



**HAL**  
open science

## Two-Frame Groups with Scalings

Paul Chauchat, Axel Barrau, Silvère Bonnabel

► **To cite this version:**

Paul Chauchat, Axel Barrau, Silvère Bonnabel. Two-Frame Groups with Scalings. 8th IFAC Workshop on Lagrangian and Hamiltonian Methods for Non Linear Control, Jun 2024, Besancon, France. hal-04691569

**HAL Id: hal-04691569**

**<https://hal.science/hal-04691569v1>**

Submitted on 8 Sep 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Two-Frame Groups with Scalings

Paul Chauchat \* Axel Barrau \*\* Silvère Bonnabel \*\*\*

\* Aix-Marseille Univ, CNRS, LIS, Marseille, France

paul.chauchat@lis-lab.fr

\*\* OFFROAD, Alfortville, France axel@offroad.works

\*\*\* Centre for Robotics, MINES Paris, PSL Research University,

Paris, France silvere.bonnabel@mines-paristech.fr

---

**Abstract:** We consider problems pertaining to the navigation and localization of mobile mechanical systems such as robots or crafts. We focus on the recently introduced theory of two-frame systems, within the field of invariant Kalman filtering for state estimation of dynamical systems. We show that we can build a two-frame group (TFG) structure not only using rotations as its basic building block, as has been done so far, but also rotations *and* scalings. This then makes the TFG a generalization of the group of similarity transformations  $\text{Sim}(d)$ . This allows for new examples to fit into the theory of two-frame systems. Namely, we consider the localization of a car equipped with odometry based on wheels having unknown radius, and inertial navigation where the accelerometers are affected by an unknown scale factor.

*Keywords:* Navigation, Localization, Nonlinear systems, Estimation and filtering, Lie groups, Geometrical tools, Kalman filtering

---

## 1. INTRODUCTION

Lie group embeddings are now accepted as powerful tools in navigation and mobile robotics, to estimate the position and orientation of a system, see e.g., Chirikjian (2011); Barfoot (2017); Sola et al. (2018). Most notably, the use of  $SO(3)$ ,  $SE(2)$  and  $SE(3)$  was pioneered by several authors in robotics, e.g. Mahony et al. (2008); Park et al. (2008); Wolfe et al. (2011); Long et al. (2013); Barfoot and Furgale (2014). These groups allowed designing non-linear filters with strong theoretical properties Mahony et al. (2008); Barrau and Bonnabel (2017, 2018), relying on the notion of autonomous error. However, practical navigation problems usually require estimating additional parameters, and their addition to the state usually comes at the price of losing some theoretical properties, but not all Lisus et al. (2023), and very satisfactory behavior is observed in practice Barrau (2015); van Der Laan et al. (2020); Hartley et al. (2020); Pavlasek et al. (2021). Another recent line of research proposes new nonlinear filters for navigation based on Lie groups, through the notion of Equivariant Filtering van Goor et al. (2023); van Goor and Mahony (2023).

Novel Lie groups have been introduced to broaden the class of systems for which properties can be guaranteed, such as  $SE_2(3)$  for IMU navigation introduced in Barrau and Bonnabel (2017); Barrau (2015), as well as the general group  $SE_K(d)$  for SLAM Barrau and Bonnabel (2015a); Barrau (2015). Recently, the first systematic way of designing systems which fit into the invariant framework was introduced, relying on the Two Frame Group (TFG) structure Barrau and Bonnabel (2022). It encompasses the previously published groups, including  $SE_2(3)$  and allows including the accelerometer bias.

This paper proposes a novel application of the TFG framework, to encompass the presence of a scale factor in the dynamics. In doing so, the paper starts with the problem of estimating the attitude and position of a mobile system thanks to odometry and position measurements, but where a scale factor on the odometry is introduced. We show that the TFG framework encompasses the Lie group of similarity transformations  $\text{Sim}(2)$ , e.g., Chirikjian (2011) and may be applied to this example. Then, we show how the method is easily extended to accelerometer measurements subject to an unknown scale factor. To do so, the TFG allows for

the introduction of a novel Lie group of multivector similitudes  $\text{Sim}_2(2)$ , in the way we first introduced  $SE_2(3)$  in Barrau (2015); Barrau and Bonnabel (2017). Estimating a scale factor using geometric tools was already done for the odometry in the context of visual navigation Engel et al. (2014); Bourmaud and Megret (2015); Mur-Artal et al. (2015), using  $\text{Sim}(d)$ . A geometric filter was also proposed to handle scaled accelerometer measurements Martin et al. (2010). But none were shown to possess the theoretical properties that the invariant filtering framework guarantees, namely having autonomous error equations, state-independent Jacobians, the log-linearity property, see Barrau and Bonnabel (2018), amongst others.

The paper is structured as follows. The goal of the first sections is to introduce in a rather tutorial way the two-frame systems framework, before extending it. Section 2 introduces simple preliminary estimation problems, while Section 3 recalls how some can be shown to fit into the standard TFG framework. Section 4 presents a new application of the TFG to handle the scale factor, recovering  $\text{Sim}(2)$ . It also contains an application to inertial navigation with unknown accelerometer scale factor, introducing  $\text{Sim}_2(2)$ . As a byproduct of having to implement invariant extended Kalman filters, we provide a formula for the exponential map of this novel Lie group. This allows to address all problems of the first sections within the invariant filtering framework. Finally, Section 5 contains numerical simulations that illustrate the benefit of this novel group based on invariant filtering to handle scale factors in state estimation problems.

## 2. CONSIDERED NAVIGATION PROBLEMS

A simple class of state estimation problems in the field of navigation aim at finding the position vector  $x_n \in \mathbb{R}^d$  and the orientation matrix  $R_n \in SO(d)$  of a mobile system given some sensor information. For the ease of understanding, we focus on 2D navigation, thus  $x_n \in \mathbb{R}^2$ , and  $R_n$  is the rotation matrix of angle  $\theta_n$ :

$$R_n = \begin{pmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix}. \quad (1)$$

We focus here on wheeled systems, or non-holonomic car Barrau and Bonnabel (2017), for which  $x_n$  denotes the middle point of the rear wheels axle.

Assuming that the robot can measure its angular increments  $\Omega_n \in SO(2)$  (e.g. thanks to a one-axis gyroscope) and its linear velocity  $U_n \in \mathbb{R}^2$ , its discrete-time dynamics write:

$$R_{n+1} = R_n \Omega_n, \quad x_{n+1} = x_n + R_n U_n, \quad (2)$$

Note that this model can also apply to visual odometry, where  $\Omega_n$  and  $U_n$  would encode the relative pose between two instants.

## 2.2 Adding a Parameter: The Scale Factor

In practice, the linear velocity may only be known up to a scale factor, for instance as soon as the wheel radius is unknown and subject to slow variations over time, due to varying tire pressure for instance. Therefore the discrete-time dynamics of such a mobile wheeled robot then write:

$$R_{n+1} = R_n \Omega_n, \quad x_{n+1} = x_n + s_n R_n U_n, \quad s_{n+1} = s_n \quad (3)$$

where the scalar  $s_n > 0$  is this scaling factor.

## 2.3 Replacing Odometry with Accelerometer

In many cases, the robot may be equipped with an accelerometer instead of an odometer, leading to an inertial navigation problem. This allows to relax the assumption that the wheels roll without slip, that underlies the two previous models. Since acceleration is then measured instead of velocity, inertial navigation problems also require extended state variables, by including the speed. Moreover, accelerometers are not perfect, due to biases (which we do not consider herein) and a scale factor. Thus, although seemingly unrelated to (3), the dynamical model of a 2D vehicle equipped with bias-free gyrometers and accelerometers having an unknown scale factor, writes:

$$\begin{aligned} R_{n+1} &= R_n \Omega_n, & v_{n+1} &= v_n + s_n R_n U_n + \mathbf{d}_n, \\ p_{n+1} &= p_n + \Delta t v_n, & s_{n+1} &= s_n \end{aligned} \quad (4)$$

where  $\mathbf{d}_n$  is the gravity vector, see Barrau and Bonnabel (2019).

## 2.4 Considered Estimation Problems

For all dynamical systems considered before, we assume that the system has access to position measurements, for instance through GNSS in case of a car driving outdoors, and which provides the world-fixed frame position measurements

$$y_n = h(g_n, x_n) := x_n \in \mathbb{R}^2, \quad (5)$$

where we recall that  $g_n$  denotes the orientation  $R_n$  or both the orientation and the scaling  $s_n$  depending on the problem. The goal of state estimation is then to use the dynamical model and those measurements to reconstruct the unknown (unmeasured directly) state  $(R_n, x_n)$  or  $(R_n, s_n, x_n)$  if there is a scale factor to be estimated.

Although seemingly different, we show in this work that the geometric tool of Two-Frame Groups allows unifying these problems and bringing them into the invariant filtering framework. In turn, Invariant Extended Kalman Filters (IEKF) can be automatically derived for each case, ensuring interesting properties Bonnabel et al. (2008); Barrau and Bonnabel (2017).

Only model (2) was known to fit into the invariant Kalman filtering framework Bonnabel et al. (2008). While the scaled odometry model (3) has been studied with Lie groups before Strasdat et al. (2010); Bourmaud and Megret (2015), the scaled accelerometer one (4) has never been shown to fit into the theory of invariant filtering of Barrau and Bonnabel (2016). In the following, we recall the TFG framework of Barrau and Bonnabel (2022), and how it applies to each case.

Invariant filtering is a framework that allows for the introduction of a Lie group structure on the state space of a class of estimation problems. While it can be applied to any system that lives on a Lie group directly (and inherently accomodates the manifold aspect of the problem, guaranteeing the estimates remain in that space), the goal of invariant filtering is not to accomodate the fact a state space is a Lie group (or part of it is). The goal is to find the right Lie group that makes the problem resemble a linear estimation problem Barrau and Bonnabel (2019). As a result, each new problem requires designing a new group, adapted to its dynamics and measurements.

The problem of fusing odometry with position measurements has long been known to fit into that framework, modeled using  $SE(2)$  Bonnabel et al. (2008). Being able to properly accomodate the problem of (unbiased) inertial navigation was one of the achievements of the invariant filtering framework Barrau and Bonnabel (2017) thanks to the introduction of the  $SE_2(3)$  group Barrau (2015). However, the introduction of the two-frames framework allowed for unification of them all as specific cases of the TFG, thus unifying all previously known models and discovering others Barrau and Bonnabel (2022). It brought a first answer to the question of including accelerometer biases while retaining the theoretical properties, for instance.

In this Section, we recall the construction of a simplified version of the TFG. We then show how scale factors can be included into this framework, which has never been done before. Finally, we detail how an IEKF can then be designed.

### 3.1 Two-Frame Groups

The idea of two-frame groups is to build upon a matrix Lie group  $G$ , which we call the underlying group of the TFG. The second ingredient are multivectors  $\mathbf{x} = (x_1^T \cdots x_k^T)^T \in \mathbb{R}^{k \times d}$  which denote stacked vectors of  $\mathbb{R}^d$ . Herein, we simply consider a component-wise product operation as follows

*Notation 1.* We denote by  $*$  the following function on  $G \times \mathbb{R}^{k \times d}$

$$g * \mathbf{x} = ((gx^1)^T \cdots (gx^k)^T)^T = \text{diag}(g, \dots, g) \mathbf{x} \quad (6)$$

We may at first define the state space of a (reduced) two-frame system to be of the form  $G \times \mathbb{R}^{k \times d}$ . A state element is then of the form  $\chi := (g, \mathbf{x})$ . The two-frame group (TFG) is a group structure on the state space, that is, a way to combine state elements. For the present state space it is defined as follows.

$$\chi_1 \bullet \chi_2 = \begin{pmatrix} g_1 \\ \mathbf{x}_1 \end{pmatrix} \bullet \begin{pmatrix} g_2 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} g_1 g_2 \\ \mathbf{x}_1 + g_1 * \mathbf{x}_2 \end{pmatrix}. \quad (7)$$

The identity element is  $(I_d, 0)$ , and the inverse is given by  $(g^{-1}, -g^{-1} * \mathbf{x})$ .

### 3.2 Group-affine dynamical systems

Invariant filtering relies on the adequacy between the group structure and the system dynamics.

*Definition 2.* A dynamics of the following form on the TFG is called natural:

$$\begin{pmatrix} g_{n+1} \\ \mathbf{x}_{n+1} \end{pmatrix} = \begin{pmatrix} g_n \\ F_n \mathbf{x}_n + \mathbf{d}_n \end{pmatrix} \bullet \begin{pmatrix} \Omega_n \\ U_n \end{pmatrix}, \quad (8)$$

where we assume that  $g * (F_n x) = F_n (g * x)$  for all  $g \in G, x \in \mathbb{R}^d$ . Using the TFG law above (7), it rewrites

$$\begin{pmatrix} g_{n+1} \\ \mathbf{x}_{n+1} \end{pmatrix} = \begin{pmatrix} g_n \Omega_n \\ F_n \mathbf{x}_n + \mathbf{d}_n + g_n * U_n \end{pmatrix}, \quad (9)$$

If  $F_n = Id$  and  $\mathbf{d}_n = 0$ , then the dynamics are called left-invariant. In this work, we'll consider that  $F_n$  is made of blocks of zero or identity matrices, guaranteeing permutation.

Such dynamics were shown in Barrau and Bonnabel (2022) to be group-affine. Group-affine dynamical systems Barrau and Bonnabel (2017); Barrau (2015) have a remarkable property, that has proved key to the success of invariant filters for state estimation. Indeed, let us define the left-invariant error between two elements of the TFG

$$E = \hat{\chi}^{-1} \bullet \chi = \left( \hat{g}^{-1} * \begin{pmatrix} \hat{g}^{-1}g \\ \mathbf{x} - \hat{\mathbf{x}} \end{pmatrix} \right) := \begin{pmatrix} E^g \\ E^x \end{pmatrix}. \quad (10)$$

$E$  provides a measure of discrepancy between both elements of the group, and a null error  $\chi = \hat{\chi}$  corresponds to  $E$  being the identity element  $(Id, \mathbf{0})$ . Let us call  $E_n = \hat{\chi}_n^{-1} \chi_n$  the error between two solutions of the dynamical system (9) at time  $n$ . The following proposition shows how the TFG is adapted to (9).

*Proposition 3.* At time  $n + 1$ , the left-invariant error equals

$$E_{n+1} = \hat{\chi}_{n+1}^{-1} \chi_{n+1} = \begin{pmatrix} \Omega_n \\ U_n \end{pmatrix}^{-1} \bullet \begin{pmatrix} E_n^g \\ F_n E_n^x \end{pmatrix} \bullet \begin{pmatrix} \Omega_n \\ U_n \end{pmatrix} \quad (11)$$

$$= \begin{pmatrix} \Omega_n^{-1} E_n^g \Omega_n \\ \Omega_n^{-1} * F_n E_n^x + \Omega_n^{-1} E_n^g * U_n - \Omega_n^{-1} * U_n \end{pmatrix}, \quad (12)$$

which depends only on  $E_n$  and the inputs  $\Omega_n, U_n$ .

This means that the error is ‘‘autonomous’’, and does not depend explicitly on  $\hat{\chi}$  and  $\chi$ , it only depends upon their discrepancy. This autonomy (or state-independence) of the error evolution plays a key role in the theory of invariant filtering, and is the basis of many of the properties of the invariant extended Kalman filter (IEKF) of Barrau and Bonnabel (2017).

### 3.3 Compatible Output Maps

The main role of a filter is to update its estimate given some measurement, in order to reduce its estimation error. The general theory of the TFG relies on the notion of compatible output maps, first introduced in Bonnabel et al. (2008). For the purpose of this work, we focus on a simple case of component extraction. Consider an output map that extracts one of the vectors contained in  $\chi$

$$y = h(\chi) = x^i. \quad (13)$$

To process this measurement in the framework of the TFG we need to define how it acts on  $\mathbb{R}^d$ . Similarly to (6), the following function denotes a group action.

*Notation 4.* We denote by  $*_y$  the following function on  $TFG \times \mathbb{R}^d$

$$\chi *_y y = x^i + gy \quad (14)$$

where  $gy$  is the standard matrix vector product, and  $x^i$  is a fixed component of  $\mathbf{x}$ .

The reason that the TFG structure is adapted to the measurement  $h$  stems from the following result

*Lemma 5.* If  $*_y$  of (14) and  $h$  of (13) consider the same component  $x^i$ , we have

$$h(\chi_1 \bullet \chi_2) = \chi_1 *_y h(\chi_2). \quad (15)$$

**Proof.** On the one hand, we have using (7) and (13)

$$h(\chi_1 \bullet \chi_2) = h(g_1 g_2, \mathbf{x}_1 + g_1 * \mathbf{x}_2) = x_1^i + g_1 x_2^i,$$

and on the other

$$\chi_1 *_y h(\chi_2) = \chi_1 *_y x_2^i = x_1^i + g_1 x_2^i,$$

proving the result.

Note that, in Lie group theory, Lemma 5 indicates that  $h$  is equivariant. The main interest of this property is to replace the standard output error, also called innovation in the context of filtering, with a more adapted one. Indeed, given a state estimate

$\hat{\chi}$  and a measured output  $y$  the standard innovation is  $Z = y - h(\hat{\chi})$ . Within the TFG we rather consider

$$Z := \hat{\chi}^{-1} *_y y. \quad (16)$$

Indeed, the update is then carried out using the exponential map of the Lie group, e.g., Chirikjian (2011), and an arbitrary gain matrix  $K$

$$\hat{\chi}_{n+1}^+ = \hat{\chi}_{n+1} \bullet \exp(KZ) \quad (17)$$

Owing to the compatibility property, this choice of innovation and update helps maintaining the ‘‘autonomous’’ behavior of the error during the update step, as the following result shows

*Proposition 6.* The error  $E_{n+1}^+ = (\hat{\chi}_{n+1}^+)^{-1} \chi_{n+1}$  depends only on  $E_{n+1}$ ,  $K$  and  $Z$ .

**Proof.** Replacing  $Z$  by (16) and using (15), we have

$$Z = \hat{\chi}^{-1} *_y h(\chi) = h(\hat{\chi}^{-1} \bullet \chi) = h(E) = E^{x^i}. \quad (18)$$

Therefore, the error becomes

$$E_{n+1}^+ = (\hat{\chi}_{n+1}^+)^{-1} \chi_{n+1} = \exp(KZ)^{-1} \bullet E_{n+1} \quad (19)$$

$$= \exp(KE^{x^i})^{-1} \bullet E_{n+1} \quad (20)$$

This behavior comes with many properties regarding the state estimation problem, and may be considered a very desirable property, see for instance Barrau and Bonnabel (2018).

## 4. INTRODUCING THE GROUP $Sim_k(d)$ AS A TWO-FRAME GROUP

It has long been known the problem of state estimation for the non-holonomic car with position measurements fits into the invariant framework Bonnabel et al. (2008). The next achievement of the invariant filtering framework was to introduce the groups  $SE_k(d)$ , so as to include the problem of unbiased inertial navigation Barrau and Bonnabel (2017) and simultaneous localisation and mapping Barrau and Bonnabel (2015b). It turns out that all these groups are special cases of the TFG, but all built upon an underlying group  $G$  that exclusively consists of rotations.

However, to our knowledge, no problem involving a scale factor, such as the scaled odometry problem, has been shown to fit into the invariant filtering framework. If we are to include the scaling factor in the model, and to estimate it online, having  $G = SO(d)$  as underlying group is insufficient. It turns out though, that we can use the theory of two-frame systems developed in Section 3 once again, but changing the underlying group  $G$ . This is interesting, as it is to our best knowledge the first application of two-frame systems where  $G$  is not a rotation matrix group.

Note that including a scale factor using geometric tools was already done in the context of visual navigation Strasdat et al. (2010); Bourmaud and Megret (2015), based on the group of similitudes  $Sim(d)$  Chirikjian (2011), but never from the invariant filtering point of view.

### 4.1 Definition of the group structure

We let  $G$  be the direct product between  $SO(d)$  and  $\mathbb{R}_{>0}$  endowed with standard product of scalars. Hence, an element of  $G$  now writes  $g = (R, s)$  with  $R$  a rotation and  $s > 0$ , and the group composition law writes  $(R_1, s_1) \cdot (R_2, s_2) = (R_1 R_2, s_1 s_2)$ .

In this context, an element of the TFG is thus of the form  $\chi = (g, \mathbf{x}) = ((R, s), \mathbf{x})$ . Moreover, function  $*$  requires a matrix form for  $g = (R, s)$ , which is simply given by the scaled rotation matrix  $sR$ . This defines a TFG group law following (7), which herein particularizes to

$$\chi_1 \bullet \chi_2 = \begin{pmatrix} (R_1 R_2, s_1 s_2) \\ \mathbf{x}_1 + (s_1, R_1) * \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} (R_1 R_2, s_1 s_2) \\ \mathbf{x}_1 + s_1 R_1 * \mathbf{x}_2 \end{pmatrix}. \quad (21)$$

The inverse element is given by  $\chi^{-1} = ((R, s), \mathbf{x})^{-1} = ((R^{-1}, \frac{1}{s}), -\frac{1}{s} R^{-1} * \mathbf{x})$ .

In the case where multivector  $\mathbf{x}$  consists of one vector, that is,  $\mathbf{x} \in \mathbb{R}^d$ , the TFG coincides with the group of similitudes  $Sim(d)$ . Since this construction mimics that of  $SE_k(d)$ , we denote this new family of groups  $Sim_k(d)$ ,  $k$  being the number of vector stacked in  $\mathbf{x}$ .

In this case, similarly to  $*$ , function  $*_y$  becomes

$$\chi *_y y = \begin{pmatrix} g \\ \mathbf{x} \end{pmatrix} *_y y := x^i + sRy. \quad (22)$$

#### 4.2 The Exponential Map on $Sim_2(d)$

We now propose to derive the exponential of the group we have introduced. The exponential map on  $Sim_2(d)$  can be easily derived thanks to the existing results on  $Sim(2)$  Gallier and Quaintance (2020) and using the same approach as for the group  $SE_2(d)$  Barrau and Bonnabel (2017).

*Lemma 7.* (Gallier and Quaintance (2020)). The exponential on  $Sim(2)$  is given by

$$\exp(\theta, s, x) = (\theta, e^s, V(\theta, s)x), \text{ where,} \quad (23)$$

$$V(s, \theta) = (\alpha I + \beta J), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (24)$$

$$\alpha = \frac{s(e^s \cos(\theta) - 1) + e^s \theta \sin(\theta)}{s^2 + \theta^2} \quad (25)$$

$$\beta = \frac{\theta(1 - e^s \cos(\theta)) + e^s s \sin(\theta)}{s^2 + \theta^2} \quad (26)$$

*Theorem 8.* The exponential on  $Sim_2(2)$  is given by

$$\exp(\theta, s, \mathbf{x}) = (\theta, e^s, V(\theta, s) * \mathbf{x}) \quad (27)$$

where  $V(\theta, s)$  is given in Lemma 7

**Proof.** We just have to show that on  $Sim_k(2)$ , the exponential takes a similar form as on  $Sim(2)$ , with the same matrix being applied to all components  $x^1, \dots, x^k$  of  $\mathbf{x}$ . We will show this in matrix form.

The matrix form of the Lie algebra element  $(\theta, s, \mathbf{x})$  is  $\Omega = \begin{pmatrix} \theta J + sI & x^1 \cdots x^k \\ 0 & 0 \end{pmatrix}$ . One can then check that

$$\Omega^k = \begin{pmatrix} (\theta J + sI)^k & (\theta J + sI)^{k-1} x^1 \cdots (\theta J + sI)^{k-1} x^k \\ 0 & 0 \end{pmatrix}.$$

Therefore, we have

$$\exp_m \begin{pmatrix} \theta J + sI & \mathbf{x} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \sum_k \frac{(\theta J + sI)^k}{k!} & V(\theta, s)x^1 \cdots V(\theta, s)x^k \\ I_k & \end{pmatrix}$$

where  $V(\theta, s) = \sum_k \frac{(\theta J + sI)^k}{(k+1)!}$ . This matrix is the same as for  $Sim(2)$ , hence the result.

#### 4.3 Application to Scaled Odometry

The scaled odometry problem aims at estimating the orientation  $R_n$ , scale  $s_n$  and position  $x_n$ . Therefore, we can model it using the TFG with underlying group  $SO(2) \times \mathbb{R}_{>0}$  and a multivector of one component, i.e.  $Sim_1(2) = Sim(2)$ . This allows casting the problem into the invariant filtering framework, as we have

*Proposition 9.* Dynamics (3) are group-affine dynamics on  $Sim(2)$ , as they are of the form (9).

**Proof.** Using the definitions of  $G$  and  $*$ , (3) rewrites

$$\begin{pmatrix} (R_{n+1}, s_{n+1}) \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} (R_n \Omega_n, s_n) \\ x_n + R_n s_n U_n \end{pmatrix} = \begin{pmatrix} g_n \cdot (\Omega_n, 1) \\ x_n + g_n * U_n \end{pmatrix}, \quad (28)$$

which is in the form of (9) indeed with  $F_n = Id$  and  $\mathbf{d}_n = 0$ .

The proposition shows the theory of invariant filtering can be readily applied. In particular one can design an IEKF and it will come with the strong theoretical properties of invariant filtering.

#### 4.4 Handling Scaled Accelerometer with $Sim_2(2)$

While including unbiased inertial navigation in the invariant filtering framework relied on the  $SE_2(d)$  group, the inclusion of biases remained a problem. The two-frames framework brought a first answer to it Barrau and Bonnabel (2022). However, unknown scale factors are also present in accelerometer measurements in inertial navigation, although often ignored in academic papers. While such scale may need to be added to the estimation problem Ouyang et al. (2021); Nemirotov et al. (2023), they were not taken into account in the TFG framework until now. Early work in geometric filtering proposed a method to estimate them Martin et al. (2010), but without particular theoretical properties (such as autonomous error equations). Herein we propose to leverage the group  $Sim_2(d)$  we have introduced to include the scaled inertial navigation problem into the TFG framework.

It turns out that, following the same reasoning as in Section 4.3, it is fairly easy to include the scaled accelerometer problem into the framework. Indeed, we consider the same underlying group  $SO(2) \times \mathbb{R}_{>0}$ , but this time with multivectors of two components  $x = (v^T \ p^T)^T \in \mathbb{R}^4$ . We are thus using  $Sim_2(2)$ , whose group law and inverse are given by

$$\chi_1 \bullet \chi_2 = \begin{pmatrix} (R_1 R_2, s_1 s_2) \\ v_1 + s_1 R_1 v_2 \\ p_1 + s_1 R_1 p_2 \end{pmatrix}, \quad \chi^{-1} = \begin{pmatrix} (R^{-1}, \frac{1}{s}) \\ -\frac{1}{s} R^{-1} v \\ -\frac{1}{s} R^{-1} p \end{pmatrix}. \quad (29)$$

The scaled accelerometer problem then directly fits into the framework of invariant filtering, as we have

*Proposition 10.* Dynamics (4) are of the form of the group-affine dynamics (9) for  $Sim_2(2)$ .

**Proof.** Using the definitions of  $Sim_2(2)$  and  $*$ , (4) rewrites

$$\begin{pmatrix} (R_{n+1}, s_{n+1}) \\ v_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} (R_n \Omega_n, s_n) \\ v_n + R_n s_n U_n \\ p_n + \Delta t v_n \end{pmatrix}, \quad (30)$$

$$= \begin{pmatrix} g_n \cdot (\Omega_n, 1) \\ \begin{pmatrix} Id & 0 \\ \Delta t Id & Id \end{pmatrix} x_n + \begin{pmatrix} \mathbf{d}_n \\ 0 \end{pmatrix} + g_n * \begin{pmatrix} U_n \\ 0 \end{pmatrix} \end{pmatrix}, \quad (31)$$

which is in the form of (9) indeed, and we can check that  $F_n(g * x) = g * (F_n x)$ .

## 5. NUMERICAL EXPERIMENT

We start by recalling how to implement an (invariant) EKF.

### 5.1 Designing an IEKF in practice: The 2D Case

Designing a Kalman filter is expliciting how, given an estimate  $\hat{\chi}_n$  and an associated covariance  $P_n$ , one obtains  $\hat{\chi}_{n+1}^+$  and  $P_{n+1}^+$  given inputs  $\Omega_n, U_n$ , output  $y_{n+1}$ , and noise covariances  $Q_n$  and  $N_{n+1}$ . This requires four steps. Let us compare how the EKF and IEKF carry them out:

- (1) Propagating the estimate  $\hat{\chi}_n$  to get  $\hat{\chi}_{n+1}$ . Both use  $\hat{\chi}_{n+1} = f_n(\hat{\chi}_n)$ , with  $f_n$  the dynamical model
- (2) Propagation the covariance  $P_n$  to get  $P_{n+1}$ . Both use

$$P_{n+1} = A_n P_n A_n^T + Q_n,$$

but with different Jacobian  $A_n$

- (3) Updating the estimate to get  $\hat{\chi}_{n+1}^+$ . The EKF uses  $\hat{\chi}_{n+1}^+ = \hat{\chi}_{n+1} + K_{n+1} Z$ , while the IEKF uses (17) (with a different innovation  $Z$ ). Both use a gain

$$K_{n+1} = P_{n+1} H_{n+1}^T (H_{n+1} P_{n+1} H_{n+1} + N_{n+1})^{-1},$$

with the same Jacobian  $H_{n+1}$ .

(4) Updating the covariance to get  $P_{n+1}^+$ . Both use

$$P_{n+1}^+ = (I - K_{n+1}H_{n+1})P_{n+1}.$$

Hence, the only difference between an EKF and an IEKF is Jacobian  $A_n$ , and the computation of the updated estimate with the Lie group exponential. The procedure to derive Jacobians in general is described in Barrau and Bonnabel (2022, 2017). The main issue is that one is looking for matrices, while  $E_n$  is a group element. Being a Lie group, the TFG is linked with a Lie algebra which is a vector space of dimension  $d_{TFG} = d_G + kd$ , where  $d_G$  is the dimension of  $G$ . For instance,  $d_G = 1$  for  $SO(2)$ . Thus,  $A_n \in \mathbb{R}^{d_{TFG} \times d_{TFG}}$  and  $H_{n+1} \in \mathbb{R}^{d \times d_{TFG}}$ .

Deriving Jacobian  $H_{n+1}$  is a good illustration, as it is easy in our case, and we can check that it coincides with that of the EKF. Since  $Z = h(E^{x^i})$ , we have

$$H_{n+1} = [\mathbf{0} \quad Id \quad \mathbf{0}]$$

where the identity block is located at the  $i$ -th component of  $\mathbf{x}$ . It is clear that the EKF uses the same Jacobian.

Deriving Jacobian  $A_n$  is more intricate in general form. In the 2D case, all underlying groups  $G$  considered herein are commutative, hence  $E_{n+1}^g = E_n^g$ . we can thus simplify the first component of (12). For the second, we use that  $\Omega_n^{-1}E_n^g * U_n = \Omega_n^{-1} * E_n^g * U_n$ , and we have

$$E_{n+1} = \left( \Omega_n^{-1} * F_n E_n^x + \Omega_n^{-1} * (E_n^g * U_n - U_n) \right) \quad (32)$$

Since  $G$  is a matrix Lie group, one can approximate the matrix  $E^g \approx Id + (e)^\wedge$ , where  $e \in \mathbb{R}^{d_G}$  is a vector and  $(\cdot)^\wedge$  linear Barfoot (2017). Since  $*$  is a component-wise product, we can further simplify

$$\left( Id_G + (e_{n+1})^\wedge \right) = \left( \Omega_n^{-1} * F_n E_n^x + \Omega_n^{-1} * (e_n)^\wedge * U_n \right) \quad (33)$$

Since  $(e)^\wedge * U_n$  is linear in  $e$ , there exists  $(U_n)_*$  such that  $(e)^\wedge * U_n = (U_n)_* e$ , but which depends on the considered TFG. Given this matrix, we have

$$A_n = \left( D(\Omega_n^{-1})(U_n)_* \quad D(\Omega_n^{-1})F_n \right) \quad (34)$$

where  $D(\Omega_n^{-1}) = \text{diag}(\Omega_n^{-1}, \dots, \Omega_n^{-1})$ .

## 5.2 Experimental Setup: 2D wheeled robot with gyrometer and scaled accelerometer

The proposed TFG-EKF is compared in a filtering framework with the imperfect IEKF and the EKF formulations.

The experiment lasts 20 seconds. A wheeled vehicle was modeled to stand still for 5 seconds, accelerate in a straight line at  $1\text{m/s}^2$  for 5 seconds, and then keep a constant speed.

The vehicle/robot is equipped with scaled accelerometers and gyrometers, leading to the dynamical model (4).

Inertial increments were received at 50Hz, and position measurements at 1Hz. They were polluted by noise of respective standard deviations  $\sigma_\omega = 1e^{-4}\text{rad/s}$ ,  $\sigma_v = 1e-4\text{m/s}^2$  and  $\sigma_y = 1\text{m}$ . The initial attitude error was sampled given  $\sigma_{att}^0 = 100^\circ$  on 20 Monte Carlo runs each.

## 5.3 Compared Filters

Three different filters were compared.

*First filter* First, the IEKF derived using the TFG  $Sim_2(2)$  is denoted as TFG-EKF. The exponential map being given in Theorem 8, in order to design an IEKF, one then only needs to compute  $A_n$ .

From (31), we obtain  $F_n = \begin{pmatrix} Id & 0 \\ \Delta t Id & Id \end{pmatrix}$ . Moreover, one obtains

$$\begin{pmatrix} U_n \\ 0 \end{pmatrix}_* = \begin{bmatrix} J U_n & U_n \\ 0 & 0 \end{bmatrix} \quad (35)$$

Hence, one has

$$A_n = \begin{pmatrix} I_2 & & \\ \Omega_n^{-1} J U_n & \Omega_n^{-1} U_n & \Omega_n^{-1} \\ & \Delta t \Omega_n^{-1} & \Omega_n^{-1} \end{pmatrix} \quad (36)$$

*Second filter* The second one is based on an IEKF on  $SE_2(2)$  that would be used in the unscaled case Barrau and Bonnabel (2017), with an additional linear parameter for the scale. It is called Imperfect IEKF. Its differences with the TFG-EKF are how the update is carried out, and the matrix  $A_n$  it uses. Let us write the state  $\chi = (s, \tilde{\chi})$  where  $\tilde{\chi} = (R, v, p)$ . Let  $\delta = (\delta_s, \delta_{\tilde{\chi}}) = KZ$ . The update writes

$$\chi^+ = \begin{pmatrix} s^+ \\ \tilde{\chi}^+ \end{pmatrix} = \begin{pmatrix} s + \delta_s \\ \tilde{\chi}^+ \exp_{SE_2(2)}(\delta_{\tilde{\chi}}) \end{pmatrix}$$

$A_n$  is given for the imperfect IEKF by

$$A_n^{\text{Imp. IEKF}} = \begin{pmatrix} I_2 & & \\ s_n \Omega_n^{-1} J U_n & \Omega_n^{-1} U_n & \Omega_n^{-1} \\ & \Delta t \Omega_n^{-1} & \Omega_n^{-1} \end{pmatrix}$$

*Third filter* Finally, for the EKF, the update writes  $\chi^+ = \chi + KZ$ , and the propagation Jacobian is

$$A_n^{\text{EKF}} = \begin{pmatrix} I_2 & & \\ s_n R_n J U_n & R_n U_n & I_2 \\ & \Delta t I_2 & I_2 \end{pmatrix}$$

We can verify that, contrary to (36) for the TFG-EKF, the imperfect IEKF and EKF involve Jacobians  $A_n$  which include elements of the estimated state:  $s_n$  and  $R_n$ . Thus, their associated errors cannot be autonomous.

## 5.4 Results

Figure 1 displays the yaw, position and scale RMSEs. Two conclusions can be drawn. First, only the TFG-EKF manages to converge at each run. Indeed, the EKF can diverge, and the imperfect IEKF sometimes converge to wrong values of the scale and yaw. This hints to a second conclusion: The TFG-EKF is the only one to escape local minima. Indeed, the imperfect IEKF exhibits position RMSE similar to the TFG-EKF, meaning that erroneous yaw and scale values it can converge to actually balance themselves. This is problematic for practical use, since the yaw can be used for other purposes (e.g. relative positioning of objects using vision).

## 6. CONCLUSION

This paper focuses on the impact of the underlying group upon which the two-frame group (TFG) is built, through simple yet instructive examples. By considering a scale factor to be estimated in the state, we have introduced the first TFG where the underlying group is not limited to rotations. This led to a novel Lie group (in the context of state estimation) we called  $Sim_2(2)$ , and highlights the versatility of the TFG approach. Numerical experiments confirmed that the proposed filter performs better in hard problems involving a large initial error than the imperfect IEKF approach (when applied to this particular problem treating the scale factor linearly), and the standard EKF.

## REFERENCES

Barfoot, T.D. (2017). *State Estimation for Robotics*. Cambridge University Press. doi:10.1017/9781316671528.

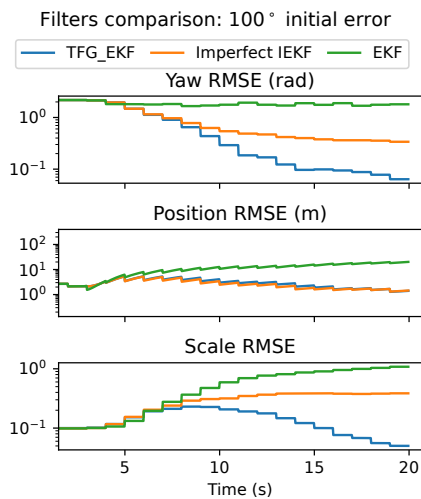


Fig. 1. Results of the alignment experiments. Yaw, position and scale RMSE of the estimates for  $\sigma_{att}^0 = 100^\circ$ , in log scale.

Barfoot, T.D. and Furgale, P.T. (2014). Associating Uncertainty with Three-Dimensional Poses for Use in Estimation Problems. *IEEE Transactions on Robotics*, 30(3), 679–693.

Barrau, A. (2015). *Non-linear state error based extended Kalman filters with applications to navigation*. Ph.D. thesis, Mines Paristech.

Barrau, A. and Bonnabel, S. (2015a). An EKF-SLAM algorithm with consistency properties. *CoRR*, abs/1510.06263. URL <http://arxiv.org/abs/1510.06263>.

Barrau, A. and Bonnabel, S. (2015b). Intrinsic filtering on lie groups with applications to attitude estimation. *IEEE Transactions on Automatic Control*, 60(2), 436 – 449.

Barrau, A. and Bonnabel, S. (2016). Navigating with highly precise odometry and noisy GPS: a case study. *IFAC Symposium on Nonlinear Control Systems (NOLCOS)*.

Barrau, A. and Bonnabel, S. (2017). The invariant extended kalman filter as a stable observer. *IEEE Transactions on Automatic Control*, 62(4), 1797–1812. doi:10.1109/TAC.2016.2594085.

Barrau, A. and Bonnabel, S. (2018). Invariant kalman filtering. *Annual Review of Control, Robotics, and Autonomous Systems*, 1, 237–257.

Barrau, A. and Bonnabel, S. (2022). The geometry of navigation problems. *IEEE Transactions on Automatic Control*, 68(2), 689–704.

Barrau, A. and Bonnabel, S. (2019). Linear observed systems on groups. *Systems and Control Letters*, 129, 36 – 42. doi:<https://doi.org/10.1016/j.sysconle.2019.05.005>. URL <http://www.sciencedirect.com/science/article/pii/S0167691119300805>.

Bonnabel, S., Martin, P., and Rouchon, P. (2008). Symmetry-Preserving Observers. *IEEE Transactions on Automatic Control*, 53(11), 2514–2526. doi:10.1109/TAC.2008.2006929.

Bourmaud, G. and Megret, R. (2015). Robust large scale monocular visual slam. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, 1638–1647.

Chirikjian, G.S. (2011). *Stochastic Models, Information Theory, and Lie Groups, Volume 2: Analytic Methods and Modern Applications*. Springer Science & Business Media.

Engel, J., Schöps, T., and Cremers, D. (2014). Lsd-slam: Large-scale direct monocular slam. In *European conference on computer vision*, 834–849. Springer.

Gallier, J. and Quaintance, J. (2020). *Differential Geometry and Lie Groups: A Computational Perspective*. doi:10.1007/978-3-030-46040-2.

Hartley, R., Ghaafari, M., Eustice, R.M., and Grizzle, J.W. (2020). Contact-aided invariant extended kalman filtering for

robot state estimation. *The International Journal of Robotics Research*, 39(4), 402–430.

Lisus, D., Cohen, M., and Forbes, J.R. (2023). Know what you don’t know: Consistency in sliding window filtering with unobservable states applied to visual-inertial slam. *IEEE Robotics and Automation Letters*, 8(6), 3382–3389.

Long, A.W., Wolfe, K.C., Mashner, M.J., Chirikjian, G.S., et al. (2013). The banana distribution is gaussian: A localization study with exponential coordinates. *Robotics: Science and Systems VIII*, 265, 1.

Mahony, R., Hamel, T., and Pfimlin, J.M. (2008). Nonlinear complementary filters on the special orthogonal group. *IEEE Transactions on automatic control*, 53(5), 1203–1218.

Martin, P., Salaün, E., et al. (2010). Generalized multiplicative extended kalman filter for aided attitude and heading reference system. In *Proc. AIAA Guid., Navigat., Control Conf.*, 1–13.

Mur-Artal, R., Montiel, J.M.M., and Tardos, J.D. (2015). Orbslam: a versatile and accurate monocular slam system. *IEEE transactions on robotics*, 31(5), 1147–1163.

Nemiroff, R., Chen, K., and Lopez, B.T. (2023). Joint on-manifold gravity and accelerometer intrinsic estimation for inertially aligned mapping. In *2023 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, 1388–1394. doi:10.1109/IROS55552.2023.10342424.

Ouyang, W., Wu, Y., and Chen, H. (2021). Ins/odometer land navigation by accurate measurement modeling and multiple-model adaptive estimation. *IEEE Transactions on Aerospace and Electronic Systems*, 57(1), 245–262. doi:10.1109/TAES.2020.3011998.

Park, W., Liu, Y., Zhou, Y., Moses, M., Chirikjian, G.S., et al. (2008). Kinematic state estimation and motion planning for stochastic nonholonomic systems using the exponential map. *Robotica*, 26(4), 419–434.

Pavasek, N., Walsh, A., and Forbes, J.R. (2021). Invariant extended kalman filtering using two position receivers for extended pose estimation. In *2021 IEEE International Conference on Robotics and Automation (ICRA)*, 5582–5588. IEEE.

Sola, J., Deray, J., and Atchuthan, D. (2018). A micro lie theory for state estimation in robotics. *arXiv preprint arXiv:1812.01537*.

Strasdat, H., Montiel, J., and Davison, A.J. (2010). Scale drift-aware large scale monocular slam. *Robotics: science and Systems VI*, 2(3), 7.

van Der Laan, N., Cohen, M., Arsenault, J., and Forbes, J.R. (2020). The invariant rauch-tung-striebel smoother. *IEEE Robotics and Automation Letters*, 5(4), 5067–5074.

van Goor, P., Hamel, T., and Mahony, R. (2023). Equivariant filter (eqf). *IEEE Transactions on Automatic Control*, 68(6), 3501–3512.

van Goor, P. and Mahony, R. (2023). Eqvio: An equivariant filter for visual-inertial odometry. *IEEE Transactions on Robotics*, 39(5), 3567–3585.

Wolfe, K.C., Mashner, M., and Chirikjian, G.S. (2011). Bayesian fusion on Lie groups. *Journal of Algebraic Statistics*, 2(1), 75–97.