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# An original approximation involving Pi, the Euler-Mascheroni constant and the Catalan constant

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## Abstract

In this note, we propose a simple numerical approximation linking three famous constants: Pi, the Euler-Mascheroni constant and the Catalan constant. An approximation of Pi as a function of the other two constants is derived. Discussions are given, also from an integral point of view.

**Keywords:** numerical approximation, Pi, Euler-Mascheroni constant, Catalan constant

**Mathematics subject classification (2000):** 11J99

## 1 Introduction

In mathematics, certain constants appear frequently in various branches, from geometry and analysis to number theory. Three particularly well-known constants are Pi, denoted  $\pi$ , the Euler-Mascheroni constant, denoted  $\gamma$ , and the Catalan constant, denoted  $G$ . Each of them plays a fundamental role in various mathematical expressions and theorems.

The constant  $\pi$  can be defined as the ratio of the circumference of a circle to its diameter. The following approximation applies:  $\pi \approx 3.14159$ . More specifically,  $\pi$  is known to be an irrational number, meaning that it cannot be expressed as a simple fraction and its decimal expansion is non-repeating and infinite. Obviously,  $\pi$  appears in a wide range of mathematical contexts, including trigonometry, calculus and geometry, as well as in various formulas involving areas, volumes and integrals. We can poetically call it "the queen of constants" to emphasize its central role and elegance in many mathematical applications. More information about  $\pi$  and its diverse approximations can be found in [4].

The Euler-Mascheroni constant  $\gamma$  is defined as the limiting difference between the truncated harmonic series and the natural logarithm:

$$\gamma = \lim_{n \rightarrow +\infty} \left( \sum_{k=1}^n \frac{1}{k} - \ln(n) \right) \approx 0.57721.$$

Although  $\gamma$  occurs naturally in many areas of analysis and number theory, including integrals, sums and special functions, it is still an open question whether  $\gamma$  is rational or irrational. We may refer to [6]. An interesting simple approximation linking  $\pi$  and  $\gamma$  is

$$\frac{\pi}{2e} - \gamma \approx 0.00064$$

(followed by: 80099939279983485345015739457185182661756422843114440860327 ...),

24 where  $e = \exp(1) \approx 2.718281$ . We refer to [8].

25 The Catalan constant  $G$  is less well known than  $\pi$  and  $\gamma$ . It is defined by the following  
26 infinite series:

$$G = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.91596.$$

27 It appears in the evaluation of integrals and sums, and has some applications in number  
28 theory. Like  $\gamma$ , it is not known whether  $G$  is rational or irrational. More details can be  
29 found in [2], [7] and [3].

30 These constants therefore have a clear theoretical meaning. They are involved in a wide  
31 variety of simple and complex series and integral formulas. See [1] and [5]. Understanding  
32 their properties and relationships remains a subject of mathematical investigation. However,  
33 to the best of our knowledge, there is no known simple formula that directly links  $\pi$ ,  $\gamma$  and  
34  $G$  in a single expression. More precisely, these constants occur in different mathematical  
35 contexts, and while they may appear together in some advanced mathematical relationships  
36 (e.g. series, integrals, or special functions), there is no simple formula that links them.

37 In this note, we do not provide such a "utopian" (or "highly unexpected") formula.  
38 However, we present a new approximation formula which combines relative simplicity, based  
39 mainly on a ratio of polynomial, power and logarithmic functions (there is no fractional  
40 exponent), with an interesting degree of precision. We then derive an original approximation  
41 to  $\pi$  as a function of  $\gamma$  and  $G$ . Some aspects of these approximations are then discussed,  
42 including an integral representation of the main term. These points will be discussed in  
43 detail in the following sections.

## 44 2 Results

45 The result below shows a formula for a function of  $\pi$ ,  $\gamma$  and  $G$  that can be "reasonably"  
46 approximated to 0.

**Proposition 2.1** *We have*

$$\frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)} - 1 \approx 0.0000748$$

(followed by: 647483592751621492905401684617601186496092312159680451228 ...).

**Proof (numerical).** Keeping 7 decimals for convenience, we have

$$\pi \approx 3.1415926, \quad \gamma \approx 0.5772156, \quad G \approx 0.9159655.$$

A basic calculus gives

$$\frac{3.1415926 \times 0.9159655 + 2 \times 0.5772156 \log(0.5772156/0.9159655)}{2(0.9159655^2 + 0.5772156^2)} - 1$$

$$\approx 1.0000749 - 1 \approx 0.0000749.$$

47 The desired result is obtained with more decimals in the approximation of  $\pi$ ,  $\gamma$  and  $G$ , which  
48 completes the numerical proof.  $\square$

49

So we have  $[\pi G + 2\gamma \log(\gamma/G)]/[2(G^2 + \gamma^2)] \approx 1.00007$ , and also, with a bit less precision, we can say that

$$\pi G + 2\gamma \log\left(\frac{\gamma}{G}\right) \approx 2(G^2 + \gamma^2).$$

50 Both quantities are approximately equal to 2.344 (the fourth decimals are slightly different).

51 Given Proposition 2.1, an exotic approximation of  $\pi$  as a function of  $\gamma$  and  $G$  is given  
52 below.

**Proposition 2.2** *We have*

$$\pi \approx \frac{2}{G} \left[ G^2 + \gamma^2 - \gamma \log\left(\frac{\gamma}{G}\right) \right] \approx 3.1414$$

(followed by: 010431635160844409294581615460625775547598940207747766211547...).

53 **Proof (numerical).** The proof follows from Proposition 2.1;  $\pi$  can be isolated to give the  
54 desired result.  $\square$

55

56 To the best of our knowledge, this result is one of the few attempts to obtain a simple  
57 numerical approximation of  $\pi$  as a function of  $\gamma$  and  $G$ . This is of pure mathematical interest.  
58 This approximation is not claimed to be very sharp; there are much more accurate numerical  
59 approximations (but these do not involve known constants). See [4].

60 Some of the secondary results in relation to the above statements are set out below.

### 61 3 Discussion

An alternative extended form of the main term in Proposition 2.1 is

$$\frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)} - 1 = \frac{1}{G^2 + \gamma^2} \left[ \frac{\pi}{2} G - \gamma \log(G) + \gamma \log(\gamma) \right] - 1$$

$$= \frac{\pi G}{2(G^2 + \gamma^2)} - \frac{\gamma \log(G)}{G^2 + \gamma^2} + \frac{\gamma \log(\gamma)}{G^2 + \gamma^2} - 1 \quad (\approx 0.0000748).$$

62 We also establish an integral formula for this term, which has the property of not depending  
63 on  $\pi$ , and use a simple linear function depending on  $\gamma$  and  $G$ , as developed below.

**Proposition 3.1** *We have*

$$\frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)} - 1 = \int_0^{+\infty} \frac{1}{(\gamma x + G)(1 + x^2)} dx - 1.$$

**Proof.** Let us introduce the following function:

$$f(x) = \frac{1}{G^2 + \gamma^2} \left\{ \gamma \log \left[ \frac{\gamma x + G}{\sqrt{1 + x^2}} \right] + G \arctan(x) \right\}.$$

Then we have

$$\begin{aligned} f'(x) &= \frac{1}{G^2 + \gamma^2} \left\{ \gamma \frac{[\gamma\sqrt{1+x^2} - (\gamma x + G)x/\sqrt{1+x^2}]/(1+x^2)}{(\gamma x + G)/\sqrt{1+x^2}} + G \frac{1}{1+x^2} \right\} \\ &= \frac{1}{G^2 + \gamma^2} \left\{ \gamma \frac{\gamma(1+x^2) - (\gamma x + G)x}{(\gamma x + G)(1+x^2)} + G \frac{1}{1+x^2} \right\} \\ &= \frac{1}{G^2 + \gamma^2} \left\{ \gamma \frac{\gamma - Gx}{(\gamma x + G)(1+x^2)} + G \frac{\gamma x + G}{(\gamma x + G)(1+x^2)} \right\} \\ &= \frac{1}{G^2 + \gamma^2} \left[ \frac{G^2 + \gamma^2}{(\gamma x + G)(1+x^2)} \right] \\ &= \frac{1}{(\gamma x + G)(1+x^2)}. \end{aligned}$$

So we have

$$\begin{aligned} \int_0^{+\infty} \frac{1}{(\gamma x + G)(1+x^2)} dx &= \int_0^{+\infty} f'(x) dx = [f(x)]_{x \rightarrow 0}^{x \rightarrow +\infty} \\ &= \lim_{x \rightarrow +\infty} \left[ \frac{1}{G^2 + \gamma^2} \left\{ \gamma \log \left[ \frac{\gamma x + G}{\sqrt{1+x^2}} \right] + G \arctan(x) \right\} \right] \\ &\quad - \lim_{x \rightarrow 0} \left[ \frac{1}{G^2 + \gamma^2} \left\{ \gamma \log \left[ \frac{\gamma x + G}{\sqrt{1+x^2}} \right] + G \arctan(x) \right\} \right] \\ &= \frac{1}{G^2 + \gamma^2} \left[ \gamma \log(\gamma) + G \frac{\pi}{2} - \gamma \log(G) \right] \\ &= \frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)}. \end{aligned}$$

64 Adding  $-1$  on both sides completes the proof. □

65

66 This result is mathematically elegant; by adopting an integral point of view, it shows  
67 how  $\pi$  appears in the resulting term and that the complex expression of  $\gamma$  and  $G$  comes from  
68 the simple linear term  $\gamma x + G$ .

We also find that

$$\int_0^{+\infty} \frac{1}{(\gamma x + G)(1+x^2)} dx \approx \frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)} \approx 1.0000748.$$

69 Note that, since the value 1 is "almost" reached, the exotic positive function  $f(x) = 1/[(\gamma x +$   
70  $G)(1+x^2)]$  depending on  $\gamma$  and  $G$  is "almost" a probability density function (but of course  
71 not in full rigour).

72 These results are related to the approximation in Proposition 2.1, but do not fully explain  
73 its sharpness. There is again room for research on this, perhaps with other technical tools,  
74 such as series and geometric techniques.

As an additional note, during our research, we found the following approximation:

$$\frac{\pi\gamma - 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)} - 1 \approx 0.000899$$

(followed by: 2679125876616077049534587844985871180473649586060425796497...).

75 So the main term is close to that in Proposition 2.1, but with two changes in the numerator:  
76 a  $\gamma$  instead of a  $G$  and a minus instead of a plus in the factor of the logarithmic term,  
77 and with a less precise approximation. However, the similarity in form is intriguing and  
78 the precision remains acceptable. For this reason, it must be regarded as a complementary  
79 remark.

80 In conclusion, we have tried to link three particularly well known constants,  $\pi$ ,  $\gamma$  and  $G$ ,  
81 through a new approximation formula that combines relative simplicity and precision.

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