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Limit theorems for a branching random walk in a random or varying environment

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In memory of Francis Comets

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Abstract

We consider a branching random walk on the real line with a stationary and ergodic environment (ξ_n) indexed by time, in which a particle of generation n gives birth to a random number of particles of the next generation, which move on the real line; the joint distribution of the number of children and their displacements on the real line depends on the environment ξ_n at time n. Let Z_n be the counting measure at time n, which counts the number of particles of generation n situated in a Borel set of the real line. For the case where the corresponding branching process is supercritical, we establish limit theorems such as large and moderate deviation principles, central and local limit theorems on the counting measures Z_n, convergence of the free energy, law of large numbers on the leftmost and rightmost positions at time n, and the convergence to infinite divisible laws. The varying environment case is also considered.

Keywords: Branching random walk, random environment, large deviation, moderate deviation, central limit theorem, local limit theorem, law of large numbers
Mathematics Subject Classification: 60J80, 60K37, 60F10, 60F05

1. Introduction and main results

As a combination of branching processes and random walks, branching random walks have been largely studied in the literature, see e.g. [48, 12, 13, 17] for early studies (in seventies), [15, 14, 20, 53, 58, 54] for later studies (in nineties), and [2, 3, 22, 66, 39, 46, 29, 8, 23, 45, 52] for some developments in the last decade. Recently there is an increasing research interest to this topic. One of the reasons is that the model has many interactions with various applied probability settings, such as fractals and Mandelbrot's cascades (cf. e.g. [47, 55, 9, 5, 18, 62]), perpetuities (see e.g. [66, 19, 44]) and branching Brownian motion (cf. e.g. [49, 21, 10, 59]). For other related works and many references, see e.g. the recent books [66, 19, 44]. Important results such as central and local limit theorems and large deviation principles for the counting measure, asymptotic properties of the fundamental martingale, and law of large numbers for the positions of the extremal particles, have been established.

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In classical branching random walks, the influence of the environments is not considered. Due to the introduction of the random environments, interesting new phenomenons and problems occur, but the study of the model becomes much more delicate. Various models, with random environments in space or in time, have been studied. For example, Baillon, Clément, Greven and den Hollander [6], Greven and den Hollander [36] studied a model in which the offspring distribution is affected by a location-dependent environment, while the displacement law is deterministic; Comets and Popov [24, 25] dealt with the case where both the offspring distribution and the displacement law are influenced by the random environment indexed by locations; Comets and Yoshida [26], Hu and Yoshida [41], Nakashima [63] and Yoshida [73] considered the case where the offspring distributions depend on a space-time environment.

Here we consider a *branching random walk with a random environment in time* (BRWRE), in which a particle of generation n gives birth to new particles of the next generation, which move on the real line; the joint distribution of the number of children and their displacements from their parent depends on the environment ξ_n at time n; the environment sequence (ξ_n) indexed by time is supposed to be stationary and ergodic. This is a natural and important extension of both the model of a branching process in a random environment introduced by Smith and Wilkinson [67] and Athreya and Karlin [4], and the classical branching random walk which corresponds to the constant environment case (that is, the case where all ξ_n are equal to the same constant). This model was proposed firstly by Biggins and Kyprianou [16] and Kuhlbusch [51] who considered the fundamental martingale; it was further studied by a number of authors, see e.g. [33, 34, 42, 60, 72, 35, 74].

Our purpose in this paper is to prove, for a BRWRE, asymptotic properties for the counting measure Z_n, which counts the number of particles of generation n situated in a given region. The study of this measure is important because it describes the configuration of the process at time n. Fundamental limit theorems, such as large and moderate deviation principles, convergence theorem on the free energy, and central limit theorems, will be established for the counting measure Z_n. A law of large numbers will also be proved for the positions of leftmost and rightmost particles. In particular, we will see that the fundamental results of Kaplin and Assmussen [48] and Biggins [13, 15] on the classical branching random walk remain valid for the random environment case, although the model becomes much richer and the approaches become significantly more delicate due to the existence of the random environment. The case of a branching random walk in a varying environment and the convergence to infinite divisible laws are also considered.

The results and the methods developed in this paper open ways for the study of related topics, such as general branching processes and multiplicative cascades in random environments; for example some of them have already been used by Hong, Hou and Zhang [74] in the study of a branching system of random walks with a random environment in location. Some of the results have been announced in the proceeding [57] and the short survey [42] without detailed proofs (only some key ideas have been mentioned therein); the complete proofs seem necessary and desired, so that they will be given here. A preliminary version of the present paper was put in arXiv [43], which has been cited in a number of papers (we found 27 citations in Google).

1.1. Model

Let us describe precisely the model that we consider in this paper. The random environment, denoted by $\xi = (\xi_n)$, indexed by the time $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$, is a stationary and ergodic sequence. The random variable ξ_n takes values in some measurable space $(\Theta, \mathcal{E})$; it represents the random environment at time n. Without loss of generality we can suppose that ξ is defined

on the product space $(\Theta^{\mathbb{N}}, \mathcal{E}^{\otimes\mathbb{N}}, \tau)$, with τ the law of ξ. Associated to each realization of ξ_n, there is a probability distribution $\eta_n = \eta(\xi_n)$ on $\mathbb{N} \times \mathbb{R}^{\mathbb{N}}$.

Given the environment ξ, the process starts from an original particle $\emptyset$ of generation 0 which lies at $S_\emptyset = 0 \in \mathbb{R}$ at time 0. At time 1, the ancestor $\emptyset$ dies and is replaced by $N = N(\emptyset)$ particles i of generation 1 which scatter on $\mathbb{R}$ with locations $L_i = L_i(\emptyset)$, $1 \le i \le N$, where the random vector $X(\emptyset) = (N, L_1, L_2, \cdots)$ is of distribution $\eta_0 = \eta(\xi_0)$. In general, each particle u in the nth generation will be replaced by $N(u)$ new particles ui of generation $n+1$ at time $n+1$; the location of ui satisfies

$$S_{ui} = S_u + L_i(u), \qquad 1 \le i \le N(u),$$

where S_u is the location of the parent particle u, $L_i(u)$ is the relative displacement of ui to its parent u, and the random vector $X(u) = (N(u), L_1(u), L_2(u), \cdots)$ has distribution $\eta_n = \eta(\xi_n)$. Note that the values $L_i(u)$ for $i > N(u)$ do not play any role for our model; we introduce them only for convenience. Conditioned on the environment ξ, all particles behave independently, meaning that the random vectors $X(u)$ indexed by all the sequences u are conditionally independent.

Denote by $\mathbb{P}_\xi$ the quenched law, i.e. the conditional probability given the environment ξ. The total probability can be expressed as $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$; it is usually called the annealed law. The expectations with respect to $\mathbb{P}_\xi$ and $\mathbb{P}$ will be denoted respectively by $\mathbb{E}_\xi$ and $\mathbb{E}$.

Let $\mathbb{U} = \cup_{n\ge 0}\mathbb{N}^{*n}$ be the set of all finite sequence $u = u_1 \cdots u_n$, where $\mathbb{N}^* = \{1, 2, \cdots\}$ and $\mathbb{N}^{*0} = \{\emptyset\}$ by convention. For $u \in \mathbb{U}$, we write $|u|$ for the length of u, and $u|n$ for the restriction to the first n terms of u, with the convention that $|\emptyset| = 0$ and $u|0 = \emptyset$. By definition, under $\mathbb{P}_\xi$, the random vectors $X(u)$, indexed by $u \in \mathbb{U}$, are independent of each other, and each $X(u)$ has distribution $\eta_n = \eta(\xi_n)$ if $|u| = n$.

Let $\mathbb{T}$ be the Galton-Watson tree with defining elements $\{N(u)\}$: by definition, we have: (i) $\emptyset \in \mathbb{T}$, (ii) if $u \in \mathbb{T}$, then for each $i \in \mathbb{N}^*$, $ui \in \mathbb{T}$ if and only if $1 \le i \le N(u)$; (iii) for all $u \in \mathbb{U}$ and $i \in \mathbb{N}^*$, $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$. Denote by $\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$ the set of particles in the nth generation. We introduce the counting measure

$$Z_n(\cdot) = \sum_{u\in\mathbb{T}_n} \delta_{S_u}(\cdot).$$

For a measurable subset A of $\mathbb{R}$, $Z_n(A)$ denotes the number of particles of generation n located in A. In this paper we study asymptotic properties of the sequence of measures $\{Z_n\}$ by establishing limit theorems.

For $u \in \mathbb{U}$, let

$$X^{(u)}(\cdot) = \sum_{i=1}^{N(u)} \delta_{L_i(u)}(\cdot)$$

be the counting measure corresponding to the random vector $X(u)$, whose increasing points are the displacements $L_i(u)$, $1 \le i \le N(u)$. For a typical representation, denote

$$X_n(\cdot) = X^{(1_n)}(\cdot),$$

where $1_n = (1, \cdots, 1) \in \mathbb{N}^{*n}$ is the sequence of length n whose components are all equal to 1. Thus for each $u \in \mathbb{N}^{*n}$, the random element $X^{(u)}$ has the same distribution as X_n, given the environment ξ. Notice that in our model, the total population of generation n, $Z_n(\mathbb{R})$, forms a branching process in a random environment (BPRE), for which we refer to [67, 4] where

fundamental results have been established. For simplicity, we introduce the following notation:

$$N_n = X_n(\mathbb{R}), \quad m_n = \mathbb{E}_\xi N_n, \quad P_0 = 1 \text{ and } P_n = \mathbb{E}_\xi Z_n(\mathbb{R}) = \prod_{i=0}^{n-1} m_i \quad \text{for } n \geq 1.$$

Let $\mathcal{F}_0 = \sigma(\xi)$ and $\mathcal{F}_n = \sigma(\xi, X(u); |u| < n)$ $(n \geq 1)$ be the σ-field containing all the information of the first n generations. It is well known that the sequence $\{Z_n(\mathbb{R})/P_n\}$ is a non-negative martingale under $\mathbb{P}_\xi$ for every ξ with respect to the filtration $\{\mathcal{F}_n\}$; hence it converges almost surely (a.s.) to a random variable denoted by W. Throughout this paper we always assume that

$$\mathbb{E} \log m_0 \in (0, \infty) \quad \text{and} \quad \mathbb{E} \frac{N}{m_0} \log^+ N < \infty. \tag{1.1}$$

The first condition means that the BPRE $Z_n(\mathbb{R})$ is supercritical; the second implies that the limit W is non-degenerate. We refer to [4, 70, 71] for more information about a BPRE. To concentrate on the intrinsic properties, we also assume that

$$N \geq 1 \ \ a.s. \quad \text{and} \quad \mathbb{E}|L_1| < \infty. \tag{1.2}$$

Together with (1.1), this implies that $Z_n(\mathbb{R}) \to \infty$ and $W > 0$ a.s.

1.2. Convergence of the free energy, large deviation principle and law of large numbers

We first establish a large deviation principle on the sequence of measures Z_n, by calculating the limit of the free energy. Let

$$\tilde{Z}_n(t) = \int e^{tx} Z_n(dx) = \sum_{u \in \mathbb{T}_n} e^{tS_u} \qquad (t \in \mathbb{R})$$

be the Laplace transform of the measure Z_n. It is also called *partition function* by physicians. The function $\frac{\log \tilde{Z}_n(t)}{n}$ is called *free energy*. We are interested in the convergence of $\frac{\log \tilde{Z}_n(t)}{n}$, which will lead to a large deviation principle for $Z_n(n\cdot)$. For $n \in \mathbb{N}$ and $t \in \mathbb{R}$, let

$$m_n(t) = \mathbb{E}_\xi \int e^{tx} X_n(dx) = \mathbb{E}_\xi \sum_{i=1}^{N(u)} e^{tL_i(u)} \quad \text{for} \quad u \in \mathbb{N}^{*n},$$

be the Laplace transform of the counting measure describing the evolution of the system at time n. In particular, $m_n(0) = m_n$. Put

$$P_0(t) = 1 \quad \text{and} \quad P_n(t) = \mathbb{E}_\xi \tilde{Z}_n(t) = \prod_{i=0}^{n-1} m_i(t) \quad \text{for } n \geq 1$$

(so that $P_n(0) = P_n$ for all $n \geq 0$). Observe that

$$W_n(t) := \frac{\tilde{Z}_n(t)}{\mathbb{E}_\xi \tilde{Z}_n(t)} = \frac{\sum_{u \in \mathbb{T}_n} e^{tS_u}}{P_n(t)}, \quad n \geq 0,$$

is a non-negative martingale, hence the limit

$$W(t) := \lim_{n \to \infty} W_n(t) \tag{1.3}$$

exists a.s. with $\mathbb{E}_\xi W(t) \le 1$. In the deterministic environment case, this martingale has been studied e.g. by Kahane and Peyrière [47], Biggins [12], Durrett and Liggett [30], Guivarc'h [37], Lyons [58], Liu [53, 54, 55, 56], Iksanov and Kabluchko [46], and Iksanov [45], in different contexts.

We assume that

$$\mathbb{E}|\log m_0(t)| < \infty \qquad \text{and} \qquad \mathbb{E}\left|\frac{m_0'(t)}{m_0(t)}\right| < \infty$$

for all $t \in \mathbb{R}$. These moment conditions imply that

$$\Lambda(t) = \mathbb{E}\log m_0(t) \qquad \text{and} \qquad \Lambda'(t) = \mathbb{E}\frac{m_0'(t)}{m_0(t)}$$

are well defined as real numbers, so that $\Lambda(t)$ is differentiable everywhere on $\mathbb{R}$, and $\Lambda'(t)$ is its derivative (this can be easily verified by the dominated convergence theorem, using the fact that the function $t \mapsto \frac{m_0'(t)}{m_0(t)}$ is increasing). Denote by $\Lambda^*(x) = \sup_{t\in\mathbb{R}}\{xt - \Lambda(t)\}$ the Legendre transform of Λ. Then

$$\Lambda^*(x) = \begin{cases} t\Lambda'(t) - \Lambda(t) & if \quad x = \Lambda'(t) \text{ for some } t \in \mathbb{R}, \\ +\infty & if \quad x \ge \Lambda'(+\infty) \text{ or } x \le \Lambda'(-\infty). \end{cases}$$

Since the function $\rho(t) = t\Lambda'(t) - \Lambda(t)$ decreases on $(-\infty, 0]$ and increases on $[0, \infty)$, it attains its minimum at 0: $\min_t \rho(t) = \rho(0) = -\Lambda(0) < 0$. We introduce two critical values t_- and t_+:

$$t_- = \inf\{t \in \mathbb{R} : \rho(t) \le 0\} \quad \text{and} \quad t_+ = \sup\{t \in \mathbb{R} : \rho(t) \le 0\}.$$

Notice that $-\infty \le t_- < 0 < t_+ \le \infty$, t_- and t_+ are two solutions of $\rho(t) = 0$ if they are finite.

Theorem 1.1 (Convergence of the free energy)**.** *Almost surely, for all $t \in \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{\log \tilde{Z}_n(t)}{n} = \tilde{\Lambda}(t) := \begin{cases} \Lambda(t) & if \quad t \in (t_-, t_+), \\ t\Lambda'(t_+) & if \quad t \ge t_+, \\ t\Lambda'(t_-) & if \quad t \le t_-. \end{cases} \tag{1.4}$$

For the deterministic environment case, see [20, 32]; see also [5] for branching random walks in $\mathbb{R}^d$.

Combining Theorem 1.1 with the Gärtner-Ellis theorem [28, p.52, Exercise 2.3.20] leads to the large deviation principle for $Z_n(n\cdot)$ as stated below. Denote by $\tilde{\Lambda}^*(x)$ the Legendre transform of $\tilde{\Lambda}(t)$. It can be seen that $\tilde{\Lambda}(t) \le \Lambda(t)$, so that $\tilde{\Lambda}^*(x) \ge \Lambda^*(x)$. More precisely,

$$\tilde{\Lambda}^*(x) = \begin{cases} \Lambda^*(x) & if \quad x \in [\Lambda'(t_-), \Lambda'(t_+)], \\ +\infty & if \quad x < \Lambda'(t_-) \text{ or } x > \Lambda'(t_+). \end{cases}$$

Theorem 1.2 (Large deviation principle)**.** *Almost surely, the sequence of finite measures $A \mapsto Z_n(nA)$ satisfies a large deviation principle with rate function $\tilde{\Lambda}^*$: for measurable set $A \subset \mathbb{R}$,*

$$-\inf_{x\in A^\circ} \tilde{\Lambda}^*(x) \le \liminf_{n\to\infty} \frac{1}{n}\log Z_n(nA) \le \limsup_{n\to\infty} \frac{1}{n}\log Z_n(nA) \le -\inf_{x\in\bar{A}} \tilde{\Lambda}^*(x),$$

where A° is the interior of A and $\bar{A}$ its closure.

Theorem 1.2 yields the following corollary. For deterministic branching random walks, similar results can be found in [13, 20].

Corollary 1.3. *It is a.s. that*

$$\lim_{n\to\infty}\frac{1}{n}\log Z_n[nx,\infty) = -\Lambda^*(x) > 0 \text{ if } x \in (\Lambda'(0), \Lambda'(t_+)),$$

$$\lim_{n\to\infty}\frac{1}{n}\log Z_n(-\infty, nx] = -\Lambda^*(x) > 0 \text{ if } x \in (\Lambda'(t_-), \Lambda'(0)).$$

We now present a law of large numbers for locations of the extremal particles. Let

$$M_n^L = \min_{u\in\mathbb{T}_n} S_u \quad (resp. \quad M_n^R = \max_{u\in\mathbb{T}_n} S_u)$$

be the position of leftmost (resp. rightmost) particles of generation n. The locations of the extremal particles have been largely studied in the literature, see e.g. [1, 2, 11, 13, 17, 39, 40, 61, 64]. We will use asymptotic properties of M_n^L and M_n^R to prove Theorem 1.1. Meanwhile, with the help of Theorem 1.1, we can see that M_n^L (resp. M_n^R) satisfies a law of large numbers:

Theorem 1.4 (Law of large numbers for leftmost and rightmost positions)**.** *It is a.s. that*

$$\lim_{n\to\infty}\frac{M_n^L}{n} = \Lambda'(t_-) \qquad \text{and} \qquad \lim_{n\to\infty}\frac{M_n^R}{n} = \Lambda'(t_+).$$

For deterministic branching random walks, the corresponding results can be found in [13, 20]. For a continuous-time counterpart on branching Brownian motion, see [65]. In the random environment case (for the model that we consider here), the the second and third order asymptotic of M_n^R have recently been explored in [60] (where the previous version [43] of our present work is cited).

Remark 1.1. In this subsection for simplicity we have assumed that the moment conditions in (1.2) hold for all $t \in \mathbb{R}$. However, if the moment conditions just hold for t in an interval, similar results can also be proved by a slight modification of the argument, as observed in [74] where the results and technics of our paper are used for the study of a branching system of a $\mathbb{N}$-valued random walk with a random environment in location. The difference is in the use of the Gärtner-Ellis theorem: instead of the use of [28, p.52, Exercise 2.3.20], we can then use [28, Theorem 2.3.6(c)] with essentially smooth condition.

1.3. Central and local limit theorems

Our second objective is to show central limit theorems and related results associated to Z_n. For the special branching random walk where the displacements $L_i(u), i = 1, 2, \cdots$, of the children ui of the same particle u are i.i.d., and independent of $N(u)$, for each fixed u, Kaplan and Asmussen [48] proved the following central limit theorem. Assume that $m = \mathbb{E}N \in (1, \infty)$ and that $\frac{\mathbb{E}X_0(\cdot)}{m}$ has mean 0 and variance 1. If $\mathbb{E}N(\log^+ N)^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$, then a.s.,

$$m^{-n}Z_n(-\infty, \sqrt{n}x] \to \Phi(x)W \qquad \forall x \in \mathbb{R}, \tag{1.5}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$ is the distribution function of the standard normal distribution $N(0, 1)$. They also gave a local version of (1.5) under the stronger moment condition that $\mathbb{E}N(\log N)^\gamma < \infty$ for some $\gamma > 3/2$. The formula (1.5), which was first conjectured by Harris [38], has been studied by many authors, see e.g. [15, 48, 50, 68]. In particular, Biggins

[15] extended the result of Kaplan and Asmussen [48] to a general branching random walk, by removing the i.i.d. condition on the family $\{L_i(u), i = 1, 2, \cdots\}$, the independence between this family and $N(u)$, and by relaxing the moment $\mathbb{E}N(\log^+ N)^{1+\varepsilon} < \infty$ to $\mathbb{E}(N \log^+ N) < \infty$. Gao et al. [33] generalized (1.5) to a special BRWRE, assuming that all the components $N(u)$ and $L_i(u)$ of the vector $X(u)$ are conditionally independent given the environment, all the displacements $L_i(u)$ $(i \geq 1)$ have the same conditional law, and that $\mathbb{E}N(\log N)^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$. Here we will extend the above cited result of Biggins [15] to the random environment case, and the result of Gao et al. [33] to a general BRWRE without the above mentioned assumptions on the components of $X(u)$, while relaxing the moment condition $\mathbb{E}N(\log N)^{2+\varepsilon} < \infty$ to the usual one $\mathbb{E}(N \log^+ N) < \infty$. Notice that, as observed by Biggins [15], for the classical branching random walk, the case $\mathbb{E}(N \log^+ N) = \infty$ is not interesting for (1.5), as in this case $m^{-n} Z_n(\mathbb{R}) \to W = 0$ a.s., so that (1.5) holds evidently.

Essential properties of the BRWRE depend on the quenched intensity measures defined by

$$\nu_n(\cdot) = \frac{\mathbb{E}_\xi X_n(\cdot)}{m_n}, \quad n \geq 0.$$

We will use the following notation about the mean and variance of ν_n:

$$\mu_n = \int x \nu_n(dx) = \frac{1}{m_n} \mathbb{E}_\xi \sum_{i=1}^{N(u)} L_i(u), \qquad \text{where } u \in \mathbb{N}^{*n},$$

$$\sigma_n^2 = \int (x - \mu_n)^2 \nu_n(dx) = \frac{1}{m_n} \mathbb{E}_\xi \sum_{i=1}^{N(u)} (L_i(u) - \mu_n)^2, \qquad \text{where } u \in \mathbb{N}^{*n},$$

$$\ell_n = \sum_{i=0}^{n-1} \mu_i \qquad \text{and} \qquad b_n = \left(\sum_{i=0}^{n-1} \sigma_i^2 \right)^{1/2}.$$

Throughout we assume that the conditional mean μ_n and variance σ_n^2 are well defined in $\mathbb{R}$ for almost every environment ξ. The following theorem is a version of (1.5) for a BRWRE.

Theorem 1.5 (Central limit theorem)**.** *Assume $\mathbb{E}\sigma_0^2 \in (0, \infty)$. Then a.s.,*

$$\frac{Z_n(-\infty, b_n x + \ell_n]}{Z_n(\mathbb{R})} \to \Phi(x) \qquad \forall x \in \mathbb{R}, \tag{1.6}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$ is the distribution function of the standard normal distribution.

In the proof, we will see that Theorem 1.5 still holds when (b_n, ℓ_n) is replaced by $(b'_n, \ell'_n) \in \mathbb{R}^2$, provided that $b'_n = b'_n(\xi)$ and $\ell'_n = \ell'_n(\xi)$ satisfy $b'_n \sim b_n$ and $\frac{\ell'_n - \ell_n}{b_n} \to 0$. See Remark 4.2.

Remark 1.2. Notice that since $\frac{Z_n(\mathbb{R})}{P_n} \to W$ a.s., we can rewrite (1.6) in the form similar to (1.5): a.s.

$$\frac{Z_n(-\infty, b_n x + \ell_n]}{P_n} \to W\Phi(x) \qquad \forall x \in \mathbb{R}. \tag{1.7}$$

Just as in the classical branching random walk, the case $\mathbb{E}(\frac{N}{m_0} \log^+ N) = \infty$ is not interesting for (1.7), as in this case $\frac{Z_n(\mathbb{R})}{P_n} \to W = 0$ a.s., so that (1.7) holds evidently.

Theorem 1.5 is an extension of the results of Kaplan and Asmussen [48, II, Theorem 1] and Biggins [15] on the classical branching random walk (with constant environment). Under a moment condition slightly stronger than the usual condition $\mathbb{E}\frac{N}{m_0}(\log^+ N) < \infty$, we can obtain a local limit theorem corresponding to Theorem 1.5, which extends the result of Biggins [15, Theorem 7] to the random environment case.

Theorem 1.6 (Local limit theorem)**.** *Assume that ν_0 is non-lattice a.s. If $\mathbb{E}\sigma_0^2 \in (0,\infty)$ and $\mathbb{E}\frac{N}{m_0}(\log^+ N)^\beta < \infty$ for some $\beta > \frac{3}{2}$, then for every $h > 0$,*

$$\sup_{x\in\mathbb{R}}\left|b_n\frac{Z_n(x,x+h)}{Z_n(\mathbb{R})} - hp\left(\frac{x-\ell_n}{b_n}\right)\right| \to 0 \quad a.s.,$$

where $p(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is the density function of the standard normal distribution.

For the classical branching random walk, the corresponding result has been established by Biggins [15, Theorem 7] under the usual moment condition $\mathbb{E}(N\log^+ N) < \infty$.

From Theorem 1.6, we immediately obtain the following corollary, which extends the result of Kaplan and Asmussen [48, Theorem 2] to the random environment case.

Corollary 1.7. *Under the conditions of Theorem 1.6, we have, a.s., for all $-\infty < a < b < \infty$,*

$$b_n\frac{Z_n(a+\ell_n, b+\ell_n)}{Z_n(\mathbb{R})} \to \frac{1}{\sqrt{2\pi}}(b-a).$$

1.4. Moderate deviation principle

Finally, we consider the moderate deviations associated to Z_n. Recently, Wang and Huang [72, Theorem 1.7] established a moderate deviation principle under the assumptions

$$\mu_0 = 0 \ \ a.s. \quad \text{and} \quad \operatorname{ess\,sup}\frac{1}{m_0}\mathbb{E}_\xi\sum_{u\in\mathbb{T}_1}e^{\delta|S_u|} < \infty \text{ for some } \delta > 0. \tag{1.8}$$

The following theorem completes their result by relaxing the boundedness assumption in (1.8) to a moment condition. Also, we consider the more general case without the centering condition $\mu_0 = 0$. Let (a_n) be a sequence of positive numbers satisfying

$$\lim_{n\to\infty}\frac{a_n}{\sqrt{n}} = \infty \qquad \text{and} \qquad \lim_{n\to\infty}\frac{a_n}{n}(\log n)^3 = 0. \tag{1.9}$$

Theorem 1.8 (Moderate deviation principle)**.** *If $\mathbb{E}\frac{1}{m_0}\sum_{u\in\mathbb{T}_1}e^{\delta|S_u-\mu_0|} < \infty$ for some $\delta > 0$ and $\mathbb{E}\log^+\mathbb{E}_\xi[(\sum_{u\in\mathbb{T}_1}e^{\delta|S_u-\mu_0|})^\gamma] < \infty$ for some $\gamma > 1$, then almost surely, the sequence of probabilities $A \mapsto Z_n(a_nA+\ell_n)/Z_n(\mathbb{R})$ satisfies a moderate deviation principle with rate function $\frac{x^2}{2\sigma^2}$ and speed $\frac{a_n^2}{n}$: for measurable set $A \subset \mathbb{R}$,*

$$-\frac{1}{2\sigma^2}\inf_{x\in A^\circ}x^2 \le \liminf_{n\to\infty}\frac{n}{a_n^2}\log\frac{Z_n(a_nA+\ell_n)}{Z_n(\mathbb{R})} \le \limsup_{n\to\infty}\frac{n}{a_n^2}\log\frac{Z_n(a_nA+\ell_n)}{Z_n(\mathbb{R})} \le -\frac{1}{2\sigma^2}\inf_{x\in\bar{A}}x^2,$$

where $\sigma^2 = \mathbb{E}\sigma_0^2$, A° is the interior of A and $\bar{A}$ its closure.

When the second condition in (1.9) is replaced by the usual condition that $\lim_{n\to\infty} a_n/n = 0$, the moderate deviation principle was proved in [72] under the stronger condition (1.8).

The remainder of the paper is organized as follows. In Section 2, after giving a large deviation principle for the quenched mean $\mathbb{E}_\xi Z_n$, we prove Theorem 1.1 about the convergence of free energy (which implies Theorem 1.2 and Corollary 1.3 on large deviations) and Theorem 1.4, the law of large numbers for leftmost and rightmost positions. Sections 3 and 4 are devoted to central limit theorems: in Section 3, we consider a branching random walk with a varying environment in time and prove the corresponding central limit theorem and local limit theorem; in Section 4 we return to the random environment case and give the proofs of Theorems 1.5 and 1.6. Finally, in Section 5, we deal with moderate deviations and prove Theorem 1.8.

2. Proofs of Theorems 1.1 and 1.4

At first, we establish a large deviation principle for the quenched means $\mathbb{E}_\xi Z_n(n\cdot)$.

Theorem 2.1. *For almost every ξ, the sequence of finite measures $A \mapsto \mathbb{E}_\xi Z_n(nA)$ satisfies a large deviation principle with rate function Λ^*: for measurable set $A \subset \mathbb{R}$,*

$$-\inf_{x\in A^\circ} \Lambda^*(x) \le \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\xi Z_n(nA) \le \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\xi Z_n(nA) \le -\inf_{x\in \bar{A}} \Lambda^*(x).$$

Proof. We consider the sequence of probability measures $\mathbb{E}_\xi Z_n(n\cdot)/\mathbb{E}_\xi Z_n(\mathbb{R})$. Notice that by the ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \log \int e^{tx} \mathbb{E}_\xi Z_n(dx) = \lim_{n\to\infty} \frac{1}{n} \log P_n(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log m_i(t) = \Lambda(t) \quad a.s.,$$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\xi Z_n(\mathbb{R}) = \lim_{n\to\infty} \frac{1}{n} \log P_n(0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log m_i = \Lambda(0) \quad a.s.$$

Using the Gärtner-Ellis theorem [28, p.52, Exercise 2.3.20] to the sequence of probability measures $\mathbb{E}_\xi Z_n(n\cdot)/\mathbb{E}_\xi Z_n(\mathbb{R})$, and after scaling, we get the desired result. □

Now let us give the proofs of Theorems 1.1 and 1.4 which are composed by several lemmas. Similar arguments have been used in [20, 32] in the constant environment case. Recall that $W(t)$ is defined in (1.3), as the a.s. limit of the fundamental martingale $W_n(t)$. Using a result of Biggins and Kyprianou [16] on a BRWRE, or that of Kuhlbusch [51] on weighted branching processes in random environment, we obtain the following lemma.

Lemma 2.1. *If $t \in (t_-, t_+)$ and $\mathbb{E}W_1(t) \log^+ W_1(t) < \infty$, then $W(t) > 0$ a.s.; if $t \le t_-$ or $t \ge t_+$, then $W(t) = 0$ a.s.*

Lemma 2.2. *For $t \in (t_-, t_+)$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \tilde{Z}_n(t) = \Lambda(t) \qquad a.s. \tag{2.1}$$

Proof. If $\mathbb{E}W_1(t) \log^+ W_1(t) < \infty$, then by Lemma 2.1, $W(t) > 0$ a.s. Consequently,

$$\frac{1}{n} \log \tilde{Z}_n(t) = \frac{1}{n} \log W_n(t) + \frac{1}{n} \log P_n(t) \to \Lambda(t) \quad a.s.$$

Now we consider the general case where $\mathbb{E}W_1(t)\log^+ W_1(t)$ may be infinite. Let $c > 0$. We construct a new BRWRE in which the point process for a particle u is

$$X^c(u) = (N^c(u), L_1(u), L_2(u), \cdots) \quad \text{with} \quad N^c(u) = N(u) \wedge c,$$

where and throughout we write $a \wedge b = \min(a, b)$.

We shall apply Lemma 2.1 to the new BRWRE. To this end, we will prove that for $c > 0$ large enough, the new BRWRE satisfies the two conditions of Lemma 2.1.

We first observe that with $m_0^c(t) = \mathbb{E}_\xi \sum_{i=1}^{N\wedge c} e^{tLi}$ for $c > 0$, we have

$$\Lambda_c(t) := \mathbb{E}\log m_0^c(t) \uparrow \Lambda(t) \quad \text{as } c \uparrow \infty. \tag{2.2}$$

In fact, on the one hand, by the monotone convergence theorem, $m_0^c(t) = \mathbb{E}_\xi \sum_{i=1}^{N\wedge c} e^{tL_i} \uparrow m_0(t)$ as $c \uparrow \infty$, which leads to $\mathbb{E}\log^+ m_0^c(t) \uparrow \mathbb{E}\log^+ m_0(t)$. On the other hand, by (1.2), we have $\log^- m_0^c(t) \le |t|\mathbb{E}_\xi|L_1|$, so that $\mathbb{E}\log^- m_0^c(t) \downarrow \mathbb{E}\log^- m_0(t)$ by the dominated convergence theorem, since $\mathbb{E}|L_1| < \infty$. This ends the proof of (2.2).

We next prove that for $c > 0$ large enough, the new BRWRE satisfies the first condition of Lemma 2.1. Let $t \in (t_-, t_+)$. We need to prove that $t \in (t_-^c, t_+^c)$ for c large enough. Notice that $t \in (t_-, t_+)$ is equivalent to $t\Lambda'(t) - \Lambda(t) < 0$, and $t \in (t_-^c, t_+^c)$ is equivalent to $t\Lambda_c'(t) - \Lambda_c(t) < 0$. We shall show that the latter holds for c large enough. Assume that $t > 0$. By the definition of $\Lambda'(t)$, there exists a $h > 0$ such that

$$t\frac{\Lambda(t+h) - \Lambda(t)}{h} - \Lambda(t) < 0.$$

Since $\Lambda_c \uparrow \Lambda$ as $c \uparrow \infty$ (cf. (2.2)), this implies that for c large enough,

$$t\frac{\Lambda_c(t+h) - \Lambda_c(t)}{h} - \Lambda_c(t) < 0. \tag{2.3}$$

The convexity of $\Lambda_c(t)$ shows that

$$\Lambda_c'(t) \le \frac{\Lambda_c(t+h) - \Lambda_c(t)}{h}. \tag{2.4}$$

Combining (2.4) with (2.3) we obtain that for c large enough,

$$t\Lambda_c'(t) - \Lambda_c(t) < 0. \tag{2.5}$$

If $t < 0$, we can also obtain (2.5) by a similar argument. So we have proved that $t \in (t_-, t_+)$ implies that $t \in (t_-^c, t_+^c)$ for c large enough.

We now prove that the second condition in Lemma 2.1 is also satisfied for the new BRWRE, that is, $\mathbb{E}W_1^c(t)\log^+ W_1^c(t) < \infty$, when c is large enough. Let $Y = W_1^c(t)$. We define a random variable X whose distribution is determined by $\mathbb{E}_\xi g(X) = \mathbb{E}_\xi Y g(Y)$ for all bounded and measurable function g. For $x \in \mathbb{R}_+ = [0, \infty)$, let

$$l(x) = \begin{cases} x/e & \text{if } x < e, \\ \log x & \text{if } x \ge e. \end{cases}$$

It is clear that l is concave and $\log^+ x \le l(x) \le 1 + \log^+ x$ for all $x \in \mathbb{R}_+$. Thus

$$\mathbb{E}_\xi Y\log^+ Y = \mathbb{E}_\xi \log^+ X \le \mathbb{E}_\xi l(X) \le l(\mathbb{E}_\xi X) \le 1 + \log^+ \mathbb{E}_\xi Y^2 \le 1 + \log^+\left(\frac{cm_0^c(2t)}{m_0^c(t)^2}\right).$$

Taking the expectation in the above inequality, we get

$$\begin{aligned}\mathbb{E}W_1^c(t)\log^+ W_1^c(t) &\leq 1+\mathbb{E}\log^+\left(\frac{cm_0^c(2t)}{m_0^c(t)^2}\right)\\ &\leq 1+\log c+\mathbb{E}\log^+ m_0(2t)+2|t|\mathbb{E}|L_1|<\infty.\end{aligned}$$

Therefore for $c>0$ large enough, the new BRWRE satisfies the two conditions of Lemma 2.1. So Lemma 2.1 implies that

$$\lim_{n\to\infty}\frac{1}{n}\log\tilde{Z}_n^c(t)=\mathbb{E}\log m_0^c(t)=\Lambda_c(t)\ \ a.s. \tag{2.6}$$

Finally, we will deduce (2.1) from (2.6). For the lower bound, since $\tilde{Z}_n(t)\geq\tilde{Z}_n^c(t)$, it follows that

$$\liminf_{n\to\infty}\frac{1}{n}\log\tilde{Z}_n(t)\geq\Lambda_c(t)\ \ a.s.$$

Letting $c\uparrow\infty$, we obtain

$$\liminf_{n\to\infty}\frac{1}{n}\log\tilde{Z}_n(t)\geq\Lambda(t)\ \ a.s.$$

For the upper bound, from the decomposition $\frac{1}{n}\log\tilde{Z}_n(t)=\frac{1}{n}\log W_n(t)+\frac{1}{n}\log P_n(t)$, we get

$$\limsup_{n\to\infty}\frac{1}{n}\log\tilde{Z}_n(t)\leq\limsup_{n\to\infty}\frac{1}{n}\log^+ W_n(t)+\limsup_{n\to\infty}\frac{1}{n}\log P_n(t)=\Lambda(t)\ \ a.s.$$

This completes the proof of (2.1). □

Lemma 2.3. *It is a.s. that* $\limsup_{n\to\infty}M_n^R/n\leq\Lambda'(t_+)$.

Proof. For $c>\Lambda'(t_+)$, we have $\Lambda^*(c)>0$. By Theorem 2.1,

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}_\xi Z_n[cn,\infty)=-\Lambda^*(c)<0\ \ a.s.$$

This leads to $\sum_n\mathbb{P}_\xi(Z_n[cn,\infty)\geq 1)<\infty$ a.s. It follows by Borel-Cantelli's lemma that, $\mathbb{P}_\xi$-a.s., $Z_n[cn,\infty)=0$ for n large enough, so that $M_n^R<cn$, which implies that a.s. $\limsup_n\frac{M_n^R}{n}\leq c$. Letting $c\downarrow\Lambda'(t_+)$ yields the result. □

Lemma 2.4. *If* $t\geq t_+$, *then a.s.,*

$$\lim_{n\to\infty}\frac{\log\tilde{Z}_n(t)}{n}=t\Lambda'(t_+).$$

Proof. Since $t\in\mathbb{R}$ and $t\geq t_+$, we have $t_+<\infty$. We first consider the upper bound. Choose $0<t_0<t_+\leq t$. Since $S_u\leq M_n^R$ for $u\in\mathbb{T}_n$, we have $\tilde{Z}_n(t)\leq\tilde{Z}_n(t_0)e^{(t-t_0)M_n^R}$. Thus

$$\frac{\log\tilde{Z}_n(t)}{n}\leq\frac{\log\tilde{Z}_n(t_0)}{n}+(t-t_0)\frac{M_n^R}{n}.$$

Letting $n\to\infty$ and using Lemma 2.3, we get that a.s.,

$$\limsup_{n\to\infty}\frac{\log\tilde{Z}_n(t)}{n}\leq\Lambda(t_0)+(t-t_0)\Lambda'(t_+).$$

Letting $t_0 \uparrow t_+$ and using $\Lambda(t_+) - t_+\Lambda'(t_+) = 0$, we obtain that a.s.,

$$\limsup_{n\to\infty} \frac{\log \tilde{Z}_n(t)}{n} \le t\Lambda'(t_+).$$

We next consider the lower bound. As $\log \tilde{Z}_n(t)$ is a convex function of t, for $t_0 < t_1 < t_+$, we have

$$\frac{\log \tilde{Z}_n(t) - \log \tilde{Z}_n(t_0)}{t - t_0} \ge \frac{\log \tilde{Z}_n(t_1) - \log \tilde{Z}_n(t_0)}{t_1 - t_0}.$$

Applying Lemma 2.2 to t_0 and t_1, we deduce that a.s.,

$$\liminf_{n\to\infty} \frac{\log \tilde{Z}_n(t)}{n} \ge \Lambda(t_0) + \frac{t - t_0}{t_1 - t_0}(\Lambda(t_1) - \Lambda(t_0)).$$

Letting $t_1 \downarrow t_0$, we get that a.s.

$$\liminf_{n\to\infty} \frac{\log \tilde{Z}_n(t)}{n} \ge \Lambda(t_0) + (t - t_0)\Lambda'(t_0).$$

Letting $t_0 \uparrow t_+$ and using $\Lambda(t_+) - t_+\Lambda'(t_+) = 0$, we obtain that a.s.,

$$\liminf_{n\to\infty} \frac{\log \tilde{Z}_n(t)}{n} \ge t\Lambda'(t_+).$$

This completes the proof of Lemma 2.4. □

Lemma 2.5. *It is a.s. that* $\liminf_{n\to\infty} \frac{M_n^R}{n} \ge \Lambda'(t_+)$.

Proof. Notice that $S_u \le M_n^R$ for $u \in \mathbb{T}_n$, we have $\tilde{Z}_n(t) \le Z_n(\mathbb{R})e^{tM_n^R}$, so that for each $0 < t < \infty$,

$$\frac{\log \tilde{Z}_n(t)}{n} \le \frac{\log Z_n(\mathbb{R})}{n} + t\frac{M_n^R}{n}. \tag{2.7}$$

If $t_+ < \infty$, then by Lemma 2.4, the above inequality gives for $t > t_+$, a.s.,

$$\Lambda'(t_+) \le \frac{1}{t}\mathbb{E}\log m_0 + \liminf_{n\to\infty} \frac{M_n^R}{n}.$$

Letting $t \uparrow \infty$, we obtain the desired result.

If $t_+ = \infty$, then by Lemma 2.2 and inequality (2.7), we see that for $t > 0$, a.s.,

$$\frac{\Lambda(t)}{t} \le \frac{1}{t}\mathbb{E}\log m_0 + \liminf_{n\to\infty} \frac{M_n^R}{n}.$$

Letting $t \uparrow \infty$, we get that $\liminf_n \frac{M_n^R}{n} \ge \Lambda'(\infty) = \Lambda'(t_+)$ a.s.. □

The conclusions for $t \le t_-$ and M_n^L can be obtained in a similar way, or by applying the obtained results for $t \ge t_+$ and M_n^R to the opposite branching random walk $-S_u$. Hence Theorem 1.4 holds, and (1.4) holds a.s. for each fixed $t \in \mathbb{R}$. So a.s. (1.4) holds for all rational t, and therefore for all real t by the convexity of $\log \tilde{Z}_n(t)$. This ends the proof of Theorem 1.1.

3. Limit theorems for a branching random walk in a varying environment

In this section we consider a branching random walk in varying environment (BRWVE), and prove that with a suitable norming, the counting measure converges to a non-degenerate law, when the intensity measures satisfy a suitable norming condition and when some moment conditions are also satisfied.

3.1. Branching random walk in varying environment

To prove limit theorems for random walks in random environments, we first establish corresponding results for a branching random walk with a varying environment (BRWVE), which is of independent interest. When the environment ξ is given, a BRWRE is a BRWVE. Therefore, we can apply the results for a BRWVE to a BRWRE for a given environment ξ.

A BRWVE is defined in the same way as a BRWRE when the environment is given. For completeness let us give its precise definition as below, with an abuse of the notation already introduced (there will be no confusion according to the context). At time 0, there is a unique ancestor particle $\emptyset$ of generation 0, which lies at $S_\emptyset = 0$; at time 1, it will be replaced by $N(\emptyset)$ particles of generation 1, located at $L_i(\emptyset)$, $1 \le i \le N$; the random vector $X(\emptyset) = (N(\emptyset), L_1(\emptyset), L_2(\emptyset), \cdots)$ takes values on $\mathbb{N} \times \mathbb{R}^{\mathbb{N}}$. In general, each particle $u = u_1 \cdots u_n$ in the nth generation (at time n), which lies at S_u, will be replaced at time $n+1$ by some new particles ui of generation $n+1$ with locations

$$S_{ui} = S_u + L_i(u), \qquad 1 \le i \le N(u);$$

the random vectors $X(u) = (N(u), L_1(u), L_2(u), \cdots)$, formulated by the number of offspring and their displacements, indexed by all finite sequences $u \in U$, are independent (which interprets the hypothesis that all particles behave independently); all $X(u)$ with $u \in \mathbb{N}^n$ have the same distribution as the random vector $X(1_n) = (N(1_n), L_1(1_n), L_2(1_n), \cdots)$ with values in $\mathbb{N} \times \mathbb{R} \times \mathbb{R} \times \cdots$. Recall that $1_n = (1, \cdots, 1)$ is the sequence of length n whose components are all equal to 1; by convention $1_0 = \emptyset$. By definition, a BRWRE with random environment ξ is just a class of BRWVE indexed by ξ.

As in the random environment case, we also use the following notation. Let

$$Z_n = \sum_{u \in \mathbb{T}_n} \delta_{S_u}$$

be the counting measure on $\mathbb{R}$, so that $Z_n(B)$ counts the number of particles of generation n situated in a Borel set B of $\mathbb{R}$. Let

$$X_n = \sum_{i=1}^{N(1_n)} \delta_{L_i(1_n)}$$

be the counting measure whose atoms are the displacements $L_i(1_n)$ of the children of 1_n. Set

$$N_n = N(1_n) = X_n(\mathbb{R}), \qquad m_n = \mathbb{E}N_n, \qquad P_0 = 1 \quad \text{and} \quad P_n = \mathbb{E}Z_n(\mathbb{R}) = \prod_{i=0}^{n-1} m_i.$$

Assume that

$$0 < m_n < \infty \ (\forall n \ge 0), \quad \liminf_{n\to\infty} \frac{1}{n} \log P_n > 0 \quad \text{and} \quad \liminf_{n\to\infty} \frac{1}{n} \log m_n = 0. \tag{3.1}$$

Thus for some $c > 1$, there exists an integer n_0 which may depend on c such that

$$P_n > c^n \qquad \text{for all } n > n_0. \tag{3.2}$$

Denote by Γ the probability space under which the process is defined. Let $\mathcal{F}_0 = \{\emptyset, \Gamma\}$ and $\mathcal{F}_n = \sigma((N(u), L_1(u), L_2(u), \cdots); |u| < n)$ $(n \geq 1)$ be the σ-field containing all the information of the first n generations. Then the sequence

$$W_n := \frac{Z_n(\mathbb{R})}{P_n}, \qquad n \geq 0,$$

forms a non-negative martingale with respect to the filtration $\{\mathcal{F}_n\}$ and converges a.s. to a random variable W with values in $[0, \infty)$. Let

$$\nu_n(\cdot) = \mathbb{E}\frac{X_n(\cdot)}{m_n}$$

be the intensity measure (on $\mathbb{R}$) at time n, and let ϕ_n be its characteristic function:

$$\phi_n(t) = \int e^{\mathbf{i}tx}\nu_n(dx) = \frac{1}{m_n}\mathbb{E}\int e^{\mathbf{i}tx}X_n(dx), \qquad t \in \mathbb{R}.$$

The characteristic function of the random normalized probability measure $\frac{Z_n}{P_n}$ is denoted by

$$\Psi_n(t) = \frac{1}{P_n}\int e^{\mathbf{i}tx}Z_n(dx) = \frac{1}{P_n}\sum_{u\in\mathbb{T}_n} e^{\mathbf{i}tS_u}, \qquad t \in \mathbb{R}.$$

It is not difficult to see that $\{\phi_n\}$ and $\{\Psi_n\}$ have the following relation:

$$\mathbb{E}\Psi_n(t) = \prod_{i=0}^{n-1}\phi_i(t), \qquad t \in \mathbb{R}.$$

As in Klebaner [50] and Biggins [15], we will use the following normalization condition on the family of intensity measures $\{\nu_n\}$.

Condition (A). *There are constants $\{b_n, c_n\}$ with $b_n \to \infty$ and a non-degenerate probability distribution L such that as $n \to \infty$, for all $t \in \mathbb{R}$,*

$$e^{-\mathbf{i}tc_n}\prod_{i=0}^{n-1}\phi_i(\frac{t}{b_n}) \to g(t) = \int e^{\mathbf{i}tx}L(dx). \tag{3.3}$$

Notice that the expression of the left-hand side is the characteristic function of the normalized probability measure $\bar{G}_n$ defined by

$$\bar{G}_n(B) = G_n(b_n(B + c_n)), \text{ where } G_n = \nu_0 * \cdots * \nu_{n-1},$$

for any Borel set B of $\mathbb{R}$. Therefore by Levy's theorem, (3.3) holds if and only if the normalized sequence $\{\bar{G}_n\}$ converges weakly to L, which is equivalent to the convergence of $G_n(b_n(x + c_n))$ to $L(x)$ for any continuity point x of the distribution function $L(x) := L(-\infty, x]$ of L, where $G_n(x) = G_n((-\infty, x])$ is the distribution function of G_n. If additionally $b_{n+1}/b_n \to 1$, then the limit

distribution is called in the class L according to Feller [31], also known as a self-decomposable distribution. If $\{\nu_n\}$ satisfies the condition of Lindeberg (see (4.13)) or Liapounoff, writing

$$\ell_n = \sum_{i=0}^{n-1} \mu_i \quad \text{and} \quad b_n = (\sum_{i=0}^{n-1} \sigma_i^2)^{1/2}, \quad \text{with } \mu_n = \int x\nu_n(dx) \text{ and } \sigma_n^2 = \int |x-\mu_n|^2 \nu_n(dx),$$

we have $G_n(b_n x + \ell_n) \to \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$ for any $x \in \mathbb{R}$, so that condition (A) holds with $b_n' = b_n$, $c_n' = \ell_n / b_n$ and $L = N(0,1)$. See Section 4 for a discussion on this point.

Throughout we use the following usual notation:

$$x_n = o(y_n), \quad x_n = O(y_n) \quad \text{and} \quad x_n \sim y_n$$

to signify that $\lim_{n\to\infty} x_n/y_n = 0$, $\limsup_{n\to\infty} |x_n|/|y_n| < \infty$ and $\lim_{n\to\infty} x_n/y_n = 1$, respectively. The same notation are used when n is replaced by $t \in \mathbb{R}_+ = [0,\infty)$. We use the symbol C to denote a positive constant whose value may change from line to line. Denote for $n \geq 1$ and $\varepsilon > 0$,

$$\upsilon_n(\varepsilon) = \sum_{i=0}^{n-1} \int |x|^{\varepsilon} \nu_i(dx).$$

The following theorem states that when the intensity measures $\mathbb{E}\frac{X_i}{m_i}$ satisfy the normalizing condition (A), then the counting measures Z_n with a suitable norming converges, under some moment conditions.

Theorem 3.1 (Convergence of Z_n with a suitable norming)**.** *For a BRWVE satisfying (3.1), assume that: (a) for some $\delta > 0$,*

$$\sum_{n=2}^{\infty} \frac{1}{m_n n (\log n)^{1+\delta}} \mathbb{E} N_n \log^+ N_n (\log^+ \log^+ N_n)^{1+\delta} < \infty; \tag{3.4}$$

(b) for some $\varepsilon > 0$ and $\gamma_1 < \infty$,

$$\upsilon_n(\varepsilon) = o(n^{\gamma_1}); \tag{3.5}$$

(c) for some $\gamma_2 > 0$,

$$b_n^{-1} = o(n^{-\gamma_2}). \tag{3.6}$$

Then it is a.s. that for all $t \in \mathbb{R}$,

$$\Psi_n(\frac{t}{b_n}) - W \prod_{i=0}^{n-1} \phi_i(\frac{t}{b_n}) \to 0. \tag{3.7}$$

If additionally the characteristic functions ϕ_i of the intensity measures $\mathbb{E}\frac{X_i}{m_i}$ satisfy the normalizing condition (A), then it is a.s. that for all $t \in \mathbb{R}$,

$$e^{-\mathbf{i}tc_n} \Psi_n(\frac{t}{b_n}) \to g(t) W, \tag{3.8}$$

and that for any continuity point x of the distribution function $L(x) = L(-\infty, x]$ of L,

$$P_n^{-1} Z_n(-\infty, b_n(x + c_n)] \to L(x) W. \tag{3.9}$$

Remark 3.1. The above conclusions were obtained by Biggins [15, Theorems 1 and 2] while condition (3.4) is replaced by $\int x \log x \, F(dx) < \infty$, where F is the measure on $[0, \infty)$ whose distribution function is $F(x) := \sum_{k=0}^{[x]} \sup_n \mathbb{P}(N_n = k)$. In the homogeneous case where all the N_n have the same law, F is simply the offspring distribution. But for the general varying environment case, F does not have such a simple expression. We will see that conditions (3.4), (3.5) and (3.6) are well adapted for our purpose in the application to the random environment case, since in this case these conditions hold for almost every environment under simple moment conditions.

Remark 3.2. From the proof of Theorem 3.1, we will see that the condition (3.4) can be relaxed to the following two ones:

$$\sum_{n=1}^{\infty} \frac{1}{m_n n^{1+\delta_1}} \mathbb{E} N_n \log^+ N_n < \infty \tag{3.10}$$

for some $\delta_1 > 0$, and

$$\sum_{n=0}^{\infty} \Big(1 - \frac{\tilde{m}_{n,\kappa}}{m_n}\Big) < \infty \tag{3.11}$$

for $\kappa > 0$ large enough, where $\tilde{m}_{n,\kappa} = \mathbb{E} N_n I_n(N_n)$ with $I_n(x) = \mathbf{1}_{\{x(\log x)^\kappa \le P_{n+1}\}}$. In fact, in the the proof, we will use (3.10) and (3.11) instead of (3.4) (cf. Lemmas 3.3, 3.4 and 3.5). The condition (3.4) in Theorem 3.1 is used only to have (3.10) and (3.11). Notice that from Lemma 3.1, we see that (3.4) implies (3.11). The fact that (3.4) also implies (3.10) can be seen as follows. Using $\log^+ N_n \le e$ for $N_n \le e^e$, $1 \le \log^+ \log^+ N_n$ for $N_n > e^e$, and $\frac{1}{n^{1+\delta_1}} \le \frac{1}{n(\log n)^{1+\delta}}$ for all $n \ge n_0$ with $n_0 > 1$ large enough, we see that for all $n \ge n_0$

$$\frac{1}{m_n n^{1+\delta_1}} \mathbb{E} N_n \log^+ N_n \le \frac{e}{n^{1+\delta_1}} + \frac{1}{m_n n (\log n)^{1+\delta}} \mathbb{E}(N_n \log^+ N_n)(\log^+ \log^+ N_n)^{1+\delta}$$

Theofore (3.4) implies (3.10).

The following result is a local limit theorem on (Z_n). For a general BRWVE, we have found no such theorems in the literature, except the local limit theorem by Biggins [15] for the homogeneous case.

Theorem 3.2 (Local limit theorem)**.** *For a BRWVE satisfying (3.1), assume that $\{\phi_i\}$ satisfies condition (A) with $b_n \sim \theta n^\gamma$ for some constants $0 < \gamma \le \frac{1}{2}$ and $\theta > 0$, and g (the characteristic function of L) integrable on $\mathbb{R}$, and that for some $\iota > 0$,*

$$\sup_i \sup_{|t| \ge \iota} |\phi_i(t)| =: \rho_\iota < 1.$$

Assume also that the distributions of $\{N_n\}$ satisfies (3.5) and that for some $\delta > 0$ and $\beta > \gamma$,

$$\sum_{n=2}^{\infty} \frac{1}{m_n n (\log n)^{1+\delta}} \mathbb{E} N_n (\log^+ N_n)^{1+\beta} < \infty.$$

Then for any $h > 0$,

$$\sup_{x \in \mathbb{R}} \big| b_n P_n^{-1} Z_n(x, x+h) - W h p_L(x/b_n - c_n) \big| \to 0 \qquad a.s., \tag{3.12}$$

where p_L denotes the continuous density function of L (which exists because its characteristic function g is supposed to be integrable).

3.2. Proof of Theorem 3.1

Notice that (3.7) together with condition (A) implies (3.8); by standard results on the convergence of measures, (3.8) holds for all $t \in \mathbb{R}$ if and only if (3.9) holds for any continuity point x of the distribution function of L. Therefore, to prove Theorem 3.1, we only need to prove that a.s. (3.7) holds for all $t \in \mathbb{R}$. Notice also that the convergence of a sequence of characteristic functions to a characteristic function for a countable dense set of t implies convergence for all t (cf. Klebaner [50, p.368]). So we only need to prove that a.s. (3.7) holds for each fixed $t \in \mathbb{R}$. Our proof below is inspired by some ideas from [15, 50].

We will use a truncation method. Let $\kappa > 0$ be a constant. Consider a new BRWVE while the branching numbers N_u are replaced by

$$\tilde{N}_u = N_u I_n(N_u) \qquad \forall u \in \mathbb{N}^{*n}, \quad n \geq 0,$$

where $I_n(x) = \mathbf{1}_{\{x(\log x)^\kappa \leq P_{n+1}\}}$. We use the notation $\tilde{Z}_{n,\kappa}, \tilde{X}_{n,\kappa}, \tilde{m}_{n,\kappa}, \tilde{\nu}_{n,\kappa}, \tilde{\phi}_{n,\kappa}$ etc. for the new BRWVE just as we used $Z_n, X_n, m_n, \nu_n, \phi_n$ etc. for the original process. Then $\tilde{m}_{n,\kappa} = \mathbb{E}\tilde{X}_n = \mathbb{E}N_n I_n(N_n)$ and

$$\tilde{\phi}_{n,\kappa}(t) = \int e^{\mathbf{i}tx}\tilde{\nu}_{n,\kappa}(dx) = \frac{1}{\tilde{m}_{n,\kappa}}\int e^{\mathbf{i}tx}\mathbb{E}\tilde{X}_{n,\kappa}(dx) = \frac{1}{\tilde{m}_{n,\kappa}}\mathbb{E}\int e^{\mathbf{i}tx}X_n(dx)\, I_n(N_n), \quad t \in \mathbb{R}.$$

The proof of (3.7) for a fixed t is composed of several lemmas.

Lemma 3.1. *Let $\beta \geq 0$. If $\sum_{n=2}^{\infty} \frac{1}{m_n n(\log n)^{1+\delta}}\mathbb{E}N_n(\log^+ N_n)^{1+\beta}(\log^+\log^+ N_n)^{1+\delta} < \infty$ for some $\delta > 0$, then for all $\kappa > 0$, $\sum_{n=0}^{\infty} n^\beta(1 - \tilde{m}_{n,\kappa}/m_n) < \infty$.*

Proof. Let $I_n^c(x) = 1 - I_n(x) = \mathbf{1}_{\{x(\log x)^\kappa > P_{n+1}\}}$. Then $m_n - \tilde{m}_{n,\kappa} = \mathbb{E}N_n I_n^c(N_n)$, so that for all $a > 0$,

$$\sum_n n^\beta\left(1 - \frac{\tilde{m}_{n,\kappa}}{m_n}\right) = \sum_n \frac{n^\beta}{m_n}\mathbb{E}N_n I_n^c(N_n)\mathbf{1}_{\{N_n > a\}} + \sum_n \frac{n^\beta}{m_n}\mathbb{E}N_n I_n^c(N_n)\mathbf{1}_{\{N_n \leq a\}}.$$

The second series converges for all $a > 0$ by (3.1) (see also (3.2)) since

$$\mathbb{E}N_n I_n^c(N_n)\mathbf{1}_{\{N_n \leq a\}} \leq \mathbb{E}N_n \frac{N_n(\log N_n)^\kappa}{P_{n+1}}\mathbf{1}_{\{N_n \leq a\}} \leq \frac{a^2(\log a)^\kappa}{P_{n+1}}.$$

It remains to prove that the first series converges for some $a > 0$. Notice that

$$f(x) := (\log x)^{1+\beta}(\log\log x)^{(1+\delta)}$$

is increasing and positive on $(a, +\infty)$ for $a > 0$ large enough, and $f(x(\log x)^\kappa) \sim f(x)$ as $x \to \infty$. Therefore from (3.2), we have for n large enough,

$$\begin{aligned}\frac{n^\beta}{m_n}\mathbb{E}N_n I_n^c(N_n)\mathbf{1}_{\{N_n > a\}} &\leq \frac{n^\beta}{m_n}\mathbb{E}\frac{N_n f(N_n(\log N_n)^\kappa)}{f(P_{n+1})}\mathbf{1}_{\{N_n > a\}} \\ &\leq \frac{C}{m_n n(\log n)^{1+\delta}}\mathbb{E}N_n(\log^+ N_n)^{1+\beta}(\log^+\log^+ N_n)^{1+\delta},\end{aligned}$$

where $C > 0$ is a constant independent of n. Thus $\sum_n \frac{n^\beta}{m_n}\mathbb{E}N_n I_n^c(N_n)\mathbf{1}_{\{N_n > a\}} < \infty$ by the condition of the lemma. □

Lemma 3.2 ([15], Lemma 3 (ii)). *For each $\kappa > 0$, if $\sum_n (1 - \tilde{m}_{n,\kappa}/m_n) < \infty$, then for each $t \in \mathbb{R}$,*

$$\prod_{i=0}^{n-1} \tilde{\phi}_{i,\kappa}(\frac{t}{b_n}) - \prod_{i=0}^{n-1} \phi_i(\frac{t}{b_n}) \to 0, \qquad \text{as } n \to \infty. \tag{3.13}$$

Proof of Theorem 3.1. As explained in the beginning of this section, we only need to prove that a.s. (3.7) holds for each fixed $t \in \mathbb{R}$. By Lemma 3.1 with $\beta = 0$ and Lemma 3.2, we see that (3.4) implies (3.13). Therefore for each fixed t, (3.7) holds a.s. if it holds a.s. with $\tilde{\phi}_{i,\kappa}$ in place of ϕ_i: that is, if

$$\Psi_n(\frac{t}{b_n}) - W \prod_{i=0}^{n-1} \tilde{\phi}_{i,\kappa}(\frac{t}{b_n}) \to 0 \qquad \text{a.s.} \tag{3.14}$$

In the following we will prove that (3.14) holds under the conditions of Theorem 3.1, when κ is large enough. For simplicity, let

$$\zeta_n(t) = \tilde{\phi}_{n,\kappa}(t) \qquad \text{and} \qquad \omega_n = \frac{\tilde{m}_{n,\kappa}}{m_n},$$

where the value of κ will be fixed later to be suitably large. Let

$$\Psi^{(1)}(u,t) = \frac{1}{m_n} \int e^{\mathbf{i}tx} X(u)(dx) \qquad \text{for } u \in \mathbb{N}^{*n} \text{ and } t \in \mathbb{R}. \tag{3.15}$$

Then

$$\begin{aligned} & \Psi_{n+1}(t) - \omega_n \zeta_n(t) \Psi_n(t) \\ = \; & \frac{1}{P_n} \sum_{u \in \mathbb{T}_n} e^{\mathbf{i}tS_u} \Psi^{(1)}(u,t) I_n^c(N(u)) + \frac{1}{P_n} \sum_{u \in \mathbb{T}_n} e^{\mathbf{i}tS_u} \left(\Psi^{(1)}(u,t) I_n(N(u)) - \omega_n \zeta_n(t) \right) \\ =: \; & A_n(t) + B_n(t). \end{aligned} \tag{3.16}$$

By iteration, we obtain for $0 \le k < n$ and $t \in \mathbb{R}$,

$$\Psi_n(t) = \Psi_k(t) \prod_{i=k}^{n-1} \omega_i \zeta_i(t) + \sum_{i=k}^{n-1} (A_i(t) + B_i(t)) \prod_{j=i+1}^{n-1} \omega_j \zeta_j(t) \tag{3.17}$$

(by convention the empty product is taken to be 1). Using this decomposition with t replaced by t/b_n, we see that the left-hand side of (3.14) can be written as:

$$\begin{aligned} \Psi_n(\frac{t}{b_n}) - W \prod_{i=0}^{n-1} \zeta_i(\frac{t}{b_n}) \quad = \quad & \sum_{i=k}^{n-1} A_i(\frac{t}{b_n}) \prod_{j=i+1}^{n-1} \omega_j \zeta_j(\frac{t}{b_n}) + \sum_{i=k}^{n-1} B_i(\frac{t}{b_n}) \prod_{j=i+1}^{n-1} \omega_j \zeta_j(\frac{t}{b_n}) \\ & + \left(\Psi_k(\frac{t}{b_n}) \prod_{i=k}^{n-1} \omega_i \zeta_i(\frac{t}{b_n}) - W \prod_{i=0}^{n-1} \zeta_i(\frac{t}{b_n}) \right). \end{aligned} \tag{3.18}$$

In the following we take $\alpha > 1$, and define

$$k = J(n) = [n^{1/\alpha}] = \text{ the integral part of } n^{1/\alpha}, \tag{3.19}$$

so that $k^\alpha \sim n$ as $n \to \infty$. For this choice of k, we will see in the following three lemmas that each of the three terms in the right-hand side of (3.18) converges to 0 a.s. when $\alpha > 1$ and $\kappa > 0$ are suitable chosen, which implies that (3.14) holds. So the proof of Theorem 3.1 is finished, once the following three lemmas are established. □

Lemma 3.3. *For each $\kappa > 0$, if $\sum_n (1 - \tilde{m}_{n,\kappa}/m_n) < \infty$, then, with $k = J(n) = [n^{1/\alpha}]$ and $\alpha > 1$,*

$$\sup_{t\in\mathbb{R}} \left| \sum_{i=k}^{n-1} A_i(\frac{t}{b_n}) \prod_{j=i+1}^{n-1} \omega_j \zeta_j(\frac{t}{b_n}) \right| \to 0 \qquad a.s., \qquad as\ n \to \infty. \tag{3.20}$$

Proof. Notice that for all $t \in \mathbb{R}$, $|A_n(t)| \le \frac{1}{P_n} \sum_{u\in\mathbb{T}_n} |\Psi_u^{(1)}(u,t)| I_n^c(N_u) \le \frac{1}{P_n} \sum_{u\in\mathbb{T}_n} \frac{N(u)}{m_n} I_n^c(N_u)$, so that

$$\left| \sum_{i=k}^{n-1} A_i(\frac{t}{b_n}) \prod_{j=i+1}^{n-1} \omega_j \zeta_j(\frac{t}{b_n}) \right| \le \sum_{i=k}^{n-1} |A_i(\frac{t}{b_n})| \le \sum_{i=k}^{n-1} \frac{1}{P_i m_i} \sum_{u\in\mathbb{T}_i} N(u) I_i^c(N(u)). \tag{3.21}$$

Since

$$\mathbb{E}\left(\sum_i \frac{1}{P_i m_i} \sum_{u\in\mathbb{T}_i} N(u) I_i^c(N(u)) \right) = \sum_i \frac{1}{m_i} \mathbb{E} N_i I_i^c(N_i) = \sum_i \left(1 - \frac{\tilde{m}_{i,\kappa}}{m_i} \right) < \infty,$$

we have $\sum_i \frac{1}{P_i m_i} \sum_{u\in\mathbb{T}_i} N(u) I_i^c(N(u)) < \infty$ a.s. Together with (3.21), this implies (3.20). □

Lemma 3.4. *If for some $\delta_1 > 0$,*

$$\sum_{n=1}^{\infty} \frac{1}{m_n n^{1+\delta_1}} \mathbb{E} N_n \log^+ N_n < \infty,$$

then when $\kappa \ge \alpha + \delta_1$, we have for each $t \in \mathbb{R}$, with $k = J(n) = [n^{1/\alpha}]$ and $\alpha > 1$,

$$C_n(t) := \sum_{i=k}^{n-1} B_i(\frac{t}{b_n}) \prod_{j=i+1}^{n-1} \omega_j \zeta_j(\frac{t}{b_n}) \to 0 \qquad a.s. \qquad as\ n \to \infty. \tag{3.22}$$

Proof. We will show that $\sum_{n=1}^{\infty} \mathbb{E}|C_n(t)|^2 < \infty$, which implies that $\sum_{n=1}^{\infty} |C_n(t)|^2 < \infty$ a.s., so that (3.22) holds a.s. Notice that $\mathbb{T}_n$ and S_u with $|u| \le n$ are $\mathcal{F}_n$ mesurable and that for each $u \in \mathbb{N}^{*n}$ (recall (3.15) for the definition of $\Psi^{(1)}(u,t)$),

$$\mathbb{E}[\Psi^{(1)}(u,t) I_n(N(u)) | \mathcal{F}_n] = \frac{1}{m_n} \mathbb{E} \sum_{i=1}^{N(u)} e^{\mathbf{i} t L_i(u)} I_n(N(u)) = \frac{\tilde{m}_{n,\kappa}}{m_n} \tilde{\phi}_{n,\kappa}(t) = \omega_n \zeta_n(t).$$

Therefore by the definition of $B_n(t)$ (cf. (3.16)), for all $t \in \mathbb{R}$,

$$\mathbb{E}(B_n(t) | \mathcal{F}_n) = 0 \quad \text{a.s.}$$

It follows that the summands in $C_n(t)$ are martingale differences (which are orthogonal in L^2), so that

$$\mathbb{E}|C_n(t)|^2 = \mathrm{Var}(C_n(t)) = \sum_{i=k}^{n-1} \mathrm{Var}\left(B_i(\frac{t}{b_n}) \prod_{j=i+1}^{n-1} \omega_j \zeta_j \right) \le \sum_{i=k}^{n-1} \mathrm{Var}(B_i(\frac{t}{b_n})).$$

Since the summands of $B_n(t)$ are conditionally independent given $\mathcal{F}_n$, we have for each $t \in \mathbb{R}$,

$$\mathbb{E}(|B_n(t)|^2 | \mathcal{F}_n) = \mathrm{Var}(B_n(t) | \mathcal{F}_n)) = \frac{1}{P_n^2} \sum_{u\in\mathbb{T}_n} \mathrm{Var}\left(\Psi^{(1)}(u,t) I_n(N(u)) \right) = \frac{Z_n(\mathbb{R})}{P_n^2} \mathrm{Var}\left(\Psi_n^{(1)}(t) I_n(N_n) \right),$$

where $\Psi_n^{(1)}(t) = m_n^{-1} \int e^{\mathbf{i}tx} X_n(dx)$. Taking the expectation, we get for each $t \in \mathbb{R}$,

$$\mathrm{Var}(B_n(t)) = \frac{1}{P_n} \mathrm{Var}(\Psi_n^{(1)} I_n(N_n)) \le \frac{1}{P_n m_n^2} \mathbb{E} N_n^2 I_n(N_n). \tag{3.23}$$

Set $J^{-1}(j) = \{n : J(n) = j\}$ and let $|J^{-1}(j)|$ be its cardinality. It is not difficult to see that $|J^{-1}(j)| = O(j^{\alpha-1})$ and $\sum_{j=1}^{i} |J^{-1}(j)| = O(i^\alpha)$. Recall that C denotes a positive constant whose value may change from line to line. From the preceding calculation for $\mathbb{E}|C_n(t)|^2$ and $\mathrm{Var}(B_n(t))$, and the estimation of $|J^{-1}(j)|$, we get for any $t \in \mathbb{R}$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{E}|C_n(t)|^2 &\le \sum_{n=1}^{\infty} \sum_{i=k}^{n-1} \frac{1}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \\
&= \sum_{j=1}^{\infty} \sum_{n \in J^{-1}(j)} \sum_{i=j}^{n-1} \frac{1}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \\
&\le \sum_{j=1}^{\infty} |J^{-1}(j)| \sum_{i=j}^{\infty} \frac{1}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{i} |J^{-1}(j)| \frac{1}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \\
&\le C \sum_{i=1}^{\infty} \frac{i^\alpha}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \\
&= C \sum_{i=1}^{\infty} \frac{i^\alpha}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \mathbf{1}_{\{N_i > a\}} + C \sum_{i=1}^{\infty} \frac{i^\alpha}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \mathbf{1}_{\{N_i \le a\}}.
\end{aligned}$$

In the last line, the second series converges since $\sum_i \frac{i^\alpha}{P_i m_i^2} < \infty$. For the first series, notice that $f(x) := x(\log x)^{-(\alpha+1+\delta_1)}$ is increasing and positive on $(a, +\infty)$ for $a > 0$ large enough, and $f(x(\log x)^\kappa) \sim x(\log x)^{\kappa-(\alpha+1+\delta_1)}$ as $x \to \infty$, so that $\frac{x^2}{f(x(\log x)^\kappa)} = O(x \log x)$ if $\kappa \ge \alpha + \delta_1$. Hence when $\kappa \ge \alpha + \delta_1$, by (3.1) we have for i large enough,

$$\frac{i^\alpha}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \mathbf{1}_{\{N_i > a\}} \le \frac{i^\alpha}{P_i m_i^2} \mathbb{E} N_i^2 \frac{f(P_{i+1})}{f(\{N_i (\log N_i)^\kappa)} \mathbf{1}_{\{N_i > a\}} \le \frac{C}{m_i i^{1+\delta_1}} \mathbb{E} N_i \log^+ N_i.$$

Thus $\sum_i \frac{i^\alpha}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i) \mathbf{1}_{\{N_i > a\}} < \infty$ by the condition of the lemma. □

Lemma 3.5. *If $\sum_n (1 - \tilde{m}_{n,\kappa}/m_n) < \infty$ and (3.5), (3.6) hold, then for each $\kappa > 0$ and each $\alpha > 1$ large enough, we have for each $t \in \mathbb{R}$, with $k = J(n) = [n^{1/\alpha}]$ and $\alpha > 1$,*

$$\Psi_k(\frac{t}{b_n}) \prod_{i=k}^{n-1} \omega_i \zeta_i(\frac{t}{b_n}) - W \prod_{i=0}^{n-1} \zeta_i(\frac{t}{b_n}) \to 0 \qquad a.s., \qquad as\ n \to \infty. \tag{3.24}$$

Proof. Since $\sum_n (1 - \omega_n) < \infty$ implies that $\prod_{i=k}^{n-1} \omega_i \to 1$, the factor $\prod_{i=k}^{n-1} \omega_i$ in (3.24) can be ignored, namely it suffices to prove that

$$\Psi_k(\frac{t}{b_n}) \prod_{i=k}^{n-1} \zeta_i(\frac{t}{b_n}) - W \prod_{i=0}^{n-1} \zeta_i(\frac{t}{b_n}) \to 0 \qquad a.s., \qquad \text{as } n \to \infty. \tag{3.25}$$

Notice that

$$\Psi_k \prod_{i=k}^{n-1} \zeta_i - W \prod_{i=0}^{n-1} \zeta_i = \left(\Psi_k - \frac{Z_k(\mathbb{R})}{P_k}\right) \prod_{i=k}^{n} \zeta_i + \left(\frac{Z_k(\mathbb{R})}{P_k} - W\right) \prod_{i=k}^{n-1} \zeta_i + W \left(\prod_{i=k}^{n-1} \zeta_i - \prod_{i=0}^{n-1} \zeta_i\right).$$

Therefore, since $W_k = \frac{Z_k(\mathbb{R})}{P_k} \to W$ a.s., to prove (3.25) it suffices to prove that

$$\Psi_k(\frac{t}{b_n}) - \frac{Z_k(\mathbb{R})}{P_k} \to 0 \qquad a.s., \qquad \text{as } k \to \infty. \tag{3.26}$$

and that

$$\prod_{i=k}^{n-1} \zeta_i(\frac{t}{b_n}) - \prod_{i=0}^{n-1} \zeta_i(\frac{t}{b_n}) \to 0, \qquad \text{as } k \to \infty. \tag{3.27}$$

We first prove (3.26). Since $|e^{\mathbf{i}tx} - 1| \le C|tx|^{\varepsilon}$, we have

$$\left|\Psi_k(\frac{t}{b_n}) - \frac{Z_k(\mathbb{R})}{P_k}\right| \le \frac{1}{P_k} \int \left|e^{\mathbf{i}tb_n^{-1}x} - 1\right| Z_k(dx) \le C|t|^{\varepsilon} b_n^{-\varepsilon} \frac{1}{P_k} \int |x|^{\varepsilon} Z_k(dx).$$

If $0 < \varepsilon \le 1$, by taking the expectation in the above inequality and using the elementary inequality $(x_1 + \cdots + x_k)^{\varepsilon} \le x_1^{\varepsilon} + \cdots + x_k^{\varepsilon}$ for $x_i \ge 0$ (which implies that $\int |x|^{\varepsilon} \nu_0 * \cdots * \nu_{k-1}(dx) \le \upsilon_k(\varepsilon)$) and conditions (3.5) and (3.6), we obtain

$$\begin{aligned} \mathbb{E}\left|\Psi_k(\frac{t}{b_n}) - \frac{Z_k(\mathbb{R})}{P_k}\right| &\le C|t|^{\varepsilon} b_n^{-\varepsilon} \int |x|^{\varepsilon} \nu_0 * \cdots * \nu_{k-1}(dx) \\ &\le C|t|^{\varepsilon} b_n^{-\varepsilon} \upsilon_k(\varepsilon) = |t|^{\varepsilon} o(k^{\gamma_1 - \alpha\varepsilon\gamma_2}). \end{aligned} \tag{3.28}$$

If $\varepsilon > 1$, by a similar argument using the inequality $(x_1 + \cdots + x_k)^{\varepsilon} \le k^{\varepsilon-1}(x_1^{\varepsilon} + \cdots + x_k^{\varepsilon})$ for $x_i \ge 0$, we get

$$\mathbb{E}\left|\Psi_k(\frac{t}{b_n}) - \frac{Z_k(\mathbb{R})}{P_k}\right| \le C|t|^{\varepsilon} b_n^{-\varepsilon} k^{\varepsilon-1} \upsilon_k(\varepsilon) = |t|^{\varepsilon} o(k^{\varepsilon - 1 + \gamma_1 - \alpha\varepsilon\gamma_2}). \tag{3.29}$$

Hence (3.26) holds if we take α large enough such that $\max(0, \varepsilon - 1) + \gamma_1 - \alpha\varepsilon\gamma_2 < -1$.

For (3.27), by combining (3.28) (when $\varepsilon \le 1$) or (3.29) (when $\varepsilon > 1$) with the fact that

$$\begin{aligned} \left|\prod_{i=k}^{n-1} \phi_i(\frac{t}{b_n}) - \prod_{i=0}^{n-1} \phi_i(\frac{t}{b_n})\right| &= \left|\prod_{i=k}^{n-1} \phi_i(\frac{t}{b_n}) \left(1 - \prod_{i=0}^{k-1} \phi_i(\frac{t}{b_n})\right)\right| \\ &\le \left|\prod_{i=0}^{k-1} \phi_i(\frac{t}{b_n}) - 1\right| \\ &= \left|\mathbb{E}\left(\Psi_k(\frac{t}{b_n}) - \frac{Z_k(\mathbb{R})}{P_k}\right)\right| \\ &\le \mathbb{E}\left|\Psi_k(\frac{t}{b_n}) - \frac{Z_k(\mathbb{R})}{P_k}\right|, \end{aligned}$$

we obtain (3.27) with ϕ_i in place of ζ_i, which implies (3.27) by Lemma 3.2. □

3.3. Proof of Theorem 3.2

We follow the method used in [69] and in [15]. Let

$$K(x) = \frac{1}{2\pi}\left(\frac{\sin(\frac{1}{2}x)}{\frac{1}{2}x}\right)^2, \quad K_a(x) = \frac{1}{a}K(\frac{x}{a}) \quad \text{for} \quad a > 0,\ x \in \mathbb{R},$$

with the convention that $K(0) = 1/(2\pi)$. Then

$$\int_{\mathbb{R}} K(x)dx = 1 \qquad \text{and} \qquad \int_{\mathbb{R}} K_a(x)dx = 1.$$

The characteristic function of K_a is denoted by k_a, which vanishes outside $(-\frac{1}{a}, \frac{1}{a})$, so that the characteristic function of $\frac{Z_n}{P_n} * K_a$ is integrable, and $\frac{Z_n}{P_n} * K_a$ has a continuous density function denoted by $D_a^{(n)}$. We will get our result through the asymptotic property of $D_a^{(n)}$.

Lemma 3.6 (see for example [27])**.** *If f is a characteristic function such that $|f(t)| \le \rho$ as soon as $b \le |t| \le 2b$, then we have for $|t| < b$,*

$$|f(t)| \le 1 - (1-\rho^2)\frac{t^2}{8b^2}.$$

Lemma 3.7. *Under the conditions of Theorem 3.2, for $\kappa > 0$ large enough,*

$$\sup_{x\in\mathbb{R}} |b_n D_a^{(n)}(b_n(x + c_n)) - Wp_L(x)| \to 0 \quad a.s. \qquad as \ \ n \to \infty. \tag{3.30}$$

Proof. Let $M > 0$ be a constant. By the Fourier inversion formula,

$$2\pi \left|b_n D_a^{(n)}(b_n(x + c_n)) - Wp_L(x)\right| = \left|\int \left(\Psi_n(\frac{t}{b_n})k_a(\frac{t}{b_n})e^{-\mathbf{i}tc_n} - Wg(t)\right)e^{-\mathbf{i}tx}dt\right|. \tag{3.31}$$

We split the integral of the right-hand side into $|t| < M$ and $|t| \ge M$. Using Theorem 3.1 (assertion (3.8)) and noticing that $\lim_n k_a(\frac{t}{b_n}) = 1$, we have, by the dominated convergence theorem,

$$\left|\int_{|t|<M} \left(\Psi_n(\frac{t}{b_n})k_a(\frac{t}{b_n})e^{-\mathbf{i}tc_n} - Wg(t)\right)e^{-\mathbf{i}tx}dt\right| \to 0 \qquad \text{a.s.} \quad \text{as} \quad n \to \infty.$$

So by (3.31), to show (3.30), it suffices to prove that

$$\limsup_{M\to\infty} \limsup_{n\to\infty} \left|\int_{|t|\ge M} \left(\Psi_n(\frac{t}{b_n})k_a(\frac{t}{b_n})e^{-\mathbf{i}tc_n} - Wg(t)\right)e^{-\mathbf{i}tx}dt\right| = 0 \quad \text{a.s.} \tag{3.32}$$

Notice that by the dominated convergence theorem and the integrability of g,

$$\int_{|t|\ge M} g(t)e^{-\mathbf{i}tx}dt \to 0 \quad \text{as} \quad M \to \infty. \tag{3.33}$$

It remains to estimate

$$I_{n,M} := \left|\int_{|t|\ge M} \Psi_n(\frac{t}{b_n})k_a(\frac{t}{b_n})e^{-\mathbf{i}tx}dt\right| = \left|\int_U b_n\Psi_n(t)k_a(t)e^{-\mathbf{i}b_n tx}dt\right|, \tag{3.34}$$

where $U = \{t : \frac{M}{b_n} \le |t| \le \frac{1}{a}\}$. In the following for simplicity we write $\Psi_n = \Psi_n(t), k_a = k_a(t)$, $\zeta_i = \zeta_i(t), A_i = A_i(t), B_i = B_i(t)$, and $e_n = e^{-\mathbf{i}b_n t x}$, where $t \in \mathbb{R}$ is arbitrary. By the decomposition (3.17), we have for $0 \le k < n$,

$$b_n \Psi_n k_a e_n = b_n \Psi_k \prod_{i=k}^{n-1} \omega_i \zeta_i k_a e_n + b_n \sum_{i=k}^{n-1} A_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j k_a e_n + b_n \sum_{i=k}^{n-1} B_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j k_a e_n. \tag{3.35}$$

Below we take $k = J(n)$ defined by (3.19) as in the proof of Theorem 3.1, and we prove that the integral over U of each of the three functions in the right-hand side of (3.35) tends to 0 a.s.

Firstly, we prove that

$$\left| b_n \sum_{i=k}^{n-1} \int_U A_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j k_a e_n dt \right| \to 0 \quad a.s. \qquad \text{as } n \to \infty \tag{3.36}$$

For n large enough,

$$\begin{aligned} \left| b_n \sum_{i=k}^{n-1} \int_U A_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j k_a e_n dt \right| &\le b_n \sum_{i=k}^{n-1} \int_U |A_i| dt \\ &\le C k^{\alpha\gamma} \sum_{i=k}^{n-1} \frac{1}{P_i m_i} \sum_{u \in \mathbb{T}_i} N(u) I_i^c(N(u)) \\ &\le C \sum_{i=k}^{n-1} \frac{i^{\alpha\gamma}}{P_i m_i} \sum_{u \in \mathbb{T}_i} N(u) I_i^c(N(u)). \end{aligned}$$

Like in the proof of Lemma 3.3, from Lemma 3.1 we obtain

$$\mathbb{E}\left(\sum_{i=0}^{\infty} \frac{i^{\alpha\gamma}}{P_i m_i} \sum_{u \in \mathbb{T}_i} N(u) I_i^c(N(u)) \right) = \sum_{i=0}^{\infty} i^{\alpha\gamma} \left(1 - \frac{\tilde{m}_{i,k}}{m_i} \right) < \infty,$$

if we take α sufficiently close to 1 such that $\alpha\gamma < \beta$. Hence (3.36) is proved.

Secondly, we prove (3.36) with B_i in place of A_i, that is:

$$C_n := b_n \sum_{i=k}^{n-1} \int_U B_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j k_a e_n dt \to 0 \quad a.s. \qquad \text{as } n \to \infty \tag{3.37}$$

We have already seen in the proof of Lemma 3.4 that $\mathbb{E}(B_i | \mathcal{F}_i) = 0$, so C_n is the sum of martingale differences. Therefore, for n large enough, using Cauchy-Schwartz's inequality and (3.23) we

get

$$\begin{aligned}
\mathbb{E}|C_n|^2 &= \operatorname{Var} C_n = b_n^2 \sum_{i=k}^{n-1} \operatorname{Var}\left(\int_U B_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j k_a e_n dt \right) \\
&= b_n^2 \sum_{i=k}^{n-1} \mathbb{E}\left| \int_U B_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j k_a e_n dt \right|^2 \\
&\le b_n^2 \sum_{i=k}^{n-1} \mathbb{E}\left(\int_U dt \right)\left(\int_U |B_i \prod_{j=i+1}^{n-1} \omega_j \zeta_j k_a|^2 dt \right) \\
&\le \frac{2}{a} b_n^2 \sum_{i=k}^{n-1} \int_U \mathbb{E}|B_i|^2 dt \\
&\le C \sum_{i=k}^{n-1} \frac{i^{2\alpha\gamma}}{P_i m_i^2} \mathbb{E} N_i^2 I_i(N_i).
\end{aligned}$$

Following the last part of the proof of Lemma 3.4, we obtain that $\sum_{n=1}^{\infty} \mathbb{E}|C_n|^2 < \infty$ provided that κ is large enough, which implies (3.37).

Finally, we consider the first term in the right-hand side of (3.35), and prove that there exists a constant $\theta_1 > 0$ (not depending on M) such that

$$\limsup_{n\to\infty} \left| b_n \int_U \Psi_k \prod_{i=k}^{n-1} \omega_i \zeta_i k_a e_n dt \right| \le W \int_{|t|\ge M} e^{-\theta_1 t^2} dt. \tag{3.38}$$

Clearly,

$$\left| b_n \int_U \Psi_k \prod_{i=k}^{n-1} \omega_i \zeta_i k_a e_n dt \right| \le \frac{Z_k(\mathbb{R})}{P_k} b_n \int_U \left| \prod_{i=k}^{n-1} \zeta_i \right| dt. \tag{3.39}$$

Since $\frac{Z_k(\mathbb{R})}{P_k} \to W$ a.s. as $k \to \infty$, it suffices to prove that

$$\limsup_{n\to\infty} b_n \int_U \left| \prod_{i=k}^{n-1} \zeta_i(t) \right| dt \le \limsup_{n\to\infty} b_n \int_U \prod_{i=k}^{n-1} |\phi_i(t)| dt, \tag{3.40}$$

and that there exists a constant $\theta_1 > 0$ (not depending on M) such that

$$\limsup_{n\to\infty} b_n \int_U \prod_{i=k}^{n-1} |\phi_i(t)| dt \le \int_{|t|\ge M} e^{-\theta_1 t^2} dt. \tag{3.41}$$

We first prove (3.40). Notice that

$$b_n \int_U \left| \prod_{i=k}^{n-1} \zeta_i \right| dt \le b_n \int_U \left| \prod_{i=k}^{n-1} \zeta_i - \prod_{i=k}^{n-1} \phi_i \right| dt + b_n \int_U \prod_{i=k}^{n-1} |\phi_i| dt.$$

By the definitions of ϕ_n and $\zeta_n = \tilde{\phi}_{n,\kappa}$, we have

$$m_n \phi_n(t) = \mathbb{E} \sum_{i=1}^{N_n} e^{\mathbf{i} t L_i(1_n)} = \tilde{m}_{n,\kappa} \zeta_n(t) + \mathbb{E} \sum_{\tilde{N}_n < i \le N_n} e^{\mathbf{i} t L_i(1_n)}.$$

Hence $|m_n\phi_n(t)-\tilde{m}_{n,\kappa}\zeta_n(t)| \le m_n-\tilde{m}_{n,\kappa}$, which implies $|\zeta_n(t)-\phi_n(t)| \le 2(1-\tilde{m}_{n,k}/m_n)$, as observed by Biggins [15] in the proof of Lemma 3 therein. Therefore we have

$$\begin{aligned} b_n \int_U \left| \prod_{i=k}^{n-1} \zeta_i - \prod_{i=k}^{n-1} \phi_i \right| dt &\le b_n \int_U \sum_{i=k}^{n-1} |\zeta_i - \phi_i| dt \\ &\le \frac{4}{a} b_n \sum_{i=k}^{n-1} \left(1 - \frac{\tilde{m}_{i,k}}{m_i}\right) \\ &\le C \sum_{i=k}^{n-1} (i+1)^{\alpha\gamma} \left(1 - \frac{\tilde{m}_{i,k}}{m_i}\right) \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

provided $\alpha\gamma < \beta$. Hence (3.40) holds. Now we turn to prove (3.41). Split the set U into two parts: $U_1 = \{t : M/b_n \le |t| \le \varepsilon\}$ and $U_2 = \{t : \varepsilon \le |t| \le \frac{1}{a}\}$. Since for some $\iota > 0$, $|\phi_i(t)| \le \rho_\iota < 1$ for all $|t| \ge \iota$, by Lemma 3.6, we have for all $|t| < \iota$,

$$|\phi_i(t)| \le 1 - \frac{1-\rho_\iota^2}{8\iota^2} t^2 \le e^{-\gamma_1 t^2},$$

where $\gamma_1 = \frac{1-\rho_\iota^2}{8\iota^2}$. Thus

$$\sup_i \sup_{|t|\ge\varepsilon} |\phi_i(t)| = \max\{e^{-\gamma_1\varepsilon^2}, \rho_\iota\} =: \rho'_\iota < 1.$$

It follows that

$$b_n \int_{U_2} \prod_{i=k}^{n-1} |\phi_i(t)| dt \le \frac{2}{a} b_n (\rho'_\iota)^{n-k-1} \to 0 \text{ as } n \to \infty, \tag{3.42}$$

and

$$b_n \int_{U_1} \prod_{i=k}^{n-1} |\phi_i(t)| dt \le \int_{|t|\ge M} \exp(-b_n^{-2}(n-k-1)\gamma_1 t^2) dt.$$

It is easy to see that

$$\lim_{n\to\infty} \frac{n-k-1}{b_n^2} = \begin{cases} \frac{1}{\theta^2} & \text{if } \gamma = \frac{1}{2}, \\ \infty & \text{if } 0 < \gamma < \frac{1}{2}. \end{cases}$$

So there exists a constant $\theta_1 > 0$ such that $b_n^{-2}(n-k-1)\gamma_1 > \theta_1$ for n large enough. Thus

$$\limsup_{n\to\infty} b_n \int_{U_1} \prod_{i=k}^{n-1} |\phi_i(t)| dt \le \int_{|t|\ge M} e^{-\theta_1 t^2} dt. \tag{3.43}$$

From (3.42) and (3.43), we obtain (3.41). Combining (3.39), (3.40) and (3.41), we get (3.38).

From the decomposition (3.35) and the assertions (3.36), (3.37) and (3.38), we see that the integral $I_{n,M}$ defined in (3.34) satisfies

$$\limsup_{n\to\infty} I_{n,M} \le \int_{|t|\ge M} e^{-\theta_1 t^2} dt.$$

This, together with (3.33), implies (3.32), which ends the proof of the lemma. □

By the proof of Stone [69, Lemma 1], we have the following result. Since the argument is very similar, we omit the details.

Lemma 3.8. *If (3.30) holds, then for every $\varepsilon > 0$, there exist $n_0 > 0$ and $\delta > 0$ such that for $n \geq n_0$ and $0 < h < \delta$, a.s.,*

$$h(Wp_L(x) - \varepsilon) \leq P_n^{-1} Z_n\Big(b_n(x + c_n), b_n(x + c_n + h)\Big) \leq h(Wp_L(x) + \varepsilon), \quad \forall x \in \mathbb{R}.$$

Now we turn to the proof of Theorem 3.2.

Proof of Theorem 3.2. Fix $h > 0$. For $\varepsilon > 0$, take $0 < \varepsilon' < \varepsilon/h$. By Lemmas 3.7 and 3.8, for this $\varepsilon' > 0$, there exist $n_0' > 0$ and $\delta' > 0$ such that for $n \geq n_0'$ and $0 < h' < \delta'$, a.s.,

$$h'(Wp_L(x) - \varepsilon') \leq P_n^{-1} Z_n\Big(b_n(x + c_n),\ b_n(x + c_n + h')\Big) \leq h'(Wp_L(x) + \varepsilon'), \quad \forall x \in \mathbb{R}.$$

Let $h' = h/b_n$. Then there exists $\tilde{n}_0 > 0$ such that $0 < h' < \delta'$ for $n \geq \tilde{n}_0$. So for $n \geq n_0 := \max\{n_0', \tilde{n}_0\}$, we have a.s.,

$$h(Wp_L(x) - \varepsilon') \leq b_n P_n^{-1} Z_n\Big(b_n(x + c_n),\ b_n(x + c_n) + h\Big) \leq h(Wp_L(x) + \varepsilon'), \quad \forall x \in \mathbb{R},$$

which implies that

$$\sup_{x \in \mathbb{R}} |b_n P_n^{-1} Z_n\Big(b_n(x + c_n), b_n(x + c_n) + h\Big) - Whp_L(x)| \leq \varepsilon' h < \varepsilon \qquad a.s.,$$

so that

$$\sup_{x \in \mathbb{R}} |b_n P_n^{-1} Z_n(x, x + h) - Whp_L(x/b_n - c_n)| < \varepsilon \qquad a.s.$$

This ends the proof of (3.12). □

From the proof, we have the following remark which will be used to relax a moment condition in the random environment case.

Remark 3.3. For a BRWVE satisfying (3.1), the conclusion (3.12) of Theorem 3.2 holds whenever the following conditions hold:

1) the convergence result (3.8) holds a.s. for all $t \in \mathbb{R}$, where $\{b_n, c_n\}$ are normalizing constants such that $b_n \sim \theta n^\gamma$ for some $0 < \gamma \leq \frac{1}{2}$ and $\theta > 0$, and g is the characteristic function of a non-degenerate probability distribution L on $\mathbb{R}$, which is integrable on $\mathbb{R}$;

2) there exists a constant $\beta > \gamma$ such that $\sum_n n^\beta (1 - \tilde{m}_{n,\kappa}/m_n) < \infty$ for κ large enough;

3) there exists a constant $\delta_1 > 0$ such that (3.10) holds;

4) there exists a constant $\theta_1 > 0$ such that (3.41) holds for all $M > 0$ large enough.

4. Proofs of Theorems 1.5 and 1.6

Let us return back to a BRWRE and establish central limit theorems for it. As already mentioned in Section 3, when the environment ξ is fixed, a BRWRE is a BRWVE. So the results of the previous section on a BRWVE can be applied to a BRWRE conditioned on the environment. We use the notation already introduced in the introduction for a BRWRE, which correspond the notation used in Section 3 but with the probability and expectation therein replaced by the quenched probability $\mathbb{P}_\xi$ and the quenched expectation $\mathbb{E}_\xi$, given the environment ξ. For example, the intensity measures ν_n are now defined as $\nu_n = \frac{\mathbb{E}_\xi X_n(\cdot)}{m_n}$, whose mean and variance are still denoted respectively by μ_n and σ_n^2; $m_n = \mathbb{E}_\xi N_n$ denotes the mean of the offspring distribution given the environment.

By Theorem 3.1, and Remark 3.2, we can obtain the following theorem about the convergence to infinitely divisible laws for a BRWRE.

Theorem 4.1. *Assume that for some $\varepsilon > 0$,*

$$\upsilon(\varepsilon) := \mathbb{E} \int |x|^\varepsilon \nu_0(dx) < \infty,$$

and that $b_n = b_n(\xi)$ satisfies

$$b_n^{-1} = o(n^{-\gamma}) \quad a.s. \quad \text{for some} \quad \gamma > 0,$$

then a.s. for all $t \in \mathbb{R}$,

$$\Psi_n(\frac{t}{b_n}) - W \prod_{i=0}^{n-1} \phi_i(\frac{t}{b_n}) \to 0. \tag{4.1}$$

If in addition (A) holds with $b_n = b_n(\xi)$, $c_n = c_n(\xi)$ and $g = g_\xi$ for almost every environment ξ, then it is a.s. that for all $t \in \mathbb{R}$,

$$e^{-\mathbf{i}tc_n}\Psi_n(\frac{t}{b_n}) \to g_\xi(t)W, \tag{4.2}$$

and for all continuity point x of the distribution function $L_\xi(x) = L_\xi(-\infty, x]$ of L_ξ,

$$P_n^{-1} Z_n(-\infty, b_n(x + c_n)] \to L_\xi(x)W.$$

Proof of Theorem 4.1. We apply Theorem 3.1 and Remark 3.2 by checking the conditions therein. We need to prove that (4.1) holds (for all t) $\mathbb{P}_\xi$ a.s. for almost every environment ξ (which implies that (4.1) holds $\mathbb{P}$ a.s.). Notice that for a BRWRE, assumption (1.1) implies that

$$\lim_{n\to\infty} \frac{1}{n} \log P_n = \mathbb{E} \log m_0 > 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{n} \log m_n = 0 \qquad a.s.$$

by the ergodic theorem. Hence the assumption (3.1) is satisfied for almost every environment ξ. The condition (3.4) now reads

$$\sum_n \frac{1}{m_n n (\log n)^{1+\delta}} \mathbb{E}_\xi N_n \log^+ N_n (\log^+ \log^+ N_n)^{1+\delta} < \infty$$

for some $\delta > 0$. Taking the expectation, we see that this condition holds for almost every environment ξ if $\mathbb{E}\frac{N}{m_0}\log^+ N(\log^+\log^+ N)^{1+\delta} < \infty$ for some $\delta > 0$. To relax this condition to the usual one $\mathbb{E}\frac{N}{m_0}\log^+ N < \infty$, we will use the conditions in Remark 3.2 instead of (3.4): namely,

$$\sum_{n=1}^{\infty} \frac{1}{m_n n^{1+\delta_1}} \mathbb{E}_\xi N_n \log^+ N_n < \infty \quad \text{a.s.} \tag{4.3}$$

for some $\delta_1 > 0$, and

$$\sum_{n=0}^{\infty} \left(1 - \frac{\tilde{m}_{n,\kappa}}{m_n}\right) < \infty \quad \text{a.s.} \tag{4.4}$$

for $\kappa > 0$ large enough. Taking the expectation at both sides of (4.3), we see that (4.3) holds a.s. for every $\delta_1 > 0$ provided that $\mathbb{E}\frac{N}{m_0}\log^+ N < \infty$. From Lemma 4.1 below, we see that this latter condition also implies that (4.4) holds a.s. for every $\kappa > 0$.

The condition (3.5) holds a.s. for each $\gamma_1 > 1$ because by the ergodic theorem,

$$\lim_n \frac{\upsilon_n(\varepsilon)}{n} = \upsilon(\varepsilon) < \infty \qquad a.s.$$

We have therefore proved Theorem 4.1, using Theorem 3.1, Remark 3.2 and the lemma below which will also be used later. □

Lemma 4.1. *Let $\beta \geq 0$. If $\mathbb{E}\frac{N}{m_0}(\log^+ N)^{1+\beta} < \infty$, then for all $\kappa > 0$, $\sum_n n^\beta(1 - \tilde{m}_{n,\kappa}/m_n) < \infty$ a.s.*

Proof. As the proof of Lemma 3.1, we have $\sum_n (1 - \tilde{m}_{n,\kappa}/m_n) = \sum_n \frac{1}{m_n}\mathbb{E}_\xi N_n I_n^c(N_n)$. By (3.2), for n large enough, $\mathbb{E}_\xi N_n I_n^c(N_n) \leq \mathbb{E}_\xi N_n \mathbf{1}_{\{N_n(\log N_n)^\kappa > c^{n+1}\}}$. Notice that

$$\begin{aligned}\mathbb{E}\left(\sum_n \frac{n^\beta}{m_n}\mathbb{E}_\xi N_n \mathbf{1}_{\{N_n(\log N_n)^\kappa > c^{n+1}\}}\right) &= \mathbb{E}\frac{N}{m_0}\sum_n n^\beta \mathbf{1}_{\{N(\log N)^\kappa > c^{n+1}\}} \\ &\leq C\mathbb{E}\frac{N}{m_0}(\log^+ N)^{1+\beta} < \infty.\end{aligned}$$

Therefore $\sum_n n^\beta(1 - \tilde{m}_{n,\kappa}/m_n) < \infty$ a.s. □

Similarly to the varying environment case, we also have a local limit theorem:

Theorem 4.2. *Assume that ν_0 is non-lattice a.s., condition (A) holds for almost every environment ξ, with $(b_n, c_n) = (b_n(\xi), c_n(\xi))$ satisfying a.s. $b_n \sim \theta n^\gamma$ for some constants $0 < \gamma \leq \frac{1}{2}$ and $\theta > 0$, and $g = g_\xi$ a.s. integrable. If $\upsilon(\varepsilon) := \mathbb{E}\int |x|^\varepsilon \nu_0(dx) < \infty$ for some $\varepsilon > 0$, and*

$$\mathbb{E}\frac{N}{m_0}(\log^+ N)^{1+\beta} < \infty$$

for some $\beta > \gamma$, then a.s. $\forall h > 0$,

$$\sup_{x\in\mathbb{R}} |b_n P_n^{-1} Z_n(x, x+h) - Whp_L(x/b_n - c_n)| \to 0, \tag{4.5}$$

where p_L is the continuous density function of L_ξ (which exists since its characteristic function g_ξ is integrable).

Notice that if we use Theorem 3.2 to the random environment case, then we need the (3.2) which is stronger than the non-lattice condition assumed in Theorem 4.2. For this reason, we will use Remark 3.3 instead. We first establish the following technical lemma.

Lemma 4.2. *Let $M > 0$ be a constant. Assume that $b_n \sim \theta n^\gamma$ a.s. for some constants $0 < \gamma \le \frac{1}{2}$. If ν_0 is non-lattice a.s.,then there exists a constant $\theta_1 > 0$ (not depending on M) such that*

$$\limsup_{n\to\infty} b_n \int_U \prod_{i=k}^{n-1} |\phi_i(t)| dt \le \int_{|t|\ge M} e^{-\theta_1 t^2} dt \quad a.s., \tag{4.6}$$

where $k = J(n)$ is defined by (3.19) as in the proof of Theorem 3.1 and $U = \{t : \frac{M}{b_n} \le |t| \le \frac{1}{a}\}$.

Proof. Take $0 < 2\varepsilon < \frac{1}{a}$. As the last part of the proof of Theorem 3.2, we split U into U_1 and U_2 defined therein, so that

$$b_n \int_U \prod_{i=k}^{n-1} |\phi_i(t)| dt = b_n \int_{U_1} \prod_{i=k}^{n-1} |\phi_i(t)| dt + b_n \int_{U_2} \prod_{i=k}^{n-1} |\phi_i(t)| dt.$$

Since ν_i is non-lattice a.s., we have

$$\sup_{\varepsilon\le|t|\le a^{-1}} |\phi_i(t)| =: \rho_i(\varepsilon, a) = \rho_i < 1 \qquad a.s. \tag{4.7}$$

Hence by Lemma 3.6, for $|t| < \varepsilon$,

$$|\phi_i(t)| \le 1 - \frac{1-\rho_i^2}{8\varepsilon^2} t^2 \le \exp\left(-\frac{1-\rho_i^2}{8\varepsilon^2} t^2\right) = e^{-\alpha_i t^2} \qquad a.s., \tag{4.8}$$

where $\alpha_i = \frac{1-\rho_i^2}{8\varepsilon^2} > 0$ a.s. Using (4.7), we immediately get

$$b_n \int_{U_2} \prod_{i=k}^{n-1} |\phi_i(t)| \le \frac{2}{a} b_n \prod_{i=k}^{n-1} \rho_i \to 0 \qquad a.s., \tag{4.9}$$

since

$$\lim_{n\to\infty} \frac{\log b_n + \sum_{i=k}^{n-1} \log \rho_i}{n} = \mathbb{E} \log \rho_0 < 0 \qquad a.s.$$

Observe that

$$\lim_{n\to\infty} \frac{\sum_{i=k}^{n-1} \alpha_i}{b_n^2} = \begin{cases} \frac{1}{\theta^2}\mathbb{E}\alpha_0 > 0 & \text{if } \gamma = \frac{1}{2} \\ \infty & \text{if } 0 < \gamma < \frac{1}{2} \end{cases} \qquad a.s.$$

So taking $0 < \theta_1 < \frac{1}{\theta^2}\mathbb{E}\alpha_0$ and using (4.8), we get a.s. for n large enough,

$$b_n \int_{U_1} \prod_{i=k}^{n-1} |\phi_i(t)| dt \le \int_{|t|\ge M} \exp\left(-b_n^{-2} \sum_{i=k}^{n-1} \alpha_i t^2\right) du \le \int_{|t|\ge M} e^{-\theta_1 t^2} dt. \tag{4.10}$$

Combining (4.9) and (4.10) yields (4.6). □

Proof of Theorem 4.2. We apply Remark 3.3 when the environment ξ is given. Below we prove that the four conditions of Remark 3.3 are satisfied for almost every environment ξ. First, by Theorem 4.1, we see that a.s. (4.2) holds for every t; in other words, (3.8) holds in the random environment case for almost every environment ξ. Second, by Lemma 4.1, $\sum_n n^\beta(1-\tilde{m}_{n,\kappa}/m_n) < \infty$ a.s. for all $\kappa > 0$. Third, in the proof of Theorem 4.1 we have already proved that (4.3) (which corresponds to (3.10) when environment ξ is given) holds a.s. for every $\delta_1 > 0$. Finally, by Lemma 4.2, there exists a constant $\theta_1 > 0$ such that (4.6) (which corresponds to (3.41) when random environment ξ is given) holds for all $M > 0$ large enough. Therefore, from Remark 3.3, we see that (4.5) holds $\mathbb{P}_\xi$ a.s. for almost every environment ξ. This implies that (4.5) holds $\mathbb{P}$ a.s. The proof of Theorem 4.2 is therefore finished. □

We now establish the convergence to the normal law, and the corresponding local limit theorem. To this end, we first prove that in the random environment case, under the usual second moment condition, the normalizing condition (A) holds a.s. with $L = N(0,1)$, as shown by the following central limit theorem for means $\mathbb{E}_\xi Z_n(\cdot)$. For the notation used below, we refer to the definitions introduced before Theorem 1.5 .

Theorem 4.3. *If $\mathbb{E}\sigma_0^2 \in (0,\infty)$, then a.s. for all $x \in \mathbb{R}$,*

$$\frac{\mathbb{E}_\xi Z_n(-\infty, b_n x + \ell_n]}{\mathbb{E}_\xi Z_n(\mathbb{R})} \to \Phi(x), \tag{4.11}$$

where $\ell_n = \sum_{i=0}^{n-1} \mu_i$, $b_n = (\sum_{i=0}^{n-1} \sigma_i^2)^{1/2}$, and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$ is the distribution function of the standard normal distribution.

Observe that $\frac{\mathbb{E}_\xi Z_n(\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})} = \nu_0 * \cdots * \nu_{n-1}(\cdot)$. So (4.11) just means that condition (A) holds for almost every environment with b_n defined above, $c_n = \ell_n/b_n$ and $L = N(0,1)$.

We notice that by using Theorem 4.3 and [7, Theorem 3.5], we can prove a weaker version of Theorem 1.5: if $\mathbb{E}\sigma_0^2 \in (0,\infty)$ and $\mathbb{E}(\log \frac{\mathbb{E}_\xi N^2}{m_0^2}) < \infty$, then for almost every ξ we have, for each $x \in \mathbb{R}$,

$$\frac{Z_n(-\infty, b_n x + \ell_n]}{Z_n(\mathbb{R})} \to \Phi(x) \quad \text{in probability under} \quad \mathbb{P}_\xi.$$

Actually we have the a.s. convergence, as seen in Theorem 1.5, assuming only $\mathbb{E}\sigma_0^2 \in (0,\infty)$ (without the additional condition $\mathbb{E}(\log \frac{\mathbb{E}_\xi N^2}{m_0^2}) < \infty$).

Remark 4.1. By Slutsky's lemma, we know that if F_n, F are probability distribution functions on $\mathbb{R}$ such that $F_n(x) \to F(x)$ for any continuity point x of F, and if $c_n, d_n \in \mathbb{R}$ are constants such that $c_n \to 1$ and $d_n \to 0$, then the same conclusion holds for $F_n(c_n x + d_n)$ instead of $F_n(x)$: as $n \to \infty$,

$$F_n(c_n x + d_n) \to F(x) \quad \text{for all continuity point x of F.}$$

With (4.11), using this for $F_n(x) = \frac{\mathbb{E}_\xi Z_n(-\infty, b_n x + \ell_n]}{\mathbb{E}_\xi Z_n(\mathbb{R})}$ and $F(x) = \Phi(x)$, for fixed ξ, and taking $c_n = b'_n/b_n$ and $d_n = (\ell'_n - \ell_n)/b_n$, we see that (4.11) still holds a.s. for all $x \in \mathbb{R}$, when (b_n, ℓ_n) is replaced by $(b'_n, \ell'_n) \in \mathbb{R}^2$, provided that $b'_n = b'_n(\xi)$ and $\ell'_n = \ell'_n(\xi)$ satisfy

$$b'_n \sim b_n \quad \text{and} \quad \frac{\ell'_n - \ell_n}{b_n} \to 0. \tag{4.12}$$

In particular, since $b_n \sim n^{1/2}(\mathbb{E}\sigma_0^2)^{1/2}$ a.s., (4.11) still holds a.s. for all $x \in \mathbb{R}$, when b_n is replaced by $n^{1/2}(\mathbb{E}\sigma_0^2)^{1/2}$. The advantage of the choice of b_n and ℓ_n is to make that the mean and variance of F_n to be 0 and 1, respectively:

$$\int x dF_n(x) = 0 \quad \text{and} \quad \int x^2 dF_n(x) = 1.$$

Proof of Theorem 4.3. Since $\frac{\mathbb{E}_\xi Z_n(\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})} = \nu_0 * \cdots * \nu_{n-1}(\cdot)$, it suffices to show that $\{\nu_n\}$ satisfies a.s. the Lindeberg condition, that is, a.s. for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \sum_{i=0}^{n-1} \int_{|x-\mu_i|>\varepsilon b_n} |x-\mu_i|^2 \nu_i(dx) = 0. \tag{4.13}$$

By the ergodic theorem,

$$\lim_{n\to\infty} \frac{b_n^2}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sigma_i^2 = \mathbb{E}\sigma_0^2 > 0 \qquad a.s. \tag{4.14}$$

So for a positive constant a satisfying $0 < a^2 < \mathbb{E}\sigma_0^2$, there exists an integer n_0 depending on a and ξ such that $b_n^2 \geq a^2 n$ for all $n \geq n_0$. Fix a constant $M > 0$. For $n \geq \max\{n_0, M\}$, we have $b_n^2 \geq a^2 n \geq a^2 M$, so that for each $\varepsilon > 0$,

$$\frac{1}{b_n^2} \sum_{i=0}^{n-1} \int_{|x-\mu_i|>\varepsilon b_n} |x-\mu_i|^2 \nu_i(dx) \leq \frac{1}{a^2 n} \sum_{i=0}^{n-1} \int_{|x-\mu_i|>\varepsilon a\sqrt{M}} |x-\mu_i|^2 \nu_i(dx).$$

Taking the superior limit in the above inequality, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{b_n^2} \sum_{i=0}^{n-1} \int_{|x-\mu_i|>\varepsilon b_n} |x-\mu_i|^2 \nu_i(dx) &\leq \frac{1}{a^2} \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_{|x-\mu_i|>\varepsilon a\sqrt{M}} |x-\mu_i|^2 \nu_i(dx) \\ &= \frac{1}{a^2} \mathbb{E} \int_{|x-\mu_0|>\varepsilon a\sqrt{M}} |x-\mu_0|^2 \nu_0(dx). \end{aligned}$$

Since $\mathbb{E}\sigma_0^2 < \infty$, it follows that $\mathbb{E}\int_{|x-\mu_0|>\varepsilon a\sqrt{M}} |x-\mu_0|^2 \nu_0(dx) \to 0$ as $M \to \infty$ by the dominated convergence theorem. So we have proved that for each fixed $\varepsilon > 0$, (4.13) holds a.s. It follows that a.s. (4.13) holds for all rational $\varepsilon > 0$. By the monotonicity in ε of the integral in (4.13) , this implies that a.s. (4.13) holds for all $\varepsilon > 0$. So the proof is completed. □

Now we prove Theorems 1.5 and 1.6 about the convergence to the normal law $N(0, 1)$.

Proof of Theorem 1.5. We can assume $\mu_0 = 0$ by centering the associated random walk, that is, by considering $\bar{L}_i(u) = L_i(u) - \mu_n$ instead of $L_i(u)$, for $|u| = n$. Then $\ell_n = 0$.

We will use Theorems 4.1 and 4.3 to prove Theorem 1.5. By (4.14), $b_n \sim \mathbb{E}\sigma_0^2\sqrt{n}$ a.s., which implies that for any $0 < \gamma < \frac{1}{2}$, $b_n^{-1} = o(n^{-\gamma})$ a.s. Theorem 4.3 shows that $\{\nu_n\}$ satisfies a.s. condition (A) with b_n defined in Theorem 4.3, $c_n = 0$, and $L = N(0, 1)$. So by Theorem 4.1, a.s. for all $x \in \mathbb{R}$,

$$P_n^{-1} Z_n(-\infty, b_n x] \to \Phi(x) W.$$

Since $Z_n(\mathbb{R})/P_n \to W$ a.s., this implies that a.s. (1.6) holds (in the centered case where $\mu_0 = 0$). □

Remark 4.2. Like in the case of Theorem 4.3 (see also Remark 4.1), Theorem 1.5 still holds when (b_n, ℓ_n) is replaced by $(b'_n, \ell'_n) \in \mathbb{R}^2$, provided that $b'_n = b'_n(\xi)$ and $\ell'_n = \ell'_n(\xi)$ satisfy (4.12). This can be seen from Remark 4.1 and the proof above.

Proof of Theorem 1.6. As in the proof of Theorem 1.5, we can first center the walk so that $\mu_n = 0$. In this case the condition that $\upsilon(\varepsilon) = \mathbb{E}\int |x|^\varepsilon \nu_0(dx) < \infty$ holds with $\varepsilon = 2$, thanks to the condition $\mathbb{E}\sigma_0^2 < \infty$. Theorem 1.6 is then a direct consequence of Theorem 4.2, noting that the conditions of Theorem 4.2 are satisfied by the conditions of Theorem 1.6 and the proof of Theorem 1.5. □

5. Proof of Theorem 1.8

The method is a refinement of the proof of [72, Theorem 5.1] where a strong boundedness condition was used. Consider the probability measures $q_n(\cdot) = \frac{\mathbb{E}_\xi Z_n(a_n \cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$. Recall that the sequence (a_n) satisfies (1.9). Let λ_n be the log Laplace transform of q_n: for $t \in \mathbb{R}$,

$$\lambda_n(t) = \log \int e^{tx} q_n(dx) = \sum_{i=0}^{n-1} \left(\log m_i(a_n^{-1} t) - \log m_i \right).$$

We first give a convergence result on the log Laplace transform of q_n with suitable norming.

Lemma 5.1. *If $\mu_0 = 0$ a.s. and $\mathbb{E}\frac{1}{m_0}\sum_{u\in\mathbb{T}_1} e^{\delta|S_u|} < \infty$ for some $\delta > 0$, then a.s. for all $t \in \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{n}{a_n^2} \lambda_n\left(\frac{a_n^2}{n} t\right) = \frac{1}{2}\sigma^2 t^2. \tag{5.1}$$

Proof. We first notice that it suffices to prove that (5.1) holds a.s. for each fixed t, since this implies that a.s. (5.1) holds for all rational t, and therefore for all $t \in \mathbb{R}$ by the convexity of λ_n and the continuity of the limit function $t \mapsto \frac{1}{2}\sigma^2 t^2$. Fix $t \in \mathbb{R}$, and set $\Delta_{n,i} = m_i(\frac{a_n}{n} t) m_i^{-1} - 1$. We shall show that for each $t \in \mathbb{R}$,

$$\sup_{0\le i\le n-1} |\Delta_{n,i}| < 1 \qquad a.s. \tag{5.2}$$

for n large enough. Let

$$Q_n^{(\delta)} = \frac{1}{m_n}\mathbb{E}_\xi \int e^{\delta|x|} X_n(dx) = \frac{1}{m_n}\mathbb{E}_\xi \sum_{i=1}^{N(u)} e^{\delta|L_i(u)|} \quad \text{for} \quad u \in \mathbb{N}^{*n}.$$

By the ergodic theorem, $\lim_n \sum_{i=0}^{n-1} Q_i^{(\delta)}/n = \mathbb{E}Q_0^{(\delta)} < \infty$ a.s., hence a.s. for any constant $C > \mathbb{E}Q_0^{(\delta)}$, there is $n_0 = n_0(\xi) \in \mathbb{N}$ large enough such that, for all $n \ge n_0$,

$$\sup_{0\le i\le n-1} Q_i^{(\delta)} \le \sum_{i=0}^{n-1} Q_i^{(\delta)} \le Cn. \tag{5.3}$$

Let $n_t \in \mathbb{N}$ be large enough such that $\frac{a_n}{n}|t| < \delta$ for all $n \ge n_t$. Then for all $n \ge n_t$,

$$\sum_{k=0}^{\infty} \frac{1}{m_i}\mathbb{E}_\xi \int \frac{1}{k!} \left|\frac{a_n}{n} tx\right|^k X_i(dx) \le \sum_{k=0}^{\infty} \frac{1}{m_i}\mathbb{E}_\xi \int \frac{1}{k!} |\delta x|^k X_i(dx) = Q_i^{(\delta)} < \infty \qquad a.s.$$

Write $\Delta_{n,i} = \sum_{k=2}^{\infty} \left(\alpha_{ik}^n + \beta_{ik}^n\right)$, where

$$\alpha_{ik}^n = \frac{1}{k!}\frac{1}{m_i}\mathbb{E}_\xi \int \left(\frac{a_n}{n}tx\right)^k \mathbf{1}_{\{|x|\le \frac{2}{\delta}\log n\}} X_i(dx) \quad \text{and} \quad \beta_{ik}^n = \frac{1}{k!}\frac{1}{m_i}\mathbb{E}_\xi \int \left(\frac{a_n}{n}tx\right)^k \mathbf{1}_{\{|x|> \frac{2}{\delta}\log n\}} X_i(dx).$$

Using the fact that $\mathbb{E}_\xi X_i(\mathbb{R}) = m_i$ and that $\frac{1}{k!}\left(\frac{\delta}{2}|x|\right)^k \le e^{\frac{\delta}{2}|x|}$ for all k, we can see that

$$\begin{aligned} |\alpha_{ik}^n| &\le \frac{1}{k!}\left(\frac{a_n}{n}|t|\right)^k \frac{1}{m_i}\mathbb{E}_\xi \int |x|^k \mathbf{1}_{\{|x|\le \frac{2}{\delta}\log n\}} X_i(dx) \\ &\le \frac{1}{k!}\left(\frac{a_n}{n}|t|\right)^k \left(\frac{2}{\delta}\log n\right)^k \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} |\beta_{ik}^n| &\le \left(\frac{2}{\delta}\frac{a_n}{n}|t|\right)^k \frac{1}{m_i}\mathbb{E}_\xi \int \frac{1}{k!}\left(\frac{\delta}{2}|x|\right)^k \mathbf{1}_{\{|x|> \frac{2}{\delta}\log n\}} X_i(dx) \\ &\le \left(\frac{2}{\delta}\frac{a_n}{n}|t|\right)^k \frac{1}{m_i}\mathbb{E}_\xi \int e^{\frac{\delta}{2}|x|} \mathbf{1}_{\{|x|> \frac{2}{\delta}\log n\}} X_i(dx) \\ &= \left(\frac{2}{\delta}\frac{a_n}{n}|t|\right)^k \frac{1}{m_i}\mathbb{E}_\xi \int e^{\delta|x|} e^{-\frac{\delta}{2}|x|} \mathbf{1}_{\{|x|> \frac{2}{\delta}\log n\}} X_i(dx) \\ &\le \left(\frac{2}{\delta}\frac{a_n}{n}|t|\right)^k \frac{1}{m_i}\mathbb{E}_\xi \int e^{\delta|x|} e^{-\log n} \mathbf{1}_{\{|x|> \frac{2}{\delta}\log n\}} X_i(dx) \\ &\le \left(\frac{2}{\delta}\frac{a_n}{n}|t|\right)^k \frac{Q_i^{(\delta)}}{n}. \end{aligned} \tag{5.5}$$

Combining (5.5) and (5.3) yields, for n large enough,

$$\sup_{0\le i\le n-1} |\beta_{ik}^n| \le \left(\frac{2}{\delta}\frac{a_n}{n}|t|\right)^k n^{-1} \sup_{0\le i\le n-1} Q_i^{(\delta)} \le C\left(\frac{2}{\delta}\frac{a_n}{n}|t|\right)^k. \tag{5.6}$$

By (5.4) and (5.6),

we see that a.s. for n large enough,

$$\sup_{0\le i\le n-1} |\Delta_{n,i}| \le \sum_{k=2}^{\infty} \sup_{0\le i\le n-1} |\alpha_{ik}^n + \beta_{ik}^n| \le C\sum_{k=2}^{\infty}\left(\frac{2}{\delta}d_n|t|\right)^k \le Md_n^2,$$

where $d_n = (a_n \log n)/n$ and $M > 0$ is a constant (depending on t but independent of n). It is clear that $\lim_n d_n = 0$, so that (5.2) holds for n sufficiently large. Using (5.2) and the fact that $\lim_n n^2 d_n^3/a_n^2 = 0$, by the same argument as in the last part of the proof of [72, Theorem 5.1], we can prove that (5.1) holds a.s. for each fixed t. This ends the proof of Lemma 5.1. □

Proof of Theorem 1.8. Let $\Gamma_n(t) = \log\left[\frac{\int e^{tx} Z_n(a_n dx + \ell_n)}{Z_n(\mathbb{R})}\right]$. Observe that

$$\frac{n}{a_n^2}\Gamma_n\left(\frac{a_n^2}{n}t\right) = \frac{n}{a_n^2}\log W_n\left(\frac{a_n}{n}t\right) + \frac{n}{a_n^2}\left[\lambda_n\left(\frac{a_n^2}{n}t\right) - \frac{a_n}{n}t\ell_n\right] - \frac{n}{a_n^2}\log W_n(0). \tag{5.7}$$

Set $\bar{L}_i(u) = L_i(u) - \mu_{|u|}$ and consider the new BRWRE with $\bar{L}_i(u)$ in place of $L_i(u)$. Define $\bar{\lambda}_n(t)$ and $\bar{W}_n(t)$ for the new BRWRE just as we defined $\lambda_n(t)$ and $W_n(t)$ for the original BRWRE. Then $\bar{W}_n(t) = W_n(t)$. Since $\mathbb{E}\frac{1}{m_0}\sum_{u\in\mathbb{T}_1} e^{\delta|S_u-\mu_0|} < \infty$, by Lemma 5.1, we have a.s. for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{n}{a_n^2}\left[\lambda_n(\frac{a_n^2}{n}t) - \frac{a_n}{n}t\ell_n\right] = \lim_{n\to\infty} \frac{n}{a_n^2}\bar{\lambda}_n(\frac{a_n^2}{n}t) = \frac{1}{2}\sigma^2t^2. \tag{5.8}$$

By [72, Theorem 1.6], we see that a.s. $\bar{W}_n(t)$ converges uniformly in a neighbourhood of $t = 0$. Thus, letting $n \to \infty$ in (5.7) yields that a.s. for all $t \in \mathbb{R}$,

$$\lim_{n\to} \frac{n}{a_n^2}\Gamma_n(\frac{a_n^2}{n}t) = \frac{1}{2}\sigma^2t^2.$$

Therefore the conclusion of the theorem follows by the Gärtner-Ellis theorem. □

Notice that, still by the Gärtner-Ellis theorem, formula (5.8) implies a moderate deviation principle for means.

Theorem 5.1. *If $\mathbb{E}\frac{1}{m_0}\sum_{u\in\mathbb{T}_1} e^{\delta|S_u-\mu_0|} < \infty$ for some $\delta > 0$, then for almost every ξ, the sequence of probabilities $A \mapsto \mathbb{E}_\xi Z_n(a_nA + \ell_n)/\mathbb{E}_\xi Z_n(\mathbb{R})$ satisfies a moderate deviation principle with rate function $\frac{x^2}{2\sigma^2}$: for measurable set $A \subset \mathbb{R}$,*

$$-\frac{1}{2\sigma^2}\inf_{x\in A^\circ} x^2 \le \liminf_{n\to\infty} \frac{n}{a_n^2}\log\frac{\mathbb{E}_\xi Z_n(a_nA+\ell_n)}{\mathbb{E}_\xi Z_n(\mathbb{R})} \le \limsup_{n\to\infty} \frac{n}{a_n^2}\log\frac{\mathbb{E}_\xi Z_n(a_nA+\ell_n)}{\mathbb{E}_\xi Z_n(\mathbb{R})} \le -\frac{1}{2\sigma^2}\inf_{x\in\bar{A}} x^2,$$

where $\sigma^2 = \mathbb{E}\sigma_0^2$, A° is the interior of A and $\bar{A}$ its closure.

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References

[1] L. Addario-Berry, B. Reed, Minima in branching random walks. Ann. Probab. 37 (2009), 1044-1079.
[2] E. Aïdékon, Convergence in law of the minimum of a branching random walk. Ann. Probab. 41 (2013), 1362-1426.
[3] E. Aïdékon, Z. Shi. The Seneta–Heyde scaling for the branching random walk. Ann. Probab. 42 (2014), 959–993.
[4] K.B. Athreya, S. Karlin, On branching processes in random environments I & II. Ann. Math. Statist. 42 (1971), 1499-1520 & 1843-1858.
[5] N. Attia, J. Barral, Hausdorff and packing spectra, large deviations and free energy for branching random walks in R^d. Commun. Math. Phys. 331 (2014), 139-187.
[6] J. Baillon, P. Clément, A. Greven, F. den Hollander, A variational approach to branching random walk in random environment. Ann. Probab. 21 (1993), 290-317.
[7] V. Bansaye, C. Huang, Weak law of large numbers for some Markov chains along non homogeneous genealogies. ESAIM: P&S 19 (2015), 307-326.
[8] J. Barral, Y. Hu, T. Madaule, The minimum of a branching random walk outside the boundary case. Bernoulli 24 (2018), no. 2, 801-841.

[9] J. Barral, X. Jin, On exact scaling log-infinitely divisible cascades. Probab. Theory Related Fields 160 (2014), no. 3-4, 521-565.
[10] J. Berestycki, N. Berestycki, J. Schweinsberg, Critical branching Brownian motion with absorption: particle configurations. Ann. Inst. H. Poincaré 51 (2015), no. 4, 1215-1250.
[11] J.D. Biggins, The first- and last-birth problems for a multitype age-dependent branching process. Adv. Appl. Probab. 8 (1976), 446-459.
[12] J.D. Biggins, Martingale convergence in the branching random walk. J. Appl. probab. 14 (1977), 25-37.
[13] J.D. Biggins, Chernoff's theorem in the branching random walk. J. Appl. Probab. 14 (1977), 630-636.
[14] J.D. Biggins, A.E. Kyprianou, Seneta-Heyde norming in the branching random walk. Ann. Probab. 25 (1997), 337-360.
[15] J.D. Biggins, The central limit theorem for the supercritical branching random walk,and related results. Stoch. Proc. Appl. 34 (1990), 255-274.
[16] J.D. Biggins, A.E. Kyprianou, Measure change in multitype branching. Adv. Appl. Probab. 36 (2004), 544-581.
[17] M. Bramson, Minimal displacement of branching random walk, Z. Wahrsch. verw. Geb. 45 (1978), 89-108.
[18] D. Buraczewski, E. Damek, Y. Guivarc'h, S. Mentemeier, On multidimensional Mandelbrot cascades. J. Difference Equ. Appl. 20 (2014), no. 11, 1523-1567.
[19] D. Buraczewski, E. Damek, T. Mikosch, Stochastic models with power-law tails. The equation X=AX+B. Springer Series in Operations Research and Financial Engineering. Springer, 2016.
[20] B. Chauvin, A. Rouault, Boltzmann-Gibbs weights in the branching random walk. In K.B. Athreya, P. Jagers, (eds.), Classical and Modern Branching Processes, IMA Vol. Math. Appl. 84, pp. 41-50, Springer-Verlag, New York, 1997.
[21] X. Chen, Exact convergence rates for the distribution of particles in branching random walks. Ann. Appl. Probab. 11 (2001), 1242-1262.
[22] X. Chen, A necessary and sufficient condition for the nontrivial limit of the derivative martingale in a branching random walk. Adv. Appl. Probab. 47(2015), 741-760.
[23] X. Chen, H. He, On large deviation probabilities for empirical distribution of supercritical branching random walks with unbounded displacements. Probab. Theory Related Fields 175 (2019), no. 1-2, 255-307.
[24] F. Comets, S. Popov, On multidimensional branching random walks in random environment, Ann. Probab. 35 (2007),68-114.
[25] F. Comets, S. Popov, Shape and local growth for multidimensional branching random walks in random environment. ALEA 3 (2007), 273-299.
[26] F. Comets, N. Yoshida, Branching random walks in space-time random environment: survival probability, global and local growth rates. J. Theor. Probab. 24 (2011), 657-687.
[27] H. Cramér, Random variables and probability distributions. Cambridge University Press, Cambridge, 1937.
23 (2018), no. 60, 12 pp.
[28] A. Dembo, O. Zeitouni, Large deviations techniques and applications. 2nd ed. Springer, New York, 1998.
[29] B. Derrida, Z. Shi. Slower deviations of the branching Brownian motion and of branching random walks. J. Phys. A 50 (2017), no. 34, 344001, 13 pp.
[30] R. Durrett, T. Liggett, Fixed points of the smoothing transformation. Z. Wahrsch. verw. Geb. 64 (1983), 275-301.
[31] W. Feller, An introduction to probability theory and its applications, Vol.II. Wiley, New York, 1971.
[32] J. Franchi, Chaos Multiplicatif: un traitement simple et complet de la fonction de partition. Séminaire de Probabilités XXIX. Springer, 1995, 194-201.
[33] Z. Gao, Q. Liu, H. Wang, Central limit theorems for a branching random walk with a random environment in time. Acta Math. Sci. 34 B (2) (2014), 501-512.
[34] Z. Gao, Q. Liu, Exact convergence rates in central limit theorems for a branching random walk with a random environment in time. Stoch. Proc. Appl. 126 (2016), 2634-2664.
[35] Z. Gao, Q. Liu, Second and third orders asymptotic expansions for the distribution of particles in a branching random walk with a random environment in time. Bernoulli 24 (2018), no.1, 772-800.
[36] A. Greven, F. den Hollander, Branching random walk in random environment: phase transitions for local and global growth rates, Probab. Theory Related Fields 91 (1992), 195-249.
[37] Y. Guivarc'h, Sur une extension de la notion de loi semi-stable. Ann. Inst. H. Poincaré. Probab. Statist. 26 (1990), 261-285.
[38] T. E. Harris, The theory of branching process. Springer, Berlin, 1963.
[39] Y. Hu, How big is the minimum of a branching random walk? Ann. Inst. H. Poincaré. Probab. Statist. 52 (2016), 233-260.
[40] Y. Hu, Z. Shi, Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. Ann. Probab. 37 (2009), 742-789.
[41] Y. Hu, N. Yoshida, Localization for branching random walks in random environment. Stoch. Proc. Appl. 119 (2009), 1632-1651.

[42] C. Huang, X. Liang, Q. Liu, Branching random walks with random environments in time. Front. Math. China 9 (2014), 835-842.
[43] C. Huang, Q. Liu, Branching random walk with a random environment in time. ArXiv.1407.7623.
[44] A. Iksanov, Renewal theory for perturbed random walks and similar processes. Probability and its Applications. Birkhäuser, Springer, 2016.
[45] A. Iksanov, K. Kolesko, M. Meiners, Stable-like fluctuations of Biggins' martingales. Stoch. Proc. Appl. 129 (2019), no. 11, 4480-4499.
[46] A. Iksanov, Z. Kabluchko, A central limit theorem and a law of the iterated logarithm for the Biggins martingale of the supercritical branching random walk. J. Appl. Probab. 53 (2016), 1178-1192.
[47] J.P. Kahane, J. Peyrière, Sur certaines martingales de Benoit Mandelbrot. Adv. Math. 22 (1976), 131-145.
[48] N. Kaplan, S. Asmussen, Branching random walks I & II. Stoch. Proc. Appl. 4 (1976), 1-13 & 15-31.
[49] H. Kesten, Branching Brownian motion with absorption. Stoch. Proc. Appl. 7 (1978), 9-47.
[50] C.F. Klebaner, Branching random walk in varying environment. Adv. Appl. Probab. 14 (1982), 359-367.
[51] D. Kuhlbusch, On weighted branching processes in random environment. Stoch. Proc. Appl. 109 (2004), 113-144.
[52] X. Liang, Q. Liu, Regular variation of fixed points of the smoothing transform. Stoch. Proc. Appl. 130 (2020), 4104-4140.
[53] Q. Liu, Sur une équation fonctionnelle et ses applications: une extension du théorème de Kesten-Stigum concernant des processus de branchement. Adv. Appl. Probab. 29 (1997), 353-373.
[54] Q. Liu, Fixed points of a generalized smoothing transformation and applications to branching processes. Adv. Appl. Probab. 30 (1998), 85-112.
[55] Q. Liu, On generalized multiplicative cascades. Stoch. Proc. Appl. 86 (2000), 61-87.
[56] Q. Liu, Asymptotic properties absolute continuity of laws stable by random weighted mean. Stoch. Proc. Appl. 95 (2001), 83-107.
[57] Q. Liu, Branching random walks in random environment. Proceedings of the 4th International Congress of Chinese Mathematicians, 2007 (ICCM 2007), Vol. II, 702-719. Eds: L. Ji, K. Liu, L. Yang, S.-T. Yau.
[58] R. Lyons, A simple path to Biggins's martingale convergence for branching random walk. Classical and Modern Branching Processes, IMA Vol. Math. Appl. 84, pp. 217-221. Eds.: K.B. Athreya, P. Jagers. Springer-Verlag, New York, 1997.
[59] P. Maillard, M. Pain, 1-stable fluctuations in branching Brownian motion at critical temperature I: the derivative martingale. Ann. Probab. 47 (2019), no.5, 2953-3002.
[60] B. Mallein, P. Miloś, Maximal displacement of a supercritical branching random walk in a time-inhomogeneous random environment. Stoch. Proc. Appl. 129 (2019), 3239-3260.
[61] C. Mcdiarmid, Minimal positions in a branching random walk. Ann. Appl. Probab. 5 (1995), 128-139.
[62] Mentemeier, S. The fixed points of the multivariate smoothing transform. Probab. Theory Relat. Fields 164 (2016), no. 1-2, 401-458.
[63] M. Nakashima, Almost sure central limit theorem for branching random walks in random environment. Ann. Appl. Probab. 21 (2011), 351-373.
[64] M. Nakashima, Minimal position of branching random walks in random environment. J. Theor. Probab. 26 (2013), 1181-1217.
[65] T. Rosati, A. Tóbiás, The wave speed of an FKPP equation with jumps via coordinated branching. Electron. J. Probab. 28 (2023), no. 71, 1-29.
[66] Z. Shi, Branching random walks. Lecture Notes in Mathematics, 2151. École d'été de Probabilités de Saint-Flour. Springer, Berlin, 2015.
[67] W. L. Smith, W. Wilkinson, On branching processes in random environments. Ann. Math. Statist. 40 (1969), 814-827.
[68] A.J. Stam, On a conjecture of Harris. Z. Wahrsch. Verw. Geb. 5 (1966), 202-206.
[69] C. Stone, A local limit theorem for nonlattice multi-dimensional distribution functions. Ann. Math. Statist. 36 (1965), 546-551.
[70] D. Tanny, Limit theorems for branching processes in a random environment. Ann. Probab. 5 (1977), 100-116.
[71] D. Tanny, A necessary and sufficient condition for a branching process in a random environment to grow like the product of its means. Stoch. Proc. Appl. 28 (1988), 123-139.
[72] X. Wang, C. Huang, Convergence of martingale and moderate deviations for a branching random walk with a random environment in time. J. Theor. Probab. 30 (2017), 961-995.
[73] N. Yoshida, Central limit theorem for random walk in random environment. Ann. Appl. Probab. 18 (2008), 1619-1635.
[74] X. Zhang, W. Hou, W. Hong, Limit theorems for the minimal position of a branching random walk in random environment. Markov Process. Related Fields 26 (2020), 839-860.