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On Non-uniqueness of Phase Retrieval in Multidimensions

by Roman G. Novikov and Tianli Xu

Abstract: We give a large class of examples of non-uniqueness for the phase retrieval problem in multidimensions. Our examples include the case of functions with strongly disconnected compact support.

Keywords: Fourier transform, phase retrieval problem, disconnected support, non-uniqueness.

AMS subject classification: 42A38, 35R30

1. Introduction

The phase retrieval problem consists in finding a function $v : \mathbb{R}^d \rightarrow \mathbb{C}$ from the magnitude $|\hat{v}|$ of its Fourier transform

$$\hat{v}(p) = Fv(p) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ipx} v(x) dx, p \in \mathbb{R}^d. \quad (1)$$

This problem naturally arises in quantum mechanics, optics, and related areas such as electron tomography and X-ray imaging; see, for example, [1] - [10] and references therein.

In general, many different functions have the same Fourier modulus. These different solutions can be obtained by multiplying $|\hat{v}|$ by measurable complex-valued functions with modulus one and taking the inverse Fourier transform; see, for example, [3].

When v is compactly supported, the degree of ambiguity is reduced. In particular, for $d = 1$, all solutions with compact support could be obtained from any one of them by flipping (conjugating) non-real zeros of its Fourier transform extended by analyticity to the complex plane; see [10], [4].

When v is a sum of functions with sufficiently disconnected compact supports, the degree of ambiguity is further reduced. In particular, for $d = 1$, this ambiguity is completely described in [3]. Roughly speaking, in this case, the phase retrieval problem almost always has essentially a unique solution.

In addition, it is also mentioned in the literature that for functions with compact support, the degree of non-uniqueness of phase retrieval is further reduced in dimension $d \geq 2$, in general, and for the case of sufficiently disconnected support in particular; see [3].

Moreover, the recent important work [7] suggests an efficient numerical phase retrieval algorithm for functions with sufficiently disconnected compact support. This algorithm works very well numerically (at least, for $d = 2$) and possible non-trivial non-uniqueness is not even discussed in [7].

Recall that the non-uniqueness in phase retrieval for compactly supported v with possible additional assumptions is non-trivial (and of interest) if it does not reduce to the functions $v_{\alpha,y}$ and $\tilde{v}_{\alpha,y}$ associated to v , where

$$v_{\alpha,y}(x) = e^{i\alpha} v(x - y) \text{ and } \tilde{v}_{\alpha,y}(x) = e^{i\alpha} \overline{v(-x + y)}, \alpha \in \mathbb{R}, y \in \mathbb{R}^d, \quad (2)$$

where bar denotes the complex conjugation; see, for example, [2].

Nevertheless, in the present article, we construct a large class of non-trivial examples of non-uniqueness for phase retrieval in multidimensions. Then we also give, in particular, interesting non-trivial examples of non-uniqueness in phase retrieval for functions with strongly disconnected compact support in multidimensions. These results are given as Theorems 1 and 2 in Section 2.

Thus, in order to have complete uniqueness in phase retrieval for functions v even with strongly disconnected compact support, and even modulo associated functions in dimension $d \geq 2$, additional a priori information is necessary. In connection with natural theoretical and numerical results in this direction, see, for example, [1], [5], [8] and references therein.

A preliminary version of this article corresponds to the preprint [9].

2. The main results

First, we construct a large class of non-trivial examples of non-uniqueness for phase retrieval in dimension $d \geq 2$.

Let $f_j, g_j \in L^2(\mathbb{R})$, $\text{supp } f_j, \text{supp } g_j \subseteq [-\epsilon, \epsilon]$, $\epsilon > 0$, $|\hat{f}_j|^2 = |\hat{g}_j|^2 \neq 0$ on \mathbb{R} , $j = 1, \dots, N$, where $\hat{\cdot}$ stands for the one-dimensional Fourier transform.

Here, f_j and g_j are constructed as follows, using results of [10] on non-uniqueness in phase retrieval in dimension $d = 1$.

Let

$$I_j \text{ denotes the set of non-real zeros of } \hat{f}_j, \text{ where each zero is repeated according to its multiplicity.} \quad (3)$$

As in [10], we assume that

$$\hat{g}_j(z) = \exp(i\alpha_j + i\beta_j z) \prod_{\zeta \in I'_j} \frac{1 - z/\bar{\zeta}}{1 - z/\zeta} \hat{f}_j(z), \quad z \in \mathbb{C}, \quad j = 1, \dots, N, \quad (4)$$

where α_j, β_j are two real constants and

$$I'_j \subset I_j, \quad I'_j \neq \emptyset, \quad I'_j \neq I_j, \quad I'_j \cap \overline{I'_j} = \emptyset, \quad I'_j \neq I_j \setminus \overline{I_j}, \quad (5)$$

where bar denotes the complex conjugation. Note that α_j in (4) are arbitrary, whereas $|\beta_j|$ are sufficiently small (maybe just zeros), so that $\text{supp } g_j \subseteq [-\epsilon, \epsilon]$.

For simplicity, we assume that I'_j are finite.

Next, we define f, g as

$$f = F^{-1} \hat{f}, \quad \hat{f}(p) = \prod_{j=1}^N \hat{f}_j(\omega_j \cdot p), \quad p \in \mathbb{R}^d, \quad (6)$$

$$g = F^{-1} \hat{g}, \quad \hat{g}(p) = \prod_{j=1}^N \hat{g}_j(\omega_j \cdot p), \quad p \in \mathbb{R}^d, \quad (7)$$

where f_j, g_j are as in formulas (3) - (5), $\omega_j \in \mathbb{S}^{d-1}$, and $\omega_j \neq \pm \omega_k$ if $j \neq k$.

Let

$$B_r = \{x \in \mathbb{R}^d : |x| \leq r\}, \quad r > 0. \quad (8)$$

Theorem 1: Let f, g be defined by (6), (7). Then:

- (i) $\text{supp } f, \text{supp } g \subseteq B_r$, where $r = N\epsilon$,
- (ii) $|\hat{f}|^2 = |\hat{g}|^2 \neq 0$ on \mathbb{R}^d ,
- (iii) $f \neq g$ in the sense of distributions on \mathbb{R}^d , moreover, f and g are not associated functions in the sense of formulas (2),

(iv) in addition, if $N \geq d$ and there are d linearly independent ω_j in (6), (7), then $f, g \in L^2(B_r)$.

Theorem 1 is of interest for $d \geq 2$, whereas it reduces to known results for $d = 1$.

Theorem 1 is proved in Section 3, using, in particular, some complex analysis in \mathbb{C}^d .

Next, we give non-trivial examples of non-uniqueness in phase retrieval for functions with strongly disconnected compact support in multidimensions.

We consider complex-valued functions v on \mathbb{R}^d of the form

$$v = \sum_{k=1}^n v_k, \quad \text{supp } v_k \subset D_k, \quad v_k \neq 0, \quad (9)$$

where

$$D_k \text{ are open convex bounded domains in } \mathbb{R}^d, \quad \text{dist}(D_i, D_j) \geq r > 0 \text{ for } i \neq j. \quad (10)$$

Here, $\text{dist}(\mathcal{A}, \mathcal{B})$ denotes the distance between sets \mathcal{A} and \mathcal{B} in \mathbb{R}^d .

Let

$$N_\epsilon(U) = \{x \in \mathbb{R}^d : \text{dist}(x, U) < \epsilon\}, \quad \epsilon > 0, \quad U \subset \mathbb{R}^d; \quad (11)$$

$$u_1 * u_2(x) = \int_{\mathbb{R}^d} u_1(x-y)u_2(y)dy, x \in \mathbb{R}^d, \quad (12)$$

where u_1, u_2 are test functions on \mathbb{R}^d .

Theorem 2: Let f, g be two complex-valued functions on \mathbb{R}^d such that $\text{supp } f, \text{supp } g \subseteq B_\delta, |\hat{f}|^2 = |\hat{g}|^2 \not\equiv 0$ on \mathbb{R}^d , but f and g are not associated functions in the sense of formulas (2). Let v be as in (9), (10), where $r > 2\delta$. Let $v_f = f * v, v_g = g * v$. Then:

- (i) v_f, v_g are of the form (9), (10) with $N_\delta(D_k)$ in place of D_k and $r_\delta = r - 2\delta$ in place of r ;
- (ii) $|\hat{v}_f|^2 = |\hat{v}_g|^2 \not\equiv 0$ on \mathbb{R}^d ,
- (iii) $v_f \neq v_g$, moreover, v_f and v_g are not associated functions in the sense of formulas (2).

Theorem 2 is proved in Section 4.

Note that Theorem 1 gives a large class of possible functions f, g for Theorem 2.

In particular, one can consider Theorem 2 assuming that $f, g \in L^2(B_\delta), v_k \in L^2(D_k)$.

In addition, Theorem 2 is of interest even when all v_k in (9) are Dirac delta functions, i.e.,

$$v_k(x) = C_k \delta(x - y_k), y_k \in D_k, C_k \in \mathbb{C}, k = 1, \dots, n. \quad (13)$$

Theorem 2, for $n \geq 2$, gives an interesting class of non-trivial examples of non-uniqueness in phase retrieval for functions with strongly disconnected compact support in multidimensions, taking also into account Theorem 1.

Theorem 2 is of some interest even when $n = 1$ in (9).

The non-uniqueness in phase retrieval for functions with strongly disconnected support given by Theorems 1 and 2 is already of interest when $N = 1$ in (6), (7) and $v_k \in L^2(D_k)$.

3. Proof of Theorem 1

Recall that

$$F^{-1}(\phi_1 \phi_2) = (2\pi)^{-d} F^{-1} \phi_1 * F^{-1} \phi_2, \quad (14)$$

$$(2\pi)^d F(u_1 * u_2) = F u_1 F u_2, \quad (15)$$

where F is defined by (1), ϕ_1, ϕ_2, u_1, u_2 are test functions on \mathbb{R}^d , and $*$ denotes the convolution defined by (12).

Recall also that if $\text{supp } u_1 \subseteq \mathcal{U}_1, \text{supp } u_2 \subseteq \mathcal{U}_2$, where $\mathcal{U}_1, \mathcal{U}_2$ are closed bounded sets in \mathbb{R}^d , then

$$\text{supp } u_1 * u_2 \subseteq \mathcal{U}_1 + \mathcal{U}_2, \quad (16)$$

where

$$\mathcal{U}_1 + \mathcal{U}_2 = \{x + y : x \in \mathcal{U}_1, y \in \mathcal{U}_2\}. \quad (17)$$

In addition,

$$B_{r_1}(a_1) + B_{r_2}(a_2) = B_{r_1+r_2}(a_1 + a_2), \quad (18)$$

where

$$B_r(a) = \{x \in \mathbb{R}^d : |x - a| \leq r\}, a \in \mathbb{R}^d, r > 0. \quad (19)$$

Item (i) follows from formulas (14), (16), (18) with $a_1 = a_2 = 0$.

Item (ii) follows from definitions (6), (7) and the property that $|\hat{f}_j|^2 = |\hat{g}_j|^2 \not\equiv 0$ on $\mathbb{R}, j = 1, \dots, N$.

The proof of item (iv) is as follows.

Without restriction of generality, we can assume that $\omega_1, \dots, \omega_d$ are linearly independent.

Let

$$\phi(p) = \prod_{j=1}^d \hat{f}_j(\omega_j \cdot p), \quad \psi(p) = \prod_{j=1}^d \hat{g}_j(\omega_j \cdot p), p \in \mathbb{R}^d. \quad (20)$$

Then using that $f_j, g_j \in L^2(\mathbb{R})$ and using the change of variables $\tilde{p}_j = \omega_j \cdot p, j = 1, \dots, d$, we get

$$\phi, \psi \in L^2(\mathbb{R}^d). \quad (21)$$

We also have that

$$\hat{f}_j(\omega_j \cdot p), \hat{g}_j(\omega_j \cdot p) \text{ are analytic and bounded on } \mathbb{R}^d, j = 1, \dots, N, \quad (22)$$

since $f_j, g_j \in L^2(\mathbb{R})$ and $\text{supp } f_j, \text{supp } g_j \subseteq [-\epsilon, \epsilon]$.

Definitions of \hat{f}, \hat{g} in (6), (7) and formulas (20) - (22) imply that $\hat{f}, \hat{g} \in L^2(\mathbb{R}^d)$. Thus, item (iv) is proved. To prove item (iii), we first rewrite formulas (2) in the Fourier domain:

$$\hat{v}_{\alpha,y}(p) = \hat{v}(p) \exp(i\alpha + iy p), \quad p \in \mathbb{R}^d, \quad (23)$$

$$\hat{\hat{v}}_{\alpha,y}(p) = \overline{\hat{v}(p)} \exp(i\alpha + iy p), \quad p \in \mathbb{R}^d. \quad (24)$$

We suppose that

$$I'_j = \{\zeta_{j,k} : k = 1, \dots, N_j\}, \quad (25)$$

where N_j is the cardinality of I'_j .

From (4), (6), (7), we have that

$$\hat{g}(p) = \hat{f}(p) \exp \left[i \sum_{j=1}^N \alpha_j + i \sum_{j=1}^N \beta_j (\omega_j \cdot p) \right] \prod_{j=1}^N \prod_{k=1}^{N_j} \frac{1 - (\omega_j \cdot p) / \bar{\zeta}_{j,k}}{1 - (\omega_j \cdot p) / \zeta_{j,k}}, \quad p \in \mathbb{C}^d. \quad (26)$$

We define the following complex hyperplanes in \mathbb{C}^d :

$$Z_{j,k} = \{p \in \mathbb{C}^d : \omega_j \cdot p = \zeta_{j,k}\}, \quad \zeta_{j,k} \in I'_j, \quad (27)$$

$$\overline{Z_{j,k}} = \{p \in \mathbb{C}^d : \omega_j \cdot p = \bar{\zeta}_{j,k}\}, \quad \zeta_{j,k} \in I'_j. \quad (28)$$

Thus, we have that

$$\dim_{\mathbb{C}} Z_{j,k} = \dim_{\mathbb{C}} \overline{Z_{j,k}} = d - 1, \quad j = 1, \dots, N, \quad k = 1, \dots, N_j. \quad (29)$$

In addition, using our assumption in (5) that $I'_j \cap \overline{I'_j} = \emptyset$, we get

$$Z_{j,k} \cap \overline{Z_{j',k'}} = \emptyset, \quad j = 1, \dots, N, \quad k, k' = 1, \dots, N_j. \quad (30)$$

Moreover, due to the condition that $\omega_j \neq \pm \omega_k$ if $j \neq k$, we get

$$\dim_{\mathbb{C}} (Z_{j,k} \cap \overline{Z_{j',k'}}) = d - 2, \quad j \neq j', \quad k = 1, \dots, N_j, \quad k' = 1, \dots, N_{j'}. \quad (31)$$

Therefore,

$$Z_{j,k} \not\subseteq \bigcup_{j'=1}^N \bigcup_{k'=1}^{N_{j'}} \overline{Z_{j',k'}}, \quad j = 1, \dots, N, \quad k = 1, \dots, N_j. \quad (32)$$

It then follows from (32) that

$$\prod_{j=1}^N \prod_{k=1}^{N_j} \frac{1 - (\omega_j \cdot p) / \bar{\zeta}_{j,k}}{1 - (\omega_j \cdot p) / \zeta_{j,k}} \neq \exp(i\tilde{\alpha} + i\tilde{y} p), \quad p \in \mathbb{C}^d, \quad \forall \tilde{\alpha} \in \mathbb{R}, \quad \forall \tilde{y} \in \mathbb{R}^d, \quad (33)$$

since the poles of the left-hand side in (33) on $Z_{j,k}$ are not compensated by zeros in the numerator.

Formulas (26), (33) imply that f, g are not associated functions in the sense of formula (23).

To prove that f, g are not associated in the sense of formula (24), we proceed as follows. We rewrite (26) as

$$\hat{g}(p) = \overline{\hat{f}(p)} \exp \left[i \sum_{j=1}^N \alpha_j + i \sum_{j=1}^N \beta_j (\omega_j \cdot p) \right] \left(\hat{f}(p) / \overline{\hat{f}(p)} \right) \prod_{j=1}^N \prod_{k=1}^{N_j} \frac{1 - (\omega_j \cdot p) / \bar{\zeta}_{j,k}}{1 - (\omega_j \cdot p) / \zeta_{j,k}}, \quad p \in \mathbb{R}^d. \quad (34)$$

According to Hadamard's factorization theorem, we have

$$\hat{f}_j(z) = \exp(\gamma_j + \eta_j z) R_j(z) \prod_{\zeta \in I_j} (1 - z/\zeta) \exp(z/\zeta), \quad z \in \mathbb{C}, \quad (35)$$

where γ_j and η_j are two complex constants, $R_j(z)$ corresponds to the product with respect to the real zeros of \hat{f}_j and is real-valued on \mathbb{R} ; see [10] and references therein.

We get, using formulas (6), (35), that

$$\hat{f}(p) = \overline{\hat{f}(p)} \exp \left[i \sum_{j=1}^N \mu_j + i \sum_{j=1}^N \tau_j (\omega_j \cdot p) \right] \prod_{j=1}^N \prod_{\zeta \in I_j} \frac{1 - (\omega_j \cdot p)/\zeta}{1 - (\omega_j \cdot p)/\bar{\zeta}}, \quad p \in \mathbb{R}^d, \quad (36)$$

$$\mu_j = 2 \operatorname{Im} \gamma_j, \quad \tau_j = 2 \left(\operatorname{Im} \eta_j - \sum_{\zeta \in I_j} \frac{\operatorname{Im} \zeta}{|\zeta|^2} \right), \quad (37)$$

where Im denotes the imaginary part and $|\cdot|$ denotes the magnitude.

We then get, using formulas (34) and (36), that

$$\hat{g}(p) = \overline{\hat{f}(p)} h(p) \exp \left[i \sum_{j=1}^N (\alpha_j + \mu_j) + i \sum_{j=1}^N (\beta_j + \tau_j) (\omega_j \cdot p) \right], \quad p \in \mathbb{R}^d, \quad (38)$$

$$h(p) = \prod_{j=1}^N \left(\prod_{\zeta \in I_j} \frac{1 - (\omega_j \cdot p)/\zeta}{1 - (\omega_j \cdot p)/\bar{\zeta}} \prod_{\zeta \in I'_j} \frac{1 - (\omega_j \cdot p)/\bar{\zeta}}{1 - (\omega_j \cdot p)/\zeta} \right), \quad p \in \mathbb{R}^d. \quad (39)$$

Note that

$$\prod_{\zeta \in I_j} \frac{1 - (\omega_j \cdot p)/\zeta}{1 - (\omega_j \cdot p)/\bar{\zeta}} = \prod_{\zeta \in I_j \setminus \bar{I}_j} \frac{1 - (\omega_j \cdot p)/\zeta}{1 - (\omega_j \cdot p)/\bar{\zeta}}. \quad (40)$$

We then get from (39) and (40) that

$$h(p) = \prod_{j=1}^N \left(\prod_{\zeta \in I_j \setminus \bar{I}_j} \frac{1 - (\omega_j \cdot p)/\zeta}{1 - (\omega_j \cdot p)/\bar{\zeta}} \prod_{\zeta \in I'_j} \frac{1 - (\omega_j \cdot p)/\bar{\zeta}}{1 - (\omega_j \cdot p)/\zeta} \right), \quad p \in \mathbb{R}^d. \quad (41)$$

Using the assumptions $I'_j \neq I_j$, $I'_j \cap \bar{I}_j = \emptyset$, $I'_j \neq I_j \setminus \bar{I}_j$ in (5), in a similar way as for formula (33), we have that

$$h(p) \neq \exp(i\tilde{\alpha} + i\tilde{y}p), \quad p \in \mathbb{C}^d, \quad \forall \tilde{\alpha} \in \mathbb{R}, \quad \forall \tilde{y} \in \mathbb{R}^d. \quad (42)$$

More precisely: if $\zeta = \zeta_{j,k} \in I'_j$ but $\zeta \notin I_j \setminus \bar{I}_j$, then the pole of h on $Z_{j,k}$ is not compensated by zeros in the numerator; if $\zeta = \zeta_{j,k} \in I_j \setminus \bar{I}_j$ but $\zeta \notin I'_j$, then the pole of h on $\bar{Z}_{j,k}$ is not compensated by zeros in the numerator.

It follows from (38), (41) and (42) that f and g are not associated functions in the sense of formula (24).

This completes the proof of Theorem 1.

4. Proof of Theorem 2

Note that

$$v_f = f * v = \sum_{k=1}^n f * v_k, \quad v_g = g * v = \sum_{k=1}^n g * v_k. \quad (43)$$

Due to formula (15), we have also that

$$\hat{v}_f = (2\pi)^{-d} \hat{f} \hat{v}, \quad \hat{v}_g = (2\pi)^{-d} \hat{g} \hat{v}. \quad (44)$$

Recall also that

$$\operatorname{mes}(\{p \in \mathbb{R}^d : \hat{u}(p) = 0\}) = 0, \quad (45)$$

where \hat{u} is the Fourier transform of a non-zero compactly supported function u on \mathbb{R}^d , and mes denotes the Lebesgue measure in \mathbb{R}^d .

Item (ii) follows from (44), (45) and the assumptions that f, g, v are compactly supported, $|\hat{f}|^2 = |\hat{g}|^2 \neq 0$ on \mathbb{R}^d

and $v \neq 0$ on \mathbb{R}^d .

Item (iii) follows from (23), (24), (44), (45) and the assumptions that f, g, v are compactly supported, \hat{f}, \hat{g} are not associated functions in the sense of formulas (2) and $v \neq 0$ on \mathbb{R}^d .

The properties that $\text{supp } f, \text{supp } g \subseteq B_\delta, \text{supp } v_k \subset D_k$ and formula (16) imply that

$$\text{supp } f * v_k \subset N_\delta(D_k), \quad \text{supp } g * v_k \subset N_\delta(D_k). \quad (46)$$

In addition, since $\text{dist}(D_i, D_j) \geq r > 2\delta, i \neq j$, we have that

$$\text{dist}(N_\delta(D_i), N_\delta(D_j)) \geq r_\delta, \quad i \neq j. \quad (47)$$

Item (i) follows from formulas (43), (46) and (47).

This completes the proof of Theorem 2.

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