + v(p+1) - v(p) \leq 0 \end{aligned}$$

Consider, for any $p \in [0, 4n - 7] \cap \mathcal{P}$,

$$\begin{aligned} p+3 &= q + \epsilon + \delta; \quad p+2 = q + \epsilon = p + \epsilon + \delta; \\ p+1 &= q = p + \delta \end{aligned}$$

For any $p \in [0, 4n - 7] \cap \mathcal{P}$, it comes:

$$\begin{aligned} v(q + \epsilon + \delta) - v(q + \epsilon) - v(q + \delta) + v(q) \\ -v(p + \epsilon + \delta) + v(p + \epsilon) + v(p + \delta) - v(p) \leq 0 \end{aligned}$$

which, for any $p \in [0, 4n - 7] \cap \mathcal{P}$, is equivalent to

$$\frac{1}{n} \begin{bmatrix} v(q + \epsilon + \delta) - v(q + \epsilon) - v(q + \delta) + v(q) \\ -v(p + \epsilon + \delta) + v(p + \epsilon) + v(p + \delta) - v(p) \end{bmatrix} \leq 0 \quad (5)$$

This implies that

$$\sum_{p=0}^{4n-4} v(p)f(p) \geq \sum_{p=0}^{4n-4} v(p)g(p)$$

in which f can be obtained from g by a finite sequence of BCTs. Indeed in such a consideration, (5) may be rewritten as:

$$\begin{aligned} &[f(q + \epsilon + \delta) - g(q + \epsilon + \delta)]v(q + \epsilon + \delta) + [f(q + \epsilon) - g(q + \epsilon)]v(q + \epsilon) \\ &+ [f(q + \delta) - g(q + \delta)]v(q + \delta) + [f(q) - g(q)]v(q) + [f(p + \epsilon + \delta) - g(p + \epsilon + \delta)]v(p + \epsilon + \delta) \\ &+ [f(p + \epsilon) - g(p + \epsilon)]v(p + \epsilon) + [f(p + \delta) - g(p + \delta)]v(p + \delta) + [f(p) - g(p)]v(p) \geq 0 \end{aligned}$$

Second step: For any $r \in [0, 4n - 6] \cap \mathcal{P}$, and for all $v \in V_3$,

$$v(r+2) - v(r+1) - v(r+1) + v(r) \leq 0$$

Consider, for any $r \in [0, 4n - 6] \cap \mathcal{P}$:

$$r+2 = r + \epsilon + \delta; \quad r+1 = r + \epsilon = r + \delta$$

For any $r \in [0, 4n - 6] \cap \mathcal{P}$, it comes:

$$v(r + \epsilon + \delta) - v(r + \epsilon) - v(r + \delta) - v(r) \leq 0$$

Taking recourse on (5), it comes for any $p \in [0, 4n - 7] \cap \mathcal{P}$ and $r \in [0, 4n - 6] \cap \mathcal{P}$:

$$\frac{1}{n} \left[\begin{array}{c} v(r + \epsilon + \delta) - v(r + \epsilon) \\ -v(r + \delta) - v(r) \end{array} \right] + \frac{1}{n} \left[\begin{array}{c} v(q + \epsilon + \delta) - v(q + \epsilon) - v(q + \delta) + v(q) \\ -v(p + \epsilon + \delta) + v(p + \epsilon) + v(p + \delta) - v(p) \end{array} \right] \leq 0$$

This implies that

$$\sum_{p=0}^{4n-4} v(p)f(p) \geq \sum_{p=0}^{4n-4} v(p)g(p)$$

in which f can be obtained from g by a finite sequence of BCTs and progressive transfers.

[(ii) \Rightarrow (iii)] Building on Theorem 3.1, if f is reached from g by a finite sequence of progressive transfers, then

$$\Delta D^2(k) \geq 0 \quad \forall k = 0, \dots, 4n - 5$$

Since

$$\Delta D^3(k) = \sum_{j=k}^{4n-6} \Delta D^2(j)$$

then

$$\Delta D^3(k) \geq 0 \quad \forall k = 0, \dots, 4n - 6$$

Moreover, assume that f is reached from g by a BCT as in Definition 4.1 such that $\delta < \epsilon$ and $q > p + \delta + \epsilon$: We have

$$\Delta D^2(k) = 0 \quad \text{for } k = 0, \dots, p - 1 \quad (\text{A1})$$

$$\Delta D^2(k) = \frac{-1}{n}(k - p + 1) \quad \text{for } k = p, \dots, p + \delta - 1 \quad (\text{A2})$$

$$\Delta D^2(k) = \frac{-\delta}{n} \quad \text{for } k = p + \delta, \dots, p + \epsilon - 1 \quad (\text{A3})$$

$$\Delta D^2(k) = \frac{1}{n}(k - p - \epsilon + 1) - \frac{\delta}{n} \quad \text{for } k = p + \epsilon, \dots, p + \delta + \epsilon - 1 \quad (\text{A4})$$

$$\Delta D^2(k) = 0 \quad \text{for } k = p + \delta + \epsilon, \dots, q - 1 \quad (\text{A5})$$

$$\Delta D^2(k) = \frac{1}{n}(k - q + 1) \quad \text{for } k = q, \dots, q + \delta - 1 \quad (\text{A6})$$

$$\Delta D^2(k) = \frac{\delta}{n} \quad \text{for } k = q + \delta, \dots, q + \epsilon - 1 \quad (\text{A7})$$

$$\Delta D^2(k) = \frac{\delta}{n} - \frac{1}{n}(k - q - \epsilon + 1) \quad \text{for } k = q + \epsilon, \dots, q + \delta + \epsilon - 1 \quad (\text{A8})$$

$$\Delta D^2(k) = 0 \quad \text{for all } k = q + \delta + \epsilon, \dots, 4n - 4 \quad (\text{A9})$$

To check whether $\Delta D^2(4n - 5) \geq 0$, three cases should be studied: For the first two cases, *i.e.* $4n - 5 > q + \delta + \epsilon$ and $4n - 5 = q + \delta + \epsilon$, it is straightforward to verify that $\Delta D^2(4n - 5) = 0$, so $\Delta D^2(4n - 5) \geq 0$. For the third case, *i.e.* $4n - 5 = q + \delta + \epsilon - 1$ we have

$$\Delta D^2(4n - 5) = \frac{\delta}{n} - \frac{1}{n}(4n - 5 - q - \epsilon + 1) = \frac{\delta}{n} - \frac{1}{n}(q + \delta + \epsilon - 1 - q - \epsilon + 1) = 0$$

It follows that $\Delta D^2(4n - 5) \geq 0$ is verified.

Now in order to check whether $\Delta D^3(k) \geq 0$ for all $k \in \{0, \dots, 4n - 6\}$, we proceed by a step-by-step analysis. The first step consists of checking for the sign of $\Delta D^3(k)$ for all $k \in \{q + \delta + \epsilon, \dots, 4n - 6\}$, which is a subset of $\{0, \dots, 4n - 6\}$. The second step consists of checking for the sign of $\Delta D^3(k)$ for all $k \in \{q + \delta, \dots, q + \delta + \epsilon - 1\}$ and so down until the last step, which consists of checking for the sign of $\Delta D^3(k)$ for all $k \in \{0, \dots, p\}$. Let us begin by the *first step*: By (A9), it is straightforward that $\Delta D^3(k) \geq 0$ for $k = q + \delta + \epsilon, \dots, 4n - 6$.

Second step: By (A8),

$$\Delta D^3(k) = \frac{1}{n} \sum_{j=k}^{q+\delta+\epsilon-1} (q + \delta + \epsilon - 1 - j) \quad \text{for } k = q + \epsilon, \dots, q + \delta + \epsilon - 1$$

Then, it is clear that $\Delta D^3(k) \geq 0$ for $k = q + \epsilon, \dots, q + \delta + \epsilon - 1$.

Third step: By (A8) and (A7),

$$\Delta D^3(k) = \frac{1}{n} \sum_{j=q+\epsilon}^{q+\delta+\epsilon-1} (q + \delta + \epsilon - 1 - j) + \frac{\delta}{n} (q + \epsilon - k) \quad \text{for } k = q + \delta, \dots, q + \epsilon - 1$$

As already stated, the first term is nonnegative. Moreover since $\epsilon \geq \delta$ it is clear that the second term is also nonnegative for $k = q + \delta, \dots, q + \epsilon - 1$. Then, $\Delta D^3(k) \geq 0$ for $k = q + \delta, \dots, q + \epsilon - 1$.

Fourth step: By (A8), (A7) and (A6),

$$\Delta D^3(k) = \frac{1}{n} \sum_{j=q+\epsilon}^{q+\delta+\epsilon-1} (q + \delta + \epsilon - 1 - j) + \frac{(\epsilon - \delta)\delta}{n} + \frac{1}{n} \sum_{j=k}^{q+\delta-1} (j - q + 1) \quad \text{for } k = q, \dots, q + \delta - 1$$

Since the last term of the above equation is nonnegative for $k = q, \dots, q + \delta - 1$, it is clear that $\Delta D^3(k) \geq 0$ for $k = q, \dots, q + \delta - 1$.

Fifth step: By (A8), (A7), (A6) and (A5),

$$\Delta D^3(k) = \frac{1}{n} \sum_{j=q+\epsilon}^{q+\delta+\epsilon-1} (q + \delta + \epsilon - 1 - j) + \frac{(\epsilon - \delta)\delta}{n} + \frac{1}{n} \sum_{j=k}^{q+\delta-1} (j - q + 1) \quad \text{for } k = p + \delta + \epsilon, \dots, q - 1$$

It is clear that $\Delta D^3(k) \geq 0$ for $k = p + \delta + \epsilon, \dots, q - 1$.

Sixth step: By (A8), (A7), (A6), (A5) and (A4),

$$\begin{aligned} \Delta D^3(k) &= \frac{1}{n} \sum_{j=q+\epsilon}^{q+\delta+\epsilon-1} (q + \delta + \epsilon - 1 - j) + \frac{(\epsilon - \delta)\delta}{n} + \frac{1}{n} \sum_{j=k}^{q+\delta-1} (j - q + 1) \\ &\quad - \frac{1}{n} \sum_{j=k}^{p+\delta+\epsilon-1} (p + \delta + \epsilon - 1 - j) \quad \text{for } k = p + \epsilon, \dots, p + \delta + \epsilon - 1 \end{aligned}$$

Making a focus on the first sum and the last one in the right-hand side of the above equation,

$$\frac{1}{n} \sum_{j=q+\epsilon}^{q+\delta+\epsilon-1} (q + \delta + \epsilon - 1 - j) - \frac{1}{n} \sum_{j=k}^{p+\delta+\epsilon-1} (p + \delta + \epsilon - 1 - j)$$

For $k = p + \delta + \epsilon - 1$, the expression becomes:

$$0 + \sum_{j=q+\epsilon}^{q+\delta+\epsilon-2} (q + \delta + \epsilon - 1 - j) - 0$$

which is nonnegative. One sub-step further, for $k = p + \delta + \epsilon - 2$, it becomes:

$$\frac{1}{n} + \sum_{j=q+\epsilon}^{q+\delta+\epsilon-3} (q + \delta + \epsilon - 1 - j) - \frac{1}{n}$$

which is nonnegative. This sub-step can be repeated up to $k = p + \epsilon$, and the expression becomes:

$$\frac{1}{n} \sum_{j=q+\epsilon}^{q+\delta+\epsilon-1} (q + \delta + \epsilon - 1 - j) - \frac{1}{n} \sum_{j=p+\epsilon}^{p+\delta+\epsilon-1} (p + \delta + \epsilon - 1 - j) = 0$$

Then $\Delta D^3(k) \geq 0$ for $k = p + \epsilon, \dots, p + \delta + \epsilon - 1$.

Seventh step: By (A8), (A7), (A6), (A5), (A4) and (A3),

$$\Delta D^3(k) = \frac{(\epsilon - \delta)\delta}{n} + \frac{1}{n} \sum_{j=q}^{q+\delta-1} (j - q + 1) - \frac{\delta(p + \epsilon - k)}{n} \text{ for } k = p + \delta, \dots, p + \epsilon - 1$$

Making a focus on the first and the last terms in the right-hand side of the above equation:

$$\frac{(\epsilon - \delta)\delta}{n} - \frac{\delta(p + \epsilon - k)}{n}$$

It is straightforward that $(\epsilon - \delta)\delta - (p + \epsilon - k)\delta \geq 0$ for $k = p + \delta, \dots, p + \epsilon - 1$ and so, $\Delta D^3(k) \geq 0$ for $k = p + \delta, \dots, p + \epsilon - 1$.

Eighth step: By (A8), (A7), (A6), (A5), (A4), (A3) and (A2),

$$\Delta D^3(k) = \frac{1}{n} \sum_{j=q}^{q+\delta-1} (j - q + 1) - \frac{1}{n} \sum_{j=k}^{p+\delta-1} (j - p + 1) \text{ for } k = p, \dots, p + \delta - 1$$

For $k = p + \delta - 1$:

$$\Delta D^3(k) = \frac{\delta}{n} + \frac{1}{n} \sum_{j=q}^{q+\delta-2} (j - q + 1) - \frac{\delta}{n}$$

which is nonnegative. If $\delta \leq 2$, then there is nothing more to show for this step. Otherwise, one sub-step further, for $k = p + \delta - 2$, the expression becomes

$$\Delta D^3(k) = \frac{\delta}{n} + \frac{\delta - 1}{n} + \frac{1}{n} \sum_{j=q}^{q+\delta-3} (j - q + 1) - \frac{\delta}{n} - \frac{\delta - 1}{n}$$

which is nonnegative. This sub-step can be repeated up to $k = p$, and the expression becomes:

$$\Delta D^3(k) = \frac{1}{n} \sum_{j=q}^{q+\delta-1} (j - q + 1) - \frac{1}{n} \sum_{j=p}^{p+\delta-1} (j - p + 1) = 0$$

Then $\Delta D^3(k) \geq 0$ for $k = p, \dots, p + \delta - 1$.

Last step: By (A1), it follows that $\Delta D^3(k) = 0$ for $k = 0, \dots, p$.

Since we have set $\delta < \epsilon$ and $q > p + \delta + \epsilon$, the BCT taken in this proof is not compelling with all the cases of Definition 4.1. All other cases fulfilling $q > p$, $0 < \delta \leq q - p$, and $\delta \leq \epsilon$ follow directly from the proof provided above. To illustrate it, consider $\delta = \epsilon = 1 = q - p$. It comes that:

$$\begin{aligned} \Delta f(k) &= 0 \quad \text{for } k = 0, \dots, p - 1 \\ \Delta f(k) &= \frac{1}{n} \quad \text{for } k = p \\ \Delta f(k) &= -\frac{3}{n} \quad \text{for } k = p + \delta = p + \epsilon = q \\ \Delta f(k) &= \frac{3}{n} \quad \text{for } k = p + \delta + \epsilon = q + \epsilon = q + \delta \\ \Delta f(k) &= -\frac{1}{n} \quad \text{for } k = q + \delta + \epsilon \\ \Delta f(k) &= 0 \quad \text{for } k = q + \delta + \epsilon + 1, \dots, 4n - 4 \end{aligned}$$

Whence:

$$\begin{aligned} \Delta D^2(k) &= 0 \quad \text{for } k = 0, \dots, p - 1 \\ \Delta D^2(k) &= -\frac{1}{n} \quad \text{for } k = p \\ \Delta D^2(k) &= \frac{1}{n} \quad \text{for } k = p + \delta = p + \epsilon = q \\ \Delta D^2(k) &= 0 \quad \text{for } k = p + \delta + \epsilon = q + \epsilon = q + \delta \\ \Delta D^2(k) &= 0 \quad \text{for } k = q + \delta + \epsilon \\ \Delta D^2(k) &= 0 \quad \text{for } k = q + \delta + \epsilon + 1, \dots, 4n - 4 \end{aligned}$$

Thus $\Delta D^2(4n - 5) \geq 0$. Moreover:

$$\begin{aligned} \Delta D^3(k) &= 0 \quad \text{for } k = p + \delta + \epsilon, \dots, 4n - 4 \\ \Delta D^3(k) &= \frac{1}{n} \quad \text{for } k = p + \delta \\ \Delta D^3(k) &= 0 \quad \text{for } k = p \\ \Delta D^3(k) &= 0 \quad \text{for } k = 0, \dots, p - 1 \end{aligned}$$

It follows that $\Delta D^3(k) \geq 0$ for $k = 0, \dots, 4n - 6$.

[(iii) \Rightarrow (i)] This implication follows directly from (4). □

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