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# Some LMIs for the design of dynamic output feedback contractive controllers for a class of nonlinear systems

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**Abstract:** This paper deals with the problem of designing an output feedback controller for the class of linear systems interconnected with a nonlinearity satisfying a monotonicity assumption. The control design objective is to ensure the incremental exponential stability of the closed loop. First, a set of sufficient conditions by relying on a design based on Linear Matrix Inequalities (LMIs) are proposed. The results are then expanded to include in the design a filter to cancel additional (unwanted) harmonics of the steady-state solution. The proposed results are validated via a simplified example of application.

Keywords: Nonlinear systems, contraction, incremental stability, LMI, monotonic, harmonic filtering

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## 1. INTRODUCTION

Contraction theory (Lohmiller and Slotine [1998], Angeli [2002], Forni and Sepulchre [2013], Andrieu et al. [2016], Bullo [2023]) is a valuable tool to provide solutions to several control problems, such as, for example, output regulation (Pavlov et al. [2006], Giaccagli et al. [2022b], Pavlov and Marconi [2008]), multiagent synchronization (Jafarpour et al. [2021], Giaccagli et al. [2024b], Pavlov et al. [2022], DeLellis et al. [2010]), and observer design (Sanfelice and Praly [2011], Yi et al. [2021]). Roughly speaking, a dynamical system is said to be contractive (also called incrementally exponentially stable) if the distance between any two trajectories of the system exponentially shrinks to zero, i.e. trajectories converge to each other.

As such, great interest has been directed into answering the question of how to make a system contractive via feedback. Existing tools can be divided into three main classes. i) Feedback designs for particular classes of systems, such as (incremental) backstepping (Zamani and Tabuada [2011]) and forwarding (Giaccagli et al. [2024a]). These designs provide tractable conditions with the drawback that they apply only to specific classes of systems. ii) Designs achieving similar but subtly different notions than contraction, such as that of convergent systems Pavlov et al. [2006], Rüffer et al. [2013]. In this case, existing tools can be applied, but each case has to be carefully addressed. iii) Data-driven and optimization-based tools, such as Control Contraction Metrics (Manchester and Slotine [2017]) and neural networks-based designs (Tsukamoto et al. [2021]). These tools usually apply to a wide range of systems, with the drawback of requiring the solution of a (possibly online) optimization problem and, commonly, the loss of analytical stability guarantees.

In this work, we consider the problem of designing a contractive feedback controller for linear systems interconnected with a nonlinearity satisfying a monotonicity assumption. As such, our design takes place in the first of the former classes. The interest in such a class of systems is that this description commonly appears in many applications, such that, for example, robotic link manipulators (Wu et al. [2015]), and medical equipment (Reinders et al. [2023]).

In particular, with the objective of deriving a set of practically tractable conditions that do not require a significant case-by-case mathematical burden, we focus on a design whose controller structure can be obtained via a single offline quadratic optimization problem, represented via a set of Linear Matrix Inequalities (LMIs).

Combining LMIs with this class of systems in the contraction framework is not new, and it has been already considered for instance in Waitman et al. [2017] with piecewise affine approximations, in Zhang et al. [2015] in the context of multiagent synchronization, and in [Scherer and Weiland, 2000, Chapter 7] in connection to dissipativity. Our result is grounded on the seminal work of Andrieu and Tarbouriech [2019] and sequentially developed in Giaccagli et al. [2022a, 2023a]. The main difference with respect to the literature is that our design considers purely dynamic output feedback controllers, while the former works mainly focus on analysis tools and state-feedback design.

To do so, we blend the tools introduced in Andrieu and Tarbouriech [2019] to handle the nonlinearity, together with the congruence transformation presented in Scherer et al. [1997]. As a consequence, we derive a set of sufficient conditions for the design of a pure dynamic output feedback controller achieving contraction for the closed loop. Then, we consider the case in which the plant is excited

by a periodic exogenous signal representing disturbances and/or references. As a consequence, if the closed loop is incremental Input-to-State Stable (ISS), then its trajectories will converge to a periodic steady-state solution with the same period, but possibly with higher harmonic content. We provide an extension of our main result, in which we present a set of LMIs for the design of a dynamic controller composed of a filter and a feedback stabilizer such that the closed-loop system is incrementally stable. In particular, the filter is designed to filter out a selected harmonic content of the (periodic) steady-state solution.

The rest of the paper is structured as follows. In Section 2.1 we review the main tools of contraction theory. Then, in Section 2.2, we present the problem statement. The main result is given in Section 3.1. The extension to harmonic filtering is provided in Section 3.2. An illustrative example is given in Section 3.3. Concluding remarks are then drawn in Section 4.

*Notation:* The identity matrix of dimension  $n$  is indicated as  $I_n$ . For any matrix  $A$ , we denote its transpose matrix as  $A^\top$  and, for square matrices,  $\text{He}\{A\}$  is the Hermitian operator  $\text{He}\{A\} := A + A^\top$ . Given two square matrices  $A_1, A_2$ , we indicate with  $\text{blkdiag}\{A_1, A_2\}$  the block-diagonal matrix which has  $A_1$  and  $A_2$  in the main diagonal and 0 everywhere else. We write that  $P \in \mathbb{S}^n$  if it is a  $n \times n$  symmetric matrix  $P = P^\top$ . We write  $P > 0$  ( $\geq 0$ ) if  $P$  is symmetric and positive definite (semi-positive definite), that is,  $x^\top P x > 0$  ( $\geq 0$ ) for all  $x \neq 0$ . We write  $P < 0$  ( $\leq 0$ ) if  $-P$  is positive definite (semi-positive definite). For symmetric matrices, the symbol  $\star$  indicates the symmetric component. We indicate with  $|\cdot|$  the vector Euclidean norm. Given two column vectors  $x_1, x_2$ , we write  $x = (x_1, x_2)$  to indicate  $x = [x_1^\top \ x_2^\top]^\top$ .

## 2. PRELIMINARIES

### 2.1 Contraction theory

Consider a system of the form

$$\dot{\chi} = f(\chi, t) \quad (1)$$

with  $\chi \in \mathbb{R}^n$  and the vector field  $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  to be sufficiently smooth. For the sake of simplicity, let trajectories be defined for all positive times and let  $\mathcal{X}(\chi_0, t, t_0)$  be the solution to (1) with initial condition  $(\chi_0, t_0)$  evaluated at time  $t \geq t_0$ .

*Definition 1.* (Incremental stability). We say that system (1) is (globally<sup>1</sup>) incrementally (exponentially) stable if there exist  $k, \lambda > 0$  such that

$$|\mathcal{X}(\chi_1, t, t_0) - \mathcal{X}(\chi_2, t, t_0)| \leq k |\chi_1 - \chi_2| \exp(-\lambda(t - t_0)) \quad (2)$$

for all  $t \geq t_0$  and all  $\chi_1, \chi_2 \in \mathbb{R}^n$ .

A sufficient condition<sup>2</sup> for system (1) to be incrementally stable, is the existence of a norm such that the flow generates trajectories for which the distance associated with the norm is monotonically decreasing forward in time.

<sup>1</sup> In this work we focus on global results, with the awareness that our conditions can be relaxed when considering compact forward invariant sets.

<sup>2</sup> Also necessary in case  $f$  has bounded first and second derivative in its first argument, see Andrieu et al. [2016].

Such a condition, corresponding to the Lie derivative of a 2-tensor to be uniformly decreasing, can be expressed via a matrix inequality according to the following theorem (see e.g. Lohmiller and Slotine [1998], Andrieu et al. [2016] and many others for a proof).

*Theorem 1.* Consider system (1). If there exist a  $C^1$  matrix function  $\mathcal{P} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{S}^n$  taking symmetric values and three positive real numbers  $\underline{p}, \bar{p}, \lambda > 0$  such that<sup>3</sup>

$$\underline{p}I_n \leq \mathcal{P}(\chi, t) \leq \bar{p}I_n \quad (3a)$$

$$L_f \mathcal{P}(\chi, t) \leq -2\lambda \mathcal{P}(\chi, t), \quad (3b)$$

for all  $(\chi, t) \in \mathbb{R}^n \times \mathbb{R}$ , then the system is (globally) incrementally exponentially stable, that is, contractive.

### 2.2 Problem statement

In this work, we consider systems of the form

$$\dot{x} = Ax + Bu + G\varphi(z) + Rw(t) \quad (4a)$$

$$y = Cx + D\varphi(z) + Qw(t) \quad (4b)$$

$$z = Hx \quad (4c)$$

with  $x \in \mathbb{R}^n$  being the state,  $u \in \mathbb{R}^m$  being a control input,  $y \in \mathbb{R}^p$  being a (measurable) output,  $z \in \mathbb{R}^q$  being a (possibly unmeasurable) linear combination of the state,  $\varphi : \mathbb{R}^q \rightarrow \mathbb{R}^q$ ,  $w$  being a (state-independent) piece-wise continuous and locally Lipschitz exogenous signal representing disturbances and/or references, and  $A, B, G, R, C, D, Q, H$  being constant matrices of suitable dimension. In particular, we assume that the nonlinearity  $\varphi$  satisfies the following monotonicity assumption.

*Assumption 1.* (Monotonic). Consider system (4). The nonlinearity  $\varphi$  is a  $C^1$  mapping<sup>4</sup>. Moreover, there exists a known  $\Gamma \in \mathbb{S}^q$  positive definite  $\Gamma = \Gamma^\top > 0$  such that

$$0 \leq \frac{\partial \varphi}{\partial z}(z) + \frac{\partial \varphi^\top}{\partial z}(z) \leq \Gamma \quad (5)$$

for all  $z \in \mathbb{R}^q$ .

Here we will consider the design of a (dynamic) output feedback controller of the form

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y \end{aligned} \quad (6)$$

with  $x_c \in \mathbb{R}^{n_c}$  such that the closed loop with extended state  $\chi = (x, x_c)$  associating (4) and (6) is defined by

$$\dot{\chi} = \mathcal{A}\chi + \mathcal{G}\varphi(\mathcal{H}\chi) + \mathcal{R}w \quad (7a)$$

and

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} A + BD_c C & BC_c \\ B_c C & A_c \end{pmatrix}, \quad \mathcal{G} = \begin{pmatrix} G + BD_c D \\ B_c D \end{pmatrix} \\ \mathcal{H} &= (H \quad 0), \quad \mathcal{R} = ((R + BD_c Q)^\top \quad (B_c Q)^\top)^\top. \end{aligned} \quad (7b)$$

<sup>3</sup> The symbol  $L_f \mathcal{P}(\chi, t)$  indicates the Lie derivative of the tensor  $\mathcal{P}$  along  $f$  defined as  $L_f \mathcal{P}(\chi, t) := \mathcal{P}(\chi, t) \frac{\partial f}{\partial \chi}(\chi, t) + \frac{\partial f^\top}{\partial \chi}(\chi, t) \mathcal{P}(\chi, t) + \mathfrak{d}_f \mathcal{P}(\chi, t)$ , where  $\mathfrak{d}_f \mathcal{P}(\chi, t) := \lim_{h \rightarrow 0} \frac{\mathcal{P}(\mathcal{X}(\chi, t+h, t), t) - \mathcal{P}(\chi, t)}{h} + \frac{\partial \mathcal{P}}{\partial t}(\chi, t)$  with coordinates  $(L_f \mathcal{P}(\chi, t))_{i,j} = \sum_k \left[ 2\mathcal{P}_{ik}(\chi, t) \frac{\partial f_k}{\partial \chi_j}(\chi, t) + \frac{\partial \mathcal{P}_{ij}}{\partial \chi_k}(\chi, t) f_k(\chi, t) \right] + \frac{\partial \mathcal{P}_{ij}}{\partial t}(\chi, t)$ .

<sup>4</sup> Some of the results in this paper (in particular, Proposition 1) can be generalized in case  $\varphi = \varphi(z, t)$ , with the mapping being piece-wise continuous and locally Lipschitz in its second argument.

The objective of this work is to derive a set of sufficient conditions to design a controller of the form (6) such that the closed-loop dynamics (7) is incrementally exponentially stable.

To do so, we would like to make use of Theorem 1. However, finding a suitable matrix function  $\mathcal{P}$  satisfying (3) might be practically difficult, as this is a matrix inequality that has to be verified in an infinite set of points (for all  $(\chi, t)$ ). To provide an algorithmic design that is easy to implement, we focus on the case in which the metric  $\mathcal{P}$  in Theorem 1 is a constant positive definite matrix, i.e.,  $\mathcal{P} \in \mathbb{S}^{n+n_c}$ ,  $\mathcal{P} = \mathcal{P}^\top > 0$ . In other words, we trade a set of possible solutions with an easy-to-compute design. As a consequence, this will allow us to represent our conditions via a set of LMIs. As a drawback, however, the conditions that we will derive will be, in general, only sufficient and not necessary. Before presenting the main contributions of the paper, let us recall the following theorem, [Giaccagli et al., 2023a, Proposition 2], for which the notations are adapted to the closed-loop dynamics (7).

*Theorem 2.* [Giaccagli et al., 2023a, Proposition 2] Consider system (7) and assume that  $\varphi$  satisfies Assumption 1. If there exist a positive definite matrix  $\mathcal{P} \in \mathbb{S}^{n+n_c}$  and a positive scalar  $\lambda$ , such that the following inequality holds

$$\begin{pmatrix} \mathcal{P}\mathbf{A} + \mathbf{A}^\top\mathcal{P} + 2\lambda\mathcal{P} & \mathcal{P}\mathbf{G} + \mathbf{H}^\top \\ \mathbf{G}^\top\mathcal{P} + \mathbf{H} & -4\Gamma^{-1} \end{pmatrix} \leq 0, \quad (8)$$

then system (7) is incrementally exponentially stable.

### 3. MAIN RESULTS

#### 3.1 Dynamic output feedback contractive design

In this section, we deal with the first result of the paper. The controller design can be obtained by the solution of a Bilinear Matrix Inequality (BMI), or more specifically, a hyperparameterized LMI, which is an LMI by fixing a scalar hyperparameter<sup>5</sup>  $\lambda > 0$  representing the contraction rate in (1), i.e.  $\lambda = \lambda$ . From now on, to help the reader, we indicate with **bold letters** the LMI optimization variables. For the time being, we focus on the case in which the dimension of the controller is the same as the plant, i.e.  $n_c = n$ . This framework allows us to consider and adapt the congruence transformation presented in Scherer et al. [1997]. We have the following result.

*Proposition 1.* Consider system (4) and let Assumption 1 hold. Assume there exist matrices  $\mathbf{X} \in \mathbb{S}^n$ ,  $\mathbf{Y} \in \mathbb{S}^n$ ,  $\mathbf{A} \in \mathbb{R}^n$ ,  $\mathbf{B} \in \mathbb{R}^{n \times p}$ ,  $\mathbf{C} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{D} \in \mathbb{R}^{m \times p}$  and a positive real number  $\lambda > 0$  such that (18) holds. Then, for all nonsingular matrices  $M, N$  satisfying

$$MN^\top = I_n - \mathbf{X}\mathbf{Y}, \quad (9)$$

the system (4) in closed loop with (6) where

$$\begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} = \begin{pmatrix} N & \mathbf{Y}\mathbf{B} \\ 0 & I_n \end{pmatrix}^{-1} \times \begin{pmatrix} \mathbf{A} - \mathbf{Y}\mathbf{A}\mathbf{X} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} M^\top & 0 \\ \mathbf{C}\mathbf{X} & I_n \end{pmatrix}^{-1} \quad (10)$$

<sup>5</sup> The bilinearity follows from the fact that the right-hand-side of (3b) contains the product of two unknowns: the contraction rate  $\lambda$  and the metric  $\mathcal{P}$ . In case one aims not to have such a bilinearity, the BMI can be rewritten as a pure LMI by noticing that (3b) can be substituted with  $L_f\mathcal{P}(\chi, t) \leq -2\sigma I_n$  for  $\sigma = \lambda p / \bar{p}$  thanks to (3a).

is globally incrementally exponentially stable.

*Proof.* Let inequalities (18) hold. Inequality (18b) implies that  $\mathbf{X}$  and  $\mathbf{Y}$  are positive definite and thus nonsingular. By taking the Schur's complement of (18b), it yields that  $\mathbf{X} - \mathbf{Y}^{-1} > 0$ , which is nonsingular. The product of two nonsingular matrices being nonsingular,  $I_n - \mathbf{X}\mathbf{Y} = -(\mathbf{X} - \mathbf{Y}^{-1})\mathbf{Y}$  is nonsingular. Therefore, there always exist square nonsingular matrices  $M, N$  such that (9) holds. Let

$$\Pi_1 := \begin{pmatrix} \mathbf{X} & I_n \\ M^\top & 0 \end{pmatrix}, \quad (11)$$

and note that  $\Pi_1$  is nonsingular since  $M$  is so. Therefore, we define the metric  $\mathcal{P}$  as

$$\mathcal{P} := \Pi_1^{-\top} \begin{pmatrix} \mathbf{X} & I_n \\ I_n & \mathbf{Y} \end{pmatrix} \Pi_1^{-1}. \quad (12)$$

In light of (18b), we have that  $\mathcal{P}$  is symmetric and positive definite:  $\mathcal{P} = \mathcal{P}^\top > 0$ . We would like to show that if (18a) holds, then this is also the case of inequality (8). For this purpose, we use a nonlinear change of variable depending on  $\mathbf{X}$  and  $\mathbf{Y}$ , and the choice for  $M$  and  $N$ , that is:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}\mathbf{A}\mathbf{X} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} N & \mathbf{Y}\mathbf{B} \\ 0 & I_n \end{pmatrix} \times \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} M^\top & 0 \\ \mathbf{C}\mathbf{X} & I_n \end{pmatrix}. \quad (13)$$

Note that the latter relation can be inverted when  $\mathbf{X}$ ,  $\mathbf{Y}$ ,  $M$  and  $N$  are known, to result in relation (10). Thanks to simple calculus and the relation (12), we obtain the following relations:

$$\Pi_1^\top \lambda \mathcal{P} \Pi_1 = \lambda \begin{pmatrix} \mathbf{X} & I_n \\ I_n & \mathbf{Y} \end{pmatrix}, \quad (14)$$

$$\Pi_1^\top \mathcal{P} \mathbf{A} \Pi_1 = \begin{pmatrix} \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{C} & \mathbf{A} + \mathbf{B}\mathbf{D}\mathbf{C} \\ \mathbf{A} & \mathbf{Y}\mathbf{A} + \mathbf{B}\mathbf{C} \end{pmatrix}, \quad (15)$$

$$\Pi_1^\top \mathcal{P} \mathbf{G} = \begin{pmatrix} \mathbf{G} + \mathbf{B}\mathbf{D}\mathbf{D} \\ \mathbf{Y}\mathbf{G} + \mathbf{B}\mathbf{D} \end{pmatrix}, \quad (16)$$

$$\Pi_1^\top \mathcal{H} = \begin{pmatrix} \mathbf{X}\mathbf{H}^\top \\ \mathbf{H}^\top \end{pmatrix}. \quad (17)$$

By identifying these terms in the blocks in (18a), multiplying inequality (18a) at right by  $\text{blkdiag}\{\Pi_1^{-1}, I_q\}$  and at left by its transpose leads to inequality (8) in Theorem 2. By invoking Theorem 2, the closed-loop system (7) is then contractive. And this concludes the proof.  $\square$

*Remark 1.* In the formulation of Proposition 1, we kept the bilinear formulation with the hyperparameter  $\lambda$  instead of having a pure LMI (see footnote 5). This because  $\lambda$  represents the contraction rate in (3), i.e.  $\lambda = \lambda$ . As such, the control engineer can fix a priori the  $\lambda$  representing the speed of convergence of trajectories depending on the desired performances (related to the requirements of the control task), and then check if the LMIs have a solution. Alternatively, in order to maximize the value  $\lambda$ , one can rely on quasi-convex Generalized Eigenvalue Minimization LMI tools available in common software, such as Matlab.

*Remark 2.* Inequality (18b) may lead to an issue of ill-conditioning of  $I_n - \mathbf{X}\mathbf{Y}$  appearing in Equation (9). To avoid ill-conditioning, it may be possible to modify Inequality (18b) by following the discussion in [Scherer et al., 1997, Section IV.B].



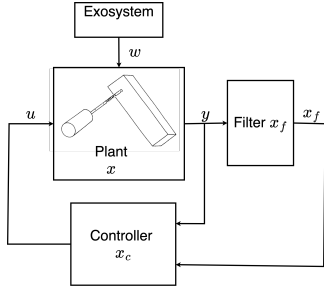


Fig. 1. Block scheme of the closed loop. The link's Figure is taken from [Spong, 1987, Fig. 1]

We let  $y = \theta_\ell$  be the measured output. This system can be represented in the form (4) where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{k}{I_m} & -\frac{C_f}{I_m} & \frac{k}{I_m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{I_\ell} & 0 & -\frac{k}{I_\ell} + \frac{mgh}{I_m} & \rho \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \frac{\kappa\tau}{I_m} \\ 0 \\ 0 \end{pmatrix},$$

$$G^\top = \begin{pmatrix} 0 & 0 & 0 & -\frac{mgh}{I_m} \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \quad D = 0, \quad \varphi(z) = \sin(z) + \rho z$$

where  $k$  is the torsional spring constant,  $I_m$  is the motor inertia,  $C_f$  is the viscous friction coefficient,  $I_\ell$  is the link inertia,  $m$  is the pointer mass,  $h$  is the link's length,  $g$  is the gravitational acceleration constant,  $\kappa\tau$  is the amplifier gain and  $\rho$  is a free-to-choose parameter. In particular, for any  $\rho > 1$ , the nonlinearity  $\varphi$  satisfies Assumption 1 with  $0 \leq 2(\rho - 1) \leq \text{He}\{\frac{\partial\varphi}{\partial z}(z)\} \leq \Gamma = 2(\rho + 1)$ . We let the system be affected by a sinusoidal exosignal  $w(t) = \sin(t)$  where we select  $R^\top = (0.01 \ 0 \ 0 \ 0)$ ,  $Q = 1$ . In particular, we first design a filter  $x_f$  rejecting the 3<sup>rd</sup> and 5<sup>th</sup> harmonic of the exogenous signal, i.e. we define (20) where we choose

$$\Phi = \text{blkdiag} \left\{ \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix} \right\}, \quad \Psi^\top = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}.$$

Note that  $(\Phi, \Psi)$  is controllable. By selecting the parameters as in [Wu et al., 2015, eq. (35)-(36)] the LMIs (18) admit a solution with e.g.  $\rho = 1.5$  and the hyperparameter  $\lambda = 0.1$ . The block scheme of the closed loop is shown in Fig. 1. Similarly, Figure 2 shows the behavior of the output  $y$  of the system with respect to three different initial conditions. The trajectories of the system converge to a single trajectory which is  $T$ -periodic which is not a pure sinusoidal but contains higher order harmonics. This is highlighted in Figure 3. To conclude, the spectral analysis of the steady-state response is shown in Fig. 4. As it is possible to see, the harmonics related to the 3<sup>rd</sup> and 5<sup>th</sup>-harmonics become negligible with respect to the fundamental one. Due to the numerical implementation and to the fact that the incremental property is an asymptotic one, and therefore the Fourier transform has been computed on a shifted finite time-window, they are not precisely canceled.

#### 4. CONCLUSION

We presented a set of parameterized LMI conditions to design an output feedback controller for a linear system coupled with a monotonic nonlinearity to be incrementally stable in closed loop. Then, we provided an extension to cancel out a desired harmonic content of the output,

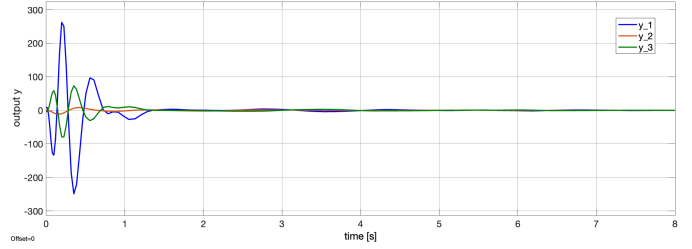


Fig. 2. Plot of the link angle  $\theta_\ell$  for three different random sets of initial conditions.

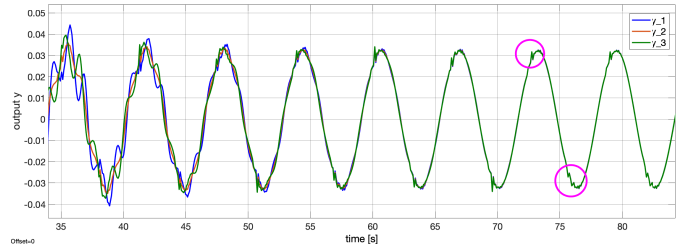


Fig. 3. Zoom of the link angle  $\theta_\ell$  for three different random sets of initial conditions. The trajectories of the system converge to a single trajectory which is  $T$ -periodic which is not a pure sinusoidal but contains higher order harmonics (highlighted with purple circles).

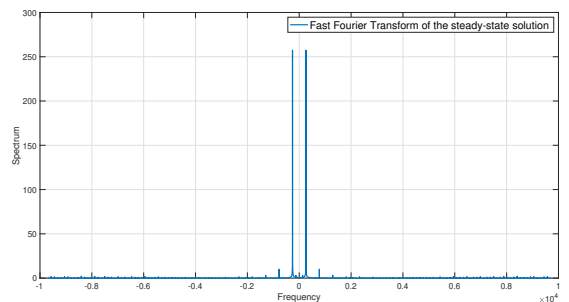


Fig. 4. Fast Fourier Transform of the steady-state output.

when the plant is excited by a periodic exosignal. We validated our result via simulation on a flexible joint manipulator affected by a sinusoidal exogenous signal. Future perspectives will include: i) the case in which the nonlinearity  $\varphi$  is not differentiable, using, for instance, the tools in Bullo [2023]; ii) the introduction of saturation in the controller and related anti-windup problems (Gomes da Silva Jr and Tarbouriech [2005]), iii) the use of dynamic controllers with state dimension higher than the plant one,  $n_c > n$ , to generalize the results and (possibly) add feasibility (see e.g. Boyd and Vandenberghe [2004]), iv) the introduction of more robustness guarantees, such as a tunable incremental gain (Manchester and Slotine [2018]).

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