



HAL
open science

An original approximation involving Pi, the Euler-Mascheroni constant and the Catalan constant

Christophe Chesneau

► **To cite this version:**

Christophe Chesneau. An original approximation involving Pi, the Euler-Mascheroni constant and the Catalan constant. 2024. hal-04692308v1

HAL Id: hal-04692308

<https://hal.science/hal-04692308v1>

Preprint submitted on 9 Sep 2024 (v1), last revised 21 Sep 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An original approximation involving Pi, the Euler-Mascheroni constant and the Catalan constant

Christophe Chesneau

Department of Mathematics, LMNO,
University of Caen-Normandie, 14032 Caen, France.
e-mail: christophe.chesneau@gmail.com

Abstract

In this note, we propose a simple numerical approximation linking three famous constants: Pi, the Euler-Mascheroni constant and the Catalan constant. An approximation of Pi as a function of the other two constants is derived. Discussions are given, also from an integral point of view.

Keywords: Pi, Euler-Mascheroni constant, Catalan constant

Mathematics subject classification (2000): 11J99

1 Introduction

In mathematics, certain constants appear frequently in various branches, from geometry and analysis to number theory. Three particularly well-known constants are Pi, denoted π , the Euler-Mascheroni constant, denoted γ , and the Catalan constant, denoted G . Each of them plays a fundamental role in various mathematical expressions and theorems.

The constant π can be defined as the ratio of the circumference of a circle to its diameter. The following approximation applies: $\pi \approx 3.14159$. More specifically, π is known to be an irrational number, meaning that it cannot be expressed as a simple fraction and its decimal expansion is non-repeating and infinite. Obviously, π appears in a wide range of mathematical contexts, including trigonometry, calculus and geometry, as well as in various formulas involving areas, volumes and integrals. We can poetically call it "the queen of constants" to emphasize its central role and elegance in many mathematical applications. More information about π and its diverse approximations can be found in [3].

The Euler-Mascheroni constant γ is defined as the limiting difference between the truncated harmonic series and the natural logarithm:

$$\gamma = \lim_{n \rightarrow +\infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln(n) \right) \approx 0.57721.$$

Although γ occurs naturally in many areas of analysis and number theory, including integrals, sums and special functions, it is still an open question whether γ is rational or irrational. We may refer to [4]. An interesting simple approximation linking π and γ is

$$\frac{\pi}{2e} - \gamma \approx 0.00064$$

(followed by: 80099939279983485345015739457185182661756422843114440860327 ...),

24 where $e = \exp(1) \approx 2.718281$. We refer to [6].

25 The Catalan constant G is less well known than π and γ , but it is important in combi-
 26 natorics and number theory. It is defined by the following infinite series:

$$G = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.91596.$$

27 It appears in the evaluation of integrals and sums, and in connection with combinatorial
 28 structures such as alternating permutations and lattice paths. Like γ , it is not known
 29 whether G is rational or irrational. More detail can be found in [1], [5] and [2].

30 These constants therefore have a clear theoretical meaning. Understanding their proper-
 31 ties and relationships remains a subject of mathematical investigation. However, to the best
 32 of our knowledge, there is no known simple formula that directly links π , γ and G in a single
 33 expression. More precisely, these constants occur in different mathematical contexts, and
 34 while they may appear together in some advanced mathematical relationships (e.g. series,
 35 integrals, or special functions), there is no simple formula that links them.

36 In this note, we do not give such a "utopian" formula. However, we present a new approx-
 37 imation formula which combines relative simplicity, based mainly on a ratio of polynomial,
 38 power and logarithmic functions (there is no fractional exponent), with an interesting degree
 39 of precision. We then derive an original approximation to π as a function of γ and G . Some
 40 aspects of these approximations are then discussed, including an integral representation of
 41 the main term. These points will be discussed in detail in the following sections.

42 2 Results

43 The result below shows a formula for a function of π , γ and G that can be "reasonably"
 44 approximated to 0.

Proposition 2.1 *We have*

$$\frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)} - 1 \approx 0.0000748$$

(followed by: 647483592751621492905401684617601186496092312159680451228 ...).

Proof (numerical). Keeping 7 decimals for convenience, we have

$$\pi \approx 3.1415926, \quad \gamma \approx 0.5772156, \quad G \approx 0.9159655.$$

A basic calculus gives

$$\frac{3.1415926 \times 0.9159655 + 2 \times 0.5772156 \log(0.5772156/0.9159655)}{2(0.9159655^2 + 0.5772156^2)} - 1$$

$$\approx 1.0000749 - 1 \approx 0.0000749.$$

The desired result is obtained with more decimals in the approximation of π , γ and G , which completes the numerical proof. \square

47

48 An exotic approximation of π as a function of γ and G is given below.

Proposition 2.2 *We have*

$$\pi \approx \frac{1}{G} \left[2(G^2 + \gamma^2) - 2\gamma \log\left(\frac{\gamma}{G}\right) \right] \approx 3.1414$$

(followed by: 010431635160844409294581615460625775547598940207747766211547...).

49 **Proof (numerical).** The proof follows from Proposition 2.1; π can be isolated to give the
50 desired result. \square

51

52 To the best of our knowledge, this result is one of the few attempts to obtain a simple
53 numerical approximation of π as a function of γ and G . This is of pure mathematical interest.
54 This approximation is not claimed to be very sharp; there are much more accurate numerical
55 approximations (but these do not involve known constants). See [3].

56 Some of the secondary results in relation to the above statements are set out below.

57 3 Discussion

An alternative extended form of the main term in Proposition 2.1 is

$$\frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)} - 1 = \frac{1}{G^2 + \gamma^2} \left[\frac{\pi}{2} G - \gamma \log(G) + \gamma \log(\gamma) \right] - 1$$

$$= \frac{\pi G}{2(G^2 + \gamma^2)} - \frac{\gamma \log(G)}{G^2 + \gamma^2} + \frac{\gamma \log(\gamma)}{G^2 + \gamma^2} - 1 \quad (\approx 0.0000748).$$

58 We also establish an integral formula for this term, which has the property of not depending
59 on π , and use a simple linear function depending on γ and G , as developed below.

Proposition 3.1 *We have*

$$\frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)} - 1 = \int_0^{+\infty} \frac{1}{(\gamma x + G)(1 + x^2)} dx - 1.$$

Proof. Let us introduce the following function:

$$f(x) = \frac{1}{G^2 + \gamma^2} \left\{ \gamma \log \left[\frac{\gamma x + G}{\sqrt{1 + x^2}} \right] + G \arctan(x) \right\}.$$

Then we have

$$\begin{aligned}
f'(x) &= \frac{1}{G^2 + \gamma^2} \left\{ \gamma \frac{[\gamma\sqrt{1+x^2} - (\gamma x + G)x/\sqrt{1+x^2}]/(1+x^2)}{(\gamma x + G)/\sqrt{1+x^2}} + G \frac{1}{1+x^2} \right\} \\
&= \frac{1}{G^2 + \gamma^2} \left\{ \gamma \frac{\gamma(1+x^2) - (\gamma x + G)x}{(\gamma x + G)(1+x^2)} + G \frac{1}{1+x^2} \right\} \\
&= \frac{1}{G^2 + \gamma^2} \left\{ \gamma \frac{\gamma - Gx}{(\gamma x + G)(1+x^2)} + G \frac{\gamma x + G}{(\gamma x + G)(1+x^2)} \right\} \\
&= \frac{1}{G^2 + \gamma^2} \left[\frac{G^2 + \gamma^2}{(\gamma x + G)(1+x^2)} \right] \\
&= \frac{1}{(\gamma x + G)(1+x^2)}.
\end{aligned}$$

So we have

$$\begin{aligned}
\int_0^{+\infty} \frac{1}{(\gamma x + G)(1+x^2)} dx &= \int_0^{+\infty} f'(x) dx = [f(x)]_{x \rightarrow 0}^{x \rightarrow +\infty} \\
&= \lim_{x \rightarrow +\infty} \left[\frac{1}{G^2 + \gamma^2} \left\{ \gamma \log \left[\frac{\gamma x + G}{\sqrt{1+x^2}} \right] + G \arctan(x) \right\} \right] \\
&\quad - \lim_{x \rightarrow 0} \left[\frac{1}{G^2 + \gamma^2} \left\{ \gamma \log \left[\frac{\gamma x + G}{\sqrt{1+x^2}} \right] + G \arctan(x) \right\} \right] \\
&= \frac{1}{G^2 + \gamma^2} \left[\gamma \log(\gamma) + G \frac{\pi}{2} - \gamma \log(G) \right] \\
&= \frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)}.
\end{aligned}$$

60 Adding -1 on both sides completes the proof. □

61

62 This result is mathematically elegant; by adopting an integral point of view, it shows
63 how π appears in the resulting term and that the complex expression of γ and G comes from
64 the simple linear term $\gamma x + G$.

We also find that

$$\int_0^{+\infty} \frac{1}{(\gamma x + G)(1+x^2)} dx \approx \frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)} \approx 1.0000748.$$

65 Note that, since the value 1 is "almost" reached, the exotic function $f(x) = 1/[(\gamma x + G)(1 +$
66 $x^2)]$ depending on γ and G is "almost" a probability density function (but of course not in
67 full rigour).

68 These results are related to the approximation in Proposition 2.1, but do not fully explain
69 its sharpness. There is again room for research on this, perhaps with other technical tools,
70 such as series and geometric techniques.

As an additional note, during our research, we found the following approximation:

$$\frac{\pi\gamma - 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)} - 1 \approx 0.000899$$

(followed by: 2679125876616077049534587844985871180473649586060425796497 ...).

71 So the main term is close to that in Proposition 2.1, but with two changes in the numerator:
72 a γ instead of a G and a minus instead of a plus in the factor of the logarithmic term,
73 and with a less precise approximation. However, the similarity in form is intriguing and
74 the precision remains acceptable. For this reason, it must be regarded as a complementary
75 remark.

76 In conclusion, we have tried to link three particularly well known constants, π , γ and G ,
77 through a new approximation formula that combines relative simplicity and precision.

78 References

- 79 [1] Arfken, G.B.: *Mathematical Methods for Physicists*, 3rd edn. Academic Press, New York
80 (1985).
- 81 [2] Bradley, D.M. Representations of Catalan's constant, CiteSeerX: 10.1.1.26.1879 (2001)
82 <http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.26.1879>
- 83 [3] Brown, P.R.: Approximations of e and π : an exploration. *International Journal of Math-*
84 *ematical Education in Science and Technology*, 48(sup1), S30-S39 (2017).
- 85 [4] Havil, J., *Gamma : Exploring Euler's Constant*, Princeton University Press, Princeton
86 (2003).
- 87 [5] Jameson, G., Lord, N.: Integrals evaluated in terms of Catalan's constant, *Math. Gazette*
88 101, 38-49 (2017).
- 89 [6] Weisstein, E.W.: OEIS A086056, online, Apr. 18, (2006).