LIMIT THEOREMS FOR MULTITYPE BRANCHING PROCESSES IN RANDOM ENVIRONMENTS AND PRODUCTS OF POSITIVE RANDOM MATRICES

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Abstract. Let $Z_n^x = (Z_n^x(1), \cdots, Z_n^x(d))$ be a supercritical d-type branching process in an independent and identically distributed random environment (ξ_n), starting with $Z_0 = x \in \mathbb{N}^d \setminus \{0\}$, whose offspring distribution at time n depends the environment ξ_n. Let $M_n = M(\xi_n)$ be the mean matrix of the offspring distribution at time n. We establish a Kesten-Stigum type theorem for the scalar product $\langle Z_n^x, y\rangle$ for any $y \in \mathbb{R}_+^d \setminus \{0\}$: under suitable conditions, $W_n^x(y) := \langle Z_n^x, y\rangle / \langle xM_0 \cdots M_{n-1}, y\rangle$ converges in probability to some $\mathbb{R}_+$-valued random variable W^x; the almost sure convergence is also established under additional moment conditions; a criterion is given for W^x to be non-degenerate. In the proof, we find (u_n) such that $(W_n^x(u_n))$ is a martingale, and prove that $W_n^x(y)$ converges uniformly for $y \in \mathbb{R}_+^d \setminus \{0\}$ to the limit W^x of $W_n^x(u_n)$. We also prove a duality of the Kesten-Stigum theorem about the convergence of $\langle Z_{n+k}, y\rangle / \langle Z_n M_n \cdots M_{n+k-1}, y\rangle$ for fixed k as $n \to \infty$, and a theorem about the convergence of the direction $Z_n^x / \|Z_n^x\|$. An important ingredient of the proofs is the Perron-Frobenius type theorem that we establish for the products of random matrices, which is of independent interest. Let $\{M_n : n \in \mathbb{Z}\}$ be a stationary and ergodic sequence of positive random matrices, and let $M_{k,n} = M_k \cdots M_n$, for $k \leqslant n$. We find unit vectors $u_n, v_n > 0$, and scalars $\lambda_n, \mu_n, a_{k,n} > 0$, such that, almost surely, for fixed k as $n \to \infty$, and for fixed n as $k \to -\infty$, $\langle xM_0 \cdots M_n, y\rangle \sim a_{k,n}\langle u_k, x\rangle\langle v_n, y\rangle$ uniformly in x and y, where $a_{k,n}$ can be taken as the product form $a_k\mu_k \cdots \mu_n$; all the sequences (u_k), (v_n), (λ_n), (μ_n) and (a_k) are stationary and ergodic. As further applications, we find new laws of large numbers and central limit theorems for $\langle Z_n^x, y\rangle$, as well as for the products of random matrices.

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1. Introduction

1.1 Perron-Frobenius theorem for products of positive random matrices. Let $d \geqslant 1$ be an integer. Denote by $\|\cdot\|$ the L^1 norm of $\mathbb{R}^d$, and by $\langle \cdot, \cdot \rangle$ the scalar product. For a matrix or vector a, we write $a > 0$ (resp. $a \geqslant 0$) to mean that all its entries are strictly positive (resp. nonnegative). Let G be the semi-group of $d \times d$ positive matrices, and let $M \geqslant 0$ be an element of G. Assume that M is primitive in the sense that there is an integer $k > 0$ such that $M^k > 0$. The famous Perron-Frobenius theorem states that the spectral radius ρ of M is a dominant simple eigenvalue, and there are unique vectors $u = (u(1), \cdots, u(d)), v = (u(1), \cdots, u(d)) \in \mathbb{R}^d_+$ such that $u, v > 0$, $\|u\| = \|v\| = 1$,

$$uM^T = \rho u \quad \text{and} \quad vM = \rho v, \tag{1.1}$$

where M^T denotes the transpose of M; moreover, the (i, j)-th entry $M^n(i, j)$ of the power matrix M^n satisfies, as $n \to \infty$,

$$M^n(i, j) \sim \rho^n \frac{\langle u, e_i \rangle \langle v, e_j \rangle}{\langle u, v \rangle}, \quad i, j = 1 \cdots, d \tag{1.2}$$

(as usual $a_n \sim b_n$ means $a_n/b_n \to 1$), where $e_i \in \mathbb{R}^d_+$ is the unit vector whose i-th component is 1, the others are 0.

An extension of (1.2) to products of random matrices has been established by Hennion [30]. Let $(M_n)_{n\in\mathbb{Z}}$ be a stationary and ergodic sequence of random elements of G, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any $k, n \in \mathbb{Z}$, let

$$M_{k,n} = M_k \dots M_n \quad \text{and} \quad M^T_{k,n} = (M_{k,n})^T \quad \text{if } k \leqslant n, \quad M_{k,n} = I_d \quad \text{if } k > n, \tag{1.3}$$

where I_d denotes the $d \times d$ identity matrix; the notation $M^T_{k,n}$ should not be confused with $M^T_k \cdots M^T_n$. For $k \leqslant n$, let $\rho_{k,n}$ be the spectral radius of $M_{k,n}$, and $u_{k,n}, v_{k,n} \in \mathbb{R}^d_+$ be such that

$$u_{k,n} M^T_{k,n} = \rho_{k,n} u_{k,n}, \quad v_{k,n} M_{k,n} = \rho_{k,n} v_{k,n}, \quad \|u_{k,n}\| = \|v_{k,n}\| = 1. \tag{1.4}$$

For the symmetry of the results that we will present below, we use the norming $\|u_{k,n}\| = \|v_{k,n}\| = 1$ rather than the more usually used norming $\|u_{k,n}\| = \langle u_{k,n}, v_{k,n} \rangle = 1$. Assuming that almost surely (a.s.) M_0 is allowable (every row and every column has at least a strictly positive entry), and $\mathbb{P}\Big(\exists n \geqslant 0 \text{ such that } M_{0,n} > 0\Big) > 0$, Hennion [30] proved that, a.s.

$$M_{0,n-1}(i, j) \sim \rho_{0,n-1} \frac{\langle u_{0,n-1}, e_i \rangle \langle v_{0,n-1}, e_j \rangle}{\langle u_{0,n-1}, v_{0,n-1} \rangle}, \quad i, j = 1, \cdots d. \tag{1.5}$$

In this paper, motivated by applications, especially to multitype branching processes in random environments, we complete Hennion's result by proving the following Perron-Frobenius type theorem: we find unit vectors $u_k, v_n > 0$, such that, a.s.,

uniformly in x and y, for all fixed $k \in \mathbb{Z}$ as $n \to \infty$, and for all fixed $n \in \mathbb{Z}$ as $k \to -\infty$,

$$u_{k,n} - u_k \to 0, \quad v_{k,n} - v_n \to 0, \tag{1.6}$$

$$\langle xM_k \cdots M_n, y\rangle \sim a_{k,n}\langle u_{k,n}, x\rangle\langle v_{k,n}, y\rangle \tag{1.7}$$

$$\sim a_{k,n}\langle u_k, x\rangle\langle v_n, y\rangle, \tag{1.8}$$

where (u_k) and (v_k) have the nice property that for all $k \in \mathbb{Z}$,

$$u_{k+1}M_k^T = \lambda_k u_k \quad \text{and} \quad v_{k-1}M_k = \mu_k v_k, \tag{1.9}$$

with $\lambda_k = \|u_{k+1}M_k^T\|$ and $\mu_k = \|v_{k-1}M_k\|$; $a_{k,n}$ can be taken as the form $a_k\mu_k \cdots \mu_n$; all the sequences (u_k), (v_n), (λ_n), (μ_n) and (a_k) are stationary and ergodic. See Theorem 2.1 where more information will be given.

Notice that the relation (1.9) can pass to the products $M_{k,n}$ by iteration (see Remark 2.2, Eq. (2.19)), just as the relation (1.1) can pass to the power M^n of M. In some sense, (u_k) and (v_k) play the same role for products of random matrices, as u, v do for the powers of the constant matrix M. We call (u_k) and (v_k) pseudo eigenvectors of (M_k), which are very useful in applications. The sequence $(u_k)_{k\geqslant 0}$ has been defined in [30]. Both sequences (u_k) and (v_k) will play important role in the study of branching processes in random environments.

Compared with Hennion's result (1.5), the new variants presented here may have advantages in applications. Indeed, from (1.8) and limit theorems about the stationary and ergodic sequence $(\log \mu_n)$ of real random variables, we can get corresponding limit theorems for the logarithme of the matrix norm $\|M_{0,n-1}\|$, the vector norm $\|xM_{0,n-1}\|$, the spectral radius $\rho_{0,n-1}$, and the scalar products $\langle xM_{0,n-1}, y\rangle$. As example, in Corollary 2.5 we get limit theorems on products of random matrices, which improve some earlier results by Hennion [30] by relaxing his first moment conditions:

1.2 Limit theorems on multitype branching processes in random environments. As main applications of the Perron-Frobenius theorem presented above, we will establish several new limit theorems for multitype branching processes in random environments. A branching process in a random environment (BPRE) is a family of non-homogeneous branching processes indexed by the environment denoted by $\xi = (\xi_0, \xi_1, \cdots)$; the environment sequence (ξ_n) is supposed to be stationary and ergodic. In such a process, given the environment ξ, the offspring distribution of particles of generation n depend on the environment ξ_n at time n.

Fundamental limit theorems on single or multitype BPRE can be found in the early papers by Smith and Wilkinson [50], Athreya and Karlin [2, 3], and Tanny [53]. For more recent results on the single type case, see e.g. Geiger, Kersting and Vatutin [20] and Afanasyev, Geiger, Kersting, and Vatutin [1] on the survival probability in the subcritical and critical cases, Bansaye and Berestycki [5], Huang and Liu [34], Grama, Liu and Miqueu [21], and Buraczewski and Dyszewski [9] on large deviations in the supercritical case. We refer the book of Kersting and Vatutin [39] for many other results.

Here we focus on the multytype case. For recent studies in this case, see e.g. Le Page, Peigné and Pham [47], Vatutin and Dyakonova [56] and Vatutin and Wachtel [57] on the survival probability for critical and subcritical processes, and Grama, Liu

and Pin [22, 23] on the Kesten-Stigum theorem and L^p convergence for supercritical processes.

Let $Z_n = (Z_n(1), \cdots, Z_n(d))$ be a d-type branching process in a stationary and ergodic environment $\xi = (\xi_0, \xi_1, \cdots)$, where $Z_n(j)$ denotes the number of type j particles of generation n. We suppose that the process starts with the initial state $Z_0 = x \in \mathbb{N}^d \setminus \{0\}$ (where $\mathbb{N} = \{0, 1, \cdots\}$), and we then write Z_n^x for Z_n; when $x = e_i$, we simply write Z_n^i instead of $Z_n^{e_i}$. Let $M_n = M(\xi_n)$ be the mean matrix of the offspring distribution at time n: its (i, j)-th entry $M_n(i, j)$ is the conditional expectation of the number of type j particles produced by a type i particle of generation n. Then (M_n) is a stationary and ergodic sequence of positive random matrices, and $\mathbb{E}_\xi Z_n^i(j) = M_{0,n-1}(i, j)$, where $\mathbb{E}_\xi$ denotes the conditional expectation given the environment ξ (and we continue to use the notation (1.3)). This explains why the study of a multitype BPRE depends much on the theory of products of random matrices.

One of the fundamental problems in the theory of branching processes is the description of the population size at time n. For a multitype branching process (Z_n), the famous theorem of Kesten-Stigum [42] tells us exactly when the population size grows at an exponential rate that we can determine explicitly, via the study of the nondegeneracy of the limit of the fundamental martingale.

Extending the Kesten-Stigum theorem to the multitype random environment case has been a challenging problem. The problem is very interesting especially due to a number of applications. Various contributions in this respect can be found in Tanny [53], Cohn [12], Jones [37], Biggins, Cohn and Nerman [6], Dolgopyat, Hebbar, Koralov and Perlman [14]. A full extension for the coordinates $Z_n^i(j)$ has been recently performed in [22]. Suppose that the multitype BPRE is supercritical in the sense that

$$\gamma := \lim_{n\to\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\| > 0, \tag{1.10}$$

where for a matrix g, $\|g\|$ denotes its operator norm induced by the L_1-norm $\|\cdot\|$ of $\mathbb{R}^d$. Under simple conditions, it has been established in [22] that for all $1 \leqslant i, j \leqslant d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \to W^i \quad \text{in probability,} \tag{1.11}$$

where W^i is a random variable with values in $[0, \infty)$, which is non-degenerate for all i if and only if

$$\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i, j)} \log^+ \frac{Z_1^i(j)}{M_0(i, j)} \right) < \infty. \tag{1.12}$$

In this paper, we consider the scalar product $\langle Z_n^x, y\rangle$ for all $x \in \mathbb{N}^d \setminus \{0\}$ and $y \in \mathbb{R}_+^d \setminus \{0\}$, instead of the coordinates $Z_n^i(j)$. This consideration is interesting, both in theory and in applications. In fact, for a full probabilistic description of the random vector Z_n^x, a usual way is to consider the scalar product; for example, the characteristic function of Z_n^x is defined by the Fourier transform of the scalar product $\langle Z_n^x, y\rangle$. If $x = e_i$ and $y = e_j$, then $\langle Z_n^x, y\rangle$ reduces to the coordinate $Z_n^i(j)$. If $y = (1, \cdots, 1)$, then $\langle Z_n^x, y\rangle = \|Z_n^x\| = Z_n^x(1) + \cdots + Z_n^x(d)$ is the population size of generation n. This consideration may also be interesting for some applied problems occurring in the society. For example, if we consider a population where individuals are classified into three types: type 1 for an individual of the upper

class, say with $y(1) = 10^8$ €; type 2 for an individual of the middle class, say with $y(2) = 10^6$ €; and type 3 for an individual of the lower class, say with $y(3) = 10^4$ €. Then $\langle Z_n^x, y\rangle = 10^8 Z_n^x(1) + 10^6 Z_n^x(2) + 10^4 Z_n^x(3)$ is the total amount of money of n-th generation.

The first objective of the present paper is to extend the Kesten-Stigum theorem on $Z_n^i(j)$ established in [22] to the scalar product $\langle Z_n^x, y\rangle$ when $Z_0 = x \in \mathbb{N}^d \setminus \{0\}$ and $y \in \mathbb{R}_+^d \setminus \{0\}$, for the multitype branching process (Z_n) in random environment. We will prove that under suitable conditions, uniformly in $y \in \mathbb{R}_+^d \setminus \{0\}$,

$$W_n^x(y) := \frac{\langle Z_n^x, y\rangle}{\langle xM_0 \cdots M_{n-1}, y\rangle} \to W^x \quad \text{in probability} \tag{1.13}$$

(see Theorem 2.6), where W^x (which depends on x but not on y) is a random variable with values in $[0, \infty)$, which is non-degenerate if and only if (1.12) holds. The convergence in probability in (1.13) can be improved to the a.s. convergence under additional moment conditions. When $x = e_i$ and $y = e_j$, (1.13) reduces to (1.11), the Kesten-Stigum theorem established in [22].

For the proof of (1.13), we first prove that, with u_n defined in (1.6), $W_n^x(u_n), n \geqslant 0$, is a martingale, thus converges a.s. to some $\mathbb{R}_+$-valued random variable W^x. In the constant environment case, for $x = e_i$, this martingale reduces to the fundamental martingale introduced by Kesten and Stigum [42]. We then prove that the limit variable W^x is non-degenerate if and only if (1.12) holds, under some simple additional assumptions. For the proof of the non-degeneracy of W^x, we use a general result of Biggins and Kyprianou [7] for the non-degeneracy of the limit of a martingale constructed from mean-harmonic functions. In particular, this leads to a simpler proof for the non-degeneracy of W^i established in [22].

To complete the proof of the convergence (1.13), we further establish a theorem about the convergence of the direction $\vec{Z}_n^x := \frac{Z_n^x}{\|Z_n^x\|}$. In particular, we find an equivalence of $\langle \vec{Z}_n, y\rangle$ uniformly for $y \in \mathbb{R}_+^d \setminus \{0\}$: see Theorem 2.9.

The Kesten-Stigum type result (1.13) is interesting because it is a good bridge between branching processes in random environments and products of random matrices. In particular, it can be used to prove important theorems such as laws of large numbers and central limit theorems: see e.g. Corollaries 2.7 and 2.10 for such results on the total population size $\|Z_n^x\|$ of generation n, the j-type population size $Z_n^x(j)$, and the scalar product $\langle Z_n^x, y\rangle$. For large deviation results and convergence rates in the Gaussian approximation, see the forthcoming paper [27].

An interesting duality result of the Kesten-Stigum type theorem is also established, showing the convergence of $\langle Z_{n+k}, y\rangle / \langle Z_n M_n \cdots M_{n+k-1}, y\rangle$ for fixed k as $n \to \infty$: see Theorem 2.8.

An important ingredient of the proofs is the Perron-Frobenius type theorem that we establish for the products of positive random matrices (see Theorem 2.1). Combining the Perron-Frobenius type theorem and the Kesten-Stigum result (1.13), we get a good description of the growth rate of $\langle Z_n^x, y\rangle$.

The rest of the paper is organized as follows. In Section 2, we present our main results. The Perron-Frobenius type theorem is proved in Section 3. In Section 4, we introduce the fundamental martingale $(W_n^x(u_n))$, and establish a criterion for the non-degeneracy of its limit W^x. In Section 5 we show some relations between the survival

event $S := \{Z_n \not\to 0\}$ and the explosion event $E := \{\|Z_n\| \to \infty\}$. The convergence of the direction $Z_n^x/\|Z_n^x\|$ is investigated in Section 6. A duality of the Kesten-Stigum theorem is established in Section 7, about the convergence of $\frac{\langle Z_{n+k}, y\rangle}{\langle Z_n M_{n,n+k-1}, y\rangle}$ for fixed k as $n \to \infty$. The uniform convergence in probability and a.s. of the normalized scalar product $W_n^x(y)$ is proved in Section 8.

2. Main results

2.1 Notation. We begin with some notation. Let $d \geqslant 1$, and let $\mathbb{R}^d$ be the d-dimensional Euclidean space endowed with the canonical orthonormal basis $(e_i)_{1\leqslant i\leqslant d}$, and with the scalar product and the L^1 norm defined for any $x = (x(1), \cdots, x(d)), y = (y(1), \cdots, y(d)) \in \mathbb{R}^d$ by

$$\langle x, y\rangle = \sum_{i=1}^{d} x(i)\, y(i) \quad \text{and} \quad \|x\| = \sum_{i=1}^{d} |x(i)|. \tag{2.1}$$

We denote by G the multiplicative semigroup of $d \times d$ matrices with non-negative entries. The subsemigroup of G with strictly positive matrices is denoted by G°. For $g \in G$, define the operator norm and the iota function:

$$\|g\| = \sup_{\|x\|=1} \|xg\| = \max_{1\leqslant i\leqslant d} \sum_{j=1}^{d} g(i,j) = \max_{1\leqslant i\leqslant d} \|e_i g\|, \tag{2.2}$$

$$\iota(g) = \inf_{\|x\|=1} \|xg\| = \min_{1\leqslant i\leqslant d} \sum_{j=1}^{d} g(i,j) = \min_{1\leqslant i\leqslant d} \|e_i g\|. \tag{2.3}$$

(As the vectors $x \in \mathbb{R}^d$ are represented in the row form, the action of a matrix g on x is denoted xg, to be consistent with the matrix multiplication; the operator norm $\|g\|$ defined here corresponds to the induced L^1-norm of the transpose g^T usually defined with column vectors, since $\sup_{\|x\|=1} \|xg\| = \sup_{\|x\|=1} \|g^T x^T\| = \sup_{\|y\|=1} \|g^T y\|$.) We shall also use the entry-wise L^1-matrix norm: for $g \in G$,

$$\|g\|_{1,1} = \langle \mathbb{1}, \mathbb{1}g\rangle = \sum_{i,j=1}^{d} g(i,j), \tag{2.4}$$

where $\mathbb{1} \in \mathbb{R}^d$ stands for the vector with all coordinates equal to 1. For a matrix $g \in G$, we write $g > 0$ to mean that each entry of g is strictly positive. The same convention applies for vectors. The transpose of a matrix g is denoted by g^T. Then, for any $x, y \in \mathbb{R}^d$ we have

$$\langle xg, y\rangle = \langle x, yg^T\rangle.$$

Let $\mathcal{S} := \{x \in \mathbb{R}^d_+ : \|x\| = 1\}$. For any matrix $g \in G$ and any $x \in \mathcal{S}$, we define the action of g on x by setting

$$x \cdot g = \frac{xg}{\|xg\|}. \tag{2.5}$$

For two probability laws α, μ, respectively on $\mathcal{S}$ and G, we denote $\alpha * \mu$ the convolution of α and μ: for each mesurable $B \subset \mathcal{S}$, we have

$$\alpha * \mu(B) = \int 1_B(a \cdot g)\alpha(da)\mu(dg). \tag{2.6}$$

As usual, for sequences of real numbers a_n, b_n, we write $a_n \sim b_n$ if $a_n/b_n \to 1$. For sequences of real numbers $a_n(\delta)$ and $b_n(\delta)$ depending on some parameter $\delta \in \Delta$, we say that, as $n \to \infty$,

$$a_n(\delta) \sim b_n(\delta) \text{ uniformly for } \delta \in \Delta \text{ if } \lim_{n\to\infty} \sup_{\delta\in\Delta} \left|\frac{a_n(\delta)}{b_n(\delta)} - 1\right| = 0. \tag{2.7}$$

Clearly, this uniform equivalence is an equivalent relation; in particular, if $a_n(\delta) \sim b_n(\delta)$ and $b_n(\delta) \sim c_n(\delta)$, both uniformly for $\delta \in \Delta$, then $a_n(\delta) \sim c_n(\delta)$, also uniformly for $\delta \in \Delta$. For a sequence of real random variables $X_n(\theta)$ depending on some parameter $\theta \in \Theta$ and for $l \in \mathbb{R}$, we say that

$$X_n(\theta) \xrightarrow{\mathbb{P}} l \text{ uniformly for } \theta \in \Theta \quad \text{if } \sup_{\theta\in\Theta} |X_n(\theta) - l| \to 0 \text{ in probability;} \tag{2.8}$$

we say that $X_n(\theta) \xrightarrow{\mathbb{P}} -\infty$ (resp. ∞) uniformly for $\theta \in \Theta$ if $\sup_{\theta\in\Theta} X_n(\theta) \to -\infty$ (resp. $\inf_{\theta\in\Theta} X_n(\theta) \to \infty$) in probability. The a.s. uniform convergence is defined in a similar way.

2.2 Perron-Frobenius theorem. Below we present a Perron-Frobenius type theorem that we will prove. This will play an important role in the proofs of limit theorems for multitype branching processes in random environments. This theorem is of interest both in theory and in applications. Let $(M_k)_{k\in\mathbb{Z}}$ be a stationary and ergodic sequence of random elements of G, defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. It turns out that the consideration of the double sided sequence (indexed by $\mathbb{Z}$) is more convenient in applications than the one sided sequence (indexed by $\mathbb{N}$).

We will use the following three conditions. The first is the moment condition on $\log^+ \|M_0\|$ under which we can define the Lyapunov exponent γ of the sequence of random matrices (M_n). The second is the positivity condition used e.g. by Tanny [53] and Hennion [30], which in the constant environment case reduces to the primitivity of the matrix. The third was introduced by Furstenberg and Kesten [19], for which we propose a weaker version. Recall the notation $M_{k,n} = M_k \dots M_n$ for $k \leqslant n$ as introduced in (1.3).

A1. *The random matrix M_0 satisfies the moment condition*

$$\mathbb{E} \log^+ \|M_0\| < \infty.$$

By sub-additivity, under **A1**, the limit

$$\gamma := \lim_{n\to\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\| \tag{2.9}$$

exists and is equal to the quantity $\inf_{k\geqslant 1} \frac{1}{k} \mathbb{E} \log \|M_{0,k-1}\|$, which lies in $\mathbb{R} \cup \{-\infty\}$. Moreover, the following strong law of large numbers has been established [19]:

$$\lim_{n\to\infty} \frac{1}{n} \log \|M_{0,n-1}\| = \gamma \quad \mathbb{P}\text{-a.s.} \tag{2.10}$$

A2. *A.s. M_0 is allowable, that is, every row and every column has at least a strictly positive entry, and*

$$\mathbb{P}\Big(\exists n \geqslant 1 \text{ such that } M_{0,n-1} > 0\Big) > 0.$$

A3. *There exists a constant $D > 1$ such that for all $i, j \in \{1, \cdots, d\}$,*

$$0 < \max M_0(i,j) \leqslant D \min M_0(i,j) \quad \mathbb{P}\text{-a.s.}$$

For some results a much weaker version of **A3** is sufficient. We will use a condition based on the following: there is some measurable function $D : G \to [1, \infty)$ such that for $n \in \mathbb{Z}$, $D_n = D(M_n)$ satisfies, for all $i = 1, \cdots, d$,

$$\max_{1 \leqslant j \leqslant d} M_n(i,j) \leqslant D_n \min_{1 \leqslant j \leqslant d} M_n(i,j), \quad \mathbb{P}\text{-a.s.} \tag{2.11}$$

For example, when $M_n > 0$ a.s. we can take $D_n = \max_{1 \leqslant i \leqslant d} \frac{\max_{1 \leqslant j \leqslant d} M_n(i,j)}{\min_{1 \leqslant j \leqslant d} M_n(i,j)}$. We will also use the same condition for the transpose of M_n, that is, for all $i = 1, \cdots, d$,

$$\max_{1 \leqslant j \leqslant d} M_n(j,i) \leqslant D_n \min_{1 \leqslant j \leqslant d} M_n(j,i), \quad \mathbb{P}\text{-a.s.} \tag{2.12}$$

Notice that when the Furstenberg-Kesten condition **A3** holds, then the conditions (2.11) and (2.12) hold with $D_n = D$ a constant.

Recall that $k \leqslant n$, $\rho_{k,n}$ denotes the spectral radius of $M_{k,n} = M_k \cdots M_n$, and $u_{k,n}, v_{k,n} \in \mathbb{R}^d_+$ are such that (1.4) holds. Recall also (2.7) for the definition of the uniform equivalence relation $\sim$.

Theorem 2.1 (Perron-Frobenius type theorem)**.** *Assume condition **A2**. Then:*
1. There are two stationary and ergodic sequences $(u_k)_{k \in \mathbb{Z}}$ and $(v_k)_{k \in \mathbb{Z}}$ such that for all k, $u_k > 0$, $v_k > 0$, $\|u_k\| = \|v_k\| = 1$, and that for each fixed $k \in \mathbb{Z}$ as $n \to \infty$, and for each fixed $n \in \mathbb{Z}$ as $k \to -\infty$, $\mathbb{P}$-a.s.

$$u_{k,n} - u_k \to 0 \quad \text{and} \quad v_{k,n} - v_n \to 0. \tag{2.13}$$

Moreover, for all $k \in \mathbb{Z}$, with $\lambda_k = \|u_{k+1} M_k^T\|$ and $\mu_k = \|v_{k-1} M_k\|$, $\mathbb{P}$-a.s.

$$u_{k+1} M_k^T = \lambda_k u_k \quad \text{and} \quad v_{k-1} M_k = \mu_k v_k, \tag{2.14}$$

*where (λ_k) and (μ_k) are two stationary and ergodic sequences. The common law of v_n (resp. u_n) is the limit law of $v_{k,n}$ (resp. $u_{k,n}$), which is the unique probability law ν (resp. ν^*) on $\mathcal{S}$ satisfying $\nu * \mu = \nu$ (resp. $\nu^* * \mu = \nu^*$), where μ (resp. μ^*) is the law of M_0 (resp. M_0^T). If additionally **A1** holds, then*

$$\mathbb{E} \log \lambda_0 = \gamma = \mathbb{E} \log \mu_0. \tag{2.15}$$

2. For each fixed $k \in \mathbb{Z}$ as $n \to \infty$, and for each fixed $n \in \mathbb{Z}$ as $k \to -\infty$, uniformly for $x, y \in \mathbb{R}^d_+ \setminus \{0\}$, $\mathbb{P}$-a.s.

$$\langle x M_{k,n}, y \rangle \sim a_{k,n} \langle u_{k,n}, x \rangle \langle v_{k,n}, y \rangle \tag{2.16}$$

$$\sim a_{k,n} \langle u_k, x \rangle \langle v_n, y \rangle, \tag{2.17}$$

where $(a_{k,n})$ is any of the following equivalent sequences:

$$a)\ a_{k,n} = \frac{\rho_{k,n}}{\langle u_{k,n}, v_{k,n} \rangle}, \quad b)\ a_{k,n} = \|M_{k,n}\|_{1,1}, \quad c)\ a_{k,n} = \frac{\prod_{j=k}^n \lambda_j}{\langle u_{n+1}, v_n \rangle} = \frac{\prod_{j=k}^n \mu_j}{\langle u_k, v_{k-1} \rangle}, \tag{2.18}$$

in which the sequence $\{(u_{n+1}, v_n) : n \in \mathbb{Z}\}$ is stationary and ergodic.

The proof will be given in Section 3. The first equality in (2.15) has been proved in [22]. A new proof based on the equivalences in (2.18) will be given.

Remark 2.2. Just as we can iterate the relation (1.1) to get similar relation for the powers of M, we can iterate (2.14) to obtain the similar relation for the products $M_{k,n}$:

$$u_{n+1}M_{k,n}^T = \left(\prod_{j=k}^{n}\lambda_j\right)u_k \quad \text{and} \quad v_{k-1}M_{k,n} = \left(\prod_{j=k}^{n}\mu_j\right)v_n \quad \mathbb{P}\text{-a.s..} \tag{2.19}$$

Hence (u_k) and (v_k) play the same role for products of matrices as u and v do for powers of M. For this reason we call u_n and v_n the pseudo-eigenvectors of M_n. The above relation indicates that the vectors u_n and v_n have the stability for products of matrices, in the sense that the relation (2.14) can pass to the products. This property is remarkable compared with the eigenvectors of M_n.

The following remark is useful to have different forms of (2.16) and (2.17), by replacing the factors $\langle u_{k,n}, x\rangle$ or $\langle v_{k,n}, y\rangle$ therein by their uniform equivalences indicated below. See the end of Section 3 for the proof of the remark.

Remark 2.3. For all fixed $k \in \mathbb{Z}$ as $n \to \infty$, and for all fixed $n \in \mathbb{Z}$ as $k \to -\infty$, $\mathbb{P}$-a.s.,

$$\langle u_{k,n}, x\rangle \sim \langle u_k, x\rangle \quad \text{and} \quad \langle v_{k,n}, x\rangle \sim \langle v_n, x\rangle \quad \text{uniformly for } x \in \mathbb{R}_+^d \setminus \{0\}; \tag{2.20}$$

consequently,

$$\langle u_{k,n}, v_{k,n}\rangle \sim \langle u_k, v_{k,n}\rangle \sim \langle u_k, v_n\rangle \sim \langle u_{k,n}, v_n\rangle.$$

Remark 2.4. Hennion [30] considered the case where $k = 0$ and $n \to \infty$: he found the sequence $(u_n)_{n\geqslant 0}$ and proved that,

$$\langle xM_{0,n-1}, y\rangle \sim \rho_{0,n-1}\frac{\langle u_{0,n-1}, x\rangle\langle v_{0,n-1}, y\rangle}{\langle u_{0,n-1}, v_{0,n-1}\rangle} \quad \text{uniformly for } x, y \in \mathbb{R}_+^d \setminus \{0\} \tag{2.21}$$

(although in [30, Theorem 1], only the case $x = e_i$ and $y = e_j$ was considered, the conclusion for general x, y was proved in its proof). In fact his statement is slightly different as he used the norming $\|u_{0,n-1}\| = \langle u_{0,n-1}, v_{0,n-1}\rangle = 1$, instead of $\|u_{0,n-1}\| = \|v_{0,n-1}\| = 1$ that we used here. We prefer the the latter for the result to be more symmetric. We remark that, with our norming, the equivalence (2.21) does not depend on the choice of eigenvalues $u_{0,n-1}, v_{0,n-1}$, as $\frac{\langle u_{0,n-1},x\rangle\langle v_{0,n-1},y\rangle}{\langle u_{0,n-1},v_{0,n-1}\rangle}$ remains unchanged while $u_{0,n-1}, v_{0,n-1}$ are replaced by any other eigenvectors (with different norming).

The new variants presented here may have advantages in applications. For example, from (2.17) and the ergodic theorem applied to the sequence $(\log \mu_n)$ or $(\log \lambda_n)$, we get immediately the laws of large numbers (for a stationary and ergodic sequence of random positive matrices), as stated in the following corollary which improves earlier results by Hennion [30].

Recall that for $g \in G$, $\|g\| = \max_{1\leqslant i\leqslant d}\sum_{j=1}^d g(i,j)$ and $\iota(g) = \min_{1\leqslant i\leqslant d}\sum_{j=1}^d g(i,j)$ (see (2.2) and (2.3)). Recall also that for $x \geqslant 0$, $\log^+ x = \max(0, \log x)$; set $\log^- x = \max(0, -\log x)$. For the uniform convergence in probability, we refer to the definition (2.8).

Corollary 2.5 (LLN for products of random matrices)**.** *Assume **A1** and **A2**. Then* $\gamma \in \mathbb{R} \cup \{-\infty\}$ *and,*

$$\frac{1}{n}\log \rho_{0,n-1} \to \gamma \quad \mathbb{P}\text{-}a.s., \tag{2.22}$$

$$\frac{1}{n}\log \|xM_{0,n-1}\| \to \gamma \quad \textit{uniformly for } x \in S, \quad \mathbb{P}\text{-}a.s., \tag{2.23}$$

$$\frac{1}{n}\log \iota(M_{0,n-1}) \to \gamma \quad \mathbb{P}\text{-}a.s., \tag{2.24}$$

$$\frac{1}{n}\log\langle xM_{0,n-1}, y\rangle \xrightarrow{\mathbb{P}} \gamma \quad \textit{uniformly for } x, y \in \mathcal{S}. \tag{2.25}$$

If, additionally, either $\mathbb{E}\log^- \iota(M_0) < \infty$ *or* $\mathbb{E}\log D_0 < \infty$ *(where* D_n *is defined as in (2.11)), then the convergence in probability in (2.25) can be improved to the a.s. convergence. In particular, in the case where* $\mathbb{E}\log^- \iota(M_0) < \infty$*, we have* $\gamma \in \mathbb{R}$ *and*

$$\sup_{x,y\in\mathcal{S}}\left|\frac{1}{n}\log\langle xM_{1,n}, y\rangle - \gamma\right| \to 0 \quad \mathbb{P}\text{-}a.s. \tag{2.26}$$

The proof will be given in Section 3.

Notice that the condition $\mathbb{E}\log D_0 < \infty$ is much weaker than the Furstenberg-Kesten condition **A3**. Actually, when **A3** holds, then D_0 can be taken as a constant.

The convergences (2.22), (2.24) and (2.26) were established in Hennion [30] under the moment condition that $\mathbb{E}|\log\|M_0\|| < \infty$ and $\mathbb{E}|\log \iota(M_0)| < \infty$. Corollary 2.5 completes Hennion's results without assuming $\mathbb{E}|\log \iota(M_0)| < \infty$.

When M_n are the mean matrices of a multi-type branching process (Z_n) in a stationary and ergodic environment with $\mathbb{P}_\xi(Z_n^i \to 0) < 1$ a.s. for all $i = 1, \cdots, d$, under the same conditions **A1** and **A2**, Tanny [53] proved a weaker version of (2.24): he established the convergence in probability instead of the a.s. convergence.

In the same spirit, from central limit theorems for the stationary and ergodic sequence $(\log \mu_n)$ of real random variables, we can easily get the corresponding results for the scalar products $\langle xM_{0,n-1}, y\rangle$. We also mention that using (2.16) or (2.17), limit theorems on $(\log \mu_n)$ also enable us to get corresponding results for the spectral radius $\rho_{0,n-1}$ and for the matrix norm $\|M_{0,n-1}\|$.

2.3 Definition of a multitype branching process in a random environment. Let $\xi = (\xi_0, \xi_1, \xi_2, \cdots)$ be a stationary and ergodic sequence of random variables taking values in some abstract space Θ. This sequence represents the environment, with ξ_n denoting the random environment at time n. Suppose that each realization of ξ_n corresponds to d probability distributions on $\mathbb{N}^d$:

$$p^r(\xi_n) = \{p_k^r(\xi_n) : \ k \in \mathbb{N}^d\}, \quad \text{where} \ \ p_k^r(\xi_n) \geqslant 0 \ \ \text{and} \ \ \sum_k p_k(\xi_n) = 1, \quad 1 \leqslant r \leqslant d,$$

called offspring distribution at time n, given the environment ξ. Let $d \geqslant 1$ be an integer. A d-type branching process $Z_n = (Z_n(1), \cdots, Z_n(d))$ in the random environment $\xi = (\xi_0, \xi_1, \ldots)$ is defined as follows: for any $n \geqslant 0$,

$$Z_{n+1} = \sum_{r=1}^{d}\sum_{l=1}^{Z_n(r)} N_{l,n}^r, \tag{2.27}$$

where, for any $j \in \{1, \dots, d\}$, the j-th component $Z_n(j)$ of Z_n is the number of type j particles of generation n, and the j-th component $N^r_{l,n}(j)$ of $N^r_{l,n}$ (row vector) denotes the number of type j children of l-th type r particle of generation n. All the random variables ξ, Z_n, $N^r_{l,n}$, are defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Given the environment ξ, all the vectors $Z_0, N^r_{l,n}$ for $n \geqslant 0$, $l \geqslant 1$, $1 \leqslant r \leqslant d$ are independent of each other, and for any $l \geqslant 1$, each $N^r_{l,n}$ has the same law $p^r(\xi_n)$, whose probability generating function is

$$f^r_{\xi_n}(s) = \mathbb{E}_\xi\left(\prod_{j=1}^d s_j^{N^r_{l,n}(j)}\right) = \sum_{k_1,\dots,k_d=0}^{\infty} p^r_{k_1,\dots,k_d}(\xi_n) s_1^{k_1} \dots s_d^{k_d}, \quad s = (s_1, \dots, s_d) \in [0,1]^d,$$

i.e. $\mathbb{P}_\xi(N^r_{l,n} = k) = p^r_k(\xi_n)$, $\forall k \in \mathbb{N}^d, l \geqslant 1, n \geqslant 0, 1 \leqslant r \leqslant d$. Here we denote by $\mathbb{P}_\xi$ the conditional probability given ξ, and by $\mathbb{E}_\xi$ the corresponding conditional expectation. The probability $\mathbb{P}_\xi$ is called quenched law, while the total probability $\mathbb{P}$ is called annealed law. The expectation with respect to $\mathbb{P}$ will be denoted by $\mathbb{E}$.

Notice that (Z_n) reduces to the classical d-type Galton-Watson process if all the ξ_n are the same constant.

As usual we suppose that $Z_0 = x$ is non random, and we write Z^x_n for Z_n when $Z_0 = x$, for each $x = (x(1), \cdots, x(d)) \in \mathbb{N}^d \setminus \{0\}$. When $x = e_i$, we write Z^i_n for $Z^{e_i}_n$.

2.4 Main results on multitype branching processes in random environments. All over the paper we assume that the environment sequence $\xi = (\xi_n)$ is stationary and ergodic unless stated otherwise, and that the means of the offspring distributions,

$$M_n(r,j) = \mathbb{E}_\xi\Big(N^r_{l,n}(j)\Big) = \mathbb{E}_\xi\left(Z_{n+1}(j)\Big|Z_n = e_r\right) = \frac{\partial f^r_{\xi_n}}{\partial s_j}(1),$$

exist and are finite a.s. for all $1 \leqslant r, j \leqslant d$ and $n \geqslant 0$. The non-negative matrix formed by the entries $M_n(r,j)$ is denoted by $M_n = M(\xi_n)$.

For convenience we extend $(\xi_k)_{k\geqslant 0}$ to a stationary and ergodic sequence $(\xi_k)_{k\in\mathbb{Z}}$ with the index set $\mathbb{Z}$. Consequently the sequence $(M_n)_{n\geqslant 0}$ is also extended to a stationary and ergodic sequence indexed by $\mathbb{Z}$. The sequence of the mean matrices (M_n) will play an important role in the sequel.

We are interested in the asymptotic properties of Z^x_n in the supercritical case where $\gamma > 0$, with γ the Lyapunov exponent defined by (2.9). We shall mainly consider the convergence of the normalized scalar product

$$W^x_n(y) := \frac{\langle Z^x_n, y\rangle}{\mathbb{E}_\xi\langle Z^x_n, y\rangle} = \frac{\langle Z^x_n, y\rangle}{\langle xM_{0,n-1}, y\rangle}, \quad n \geqslant 0,\ y \in \mathbb{R}^d_+ \setminus \{0\}. \tag{2.28}$$

We will use the following $L\log L$ condition. In the constant environment case, it was first introduced by Kesten and Stigum [40].

H1. *For all* $1 \leqslant r, j \leqslant d$,

$$\mathbb{E}\left[\frac{Z^r_1(j)}{\langle e_r M_0, e_j\rangle}\log^+\frac{Z^r_1(j)}{\langle e_r M_0, e_j\rangle}\right] < \infty.$$

Our first main result is a Kesten-Stigum type theorem for the scalar products $\langle Z_n^x, y\rangle$. We will use the natural filtration

$$\mathscr{F}_0 = \sigma\{\xi\} \text{ and } \mathscr{F}_n = \sigma\{\xi, N_{l,j}^r : l \geqslant 1, 0 \leqslant j < n, 1 \leqslant r \leqslant d\} \text{ for } n \geqslant 1. \tag{2.29}$$

Theorem 2.6 (Kesten-Stigum type theorem for $\langle Z_n^x, y\rangle$)**.** *Let $x \in \mathbb{N}^d \setminus \{0\}$.*

*(1) Assume **A2**. Let $(u_n), (\lambda_n)$ be defined as in Theorem 2.1. Then, the sequence*

$$W_n^x(u_n) = \frac{\langle Z_n^x, u_n\rangle}{\langle xM_{0,n-1}, u_n\rangle} = \frac{\langle Z_n^x, u_n\rangle}{\langle u_0, x\rangle \prod_{j=0}^{n-1} \lambda_j}, \quad n \geqslant 0, \tag{2.30}$$

is a martingale with respect to the filtration $(\mathscr{F}_n)$, both under $\mathbb{P}_\xi$ (for almost every ξ) and under $\mathbb{P}$, so that the limit

$$W^x = \lim_{n\to\infty} W_n^x(u_n) \text{ exists with } \mathbb{E}_\xi W^x \leqslant 1 \quad \mathbb{P}\text{-a.s.}$$

*(2) Assume **A1**, **A2** and $\gamma > 0$. If **H1** holds, then $\mathbb{E}_\xi W^x = 1$ $\mathbb{P}$-a.s., so that W^x is non-degenerate, that is, $\mathbb{P}_\xi(W^x > 0) > 0$ $\mathbb{P}$-a.s. Moreover, when W^x is non-degenerate, then*

$$\{W^x > 0\} = \{\|Z_n^x\| \to \infty\} = \{Z_n^x \not\to 0\} \quad \mathbb{P}\text{-a.s.} \tag{2.31}$$

*Assume additionally that the Furstenberg-Kesten condition **A3** holds, and the random environment sequence (ξ_n) is i.i.d. Then W^x is non-degenerate if and only if **H1** holds.*

*(3) Assume **A1**, **A3**, $\gamma > 0$, and that the random environment (ξ_n) is i.i.d. Then as $n \to \infty$,*

$$\sup_{y\in\mathbb{R}_+^d\setminus\{0\}} |W_n^x(y) - W^x| \to 0 \quad \text{in probability under } \mathbb{P}. \tag{2.32}$$

If additionally for some $p > 1$,

$$\max_{1\leqslant r,j\leqslant d} \mathbb{E}\left(\frac{Z_1^r(j)}{M_0(r,j)}\right)^p < \infty \quad \text{and} \quad \mathbb{E}\|M_0\|^{1-p} < \infty, \tag{2.33}$$

then the convergence in probability in (2.32) can be improved to the a.s. convergence:

$$\sup_{y\in\mathbb{R}_+^d\setminus\{0\}} |W_n^x(y) - W^x| \underset{n\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.} \tag{2.34}$$

Recall that by Theorem 2.1, the sequences (u_n) and (λ_j) are stationary and ergodic; the common law of u_n is the unique probability measure ν^* on $\mathcal{S}$ satisfying $\nu^* = \nu^* * \mu^*$, where μ^* is the law of M_0^T; the common law of λ_j satisfies $\mathbb{E}\log\lambda_0 = \gamma$.

The proof of Theorem 2.6 will be done in Sections 4 and 8: see Lemma 4.2, Theorem 4.4, Proposition 4.8, and Theorem 8.1. When $x = e_i$ and $y = e_j$, the convergence (in probability and a.s.) of $W_n^{e_i}(e_j) = Z_n^i(j)/M_{0,n-1}(i,j)$ and the non-degeneracy of its limit W^i together with the properties (2.31) have been established in [22].

Notice that the convergence of $W_n^x(y)$ with $y = \mathbb{1}$ improves Tanny's result that $\|Z_n^x\| = O(\|xM_{0,n-1}\|)$ [53, Theorem 9.11].

Theorem 2.6 gives a good bridge between branching processes in random environment and products of random matrices. It can be applied to establish interesting limit theorems on branching processes. For example, from it we obtain easily the following:

Corollary 2.7 (LLN and CLT for $\|Z_n^x\|$)**.** *Assume* ***A1****,* ***A2****,* $\gamma > 0$ *and* $q^{e_i}(\xi) := \mathbb{P}_\xi(Z_n^i \to 0) < 1$ *a.s. for all* $i \in \{1, \cdots, d\}$*. Then*

$$\frac{1}{n} \log \|Z_n^x\| \to \gamma \quad \mathbb{P}\text{-a.s. on } \{Z_n \not\to 0\}. \tag{2.35}$$

If additionally

$$\mathbb{E}[(\log^+ \|M_0\|)^2] + \mathbb{E}[(\log^- \iota(M_0))^2] < \infty, \tag{2.36}$$

and $Z_n \not\to 0$ *a.s., then for some* $\sigma \in [0, \infty)$*,*

$$\frac{\log \|Z_n^x\| - n\gamma}{\sqrt{n}} \to N(0, \sigma^2) \quad \text{in distribution.} \tag{2.37}$$

For $x = e_i$, the law of large numbers (2.35) is due to Tanny [53]. Actually, Tanny [53, Proposition 9.7] first proved that if (Z_n) satisfies **A1**, **A2** and $q^{e_i}(\xi) := \mathbb{P}_\xi(Z_n^i \to 0) < 1$ a.s., then the process Z_n is *stable* in the sense that

$$\frac{1}{n} \log \iota(M_{0,n-1}) \to \gamma \quad \text{in probability} \tag{2.38}$$

(notice that the M_n defined in [53] corresponds to M_n^T defined in this paper). In fact the convergence in (2.38) holds a.s. whenever **A1** and **A2** hold: see (2.24) of Corollary 2.5 below. He then proved that the condition $\gamma > 0$, together with the stability implies (2.35) for $x = e_i$: see Tanny [53, Theorem 9.6]. Based on Parts 1 and 2 of Theorem 2.6, we can give a short proof of (2.35) for any x, assuming the non-degeneracy condition **H1** and following the proof of Corollary 2.9 of [22] where the case $x = e_i$ was treated.

Also for $x = e_i$, the central limit theorem (2.37) has been established in [24, Theorem 2.4], using the fundamental martingale for $x = e_i$, and the central limit theorem for products of positive random matrices [30]. Using Parts 1 and 2 of Theorem 2.6 we can prove (2.37) in the same way for arbitrary $x \in \mathbb{N}^d \setminus \{0\}$.

Notice that if we assume the stronger conditions **A3** and (2.33), then using Part 3 of Theorem 2.6 with $y = \mathbb{1}$, we can obtain directly (2.35) and (2.37) from the law of large numbers and the central limit theorem on $\log \|xM_{0,n-1}\|$ established in [30].

We now state a duality of the Kesten-Stigum theorem. Let

$$E = \{\|Z_n^x\| \to \infty\} \quad \text{and} \quad S = \{\|Z_n^x\| \not\to 0\}$$

be the explosion and survival events. Write $\mathbb{P}_E(\cdot) = \mathbb{P}(\cdot|E)$ for the conditional probability given E, and $\xrightarrow[n\to\infty]{\mathbb{P}_E}$ (resp. $\xrightarrow[n\to\infty]{d(\mathbb{P}_E)}$) for the convergence in probability (resp. in law) under $\mathbb{P}_E$. The same notation will be used for the conditional probability $\mathbb{P}_S$ given S, as well as for the annealed and quenched probabilities $\mathbb{P}$ and $\mathbb{P}_\xi$.

Theorem 2.8 (Duality of the Kesten-Stigum theorem)**.** *Assume conditions* ***A1****,* ***A2****,* $\gamma > 0$*, and that the random environment sequence* $\xi = (\xi_0, \xi_1, \cdots)$ *is i.i.d. Assume also that the explosion event* E *has positive probability. Then, for any fixed integer* $k \geqslant 0$*, as* $n \to \infty$*,*

$$\sup_{y \in \mathbb{R}_+^d \setminus \{0\}} \left| \frac{\langle Z_{n+k}, y \rangle}{\langle Z_n M_{n,n+k-1}, y \rangle} - 1 \right| \xrightarrow{\mathbb{P}_E} 0; \tag{2.39}$$

Assume additionally that for some $p > 1$,

$$\max_{1\leqslant r,j\leqslant d}\mathbb{E}\left(\frac{Z_1^r(j)}{M_0(r,j)}\right)^p<\infty \quad \textit{and} \quad \max_{1\leqslant i,j\leqslant d}\mathbb{E}\Big(M_0(i,j)^{1-p}\Big)<\infty. \tag{2.40}$$

Then the convergence in probability in (2.39) can be improved to the a.s. convergence conditionally on S: for any fixed $k\geqslant 0$, as $n\to\infty$,

$$\sup_{y\in\mathbb{R}^d_+\setminus\{0\}}\left|\frac{\langle Z_{n+k},y\rangle}{\langle Z_nM_{n,n+k-1},y\rangle}-1\right|\underset{n\to\infty}{\longrightarrow}0 \quad \mathbb{P}_S \textit{ a.s.} \tag{2.41}$$

Our main result Theorem 2.6 states that the convergence of the ratio $\frac{\langle Z_{n+k},y\rangle}{\langle Z_nM_{n,n+k-1},y\rangle}$ in (2.39) or (2.41) also holds for $n=0$ as $k\to\infty$, but with the limit W^x instead of the limit 1 here. Applying this to the new process $(Z_m)_{m\geqslant n}$ starting at time n (with initial state Z_n, which is independent of the branching elements $\{N^r_{l,m}: r\in\{1,\cdots,d\}, m\geqslant n, l\geqslant 1\}$, conditional on ξ), we see that the convergence remains valid for fixed $n\geqslant 0$ as $k\to\infty$, conditioned on $Z_n=z$, for any given $z\in\mathbb{N}^d$. In this sense Theorem 2.8 is a duality of Theorem 2.6.

An important ingredient in the proof of the Kesten-Stigum type theorem (Theorem 2.6) is the following theorem about the convergence of the direction $\vec{Z}^x_n:=\frac{Z^x_n}{\|Z^x_n\|}$. Recall that, for $k\leqslant n$, $\rho_{k,n}$ is the spectral radius of $M_{k,n}=M_k\cdots M_n$, and $u_{k,n},v_{k,n}\in\mathbb{R}^d_+$ are such that (1.4) holds.

Theorem 2.9 (Convergence of the direction)**.** *Assume conditions **A1**, **A3**, $\gamma>0$, and that the random environment sequence (ξ_n) is i.i.d. Assume also that the explosion event E has positive probability. Let $x\in\mathbb{N}^d\setminus\{0\}$, and $\vec{Z}^x_n=\frac{Z^x_n}{\|Z^x_n\|}$. Then:*

(1) Let (v_n) be defined as in Theorem 2.1. For each $k\in\mathbb{N}$,

$$\lim_{n\to\infty}\vec{Z}^x_n-v_{k,n-1}=\lim_{n\to\infty}\vec{Z}^x_n-v_{n-1}=0 \quad \textit{in probability under } \mathbb{P}_E. \tag{2.42}$$

(2) Conditional on E, $\vec{Z}^x_n$ converges in law to the common law ν of v_n, which is the unique μ-invariant measure: $\nu*\mu=\nu$, where μ is the law of M_0.

(3) For each $k\in\mathbb{N}$, the following convergences hold in probability under $\mathbb{P}_E$:

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^d_+\setminus\{0\}}\left|\frac{\langle\vec{Z}^x_n,y\rangle}{\langle v_{k,n-1},y\rangle}-1\right|=\lim_{n\to\infty}\sup_{y\in\mathbb{R}^d_+\setminus\{0\}}\left|\frac{\langle\vec{Z}^x_n,y\rangle}{\langle v_{n-1},y\rangle}-1\right|=0. \tag{2.43}$$

(4) If additionally (2.33) holds for some $p>1$, then the convergence in probability on E in (2.42) and (2.43) can be improved to the a.s. convergence on S.

For $x=e_i$ and $k=0$, the convergence of $\vec{Z}^x_n-v_{0,n-1}$ in (2.42), and the convergence in law of $\vec{Z}^x_n$, have been proved in [22]. It is interesting to notice that the convergence in (2.42) does not depend on the history before the time k, for any $k>0$. This *memoryless property* can be useful in the numerical calculation of the direction $\vec{Z}^x_n$ when the data M_i is missing for $i<k$. For the constant environment case, see e.g. [55] for the result corresponding to Part 1.

Theorem 2.9 can be applied to establish limit theorems for $\langle Z^x_n,y\rangle$. For example, from this theorem we can deduce the following:

Corollary 2.10 (LLN and CLT for $\langle Z_n^x, y\rangle$)**.** *Assume that the random environment sequence (ξ_n) is i.i.d.*

*(1) Assume **A1**, **A3**, $\gamma > 0$ and $q^{e_i}(\xi) := \mathbb{P}_\xi(Z_n^i \to 0) < 1$ a.s. Then for each starting state $x \in \mathbb{N}^d \setminus \{0\}$, we have: as $n \to \infty$,*

$$\frac{1}{n} \sup_{y\in\mathcal{S}} \Big| \log\langle Z_n^x, y\rangle - \gamma \Big| \to 0 \quad \text{in probability conditional on } E. \tag{2.44}$$

If additionally (2.33) holds for some $p > 1$, then the above convergence in probability conditional on E can be improved to the a.s. convergence conditional on S.

*(2) Assume **A3**, **H1**, (2.36), $\gamma > 0$ and $\|Z_n\| \to \infty$ a.s. Then then for some $\sigma \in [0, \infty)$ and all sequences (y_n) in $\mathbb{R}_+^d$ such that $\frac{\log\|y_n\|}{\sqrt{n}} \to 0$, we have*

$$\frac{\log\langle Z_n^x, y_n\rangle - n\gamma}{\sqrt{n}} \to N(0, \sigma^2) \quad \text{in distribution.} \tag{2.45}$$

In particular, taking $y = e_j$ in (2.44) and $y_n = e_j$ (for each n) in (2.45), we obtain a law of large numbers and a central limit theorem on $Z_n^x(j)$.

For the proof of Corollary 2.10, we first remark that, from Theorem 2.9, we have

$$\log\langle Z_n^x, y\rangle = \log\|Z_n^x\| + \log\langle v_{n-1}, y\rangle + \varepsilon_n^x(y), \tag{2.46}$$

where $\sup_{y\in\mathbb{R}_+^d\setminus\{0\}} |\varepsilon_n^x(y)| \to 0$, conditional on E or S, in probability or a.s. under the corresponding conditions. We also remark that $v_{n-1} \geqslant 1/(dD)$ under **A3** (see Lemma 3.6 of Section 3, or [22, Lemma 7.1]), so that

$$\frac{1}{dD}\|y\| \leqslant \langle v_{n-1}, y\rangle \leqslant \|y\|. \tag{2.47}$$

With (2.46) and (2.47), we can then deduce the desired conclusions on $\langle Z_n^x, y\rangle$ from the law of large numbers and central limit theorem on $\|Z_n\|$ of Corollary 2.7.

3. Perron-Frobenius theorem and proof of Theorem 2.1

The objective of this section is to establish the Perron-Frobenius theorem and its corollary, namely Theorem 2.1 together with Remark 2.3., and Corollary 2.5.

We begin with some preliminaries for products of random matrices. Let $\mathbb{R}_+^d = \{x \in \mathbb{R}^d, x \geqslant 0\}$ be the non-negative quadrant, where $x \geqslant 0$ means that each component of x is non-negative. Let $\mathcal{S} = \{v \in \mathbb{R}_+^d : \|v\| = 1\}$. The space $\mathcal{S}$ will play the role of the projective space. Recall that G is the set of $d \times d$ non-negative matrices.

Following [30], we equip the projective space $\mathcal{S}$ with the Hilbert cross-ratio metric $\mathbf{d}$ defined as follows: for any $x, y \in \mathcal{S}$,

$$\mathbf{d}(x, y) = \frac{1 - m(x, y)m(y, x)}{1 + m(x, y)m(y, x)}, \tag{3.1}$$

where $m(x, y) = \sup\{\lambda > 0 : \ \lambda y(i) \leqslant x(i), \ \forall i = 1, \ldots, d\}$ for $x = (x(1), \ldots, x(d)) \in \mathbb{R}^d$ and $y = (y(1), \ldots, y(d)) \in \mathbb{R}^d$. By Proposition 3.1 in [30], the distance $\mathbf{d}$ is bounded, $\mathbf{d}(x, y) \leqslant 1$, for any $x, y \in \mathcal{S}$, and has the important property that for any

matrix $g \in G$ the action (2.5) on $\mathcal{S}$ is a contraction with respect to $\mathbf{d}$, that is, for any g there exists $\mathfrak{c}(g) \leqslant 1$ such that, for any $x, y \in \mathcal{S}$, it holds

$$\mathbf{d}(x \cdot g, y \cdot g) \leqslant \mathfrak{c}(g)\mathbf{d}(x, y) \leqslant \mathfrak{c}(g). \tag{3.2}$$

The number $\mathfrak{c}(g)$ is called contraction coefficient of the matrix g. An explicit calculation of the coefficient $\mathfrak{c}(g)$ in terms of the matrix is performed in [30], where it is shown that

$$\mathfrak{c}(g) = \max_{i,j,k,l \in \{1,\dots,d\}} \frac{|g(k,i)g(l,j) - g(k,j)g(l,i)|}{g(k,i)g(l,j) + g(k,j)g(l,i)}.$$

The contraction coefficient $\mathfrak{c}(\cdot)$ has the following properties: (i) $\mathfrak{c}(g) < 1$ if and only if $g > 0$; (ii) $\mathfrak{c}(g) = \mathfrak{c}(g^T)$; and (iii) $\mathfrak{c}(g, g') \leqslant \mathfrak{c}(g)\mathfrak{c}(g')$ for $g, g' \in G$. The distance $\mathbf{d}$ satisfies, for any $x, y \in \mathcal{S}$,

$$\frac{1}{2}\|x - y\| \leqslant \mathbf{d}(x, y) \leqslant 1. \tag{3.3}$$

Our goal of this section is to prove an analog of the Perron-Frobenius theorem (Theorem 2.1) for products of random matrices. The central point of the proof is the following contracting property due to Hennion [30]. In this section, we assume that $(M_n)_{n\in\mathbb{Z}}$ is a stationary and ergodic sequence of random $d \times d$ non-negative matrices, defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Recall that $\mathfrak{c}(g)$ denotes the contraction coefficient of the matrix g.

Lemma 3.1. *Under condition **A2**, for any fixed $k \in \mathbb{Z}$ as $n \to \infty$, and for any fixed $n \in \mathbb{Z}$ as $k \to -\infty$,*

$$\mathfrak{c}(M_{k,n}) = \mathfrak{c}((M_{k,n})^T) \to 0 \quad \mathbb{P}\text{-a.s.} \tag{3.4}$$

The assertion for the case where $k = 0$ and $n \to \infty$ was established in [30, Lemma 3.2]. The assertion for each fixed $k \in \mathbb{Z}$ as $n \to \infty$ can be proved in a similar way. The assertion for fixed n as $k \to -\infty$ can be obtained from the previous assertion applied to the transposed and reversed sequence $\{M_{-k}^T, k \in \mathbb{Z}\}$.

Denote

$$\theta_k = \inf\{n \geqslant k : M_{k,n} \in \mathcal{S}^\circ\}.$$

Recall that $\rho_{k,n}$ denotes the spectral radius of $M_{k,n} = M_k \cdots M_n$, $u_{k,n}$ and $v_{k,n}$ are associated unit eigenvectors of $M_{k,n}^T$ and $M_{k,n}$ (cf. (1.4)). It was proved in [30] that under condition **A2**, we have $\theta_k < \infty$ $\mathbb{P}$-a.s. for all $n \geqslant 0$. Thus, for $n \geqslant \theta_k$, the eigenvalue $\rho_{k,n}$ is strictly positive, and the unit eigenvectors $u_{k,n}$ and $v_{k,n}$ are strictly positive and unique.

In the following lemma we state some properties of the eigenvector $u_{k,n}$. We will need the following σ-algebras: for any $k \in \mathbb{Z}$,

$$\mathscr{F}_{-\infty,k} = \sigma\{M_j, j \leqslant k\}, \quad \mathscr{F}_{k,\infty} = \sigma\{M_j, j \geqslant k\}. \tag{3.5}$$

Lemma 3.2. *Assume condition **A2**. Then there exists a sequence $\{u_k : k \in \mathbb{Z}\}$ with $u_k \in \mathcal{S}$ and $u_k > 0$ such that, for each fixed $k \in \mathbb{Z}$ as $n \to \infty$, and for each fixed $n \in \mathbb{Z}$ as $k \to -\infty$,*

$$\sup_{y \in \mathcal{S}} d(y \cdot (M_{k,n})^T, u_k) \to 0 \quad \mathbb{P}\text{-a.s.} \tag{3.6}$$

The limit variable u_k is $\mathcal{F}_{k,\infty}$-measurable and satisfies

$$u_{k+1} \cdot M_k^T = u_k, \quad k \in \mathbb{Z}, \quad \mathbb{P}\text{-}a.s. \tag{3.7}$$

Moreover, for any fixed $k \in \mathbb{Z}$ as $n \to \infty$, and for any fixed $n \in \mathbb{Z}$ as $k \to -\infty$,

$$d(u_{k,n}, u_k) \to 0 \quad \mathbb{P}\text{-}a.s.. \tag{3.8}$$

The sequence (u_k) is stationary and ergodic; each u_k has the same law ν^*, which is the unique μ^T-invariant law: that is, ν^* is the unique probability law on $\mathcal{S}$ satisfying $\nu^* * \mu^T = \nu^*$, where μ^T is the law of M_0^T.

The results of Lemma 3.2 for the case where $k = 1$ and $n \to \infty$ were proved in Hennion [30]. The conclusion presented here can be proved in the same way. For convenience of readers, we give a proof below, following Hennion [30].

Proof of Lemma 3.2. By Lemma 3.1, the event

$$\Omega_1 = \left\{ \lim_{n\to\infty} \mathfrak{c}((M_{k,n})^T) = 0 \ \forall k \in \mathbb{Z}, \lim_{k\to-\infty} \mathfrak{c}((M_{k,n})^T) = 0 \ \forall n \in \mathbb{Z} \right\}, \tag{3.9}$$

has probability 1. For any $k \leqslant n$ in $\mathbb{Z}$, introduce the event $C_{k,n} = \mathcal{S} \cdot (M_{k,n})^T$. Since,

$$\mathcal{S} \cdot M_{n+1}^T \subseteq \mathcal{S}. \tag{3.10}$$

we have $C_{k,n+1} \subset C_{k,n}$. Moreover, the sets $C_{k,n}$ are compact. So $C_k := \cap_{m=k}^{\infty} C_{k,m} \neq \emptyset$. On Ω_1, the diameter $|C_k|$ of C_k is 0 because

$$|C_k| \leqslant \sup_{y,y' \in \mathcal{S}} d(y \cdot (M_{k,n})^T, y' \cdot (M_{k,n})^T) \leqslant \mathfrak{c}((M_{k,n})^T) \tag{3.11}$$

which goes to 0 as $n \to \infty$. Therefore, on Ω_1, the set C_k consists of one single point, which we denote by u_k. From condition **A2** we know that a.s. $M_{k,n} > 0$ for n large enough (see [30, Lemma 3.1]). Since $u_k \in C_k \subset C_{k,n}$, this implies that $u_k > 0$ a.s.

Using again the fact that $u_k \in C_{k,n}$, we know that there exists $y' \in \mathcal{S}$ such that $u_k = y' \cdot (M_{k,n})^T$. Therefore, by (3.11), we have that, for any $k \leqslant n$ in $\mathbb{Z}$ and any $y \in \mathcal{S}$,

$$d(y \cdot (M_{k,n})^T, u_k) = d\Big(y \cdot (M_{k,n})^T, y' \cdot (M_{k,n})^T\Big) \leqslant \mathfrak{c}\Big((M_{k,n})^T\Big). \tag{3.12}$$

So on Ω_1, $\sup_{y\in\mathcal{S}} d(y \cdot (M_{k,n})^T, u_k) \to 0$. This proves (3.6). In particular, for any fixed $k \in \mathbb{Z}$, uniformly in $y \in \mathcal{S}$,

$$\lim_{n\to\infty} y \cdot (M_{k,n})^T = u_k \quad \mathbb{P}\text{-a.s.}, \tag{3.13}$$

which implies that u_k is $\mathcal{F}_{k,\infty}$-measurable.

To see (3.7), we just need to notice that

$$u_k = \lim_{n\to\infty} y \cdot (M_k M_{k+1} \dots M_n)^T = (\lim_{n\to\infty} y \cdot (M_{k+1,n})^T) \cdot M_k^T = u_{k+1} \cdot M_k^T.$$

For the proof of (3.8), we just need to use again (3.6): choosing in it $y = u_{k,n}$ and using the fact that $u_{k,n} \cdot M_{k,n}^T = u_{k,n}$, we get (3.8).

As u_k depends only on $(M_k, M_{k+1}, \cdots)$, and (M_k) is stationary and ergodic, the sequence (u_k) remains to be stationary and ergodic. In particular, all u_k have the same law. Therefore, from (3.7), we see that the common law ν^* of u_k satisfies the equation $\nu^* = \nu^* * \mu^*$.

We now prove that ν^* is the unique solution of the equation $\nu^* = \nu^* * \mu^*$. To see this, let ν' be a probability law on $\mathcal{S}$ satisfying $\nu' = \nu' * \mu^*$, and let U_0 be a random

variable with law ν', independent of the sequence (M_k). Then by the equation on ν', for each $n \geqslant 1$, $U_0 \cdot M_{1,n}^T$ has the same law as ν'. On the other hand, from (3.6), $U_0 \cdot M_{1,n}^T$ converges a.s. to u_1 as $n \to \infty$, so that it converges in law to the law ν^* of u_1. This shows that $\nu' = \nu^*$.

The proof of the lemma is thus finished. □

The properties of the eigenvector $v_{k,n}$ are similar to those of $u_{k,n}$, as indicated below. They can be deduced from the previous Lemma 3.2 by considering the transposed and reversed sequence $(M_{-n}^T)_{n\in\mathbb{Z}}$. They can also be proved in a similar way as in the proof of Lemma 3.2.

Lemma 3.3. *Assume condition **A2**. Then there exists a sequence $\{v_n : n \in \mathbb{Z}\}$ with $v_n \in \mathcal{S}$ and $v_n > 0$ such that, for each fixed $n \in \mathbb{Z}$ as $k \to -\infty$, and for each fixed $k \in \mathbb{Z}$ as $n \to \infty$,*

$$\sup_{y\in\mathcal{S}} d(y \cdot M_{k,n}, v_n) \to 0 \quad \mathbb{P}\text{-a.s.} \tag{3.14}$$

The limit variable v_n is $\mathcal{F}_{-\infty,n}$-measurable and satisfies

$$v_n \cdot M_{n+1} = v_{n+1}, \quad n \in \mathbb{Z}, \quad \mathbb{P}\text{-a.s.} \tag{3.15}$$

Moreover, for any fixed $n \in \mathbb{Z}$ as $k \to -\infty$, and for any fixed $k \in \mathbb{Z}$ as $n \to \infty$,

$$d(v_{k,n}, v_n) \to 0 \quad \mathbb{P}\text{-a.s.} \tag{3.16}$$

The sequence (v_n) is stationary and ergodic; each v_n has the same law ν, which is the unique μ-invariant law: $\nu * \mu = \nu$.

The following result is the key for the proof of the Perron-Frobenius theorem, Theorem 2.1.

Lemma 3.4. *Assume condition **A2**. Then, for any $k \in \mathbb{Z}$, we have $\mathbb{P}$-a.s.,*

$$\lim_{n\to\infty} \sup_{x,y\in\mathcal{S}} \left| \frac{\langle y \cdot (M_{k,n})^T, x\rangle}{\langle u_k, x\rangle} - 1 \right| = 0, \tag{3.17}$$

and

$$\lim_{n\to\infty} \sup_{x,y\in\mathcal{S}} \left| \frac{\langle y \cdot (M_{k,n})^T, x\rangle}{\langle u_{k,n}, x\rangle} - 1 \right| = 0. \tag{3.18}$$

Proof. Let $k \in \mathbb{Z}$ be fixed. Notice that for any $x \in \mathcal{S}$, $\mathbb{P}$-a.s.,

$$\langle u_k, x\rangle \geqslant \min_{1\leqslant j\leqslant d} u_k(j)\|x\| = \min_{1\leqslant j\leqslant d} u_k(j) =: \underline{u_k} > 0, \tag{3.19}$$

so that for $n \geqslant k$,

$$\begin{aligned}
&\sup_{x,y\in\mathcal{S}} \left| \frac{\langle y \cdot (M_{k,n})^T, x\rangle}{\langle u_k, x\rangle} - 1 \right| = \sup_{x,y\in\mathcal{S}} \left| \frac{\langle y \cdot (M_{k,n})^T, x\rangle - \langle u_k, x\rangle}{\langle u_k, x\rangle} \right| \\
&= \sup_{x,y\in\mathcal{S}} \left| \frac{\langle y \cdot (M_{k,n})^T - u_k, x\rangle}{\langle u_k, x\rangle} \right| \leqslant \sup_{y\in\mathcal{S}} \frac{1}{b_k} \|y \cdot (M_{k,n})^T - u_k\| \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}$$

where the last step holds by (3.6) of Lemma 3.2 and (3.3). This proves (3.17).

The proof of (3.18) is similar: we just need to replace u_k by $u_{k,n}$, using the fact that (see (3.8)) as $n \to \infty$,

$$\sup_{x\in\mathcal{S}} |\langle u_{k,n}, x\rangle - \langle u_k, x\rangle| \to 0,$$

so that for $\mathbb{P}$-almost all ω, there is n_ω large enough, such that for all $n \geqslant n_\omega$ and all $x \in \mathcal{S}$, with $\varepsilon = \underline{u_k}/2$ (see (3.19) for the definition of $\underline{u_k}$),

$$\langle u_{k,n}, x\rangle \geqslant \langle u_k, x\rangle - \varepsilon \geqslant \min_{1\leqslant j\leqslant d} u_k(j) - \varepsilon = \underline{u_k}/2 > 0.$$

□

The following result is the counterpart of Lemma 3.4 for v_n and $v_{k,n}$. It can be proved in a similar way, and is also a direct consequence of Lemma 3.4 applied to the transposed and reversed sequence $\{M_{-k}^T : k \in \mathbb{Z}\}$.

Lemma 3.5. *Assume condition **A2**. Then, for any fixed $n \in \mathbb{Z}$, we have $\mathbb{P}$-a.s.,*

$$\lim_{k\to-\infty} \sup_{x,y\in\mathcal{S}} \left| \frac{\langle y \cdot M_{k,n}, x\rangle}{\langle v_n, x\rangle} - 1 \right| = 0, \tag{3.20}$$

and

$$\lim_{k\to-\infty} \sup_{x,y\in\mathcal{S}} \left| \frac{\langle y \cdot M_{k,n}, x\rangle}{\langle v_{k,n}, x\rangle} - 1 \right| = 0. \tag{3.21}$$

Proof of Theorem 2.1. For Part 1, the assertions (2.13) and (2.14), together with the ergodicity of the sequences $\{u_k\}$ and $\{v_k\}$, have already been proved in Lemmas 3.2 and 3.3. The assertion (2.15) will be proved after the proof of (2.18) of Part 2.

We now prove Part 2. We first consider the case where $k \in \mathbb{Z}$ is fixed and $n \to \infty$. The proof will be based on Lemma 3.4. Fix $k \in \mathbb{Z}$, and let $n \geqslant k$.

Step 1. We begin with the proof of (2.16) with $a_{k,n} = \frac{\rho_{k,n}}{\langle u_{k,n}, v_{k,n}\rangle}$. Notice that

$$\langle y \cdot (M_{k,n})^T, x\rangle = \frac{\langle y(M_{k,n})^T, x\rangle}{\|y(M_{k,n})^T\|} = \frac{\langle xM_{k,n}, y\rangle}{\langle \mathbb{1}M_{k,n}, y\rangle}.$$

So from (3.17) of Lemma 3.4, we get that, $\mathbb{P}$-a.s., uniformly for $x, y \in \mathbb{R}_+^d \setminus \{0\}$,

$$\frac{\langle xM_{k,n}, y\rangle}{\langle \mathbb{1}M_{k,n}, y\rangle\langle u_k, x\rangle} \to 1, \quad \text{as } n \to \infty. \tag{3.22}$$

Substituting $x = v_{k,n}$ into (3.22), we get that, $\mathbb{P}$-a.s., uniformly for $y \in \mathbb{R}_+^d \setminus \{0\}$,

$$\frac{\rho_{k,n}\langle v_{k,n}, y\rangle}{\langle \mathbb{1}M_{k,n}, y\rangle\langle u_k, v_{k,n}\rangle} \to 1, \quad \text{as } n \to \infty. \tag{3.23}$$

Substituting $y = u_{k,n}$ into (3.22), and using the fact that $\langle \mathbb{1}M_{k,n}, u_{k,n}\rangle = \langle \mathbb{1}, u_{k,n}M_{k,n}^T\rangle = \|u_{k,n}M_{k,n}^T\| = \rho_{k,n}$, we get that, $\mathbb{P}$-a.s., uniformly for $x \in \mathbb{R}_+^d \setminus \{0\}$,

$$\frac{\rho_{k,n}\langle u_{k,n}, x\rangle}{\langle \mathbb{1}M_{k,n}, u_{k,n}\rangle\langle u_k, x\rangle} = \frac{\langle u_{k,n}, x\rangle}{\langle u_k, x\rangle} \to 1, \quad \text{as } n \to \infty. \tag{3.24}$$

Multiplying (3.23) with (3.24), we obtain that, $\mathbb{P}$-a.s., uniformly for $x, y \in \mathbb{R}^d_+ \setminus \{0\}$,

$$\frac{\rho_{k,n}\langle u_{k,n}, x\rangle\langle v_{k,n}, y\rangle}{\langle \mathbb{1} M_{k,n}, y\rangle\langle u_k, v_{k,n}\rangle\langle u_k, x\rangle} \to 1, \quad \text{as } n \to \infty. \tag{3.25}$$

Dividing (3.22) by (3.25), we see that, $\mathbb{P}$-a.s., uniformly for $x, y \in \mathbb{R}^d_+ \setminus \{0\}$,

$$\frac{\langle x M_{k,n}, y\rangle}{\rho_{k,n}\frac{\langle u_{k,n}, x\rangle\langle v_{k,n}, y\rangle}{\langle u_k, v_{k,n}\rangle}} \to 1, \quad \text{as } n \to \infty. \tag{3.26}$$

Taking $x = v_{k,n}$ in (3.24) implies

$$\langle u_k, v_{k,n}\rangle \sim \langle u_{k,n}, v_{k,n}\rangle \quad \text{as} \quad n \to \infty. \tag{3.27}$$

Therefore, (3.26) implies that, $\mathbb{P}$-a.s., uniformly for $x, y \in \mathbb{R}^d_+ \setminus \{0\}$,

$$\langle x M_{k,n}, y\rangle \sim \rho_{k,n}\frac{\langle u_{k,n}, x\rangle\langle v_{k,n}, y\rangle}{\langle u_{k,n}, v_{k,n}\rangle}, \quad \text{as } n \to \infty. \tag{3.28}$$

This ends the proof of (2.16) with $a_{k,n} = \frac{\rho_{k,n}}{\langle u_{k,n}, v_{k,n}\rangle}$.

Step 2. We next prove (2.17) with $a_{k,n} = \|M_{k,n}\|_{1,1}$. Taking $x = v_{k-1}$ in (3.22) gives that, $\mathbb{P}$-a.s., uniformly for $x, y \in \mathbb{R}^d_+ \setminus \{0\}$,

$$\frac{\langle v_n, y\rangle \mu_k \dots \mu_n}{\langle u_k, v_{k-1}\rangle\langle \mathbb{1} M_{k,n}, y\rangle} \to 1, \quad \text{as } n \to \infty, \tag{3.29}$$

where $\mu_k = \|v_{k-1} M_k\|$. Dividing (3.22) by (3.29), we get, uniformly for $x, y \in \mathbb{R}^d_+ \setminus \{0\}$, $\mathbb{P}$-a.s.,

$$\frac{\langle x M_{k,n}, y\rangle}{\langle u_k, x\rangle\langle v_n, y\rangle}\frac{\langle u_k, v_{k-1}\rangle}{\mu_k \dots \mu_n} \to 1, \quad \text{as } n \to \infty. \tag{3.30}$$

Let $x = y = \mathbb{1}$ into (3.30), we obtain, $\mathbb{P}$-a.s.,

$$\|M_{k,n}\|_{1,1}\frac{\langle u_k, v_{k-1}\rangle}{\mu_k \dots \mu_n} \to 1, \quad \text{as } n \to \infty. \tag{3.31}$$

Combining (3.30) and (3.31), we get (2.17) with $a_{k,n} = \|M_{k,n}\|_{1,1}$.

Step 3. We now prove that $\mathbb{P}$-a.s., for $k \leqslant n$ and as $n \to \infty$,

$$\|M_{k,n}\|_{1,1} \sim \frac{\rho_{k,n}}{\langle u_{k,n}, v_{k,n}\rangle}, \quad \|M_{k,n}\|_{1,1} \sim \frac{\prod_{j=k}^n \mu_j}{\langle u_k, v_{k-1}\rangle} \quad \text{and} \quad \frac{\prod_{j=k}^n \lambda_j}{\langle u_{n+1}, v_n\rangle} = \frac{\prod_{j=k}^n \mu_j}{\langle u_k, v_{k-1}\rangle}. \tag{3.32}$$

This gives the equivalences of the different norming factors in (2.18). Indeed, the first assertion in (3.32) follows from (3.28) with $x = y = \mathbb{1}$, and the second from (3.31). The third follows from the identity that

$$\langle u_{n+1} M_{k,n}^T, v_{k-1}\rangle = \langle u_{n+1}, v_{k-1} M_{k,n}\rangle \quad \mathbb{P}\text{-a.s.} \tag{3.33}$$

(recall that $M_{k,n}^T = (M_{k,n})^T$ by our notation), together with the identity (2.19).

Combining the results proved in Steps 1-3, we see that the proof of all the equivalences of Part 2 is finished in the case where $k \in \mathbb{Z}$ is fixed and $n \to \infty$.

For the case where $n \in \mathbb{Z}$ is fixed and $k \to -\infty$, the corresponding results can be obtained from those in the previous case applied to the transposed and reversed

sequence $\{M_{-k}^T : k \in \mathbb{Z}\}$; they can also be proved in a similar way as in the previous case, using Lemma 3.5 instead of Lemma 3.4.

Finally we remark that $\{(u_{n+1}, v_n) : n \in \mathbb{Z}\}$ is a stationary and ergodic sequence. To see this, assume without loss of generality that the underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is the canonical one, so that $\Omega = G^{\mathbb{Z}}$, $\mathbb{P}$ is the law of the sequence $\{M_k : k \in \mathbb{Z}\}$, and M_k is just the k-th coordinate function $\omega \mapsto \omega_k$. For each $n \in \mathbb{Z}$, let T^n be the n-th fold shift operator: by definition the k-th coordinate of $(T^n\omega)$ is $(T^n\omega)_k = \omega_{n+k}$ for all $k \in \mathbb{Z}$; in particular $T = T^1$ is the shift operator, which is assumed to be measure preserving and ergodic. Then we can write (u_1, v_0) in the form $(u_1, v_0) = X(\omega)$, where X is a measurable function with values in $\mathbb{R}_+^d \times \mathbb{R}_+^d$. Consequently, $(u_{n+1}, v_n) = X(T^n\omega)$, for each $n \in \mathbb{Z}$, so that they constitute a stationary and ergodic sequence.

This ends the proof of Part 2.

It remains the proof of assertion (2.15) of Part 1. We will prove it as a consequence of the equivalences of the sequences in (2.18) of Part 2. Indeed, under the condition $\mathbb{E}(\log^+ \|M_0\|) < \infty$, we have $\mathbb{P}$-a.s.

$$\frac{1}{n} \log \|M_{0,n-1}\|_{1,1} \to \gamma, \quad \frac{1}{n} \log \prod_{j=0}^{n-1} \lambda_j \to \mathbb{E} \log \lambda_0, \quad \frac{1}{n} \log \prod_{j=0}^{n-1} \mu_j \to \mathbb{E} \log \mu_0, \tag{3.34}$$

where the first convergence is the law of large numbers established by Furstenberg-Kesten [19] , and is also a consequence of the sub-additive ergodic theorem; the second and the third are consequences of the ergodic theorem, remarking that

$$\mathbb{E} \log^+ \lambda_0 \leqslant \mathbb{E} \log^+ \|M_0^T\| < \infty \quad \text{and} \quad \mathbb{E} \log^+ \mu_0 \leqslant \mathbb{E} \log^+ \|M_0\| < \infty. \tag{3.35}$$

(It can be easily seen that each of the conditions $\mathbb{E} \log^+ \|M_0^T\| < \infty$ and $\mathbb{E} \log^+ \|M_0\| < \infty$ is equivalent to $\mathbb{E} \log^+ \|M_0\|_{1,1} < \infty$.)

On the other hand, since $\langle u_n, v_{n-1} \rangle$ has the same law for all $n \in \mathbb{Z}$, from Slutsky's lemma we know that

$$\frac{1}{n} \log \langle u_n, v_{n-1} \rangle \to 0 \quad \text{in probability} \tag{3.36}$$

From the equivalences of the sequences in (2.18), together with (3.34), (3.36) and the uniqueness of the limit in probability, we get (2.15). This ends the proof of the theorem. □

Proof of Remark 2.3. For fixed k as $n \to \infty$, the fact that $\mathbb{P}$-a.s.

$$\langle u_{k,n}, x \rangle \sim \langle u_k, x \rangle \quad \text{uniformly for } x \in \mathbb{R}_+^d \setminus \{0\} \tag{3.37}$$

has been seen in (3.24). Consequently, from (2.16) and (2.17), we see that for fixed k as $n \to \infty$, $\mathbb{P}$-a.s.

$$\langle v_{k,n}, y \rangle \sim \langle v_n, y \rangle \quad \text{uniformly for } y \in \mathbb{R}_+^d \setminus \{0\} \tag{3.38}$$

So we have proved that (2.20) holds $\mathbb{P}$-a.s. for fixed k as $n \to \infty$. Applying this to the transposed and reversed sequence $\{M_{-k}^T : k \in \mathbb{Z}\}$ or using a similar argument, we see that (2.20) still holds $\mathbb{P}$-a.s. for fixed n as $k \to -\infty$. □

Proof of Corollary 2.5 . Assume **A1** and **A2**. We have already seen that under **A1**, $\gamma \in \mathbb{R} \cup \{-\infty\}$. From (2.17) with $k = 1 \leqslant n$ and $a_{1,n} = \|M_{1,n}\|_{1,1}$, we get $\mathbb{P}$-a.s.,

$$\sup_{x,y\in\mathcal{S}} \Big| \log\langle xM_{1,n}, y\rangle - \log \|M_{1,n}\|_{1,1} - \log\langle u_1, x\rangle - \log\langle v_n, y\rangle \Big| \to 0. \tag{3.39}$$

1) We first prove (2.23). Using (3.39) with $y = \mathbb{1}$, together with the law of large numbers on $\log \|M_{1,n}\|_{1,1}$, and the fact that $\mathbb{P}$-a.s., $\langle v_n, \mathbb{1}\rangle = 1$ and, as $n \to \infty$,

$$\frac{1}{n} \sup_{x\in\mathcal{S}} |\log\langle u_1, x\rangle| \to 0, \tag{3.40}$$

we obtain (2.23), the a.s. convergence of $\frac{1}{n} \log \|xM_{1,n}\|$ to γ, uniformly for $x \in \mathcal{S}$.

2) We next prove (2.22). Taking $x = v_{1,n}$ in (2.23) and remarking that $\rho_{1,n} = \|v_{1,n}M_{1,n}\|$ (since $v_{1,n}M_{1,n} = \rho_{1,n}v_{1,n}$), we obtain (2.22), the law of large numbers on the spectral radius $\rho_{1,n}$.

3) We now prove (2.25). Using again (3.39) and the fact that

$$\frac{1}{n} \sup_{y\in\mathcal{S}} |\log\langle v_n, y\rangle| \to 0 \quad \text{ in probability} \tag{3.41}$$

(which is a consequence of Slutsky's lemma and the fact that $\sup_{y\in\mathcal{S}} |\log\langle v_n, y\rangle|$ have the same law), we obtain (2.25), the weak law of large numbers for $\log\langle xM_{1,n}, y\rangle$.

4) We finally consider the uniform strong law of large numbers for the scalar product, under the additional assumption $\mathbb{E}\log^- \iota(M_0) < \infty$ or $M_0 > 0$ a.s. and $\mathbb{E}\log D_0 < \infty$.

Assume first $\mathbb{E}\log^- \iota(M_0) < \infty$. Together with the condition $\mathbb{E}\log^+ \|M_0\| < \infty$, this implies $\mathbb{E}\log^+ \iota(M_0) < \infty$ and $\mathbb{E}\log^- \|M_0\| < \infty$, so that $\mathbb{E}|\log \|M_0\|| + \mathbb{E}|\log \iota(M_0)| < \infty$. Under this condition, Hennion [30] proved that $\gamma \in \mathbb{R}$ and that (2.26) holds. The fact that $\gamma \in \mathbb{R}$ can be seen as follows. We already saw that $\mathbb{E}\log^+ \|M_0\| < \infty$ implies $\gamma < +\infty$. We now prove that $\mathbb{E}\log^- \iota(M_0) < \infty$ implies $\gamma > -\infty$. In fact, since $\mu_0 = \|v_{-1}M_0\| \geqslant \iota(M_0)$, we have $\mathbb{E}\log^- \mu_0 \leqslant \mathbb{E}\log^- \iota(M_0) < \infty$. Hence from (2.15), it follows that $\gamma = \mathbb{E}\log \mu_0 > -\infty$.

Assume now $\mathbb{E}\log D_0 < \infty$. Since $\langle v_n, y\rangle \sim \langle v_{1,n}, y\rangle$ uniformly for $y \in \mathcal{S}$ (see (2.20) or (3.38)), we see that (3.39) still holds when $\langle v_n, y\rangle$ is replaced by $\langle v_{1,n}, y\rangle$: $\mathbb{P}$-a.s.,

$$\sup_{x,y\in\mathcal{S}} \Big| \log\langle xM_{1,n}, y\rangle - \log \|M_{1,n}\|_{1,1} - \log\langle u_1, x\rangle - \log\langle v_{1,n}, y\rangle \Big| \to 0. \tag{3.42}$$

From this and the strong law of large numbers for $\log \|M_{1,n}\|_{1,1}$, together with (3.40), we see that if

$$\frac{1}{n} \sup_{y\in\mathcal{S}} |\log\langle v_{1,n}, y\rangle| \to 0 \quad \mathbb{P}\text{-a.s.} \tag{3.43}$$

then

$$\frac{1}{n} \log\langle xM_{1,n}, y\rangle \to \gamma \quad \text{uniformly for } x, y \in \mathcal{S}, \quad \mathbb{P}\text{-a.s.} \tag{3.44}$$

So it remains to prove (3.43). Notice that for any $y \in \mathcal{S}$,

$$1 \geqslant \langle v_{1,n}, y\rangle \geqslant \min_{1\leqslant j\leqslant d} v_{1,n}(j),$$

so (3.43) is a consequence of (3.47) of the following lemma. □

Lemma 3.6. *For $n \in \mathbb{Z}$, let D_n be defined as in (2.11). Then for any $k, n \in \mathbb{Z}$ with $k \leqslant n$ and all $i, j, l \in \{1, \cdots, d\}$,*

$$M_{k,n}(i,j) \leqslant D_n M_{k,n}(i,l) \tag{3.45}$$

and

$$\min_{1\leqslant i\leqslant d} v_{k,n}(i) \geqslant 1/(dD_n), \quad \min_{1\leqslant i\leqslant d} v_n(i) \geqslant 1/(dD_n), \tag{3.46}$$

If $\mathbb{E}\log D_0 < \infty$, then for each fixed $k \in \mathbb{Z}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \min_{1\leqslant i\leqslant d} v_{k,n}(i) = \lim_{n\to\infty} \frac{1}{n} \log \min_{1\leqslant i\leqslant d} v_n(i) = 0 \quad \mathbb{P}\text{-a.s.} \tag{3.47}$$

Similarly, if we define D_n as in (2.12), then same conclusions hold for $u_{k,n}$ and u_n, instead of $v_{k,n}$ and v_n.

Under the Furstenberg-Kesten condition (so that (2.11) and (2.12) hold with $D_n = D$ a constant), the bound for u_n was obtained in [22, Lemma 7.1].

Proof. For the proof of (3.45), we just need to prove the following implication: for two matrices g_1, g_2, if for some $c_2 \in [1, \infty)$ and all $i, j, l \in \{1, \cdots, d\}$,

$$g_2(i,j) \leqslant c_2 g_2(i,l), \tag{3.48}$$

then for all $i, j, l \in \{1, \cdots, d\}$,

$$(g_1 g_2)(i,j) \leqslant c_2\ (g_1 g_2)(i,l). \tag{3.49}$$

Indeed, we have, for all $i, j, l \in \{1, \cdots, d\}$,

$$\begin{aligned}(g_1 g_2)(i,j) &= \sum_{m=1}^{d} g_1(i,m) g_2(m,j) \\ &\leqslant \sum_{l=1}^{d} g_1(i,l)\ c_2 g_2(m,l) = c_2\ (g_1 g_2)(i,l).\end{aligned} \tag{3.50}$$

From (3.45) and the identity $v_{k,n} M_{k,n} = \rho_{k,n} v_{k,n}$, we get, for all $i, j \in \{1, \cdots, d\}$

$$\begin{aligned}\rho_{k,n} v_{k,n}(i) &= \sum_{m=1}^{d} v_{k,n}(m) M_{k,n}(m,i) \\ &\leqslant \sum_{m=1}^{d} v_{k,n}(m) D_n M_{k,n}(m,j) \\ &= D_n \rho_{k,n} v_{k,n}(j).\end{aligned}$$

It follows that for all $i, j \in \{1, \cdots, d\}$,

$$v_{k,n}(i) \leqslant D_n v_{k,n}(j). \tag{3.51}$$

Summing this equation for i from 1 to d, we get for all $j \in \{1, \cdots, d\}$,

$$1 \leqslant d D_n v_{k,n}(j). \tag{3.52}$$

This gives (3.46). From (3.46), we get

$$0 \geqslant \log \min_{1\leqslant i\leqslant d} v_{k,n}(i) \geqslant -\log(dD_n) \tag{3.53}$$

Since $(D_n) = (D(M_n))$ is a stationary and ergodic sequence, by the ergodic theorem, we have

$$\frac{1}{n}\sum_{j=1}^{n} \log D_j \to \mathbb{E}\log D_0 \in [0,\infty) \quad \mathbb{P}\text{-a.s.}$$

Consequently,

$$\frac{\log D_n}{n} \to 0 \quad \mathbb{P}\text{-a.s.}$$

So from (3.53), we get $\lim_{n\to\infty} \frac{1}{n}\log \min_{1\leqslant i\leqslant d} v_{k,n}(i) = 0$ a.s. The same argument applies for v_n instead of $v_{k,n}$, yielding the same conclusion for v_n. So (3.47) is proved.

The conclusions for $u_{k,n}$ and u_n can be proved in the same manner. So the proof of the lemma is finished. □

4. The fundamental martingale and the non-degeneracy of the limit

4.1 The triangular array of martingales and the fundamental martingale

For a good comprehension of the Kesten-Stigum type theorem, we first introduce a triangular array of martingales, of which the terminal values will be the main object of our study. As in the single type case, the construction follows from the basic multiplicative property (see Part (1) of Lemma 4.1). However, the convergence of the terminal values of the triangular array in the multitype case is not automatic contrary to the single type case; it is our first goal to show their convergence under appropriate assumptions.

For any vector $y \in \mathbb{R}^d_+ \setminus \{0\}$ and any $n \geqslant 0$, consider the triangular array

$$W_{n,k}(y) = \frac{\langle Z_k M_{k,n-1}, y\rangle}{\langle Z_0 M_{0,n-1}, y\rangle}, \quad 0 \leqslant k \leqslant n,\ n \geqslant 0, \tag{4.1}$$

with the convention that $M_{k,n-1}$ is the identity matrix when $k > n-1$ (so that $W_{n,0}(y) = 1$ and $W_{n,n}(y) = \frac{\langle Z_n, y\rangle}{\langle Z_0 M_{0,n-1}, y\rangle}$, for all $n \geqslant 0$.) The following lemma states that each line of this triangular array is a martingale. Recall that the filtration $(\mathscr{F}_n)$ was defined in (2.29).

Lemma 4.1 (The multiplicative property and the triangular array of martingales)**.**

(1) For any $n \geqslant 0$, we have

$$\mathbb{E}_\xi[Z_{n+1}|\mathscr{F}_n] = Z_n M_n. \tag{4.2}$$

(2) For any $n \geqslant 0$ and $y \in \mathbb{R}^d_+ \setminus \{0\}$, the sequence $(W_{n,k}(y)), 0 \leqslant k \leqslant n$, is a non-negative martingale under the probability measures $\mathbb{P}_\xi$ and $\mathbb{P}$ with respect to the filtration $\{\mathscr{F}_k : 0 \leqslant k \leqslant n\}$, that is, for $0 \leqslant k \leqslant n$,

$$\mathbb{E}_\xi[W_{n,k}(y)|\mathscr{F}_{n-1}] = W_{n,k-1}(y).$$

Proof. Let $M_n(r,\cdot)$ be the r-th row-vector of the matrix M_n. Since $\mathbb{E}_\xi(N^r_{l,n}|\mathscr{F}_n) = M_n(r,\cdot)$, from (2.27) we get for $n \geqslant 0$.

$$\mathbb{E}_\xi[Z_{n+1}|\mathscr{F}_n] = \sum_{r=1}^{d} Z_n(r) M_n(r,\cdot) = Z_n M_n.$$

It is clear that for any $0 \leqslant k \leqslant n$, Z_k and $W_{n,k}$ are $\mathscr{F}_k$-measurable. From the part (1) we see that $\mathbb{E}_\xi(\langle Z_k, z\rangle|\mathscr{F}_{k-1}) = \langle Z_{k-1}M_{k-1}, z\rangle$, for any $z \in \mathbb{R}^d$, so that for $0 \leqslant k < n$,

$$\begin{aligned}\mathbb{E}_\xi[W_{n,k}(y)|\mathscr{F}_{k-1}] &= \frac{\mathbb{E}_\xi[\langle Z_k M_{k,n-1}, y\rangle|\mathscr{F}_{k-1}]}{\langle Z_0 M_{0,n-1}, y\rangle} \\ &= \frac{\langle Z_{k-1}M_{k-1,n-1}, y\rangle}{\langle Z_0 M_{0,n-1}, y\rangle} = W_{n,k-1}.\end{aligned}$$

Therefore $\{(W_{n,k}(y), \mathscr{F}_k) : 0 \leqslant k \leqslant n\}$ is a martingale sequence under $\mathbb{P}_\xi$. □

For each $x \in \mathbb{R}^d_+ \setminus \{0\}$, when $Z_0 = x$, we write respectively $Z^x_n, W^x_n(y), W^x_n(u_n))$ for $Z_n, W_n(y), W_n(u_n)$, to indicate the dependence on the initial state x.

We now introduce the fundamental martingale for the scalar product, with the help of Theorem 2.1. The result states that with the special choice $y = u_n$, the terminal sequence $(W_n(u_n))_{n\geqslant 0}$ of the above triangular arrays is a martingale. In the particular case where $Z_0 = e_i$, it has been established in [22].

Lemma 4.2 (Fundamental martingale)**.** *Assume condition **A2**. Then the sequence*

$$W^x_n(u_n) = \frac{\langle Z^x_n, u_n\rangle}{\langle xM_{0,n-1}, u_n\rangle}, \quad n \geqslant 0 \tag{4.3}$$

(by convention $W_0(u_0) = 1$) is a non-negative martingale with respect to the filtration $(\mathscr{F}_n)_{n\geqslant 0}$, both under $\mathbb{P}_\xi$ and under $\mathbb{P}$. In particular, the limit

$$W^x := \lim_{n\to\infty} W^x_n(u_n) \tag{4.4}$$

exists with $W^x \in [0, \infty)$, $\mathbb{P}$-almost surely.

Proof. Using equation (4.2) of Lemma 4.1, we have for all $n \geqslant 0$,

$$\begin{aligned}\mathbb{E}_\xi[W^x_{n+1}|\mathscr{F}_n] &= \frac{\mathbb{E}_\xi[\langle Z^x_{n+1}, u_{n+1}\rangle|\mathscr{F}_n]}{\langle xM_{0,n}, u_{n+1}\rangle} = \frac{\langle Z^x_n M_n, u_{n+1}\rangle}{\langle xM_{0,n}, u_{n+1}\rangle} \\ &= \frac{\langle Z^x_n, u_{n+1}M^T_n\rangle}{\langle xM_{0,n-1}, u_{n+1}M^T_n\rangle} = \frac{\lambda_n\langle Z^x_n, u_n\rangle}{\lambda_n\langle xM_{0,n-1}, u_n\rangle} = W^x_n.\end{aligned}$$

This proves that the sequence $(W^x_n(u_n), \mathscr{F}_n)_{n\geqslant 0}$ is a martingale under $\mathbb{P}_\xi$; it follows that it is also a martingale under $\mathbb{P}$. The a.s. convergence of $(W^x_n(u_n))_{n\geqslant 0}$ follows from the martingale convergence theorem. □

4.2 The non-degeneracy of the martingale limit. For the non-degeneracy of the martingale limit W^x, we will use the following conditions:

H2. *For all $1 \leqslant r \leqslant d$,*

$$\mathbb{E}\left(\frac{\langle N^r_{1,0}, u_1\rangle}{\lambda_0\langle u_0, e_r\rangle}\log^+\langle N^r_{1,0}, u_1\rangle\right) < \infty.$$

H3. *There exists a constant $C > 1$ such that, for all $r \in \{1, \dots, d\}$, $\mathbb{P}$-a.s.*

$$\sum_{n=0}^{\infty}\mathbb{E}_\xi\left(\frac{\langle N^r_{1,n}, u_{n+1}\rangle}{\lambda_n\langle u_n, e_r\rangle}\mathbb{1}_{\left\{\langle N^r_{1,n}, u_{n+1}\rangle \geqslant C^n\right\}}\right) < \infty.$$

The following remark gives some relations among the conditions **H1**, **H2** and **H3**.

Remark 4.3 (Remark 2.8, [22])**.** Under conditions **A1** and **A2**, we have the implications **H1** $\Rightarrow$ **H2** $\Rightarrow$ **H3**. If the environment is i.i.d. and conditions **A1** and **A3** hold, then **H3** $\Leftrightarrow$ **H2** $\Leftrightarrow$ **H1**.

The theorem below gives conditions for the non-degeneracy of W^x.

Theorem 4.4. *Assume conditions **A1**, **A2** and $\gamma > 0$. Then, for each initial state $Z_0 = x \in \mathbb{N}_+^d \setminus \{0\}$, we have:*

*(1) If **H1** holds, then the random variable W^x defined in (4.4) satisfies $\mathbb{E}_\xi W^x = 1$ $\mathbb{P}$-a.s. so that W^x is non degenerate in the sense that $\mathbb{P}_\xi(W^x > 0) > 0$ a.s.*

*(2) Assume also that the environment (ξ_n) is i.i.d. and that the Furstenberg-Kesten condition **A3** holds. Then condition **H1** is necessary and sufficient for W^x to be non-degenerate.*

For $x = e_i$ and $y = e_j$, the results have been established in [22].

4.0.1. *Preliminaries.* We shall use the notion of mean-harmonic function introduced in [7]. To stick with the notation in [7], we introduce the type space $\mathbb{S} = \mathbb{N} \times \{\partial, 1, \cdots, d\}$, so that a particle of type r of generation n is identified as type (n, r). The symbol ∂ stands for a 'ghost' particle. Following [7], it is convenient to consider that every particle has infinitely many children including those of type ∂; individuals of type ∂ have children only of type ∂, and these individuals are interpreted as being absent. Consider the positive function

$$H(n, \partial) = 0, \quad H(n, r) = \frac{\langle e_r, u_n \rangle}{\lambda_{0,n-1}}, \quad n \geqslant 0,\ r \in \{1, \ldots, d\}. \tag{4.5}$$

We will prove that H is a mean-harmonic function in the sense of [7], which in our setting means that, for each particle of type $(n, r) \in \mathbb{S}^* := \mathbb{N} \times \{1, \cdots, d\}$ in the sense of [7], that is, for each type r particle of generation n, say the l-th one, denoting by $N_{l,n}^r(j)$ the number of its direct children of type j, we have

$$\mathbb{E}_\xi\Big[\sum_{j=1}^d H(n+1, j) N_{l,n}^r(j)\Big] = H(n, r). \tag{4.6}$$

Indeed, since $u_{n+1} M_n^T = \lambda_n u_n$, we have:

$$\begin{aligned}
\mathbb{E}_\xi\Big[\sum_{j=1}^d H(n+1, j) N_{l,n}^r(j)\Big] &= \sum_{j=1}^d \frac{\langle e_j, u_{n+1} \rangle}{\lambda_{0,n}} \mathbb{E}_\xi(N_{l,n}^r(j)) \\
&= \sum_{j=1}^d \frac{u_{n+1}(j) M_n(r, j)}{\lambda_{0,n}} \\
&= \frac{(u_{n+1} M_n^T)(r)}{\lambda_{0,n}} \\
&= \frac{\langle e_r, u_n \rangle}{\lambda_{0,n-1}}.
\end{aligned}$$

With the harmonic function H, we recover the martingale (W_n) up to a scaling term with the help of the martingale:

$$W_n^H = \sum_{j=1}^{d} Z_n(j)H(n,j) = \frac{\langle Z_n, u_n\rangle}{\lambda_{0,n-1}}, \quad n \geqslant 0.$$

We will use a result from [7] to study the non-degeneracy of the limit

$$W^H = \lim_{n\to\infty} W_n^H.$$

Before stating this result we introduce the necessary notation. For $s \in \mathbb{S}^* := \mathbb{N} \times \{1,\dots,d\}$, let

$$\begin{aligned} X(s) &:= \frac{1}{H(n,r)} \sum_{j=1}^{d} H(n+1,j) N_{1,n}^r(j) \\ &= \frac{1}{\frac{\langle u_n, e_r\rangle}{\lambda_{0,n-1}}} \sum_{j=1}^{d} \frac{\langle u_{n+1}, e_j\rangle N_{1,n}^r(j)}{\lambda_{0,n}} \\ &= \frac{\langle N_{1,n}^r, u_{n+1}\rangle}{\lambda_n \langle u_n, e_r\rangle} \end{aligned}$$

(which corresponds to the variable $X(s,f)$ defined in (1.1) of [7]), and

$$\begin{aligned} A(s) &:= \mathbb{E}_\xi \left[\frac{\langle N_{1,n}^r, u_{n+1}\rangle}{\lambda_n \langle u_n, e_r\rangle} \min\left\{ \frac{\langle u_n, e_r\rangle}{\lambda_{0,n-1}} \frac{\langle N_{1,n}^r, u_{n+1}\rangle}{\lambda_n \langle u_n, e_r\rangle}, 1 \right\} \right] \\ &= \mathbb{E}_\xi \left[\frac{\langle N_{1,n}^r, u_{n+1}\rangle}{\lambda_n \langle u_n, e_r\rangle} \min\left\{ \frac{\langle N_{1,n}^r, u_{n+1}\rangle}{\lambda_{0,n}}, 1 \right\} \right] \end{aligned} \tag{4.7}$$

(which corresponds to the expression $\mathbf{E}_s\Big[X(s)\min\Big\{H(s)X(s),1\Big\}\Big]$ used in (2.1) of [7] with $s = \zeta_n$). For $s = (n,r) \in \mathbb{S}^*$ and $y > 0$, write

$$B_y(s) := \mathbb{E}_\xi \left[\frac{\langle N_{1,n}^r, u_{n+1}\rangle}{\lambda_n \langle u_n, e_r\rangle} \mathbb{1}_{\left\{ \frac{\langle N_{1,n}^r, u_{n+1}\rangle}{\lambda_{0,n}} > y \right\}} \right].$$

Let $(\zeta_n)_{n\geqslant 0}$ be a homogenous Markov chain with the state space $\mathbb{S}^* = \mathbb{N}\times\{1,\dots,d\}$ and the following transition kernel: for any $(n,r) \in \mathbb{S}^*$ and $j \in \{1,\dots,d\}$,

$$\mathbf{Q}_\xi\Big(\zeta_{n+1} = (n+1,j)\Big|\zeta_n = (n,r)\Big) = \frac{1}{H(n,r)} \mathbb{E}_\xi\left(H(n+1,j)N_{1,n}^r(j)\right) \tag{4.8}$$

and, for $m \neq n+1$,

$$\mathbf{Q}_\xi\Big(\zeta_{n+1} = (m,j)\Big|\zeta_n = (n,r)\Big) = 0. \tag{4.9}$$

The two formulas above define a Markov kernel since H is a mean-harmonic function (cf. (4.6)). Denote by $\mathbb{Q}_\xi^*$ the probability measure on the space of trajectories $\Omega^* = (\mathbb{S}^*)^{\mathbb{N}^*}$ of this Markov chain, for fixed environment ξ. Notice that ζ_n is of the form $\zeta_n = (n, \tilde{\zeta}_n)$ defined on the probability space $(\Omega^*, \mathbb{Q}_\xi^*)$, where $(\tilde{\zeta}_n)_n$ is also a Markov chain with state space $\{1,\dots,d\}$. The following proposition is a translation of [7, Theorem 2.1] in our context.

Lemma 4.5. *For almost every environment ξ, we have:*

(i) $\mathbb{E}_\xi[W^H] = \mathbb{E}_\xi[W_0^H] > 0$ *if*

$$\sum_{n=0}^{\infty} A(\zeta_n) < \infty \quad \mathbb{Q}_\xi^*\text{-a.s.}. \tag{4.10}$$

(ii) $\mathbb{E}_\xi[W^H] = 0$ *if for all* $y > 0$,

$$\sum_{n=0}^{\infty} B_y(\zeta_n) = \infty \quad \mathbb{Q}_\xi^*\text{-a.s.}. \tag{4.11}$$

We also need the following assertion.

Lemma 4.6. *Assume condition **A3**. For almost every environment ξ and for any $n \geqslant 1$ and $j \in \{1, \dots, d\}$, $\mathbb{Q}_\xi^*$-a.s.*

$$\mathbb{Q}_\xi^*\Big(\tilde{\zeta}_n = j \Big| \tilde{\zeta}_{n-1}\Big) \geqslant \frac{1}{dD^2}. \tag{4.12}$$

Proof. By the definition of $H(n,j)$, the Furstenberg-Kesten condition **A3**, and the fact that $E_\xi N_{1,n-1}^r(j) = M_{n-1}(r,j)$ and $u_n(j) \geqslant \frac{1}{dD}$ (see [22, Lemma 7.1] or Lemma 3.6), we have

$$\begin{aligned}
\mathbb{Q}^*\Big(\tilde{\zeta}_n = j \Big| \tilde{\zeta}_{n-1} = r\Big) &= \frac{1}{H(n-1,r)} \mathbb{E}_\xi\Big(H(n,j) N_{1,n-1}^r(j)\Big) \\
&= \frac{u_n(j) M_{n-1}(r,j)}{\lambda_{n-1} u_{n-1}(r)} \\
&= \frac{u_n(j) M_{n-1}(r,j)}{\sum_{k=1}^d u_n(k) M_{n-1}(r,k)} \\
&\geqslant \frac{1}{dD}\Big(\sum_{k=1}^d \frac{u_n(k) M_{n-1}(r,k)}{M_{n-1}(r,j)}\Big)^{-1} \\
&\geqslant \frac{1}{dD^2}\Big(\sum_{k=1}^d u_n(k)\Big)^{-1} = \frac{1}{dD^2}.
\end{aligned} \tag{4.13}$$

□

4.0.2. *Non degeneracy of W^x.*

Proof of Theorem 4.4. (1) *Sufficient condition.* We assume that $\gamma > 0$ and that conditions **A1**, **A2**, **H1** hold true. We will prove (4.10), which, by Lemma 4.5, will imply that $\mathbb{E}_\xi[W^H] = \mathbb{E}_\xi[W_0^H] > 0$, so that $\mathbb{E}_\xi[W^x] = 1$. Since

$$\sum_{n=0}^{\infty} A(\zeta_n) \leqslant \sum_{n=0}^{\infty} \sum_{r=1}^{d} A(n,r), \tag{4.14}$$

to prove (4.10) it is enough to prove that for almost every ξ and for any $r \in \{1, \cdots, d\}$,

$$\sum_{n=0}^{\infty} A(n,r) < \infty. \tag{4.15}$$

Indeed, since $\frac{1}{n}\log\lambda_{0,n}\to\gamma>0$ a.s., for any $C\in(1,e^\gamma)$, and for almost every ξ, there is some $n_0=n_0(\xi)\in\mathbb{N}$, such that $\lambda_{0,n}\geqslant C^n$ for any $n\geqslant n_0$. Consequently,

$$\begin{aligned}\sum_{n=n_0}^{\infty}A(n,r)&=\sum_{n=n_0}^{\infty}\mathbb{E}_\xi\Big[\frac{\langle N_{1,n}^r,u_{n+1}\rangle}{\lambda_n\langle u_n,e_r\rangle}\min\Big\{\frac{\langle N_{1,n}^r,u_{n+1}\rangle}{\lambda_{0,n}},1\Big\}\Big]\\&\leqslant\sum_{n=n_0}^{\infty}\mathbb{E}_\xi\left[\frac{\langle N_{1,n}^r,u_{n+1}\rangle}{\lambda_n\langle u_n,e_r\rangle}\min\left\{\frac{\langle N_{1,n}^r,u_{n+1}\rangle}{C^n},1\right\}\right].\end{aligned}$$

As the process $\{(\xi_n,\xi_{n+1},\ldots):n\geqslant 0\}$ is stationary, it follows that

$$\mathbb{E}\sum_{n=n_0}^{\infty}A(n,r)\leqslant\sum_{n=0}^{\infty}\mathbb{E}\left[\frac{\langle N_{1,0}^r,u_1\rangle}{\lambda_0\langle u_0,e_r\rangle}\min\left\{\frac{\langle N_{1,0}^r,u_1\rangle}{C^n},1\right\}\right].\tag{4.16}$$

Let $1<c<C$. Then

$$\begin{aligned}\mathbb{E}\left[\frac{\langle N_{1,0}^r,u_1\rangle}{\lambda_0\langle u_0,e_r\rangle}\min\left\{\frac{\langle N_{1,0}^r,u_1\rangle}{C^n},1\right\}\right]&\leqslant\mathbb{E}\left[\frac{\langle N_{1,0}^r,u_1\rangle}{\lambda_0\langle u_0,e_r\rangle}\min\left\{\frac{\langle N_{1,0}^r,u_1\rangle}{C^n},1\right\}\right]\mathbb{1}_{\left\{\langle N_{1,0}^r,u_1\rangle\leqslant c^n\right\}}\\&\quad+\mathbb{E}\left[\frac{\langle N_{1,0}^r,u_1\rangle}{\lambda_0\langle u_0,e_r\rangle}\min\left\{\frac{\langle N_{1,0}^r,u_1\rangle}{C^n},1\right\}\right]\mathbb{1}_{\left\{\langle N_{1,0}^r,u_1\rangle>c^n\right\}}\\&\leqslant\mathbb{E}\left[\frac{\langle N_{1,0}^r,u_1\rangle}{\lambda_0\langle u_0,e_r\rangle}\frac{c^n}{C^n}\right]+\mathbb{E}\left[\frac{\langle N_{1,0}^r,u_1\rangle}{\lambda_0\langle u_0,e_r\rangle}\right]\mathbb{1}_{\left\{\langle N_{1,0}^r,u_1\rangle>c^n\right\}}.\end{aligned}\tag{4.17}$$

Therefore by (4.16),

$$\mathbb{E}\sum_{n=n_0}^{\infty}A(n,r)\leqslant\mathbb{E}\left[\frac{\langle N_{1,0}^r,u_1\rangle}{\lambda_0\langle u_0,e_r\rangle}\right]\sum_{n=0}^{\infty}\left(\frac{c}{C}\right)^n+\sum_{n=0}^{\infty}\mathbb{E}\left[\frac{\langle N_{1,0}^r,u_1\rangle}{\lambda_0\langle u_0,e_r\rangle}\right]\mathbb{1}_{\left\{\langle N_{1,0}^r,u_1\rangle>c^n\right\}}.\tag{4.18}$$

In the right-hand side, the first term is finite because $c/C<1$ and

$$\mathbb{E}_\xi\left[\frac{\langle N_{1,0}^r,u_1\rangle}{\lambda_0\langle u_0,e_r\rangle}\right]=\frac{M_0(r,.)u_1^T}{\lambda_0\langle u_0,e_r\rangle}=\frac{\lambda_0u_0(r)}{\lambda_0\langle u_0,e_r\rangle}=1;$$

the second term is also finite because

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{E}\left[\frac{\langle N_{1,0}^r,u_1\rangle}{\lambda_0\langle u_0,e_r\rangle}\mathbb{1}_{\left\{\frac{\langle N_{1,0}^r,u_1\rangle}{c^n}\geqslant 1\right\}}\right]&=\mathbb{E}\left[\frac{\langle N_{1,0}^r,u_1\rangle}{\lambda_0\langle U_0,e_r\rangle}\sum_{n=0}^{\infty}\mathbb{1}_{\left\{\log^+\langle N_{1,0}^r,u_1\rangle>n\log c\right\}}\right]\\&\leqslant\mathbb{E}\Big[\frac{\langle N_{1,0}^r,u_1\rangle}{\lambda_0\langle u_0,e_r\rangle}\Big(\frac{\log^+\langle N_{1,0}^r,u_1\rangle}{\log c}+1\Big)\Big]<\infty.\end{aligned}\tag{4.19}$$

Therefore from (4.19) and the implication **H1** $\Rightarrow$ **H2**, we get $\sum_{n=n_0}^{\infty}A(n,r)<\infty$ a.s., so that

$$\sum_{n=0}^{\infty}A(n,r)<\infty\qquad\text{a.s.}$$

By Lemma 4.5, this implies that $\mathbb{E}_\xi[W^H]=\mathbb{E}_\xi[W_0^H]>0$ and $\mathbb{E}_\xi W^x=1$.

(2) *Necessary condition.* From Part 1, we just need to prove that **H1** is necessary for the non-degeneracy of W^H. To this end, it is enough to prove that if **H1** fails, then

W^H is degenerate. Assume that **H1** fails. By virtue of the equivalence **H3** $\Leftrightarrow$ **H1** of Remark 4.3, we see that **H3** also fails. Therefore, for any fixed $C > 1$, there exists $1 \leqslant r_0 \leqslant d$ such that, with positive probability,

$$\sum_{n=0}^{\infty} \mathbb{E}_\xi \left(\frac{\langle N_{1,n}^{r_0}, u_{n+1} \rangle}{\lambda_n \langle u_n, e_{r_0} \rangle} \mathbb{1}_{\left\{ \langle N_{1,n}^{r_0}, u_{n+1} \rangle \geqslant C^n \right\}} \right) = \infty. \tag{4.20}$$

We now introduce a new probability $\tilde{\mathbb{P}}_\xi$ on $(\Omega, \mathscr{F})$, such that, for every $n \geqslant 0$ and $\mathscr{F}_{n+1}$-mesurable set B,

$$\tilde{\mathbb{P}}_\xi(B) = \mathbb{E}_\xi \left(\frac{\langle N_{1,n}^{r}, u_{n+1} \rangle}{\lambda_n \langle u_n, e_{r} \rangle} \mathbb{1}_B \right). \tag{4.21}$$

The existence of the probability mesure $\tilde{\mathbb{P}}_\xi$ is ensured by Kolmogorov's extension theorem and the fact that $\{(\frac{\langle N_{1,n}^{r}, u_{n+1} \rangle}{\lambda_n \langle u_n, e_r \rangle}, \mathscr{F}_{n+1}) : n \geqslant 0\}$ is a martingale under $\mathbb{P}_\xi$. Indeed, if we write $\tilde{\mathbb{P}}_{n+1}(B)$ for the right-hand side, then the martingale property implies that $\{\tilde{\mathbb{P}}_{n+1} : n \geqslant 0\}$ is a consistent system of probabilities: for all $n \geqslant 0$ and any $B \in \mathscr{F}_n$, we have

$$\tilde{\mathbb{P}}_{n+1}(B) = \mathbb{E}\left[\mathbb{E}_\xi \Big(\frac{\langle N_{1,n}^{r}, u_{n+1} \rangle}{\lambda_n \langle u_n, e_r \rangle} \mathbb{1}_B \Big| \mathscr{F}_n \Big) \right] = \mathbb{E}_\xi \left(\frac{\langle N_{1,n-1}^{r}, u_{n} \rangle}{\lambda_{n-1} \langle u_{n-1}, e_r \rangle} \mathbb{1}_B \right) = \tilde{\mathbb{P}}_n(B).$$

With the new measure $\tilde{\mathbb{P}}_\xi$,

$$\sum_{n \in \mathbb{N}} \tilde{\mathbb{P}}_\xi \Big(\log^+ \langle N_{1,n}^{r_0}, u_{n+1} \rangle \geqslant n \log C \Big) = \infty. \tag{4.22}$$

Recall that $(\tilde{\zeta}_n)_n$ is a Markov chain with state space $\{1, \cdots, d\}$, and transition probabilities given by (4.8). Note that

$$N_{1,n}^{\tilde{\zeta}_n} = \sum_{r=1}^{d} N_{1,n}^{r} \mathbb{1}\Big(\tilde{\zeta}_n = r \Big),$$

which is a random variable defined on the product space $(\Omega \times \Omega^*, \tilde{\mathbb{P}}_\xi^*)$, where $\tilde{\mathbb{P}}_\xi^* = \tilde{\mathbb{P}}_\xi \otimes \mathbb{Q}^*$ with $\mathbb{Q}^*$ defined in Section 4.0.1. We will prove that

$$\sum_{n \in \mathbb{N}} \tilde{\mathbb{P}}_\xi^* \Big(\log^+ \langle N_{1,n}^{\tilde{\zeta}_n}, u_{n+1} \rangle \geqslant n \log C \;\Big|\; \tilde{\zeta}_n \Big) = \infty. \tag{4.23}$$

This will imply that W^H is degenerate by a Borel-Cantelli argument.

Let $(\mathscr{F}_n^*)_{n \geqslant 0}$ be the filtration defined by $\mathscr{F}_0^* = \sigma(\xi)$, and for $n \geqslant 1$,

$$\mathscr{F}_n^* = \sigma\Big(\xi, N_{l,k}^{r}, \tilde{\zeta}_k, 0 \leqslant k < n, 1 \leqslant r \leqslant d, l \geqslant 1 \Big) = \mathscr{F}_n \vee \sigma\Big(\xi, \tilde{\zeta}_k, 0 \leqslant k < n \Big).$$

We shall prove that $\{N_{1,n}^{r} : 1 \leqslant r \leqslant d\}$ and $\mathscr{F}_n^*$ are independent under $\tilde{\mathbb{P}}_\xi^* = \tilde{\mathbb{P}}_\xi \otimes \mathbb{Q}^*$. Indeed, denoting by $\tilde{\mathbb{E}}_\xi^*$ the expectation with respect to $\tilde{\mathbb{P}}_\xi^*$, we have for any positive

Borel function f and $\mathscr{F}_n^*$ measurable set B of the form $B = B_1 \times B_2$, where $B_1 \in \mathscr{F}_n$ and $B_2 \in \sigma\big(\xi, \tilde{\zeta}_k, 0 \leqslant k < n\big)$,

$$\begin{aligned}\tilde{\mathbb{E}}_\xi^*\left[f(N_{1,n}^r : 1 \leqslant r \leqslant d)1_B\right] &= \int 1_{B_2}\left(\int\left[\frac{\langle N_{1,n}^r, u_{n+1}\rangle}{\lambda_n\langle u_n, e_r\rangle} f(N_{1,n}^r : 1 \leqslant r \leqslant d)1_{B_1}\right] d\mathbb{P}_\xi\right) d\mathbb{Q}^* \\ &= \mathbb{E}_\xi\left[\frac{\langle N_{1,n}^r, u_{n+1}\rangle}{\lambda_n\langle u_n, e_r\rangle} f(N_{1,n}^r : 1 \leqslant r \leqslant d)\right] \mathbb{P}_\xi(B_1)\mathbb{Q}^*(B_2) \\ &= \tilde{\mathbb{E}}_\xi^*\left[f(N_{1,n}^r : 1 \leqslant r \leqslant d)\right] \tilde{\mathbb{E}}_\xi^* 1_B,\end{aligned}$$

where the second step holds because $N_{1,n}^r$ is independent of 1_{B_1} under $\mathbb{P}_\xi$, and the third one holds since

$$\begin{aligned}\tilde{\mathbb{E}}_\xi^* 1_B &= \int 1_{B_2}\left(\int \frac{\langle N_{1,n}^r, u_{n+1}\rangle}{\lambda_n\langle u_n, e_r\rangle} 1_{B_1} d\mathbb{P}_\xi\right) d\mathbb{Q}^* \\ &= \mathbb{E}_\xi\left(\frac{\langle N_{1,n}^r, u_{n+1}\rangle}{\lambda_n\langle u_n, e_r\rangle}\right)\mathbb{P}_\xi(B_1)\mathbb{Q}^*(B_2) = \mathbb{P}_\xi(B_1)\mathbb{Q}^*(B_2).\end{aligned}$$

Using the independence established before, we get

$$\begin{aligned}&\sum_{n=1}^\infty \tilde{\mathbb{P}}_\xi^*\left(\log^+\langle N_{1,n}^{\tilde{\zeta}_n}, u_{n+1}\rangle \geqslant n\log C \middle| \mathscr{F}_n^*\right) \\ &\quad= \sum_{n=1}^\infty \tilde{\mathbb{E}}_\xi^*\left(\sum_{r=1}^d \mathbb{1}_{\{\log^+\langle N_{1,n}^r, u_{n+1}\rangle \geqslant n\log C\}} \mathbb{1}_{\{\tilde{\zeta}_n = r\}} \middle| \mathscr{F}_n^*\right) \\ &\quad= \sum_{n=1}^\infty \sum_{r=1}^d \tilde{\mathbb{P}}_\xi\left(\log^+\langle N_{1,n}^r, u_{n+1}\rangle \geqslant n\log C \middle| \mathscr{F}_n^*\right) \tilde{\mathbb{P}}_\xi^*\left(\tilde{\zeta}_n = r \middle| \mathscr{F}_n^*\right) \\ &\quad= \sum_{n=1}^\infty \sum_{r=1}^d \tilde{\mathbb{P}}_\xi\left(\log^+\langle N_{1,n}^r, u_{n+1}\rangle \geqslant n\log C\right) \tilde{\mathbb{P}}_\xi^*\left(\tilde{\zeta}_n = r \middle| \mathscr{F}_n^*\right). \qquad (4.24)\end{aligned}$$

Note that, for any $n \geqslant 1$, under $\tilde{\mathbb{P}}_\xi^*$, $\tilde{\zeta}_n$ is independent of the family $\{N_{l,k}^r\}$ with $0 \leqslant k \leqslant n-1$, $1 \leqslant r \leqslant d$ and $l \geqslant 1$. Therefore, by Lemma 4.6, for $1 \leqslant r \leqslant d$ and $n \geqslant 1$, $\mathbb{Q}^*$-a.s.

$$\begin{aligned}\tilde{\mathbb{P}}_\xi^*\left(\tilde{\zeta}_n = r \middle| \mathscr{F}_n^*\right) &= \tilde{\mathbb{P}}_\xi^*\left(\tilde{\zeta}_n = r \middle| \ \tilde{\zeta}_k,\ 1 \leqslant k \leqslant n-1\right) \\ &= \mathbb{Q}^*\left(\tilde{\zeta}_n = r \middle| \tilde{\zeta}_{n-1}\right) \geqslant \frac{1}{dD^2}. \qquad (4.25)\end{aligned}$$

Therefore, from (4.24), we obtain that, for any $C > 1$, $\mathbb{Q}^*$-a.s. (and therefore also $\tilde{\mathbb{P}}_\xi^*$-a.s.),

$$\begin{aligned}&\sum_{n=1}^\infty \tilde{\mathbb{P}}_\xi^*\left(\log^+\langle N_{1,n}^{\tilde{\zeta}_n}, u_{n+1}\rangle \geqslant n\log C \middle| \mathscr{F}_n^*\right) \\ &\qquad\geqslant \frac{1}{dD^2}\sum_{n=1}^\infty\sum_{r=1}^d \tilde{\mathbb{P}}_\xi\left(\log^+\langle N_{1,n}^r, u_{n+1}\rangle \geqslant n\log C\right). \qquad (4.26)\end{aligned}$$

Hence, from (4.22) it follows that, for any ξ, $\tilde{\mathbb{P}}^*_\xi$-a.s.

$$\sum_{n=1}^{\infty} \tilde{\mathbb{P}}^*_\xi \left(\log^+ \langle N^{\tilde{\zeta}_n}_{1,n}, u_{n+1} \rangle \geqslant n \log C \middle| \mathscr{F}^*_n \right) = \infty. \tag{4.27}$$

By the conditional Borel-Cantelli lemma [15, Theorem 5.3.2], we have $\tilde{\mathbb{P}}^*_\xi$-a.s.

$$\left\{ \log^+ \langle N^{\tilde{\zeta}_n}_{1,n}, u_{n+1} \rangle \geqslant n \log C \text{ i.o.} \right\} = \left\{ \sum_{n=1}^{\infty} \tilde{\mathbb{P}}^*_\xi \left(\log^+ \langle N^{\tilde{\zeta}_n}_{1,n}, u_{n+1} \rangle \geqslant n \log C \middle| \mathscr{F}^*_n \right) = \infty \right\}. \tag{4.28}$$

From (4.27) and (4.28), it follows that

$$\tilde{\mathbb{P}}^*_\xi \left(\log^+ \langle N^{\tilde{\zeta}_n}_{1,n}, u_{n+1} \rangle \geqslant n \log C \text{ for infinitely many } n \right) = 1.$$

This implies that for $\mathbb{Q}^*_\xi$-almost all $\omega^* \in \Omega^*$,

$$\tilde{\mathbb{P}}_\xi \left(\log^+ \langle N^{\tilde{\zeta}_n(\omega^*)}_{1,n}, u_{n+1} \rangle \geqslant n \log C \text{ for infinitely many } n \right) = 1.$$

As in (4.28), by the conditional Borel-Cantelli lemma [15, Theorem 5.3.2], we have that, for $\mathbb{Q}^*_\xi$-almost every ω^*,

$$\sum_{n=1}^{\infty} \tilde{\mathbb{P}}_\xi \left(\log^+ \langle N^{\tilde{\zeta}_n(\omega^*)}_{1,n}, u_{n+1} \rangle \geqslant n \log C \right) = \infty. \tag{4.29}$$

Now, for any $y > 0$ there exists $n_0 = n_0(\xi, y)$ such that for all $r \in \{1, \dots, d\}$ and $n \geqslant n_0$,

$$\begin{aligned}
&\tilde{\mathbb{P}}_\xi \left(\log^+ \langle N^r_{1,n}, u_{n+1} \rangle \geqslant n \log C \right) \\
&= \mathbb{E}_\xi \frac{\langle N^r_{1,n}, u_{n+1} \rangle}{\lambda_n \langle u_n, e_r \rangle} \mathbb{1} \left(\log^+ \langle N^r_{1,n}, u_{n+1} \rangle \geqslant n \log C \right) \\
&= \mathbb{E}_\xi \frac{\langle N^r_{1,n}, u_{n+1} \rangle}{\lambda_n \langle u_n, e_r \rangle} \mathbb{1} \left(e^{\log^+ \langle N^r_{1,n}, u_{n+1} \rangle} \geqslant e^{n \log C} \right) \\
&= \mathbb{E}_\xi \frac{\langle N^r_{1,n}, u_{n+1} \rangle}{\lambda_n \langle u_n, e_r \rangle} \mathbb{1} \left(\langle N^r_{1,n}, u_{n+1} \rangle \geqslant e^{n \log C} \right) \\
&\leqslant \mathbb{E}_\xi \left[\frac{\langle N^r_{1,n}, u_{n+1} \rangle}{\lambda_n \langle u_n, e_r \rangle} \mathbb{1}_{\left\{ \frac{\langle N^r_{1,n}, u_{n+1} \rangle}{\lambda_{0,n} \langle Z_0, u_0 \rangle} > y \right\}} \right] = B_y(n, r),
\end{aligned}$$

where the last inequality holds since $\lim_{n\to\infty} \frac{1}{n} \log \left(\lambda_{0,n} \langle Z_0, u_0 \rangle y \right) = \lim_{n\to\infty} \frac{1}{n} \log \lambda_{0,n}$ $= \gamma > 0$ for $\mathbb{P}$-almost every ξ, so that $\lambda_{0,n} \langle Z_0, u_0 \rangle y < e^{n \log C}$, when $\log C > \gamma$ and $n \geqslant n_0$. This, together with (4.29), shows that for any $y > 0$ and $\mathbb{Q}^*_\xi$-almost every ω^*,

$$\sum_{n=n_0}^{\infty} B_y(\zeta_n(\omega^*)) \geqslant \sum_{n=n_0}^{\infty} \tilde{\mathbb{P}}_\xi \left(\log^+ \langle N^{\tilde{\zeta}_n(\omega^*)}_{1,n}, u_{n+1} \rangle \geqslant n \log C \right) = \infty,$$

so that (4.11) is satisfied. Then, by Lemma 4.5, it follows that $\mathbb{E}_\xi[W^H] = 0$ for $\mathbb{P}$-almost every ξ. □

Remark 4.7. The non degeneracy criterion of W^x in Theorem 4.4 can also be deduced from that for W^i proved in [22], as shown in the following. (In the preceding we have given a direct proof based on the general result of Biggins and Kyprianou [7, Theorem 2.1], as the proof in [22] is quite long and somehow complicated.)

Let $Z_0 = x$ and $x = x_1 + x_2$, with $x_1, x_2 \in \mathbb{N}^d \setminus \{0\}$. Notice that the process (Z_n^x) can be written as the form

$$Z_n^x = Z_n^{x_1} + \tilde{Z}_n^{x_2}, \quad n \geqslant 0, \tag{4.30}$$

where conditional on ξ the processes $(Z_n^{x_1})$ and $(\tilde{Z}_n^{x_2})$ are independent, and $(\tilde{Z}_n^{x_2})$ has the same law as $(Z_n^{x_2})$. Using Theorem 2.1, we then have

$$\begin{aligned}
W_n^x(u_n) &= \frac{\langle Z_n^x, u_n\rangle}{\langle xM_{0,n-1}, u_n\rangle} \\
&= \frac{\langle Z_n^{x_1} + \tilde{Z}_n^{x_2}, u_n\rangle}{\langle xM_{0,n-1}, u_n\rangle} \\
&= \frac{\langle Z_n^{x_1}, u_n\rangle}{\langle xM_{0,n-1}, u_n\rangle} + \frac{\langle \tilde{Z}_n^{x_2}, u_n\rangle}{\langle xM_{0,n-1}, u_n\rangle} \\
&= \frac{\langle Z_n^{x_1}, u_n\rangle}{\langle x_1M_{0,n-1}, u_n\rangle}.\frac{\langle x_1M_{0,n-1}, u_n\rangle}{\langle xM_{0,n-1}, u_n\rangle} + \frac{\langle \tilde{Z}_n^{x_2}, u_n\rangle}{\langle x_2M_{0,n-1}, u_n\rangle}.\frac{\langle x_2M_{0,n-1}, u_n\rangle}{\langle xM_{0,n-1}, u_n\rangle} \\
&= W_n^{x_1}(u_n).\frac{\|M_{0,n-1}\|_{1,1}\langle x_1, u_0\rangle\langle v_{n-1}, u_n\rangle}{\|M_{0,n-1}\|_{1,1}\langle x, u_0\rangle\langle v_{n-1}, u_n\rangle}(1 + o(1)) \\
&+ W_n^{x_2}(u_n).\frac{\|M_{0,n-1}\|_{1,1}\langle x_2, u_0\rangle\langle v_{n-1}, u_n\rangle}{\|M_{0,n-1}\|_{1,1}\langle x, u_0\rangle\langle v_{n-1}, u_n\rangle}(1 + o(1)) \\
&= W_n^{x_1}(u_n).\frac{\langle x_1, u_0\rangle}{\langle x, u_0\rangle}(1 + o(1)) + W_n^{x_2}(u_n).\frac{\langle x_2, u_0\rangle}{\langle x, u_0\rangle}(1 + o(1)).
\end{aligned} \tag{4.31}$$

Passing to the limit as $n \to \infty$, we get the decomposition

$$W^x = \frac{\langle x_1, u_0\rangle}{\langle x, u_0\rangle}W^{x_1} + \frac{\langle x_2, u_0\rangle}{\langle x, u_0\rangle}\tilde{W}^{x_2} \quad \text{a.s.} \tag{4.32}$$

where $\tilde{W}^{x_2}$ is independent of W^x, and has the same distribution as W^{x_2}. Consequently (or by the same argument), if $x = \sum_{i=1}^d x_i e_i$, then

$$W^x = \sum_{i=1}^d \sum_{j=1}^{x_i} \frac{\langle e_i, u_0\rangle}{\langle x, u_0\rangle}\tilde{W}_j^i \quad \text{a.s.} \tag{4.33}$$

(by convention the empty sum is 0), where $\{\tilde{W}_j^i : i, j = 1, \cdots, d\}$ is a family of independent random variables, each $\tilde{W}_j^i$ has the same distribution as W^i.

It follows that W^x is non-degenerate if and only if there is $i \in \{1, \cdots, d\}$ with $x_i > 0$ such that W^i is non degenerate. So the non-degeneracy of W^x can be deduced from that of W^i established in [22]. However, as the proof for the non-degeneracy of W^i in [22] is quite long, the different proof presented in this paper for the non-degeneracy of W^x remains of interest.

As an application of the decomposition (4.33), we obtain the following:

Proposition 4.8. *Assume conditions **A1**, **A2**, **H1**, and $\gamma > 0$. For each initial state $x \in \mathbb{N}_+^d \setminus \{0\}$, if W^x is non-degenerate, then*

$$\mathbb{P}_\xi(W^x = 0) = \mathbb{P}_\xi(Z_n^x \not\to \infty) = \mathbb{P}_\xi(Z_n^x \to 0) \quad \mathbb{P}\text{-}a.s. \tag{4.34}$$

Proof. Notice that under the given conditions,

$$\{Z_n^x \to 0\} \subset \{W^x = 0\}.$$

As shown in [22], when $x = e_i$, this together with the same functional equation satisfied by $\mathbb{P}_\xi(Z_n^i \to 0)$ and $\mathbb{P}_\xi(W^i = 0)$ implies that $\mathbb{P}_\xi(Z_n^i \to 0) = \mathbb{P}_\xi(W^i = 0)$, so that

$$\{Z_n^i \to 0\} = \{W^i = 0\} \quad \mathbb{P}\text{-a.s.} \quad \forall i = 1, \cdots, d.$$

Therefore, by the decomposition (4.33), for each $x \in \mathbb{N}^d \setminus \{0\}$,

$$\begin{aligned}\{W^x > 0\} &= \cup_{i=1}^d \{W^i > 0\} = \cup_{1 \leqslant i \leqslant d, x_i > 0} \{Z_n^i \not\to 0\} \\ &= \{Z_n^x \not\to 0\}, \qquad \mathbb{P}\text{-a.s.}\end{aligned}$$

Since $\{W^x > 0\} \subset \{Z_n^x \to \infty\} \subset \{Z_n^x \not\to 0\}$ a.s., the above equality implies that

$$\{W^x > 0\} = \{Z_n^x \to \infty\} = \{Z_n^x \not\to 0\} \quad \mathbb{P}\text{-a.s.}$$

This gives the desired conclusion. □

5. Relations between the explosion and survival events

In this section we show some relations between the survival and explosion events:

$$S = \{Z_n \not\to 0\}, \quad E := \{\|Z_n\| \to \infty\}. \tag{5.1}$$

Their complements are denoted respectively by S^c and E^c. So S^c is the extinction event. Notice that

$$S = \{Z_n \not\to 0\} = \cap_{n=0}^\infty \{Z_n \neq 0\} = \lim_{n\to\infty} \downarrow \{Z_n \neq 0\},$$

$$S^c = \{Z_n \to 0\} = \cup_{n=0}^\infty \{Z_n = 0\} = \lim_{n\to\infty} \uparrow \{Z_n = 0\}.$$

The following proposition shows that conditioned on the explosion E, each component $Z_n(r)$ of Z_n tends to ∞ in probability.

Proposition 5.1. *Assume conditions **A1**, **A2** and $\gamma > 0$. Then, for all $1 \leqslant r \leqslant d$, conditional on E, $Z_n(r) \to \infty$ in probability, namely,*

$$Z_n(r) \xrightarrow[n\to\infty]{\mathbb{P}_E} \infty,$$

where $\xrightarrow{\mathbb{P}_E}$ denotes the convergence in probability under the conditional probability $\mathbb{P}_E(\cdot) = \mathbb{P}(\cdot|E)$.

Proof. It suffices to prove that for all $1 \leqslant r \leqslant d$ and $K > 0$,

$$\mathbb{P}\Big(Z_n(r) \geqslant K, E\Big) \underset{n\to\infty}{\longrightarrow} \mathbb{P}(E). \tag{5.2}$$

By the definition of the branching process, we have the decomposition:

$$Z_{n+k} = \sum_{j=1}^d \sum_{l=1}^{Z_n(j)} Z_{l,n,k}^j, \qquad n \geqslant 0;\ k \geqslant 1.$$

Let $K_1, K_2 > 0$. For $n, k \geqslant 1$ and $1 \leqslant r \leqslant d$, we have, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\mathbb{P}_\xi\left(Z_{n+k}(r) \leqslant K_1, \|Z_n\| \geqslant K_2\right)\\
=\ &\mathbb{P}_\xi\left(\sum_{j=1}^{d}\sum_{l=1}^{Z_n(j)} Z^j_{l,n,k}(r) \leqslant K_1, \|Z_n\| \geqslant K_2\right)\\
\leqslant\ &\mathbb{P}_\xi\left(Z^j_{l,n,k}(r) \leqslant K_1, \|Z_n\| \geqslant K_2, 1 \leqslant l \leqslant Z_n(j), 1 \leqslant j \leqslant d\right)\\
=\ &\mathbb{E}_\xi\left[\mathbb{P}_{T^n\xi}\left(Z^1_k(r) \leqslant K_1\right)^{Z_n(1)} \cdots\ \mathbb{P}_{T^n\xi}\left(Z^d_k(r) \leqslant K_1\right)^{Z_n(d)} \mathbb{1}_{\{\|Z_n\| \geqslant K_2\}}\right].
\end{aligned}$$

It follows that, $\mathbb{P}$-a.s.,

$$\mathbb{P}_\xi\left(Z_{n+k}(r) \leqslant K_1, \|Z_n\| \geqslant K_2\right) \leqslant \left(\max_{1\leqslant j\leqslant d} \mathbb{P}_{T^n\xi}\left(Z^j_k(r) \leqslant K_1\right)\right)^{K_2}. \tag{5.3}$$

Notice that

$$\limsup_{n\to\infty} \mathbb{P}\{E, \|Z_n\| < K_2\} \leqslant \mathbb{P}(E \cap \limsup_{n\to\infty}\{\|Z_n\| < K_2\}) = 0$$

(in fact $E \cap \limsup_{n\to\infty}\{\|Z_n\| < K_2\} = \emptyset$). From this and (5.3) we get that

$$\begin{aligned}
\limsup_{n\to\infty} \mathbb{P}(Z_n(r) \leqslant K_1, E) &\leqslant \limsup_{n\to\infty} \mathbb{P}\left(Z_{n+k}(r) \leqslant K_1, \|Z_n\| \geqslant K_2\right)\\
&\leqslant \mathbb{E}\Big(\max_{1\leqslant j\leqslant d} \mathbb{P}_\xi\left(Z^j_k(r) \leqslant K_1\right)\Big)^{K_2}.
\end{aligned}$$

Letting $K_2 \to \infty$, it follows that

$$\begin{aligned}
\limsup_{n\to\infty} \mathbb{P}(Z_n(r) \leqslant K_1, E) &\leqslant \mathbb{P}\Big(\max_{1\leqslant j\leqslant d} \mathbb{P}_\xi(Z^j_k(r) \leqslant K_1) = 1\Big)\\
&\leqslant \sum_{j=1}^{d} \mathbb{P}\Big(\mathbb{P}_\xi(Z^j_k(r) \leqslant K_1) = 1\Big).
\end{aligned} \tag{5.4}$$

By Corollary 2.5 (equation (2.25) with $x = e_j$ and $y = e_r$) and the condition $\gamma > 0$, we know that for all $K_1 \geqslant 0$,

$$\mathbb{P}\Big(M_{0,k-1}(j, r) \leqslant K_1\Big) \underset{k\to\infty}{\longrightarrow} 0, \tag{5.5}$$

which implies that for all $K_1 \geqslant 0$,

$$\mathbb{P}\Big(\mathbb{P}_\xi(Z^j_k(r) \leqslant K_1) = 1\Big) \leqslant \mathbb{P}\Big(\mathbb{E}_\xi Z^j_k(r) \leqslant K_1\Big) = \mathbb{P}\Big(M_{0,k-1}(j, r) \leqslant K_1\Big) \underset{k\to\infty}{\longrightarrow} 0.$$

Therefore from (5.4), we conclude that for all $K_1 > 0$,

$$\mathbb{P}\Big(Z_n(r) \leqslant K_1, E\Big) \underset{n\to\infty}{\longrightarrow} 0, \tag{5.6}$$

which implies (5.2) and ends the proof of Proposition 5.1. □

The theorem below shows that, under suitable conditions, the survival event S coincides a.s. with the explosion event $E = \{\|Z_n\| \to \infty\}$. It also gives similar results when a fixed type is considered.

Theorem 5.2. *Assume conditions **A1**, **A2** and $\gamma > 0$. Assume also that one of the conditions **H1**, **H2**, **H3** holds. The following assertions hold for each $m \in \{1, \cdots, d\}$:*

(1) $\{\limsup_{n\to\infty} Z_n(m) = \infty\} = E = S$ *a.s.*

(2) $\{\limsup_{n\to\infty} Z_n(m) > 0\} = \limsup_{n\to\infty}\{Z_n(m) > 0\} = S$ *a.s.*

(3) $\{Z_n(m) \to 0\} = \liminf_{n\to\infty}\{Z_n(m) = 0\} = S^c$ *a.s.*

As a consequence of Theorem 5.2 and Proposition 5.1, we get the following properties, including the dichotomy (Part (1)) and the weak transience (Part (3)). We will consider the extinction probabilities

$$q = \mathbb{P}\Big(\lim_n Z_n = 0\Big), \quad q(\xi) = \mathbb{P}\Big(\lim_n Z_n = 0\Big). \tag{5.7}$$

When we consider the dependence on the starting state $x \in \mathbb{N}^d \setminus \{0\}$, we write

$$q^x = \mathbb{P}\Big(\lim_n Z_n^x = 0\Big), \quad q^x(\xi) = \mathbb{P}_\xi\Big(\lim_n Z_n^x = 0\Big). \tag{5.8}$$

Corollary 5.3. *Under the conditions of Theorem 5.2, we have:*

(1) $\mathbb{P}\Big(\limsup_{n\to\infty} Z_n(m) = 0 \text{ or } \infty\Big) = \mathbb{P}\Big(\lim_{n\to\infty} \|Z_n\| = 0 \text{ or } \infty\Big) = 1.$

(2) $\lim_{n\to\infty} \mathbb{P}\Big(Z_n(m) = 0\Big) = q$ *and* $\lim_{n\to\infty} \mathbb{P}\Big(Z_n(m) > 0\Big) = 1 - q.$

(3) For each $k \in \mathbb{N}^*$ *and each* $j \in \mathbb{N}^d \setminus \{0\}$,

$$\lim_{n\to\infty} \mathbb{P}\Big(Z_n(m) = k\Big) = 0 \text{ and } \lim_{n\to\infty} \mathbb{P}\Big(Z_n = j\Big) = 0.$$

Remark 5.4. Tanny [53] proved the dichotomy that a.s. $\|Z_n\| \to 0$ or ∞, for a multitype branching process in a stationary and ergodic environment which is *strongly regular* in the sense that

$$\exists n \geqslant 1 \text{ such that } \mathbb{P}\Big(\min_{1\leqslant i\leqslant d} \mathbb{P}_\xi(\|Z_n^i\| > 1) > 0\Big) > 0 \tag{5.9}$$

and which satisfies the *condition Q*:

$$\text{a.s. } q^{e_i}(\xi) < 1 \ \ \forall i = 1, \cdots, d, \text{ or a.s. } q^{e_i}(\xi) = 1 \ \ \forall i = 1, \cdots, d. \tag{5.10}$$

The branching process (Z_n) is called *regular* if

$$\exists n \geqslant 1 \text{ such that } \min_{1\leqslant i\leqslant d} \mathbb{P}(\|Z_n^i\| > 1) > 0. \tag{5.11}$$

A strongly regular branching process is necessarily regular (see [53]). Our results show that under the conditions for the non-degeneracy of W^x, we do not need the strong regularity condition, nor the condition Q.

Proof of Theorem 5.2. (1) Recall the notation $E = \{\|Z_n\| \to \infty\}$. We first prove that

$$E = S \text{ a.s.} \tag{5.12}$$

Since $E \subset S$ and $\mathbb{P}_\xi(S) = 1 - q(\xi)$, to show (5.12) we only need to prove that

$$\mathbb{P}_\xi(E) = 1 - q(\xi) \quad \mathbb{P}\text{-a.s.} \tag{5.13}$$

We write E^x for E when $Z_0 = x$. From [22, Corollary 2.9], we know that, under the conditions of Theorem 5.2, (5.13) is true for $Z_0 = e_i$:

$$\mathbb{P}_\xi(E^{e_i}) = 1 - q^{e_i}(\xi), \quad i = 1, \cdots d. \tag{5.14}$$

As any $x \in \mathbb{N}^d \setminus \{0\}$ is written as the form $x = \sum_{i=1}^d x(i)e_i$, using (5.14), to prove that (5.13) holds for all $x \in \mathbb{N}^d \setminus \{0\}$, we just need to prove that if it holds for x_1 and x_2 in $\mathbb{N}^d \setminus \{0\}$, then it also holds for $x = x_1 + x_2$. Assume that (5.13) holds for

$x_1, x_2 \in \mathbb{N}^d \setminus \{0\}$, and let $x = x_1 + x_2$. Notice that the process (Z_n^x) can be written as the form

$$Z_n^x = Z_n^{x_1} + \tilde{Z}_n^{x_2}, \quad n \geqslant 0, \tag{5.15}$$

where conditional on ξ the processes $(Z_n^{x_1})$ and $(\tilde{Z}_n^{x_2})$ are independent, and $(\tilde{Z}_n^{x_2})$ has the same law as $(Z_n^{x_2})$. Therefore

$$E^x = E^{x_1} \cup \tilde{E}^{x_2} \text{ and } q^x(\xi) = q^{x_1}(\xi) q^{x_2}(\xi),$$

where $\tilde{E}^{x_2} = \{\tilde{Z}_n^{x_2} \not\to 0\}$ is independent of E^{x_1}, and has the same probability as E^{x_2}, under $\mathbb{P}_\xi$. Since (5.13) holds for x_1 and x_2, it follows that

$$\begin{aligned}
\mathbb{P}_\xi(E^x) &= \mathbb{P}_\xi(E^{x_1}) + \mathbb{P}_\xi(\tilde{E}^{x_2}) - \mathbb{P}_\xi(E^{x_1} \cap \tilde{E}^{x_2}) \\
&= \mathbb{P}_\xi(E^{x_1}) + \mathbb{P}_\xi(E^{x_2}) - \mathbb{P}_\xi(E^{x_1}) \mathbb{P}_\xi(E^{x_2}) \\
&= (1 - q^{x_1}(\xi)) + (1 - q^{x_2}(\xi)) - (1 - q^{x_1}(\xi))(1 - q^{x_2}(\xi)) \\
&= 1 - q^{x_1}(\xi) q^{x_2}(\xi) \\
&= 1 - q^x(\xi).
\end{aligned} \tag{5.16}$$

Therefore (5.13) still holds for x. This shows that (5.13) holds for all $x \in \mathbb{N}^d \setminus \{0\}$. So we have proved (5.12), namely the second equality in Part 1.

We next prove the first equality, that is,

$$\{\limsup_n Z_n(m) = \infty\} = E \quad \text{a.s.} \tag{5.17}$$

Notice that

$$\{\limsup_n Z_n(m) = \infty\} = \cap_{k=1}^\infty \limsup_{n\to\infty} \{Z_n(m) \geqslant k\}.$$

For any integer $k > 0$, we know that

$$\mathbb{P}(\limsup_{n\to\infty} \{Z_n(m) \geqslant k\},\ E) \geqslant \limsup_{n\to\infty} \mathbb{P}(Z_n(m) \geqslant k, E) = \mathbb{P}(E), \tag{5.18}$$

where the last step holds by Proposition 5.1 or (5.2). This shows that for any integer $k > 0$,

$$\mathbb{P}_E(\limsup_{n\to\infty} \{Z_n(m) \geqslant k\}) = 1.$$

It follows that

$$\mathbb{P}_E\{\limsup_n Z_n(m) = \infty\} = 1.$$

Since $\{\limsup_n Z_n(m) = \infty\} \subset E$, this implies (5.17). So we have finished the proof of Part 1.

(2) Since the first equality is evident, we just need to prove the second one. Because $\{Z_n(m) > 0\} \subset \{Z_n \neq 0\}$, we have

$$\limsup_n \{Z_n(m) > 0\} \subset \limsup_n \{Z_n \neq 0\} = S. \tag{5.19}$$

On the other hand, by (5.18) and the fact that $E = S$ a.s. (see the second equality in Part 1), we have for any integer $k > 0$,

$$\begin{aligned}
\mathbb{P}\Big(\limsup_{n\to\infty} \{Z_n(m) > 0\}, S\Big) &\geqslant \limsup_{n\to\infty} \mathbb{P}\Big(Z_n(m) > k, E\Big) \\
&= \mathbb{P}(E) = \mathbb{P}(S).
\end{aligned} \tag{5.20}$$

From (5.21) and (5.20), we obtain

$$\limsup_{n\to\infty}\{Z_n(m)>0\}=S \quad a.s. \tag{5.21}$$

This ends the proof of Part 2.

(3) Taking the complements of the events in Part 2, we get the conclusion of Part 3. □

Proof of Corollary 5.3. (1) Part 1 is a direct consequence of Theorem 5.2:

$$\begin{aligned}
&\mathbb{P}\Big(\limsup_{n\to\infty} Z_n(m)\to 0 \text{ or } \infty\Big)\\
&=\mathbb{P}\Big(Z_n\to 0 \text{ or } \infty\Big) \quad \text{(by Part 2 or 3 of Theorem 5.2)}\\
&=\mathbb{P}(S^c)+\mathbb{P}(S) \qquad \text{(by Part 1 of Theorem 5.2)}\\
&=1.
\end{aligned}$$

(2) For Part 2, we first notice that by Part 2 of Theorem 5.2,

$$\limsup_{n\to\infty}\mathbb{P}(Z_n(m)>0)\leqslant \mathbb{P}(\limsup_{n\to\infty}\{Z_n(m)>0\})=\mathbb{P}(S). \tag{5.22}$$

Next, by Proposition 5.1 and Part 1 of Theorem 5.2, we have for each $k>0$,

$$\liminf_{n\to\infty}\mathbb{P}(Z_n(m)>0)\geqslant \liminf_{n\to\infty}\mathbb{P}(Z_n(m)>k,E)=\mathbb{P}(E)=\mathbb{P}(S). \tag{5.23}$$

Therefore, $\lim_{n\to\infty}\mathbb{P}(Z_n(m)>0)=\mathbb{P}(S)=1-q$. This proves the second conclusion of Part 2, which implies the first one.

(3) Let $k\in\mathbb{N}^*$ be arbitrarily fixed. By Proposition 5.1 (or (5.6)) and the fact that $E=S$ a.s. (see Theorem 5.2), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\mathbb{P}\Big(Z_n(m)=k\Big)\leqslant \limsup_{n\to\infty}\mathbb{P}\Big(Z_n(m)=k,E\Big)+\limsup_{n\to\infty}\mathbb{P}\Big(Z_n(m)=k,S^c\Big)\\
&\leqslant \mathbb{P}\Big(\limsup_{n\to\infty}\Big(Z_n(m)=k\Big),E\Big)+\mathbb{P}\Big(\limsup_{n\to\infty}\{Z_n(m)=k\},S^c\Big)\\
&=0+\mathbb{P}(\emptyset)=0.
\end{aligned}$$

This gives the first conclusion of Part 3. The second conclusion follows from the first one, because for any $j=(j_1,\dots,j_d)\in\mathbb{N}^d\setminus\{0\}$, there is m such that $j_m>0$, so that

$$\mathbb{P}\Big(Z_n=j\Big)\leqslant \mathbb{P}\Big(Z_n(m)=j_m\Big)\underset{n\to\infty}{\longrightarrow}0.$$

□

6. Convergence of the direction and proof of Theorem 2.9

Proof of Theorem 2.9. (1) We begin with the proof of (2.42). For $x=e_i$, $i=1,\cdots,d$, the result concerning $v_{0,n-1}$ has been proved in [22, Theorem 2.11]. For the general case, we do a decomposition as follows. Let $x=\sum_{i=1}^d x_ie_i$. By the definition of the branching process, we can decompose Z_n^x as follows:

$$Z_n^x=\sum_{i=1}^d\sum_{k=1}^{x_i}\tilde{Z}_{n,k}^i \tag{6.1}$$

(by convention the empty sum is equal to 0), where $(\tilde{Z}^i_{n,k})_n$ denotes the branching process starting with the k-th type i particle. Conditional on ξ, the processes $(\tilde{Z}^i_{n,k})_{n\geqslant 0}$, for $1\leqslant i\leqslant d, 1\leqslant k\leqslant x_i$, are independent of each other, each $(\tilde{Z}^i_{n,k})_n$ has the same law as $(Z^i_n)_n$. Using (2.42) for $x=e_i$, we obtain, for $i=1,\cdots,d$,

$$\tilde{Z}^i_{n,k}=\|\tilde{Z}^i_{n,k}\|(v_{0,n-1}+\varepsilon^i_{k,n}),$$

where $\|\varepsilon^i_{k,n}\|\to 0$ in probability under $\mathbb{P}_E$, as $n\to\infty$. Therefore

$$\begin{aligned} Z^x_n&=\sum_{i=1}^d\sum_{k=1}^{x_i}\|\tilde{Z}^i_{n,k}\|(v_{0,n-1}+\varepsilon^i_{k,n})\\ &=\Big(\sum_{i=1}^d\sum_{k=1}^{x_i}\|\tilde{Z}^i_{n,k}\|\Big)\big(v_{0,n-1}+\varepsilon^x_n\big), \end{aligned}\tag{6.2}$$

where

$$\begin{aligned} \|\varepsilon^x_n\|&\leqslant\frac{\sum_{i=1}^d\sum_{k=1}^{x_i}\|\tilde{Z}^i_{n,k}\|\,\|\varepsilon^i_{k,n}\|}{\sum_{i=1}^d\sum_{k=1}^{x_i}\|\tilde{Z}^i_{n,k}\|}\\ &\leqslant\max_{1\leqslant i\leqslant d,1\leqslant k\leqslant x_i}\|\varepsilon^i_{k,n}\|\xrightarrow{\mathbb{P}_E}0. \end{aligned}\tag{6.3}$$

Notice that $\|Z^x_n\|=\sum_{i=1}^d\sum_{k=1}^{x_i}\|\tilde{Z}^i_{n,k}\|$. So it follows that

$$\frac{Z^x_n}{\|Z^x_n\|}-v_{0,n-1}\xrightarrow{\mathbb{P}_E}0.\tag{6.4}$$

Since a.s. $v_{k,n}-v_n\to 0$ for each fixed $k\in\mathbb{N}$ as $n\to\infty$ (see (2.13)), this implies that

$$\frac{Z^x_n}{\|Z^x_n\|}-v_{n-1}\xrightarrow[n\to\infty]{\mathbb{P}_E}0\quad\text{and}\quad\frac{Z^x_n}{\|Z^x_n\|}-v_{k,n-1}\xrightarrow[n\to\infty]{\mathbb{P}_E}0\quad\forall k\in\mathbb{N}.\tag{6.5}$$

This is just (2.42).

(2) The convergence in distribution of $\frac{Z^x_n}{\|Z^x_n\|}$ is a direct consequence of (2.42), using the fact that all v_n have the same law ν (see Theorem 2.1).

(3) For the proof of (2.43) about the convergence in probability under $\mathbb{P}_E$, by homogeneity, we just need to prove the convergence uniformly for $y\in\mathcal{S}$. To this end, we notice that by Lemma 3.6 and the condition **A3**, we know that $v_{k,n}(j)\geqslant 1/(dD)$ and $v_n\geqslant 1/(dD)$ for all k and n with $k\leqslant n$. Therefore, for all $y\in\mathcal{S}$, we have $\langle\bar{v}_{n-1},y\rangle\geqslant\min_{1\leqslant j\leqslant d}v_{n-1}(j)\|y\|\geqslant 1/(dD)$, so that

$$\frac{|\langle\vec{Z}^x_n,y\rangle-\langle v_{n-1},y\rangle|}{\langle v_{n-1},y\rangle}\leqslant dD|\langle\vec{Z}^x_n,y\rangle-\langle v_{n-1},y\rangle|\leqslant dD\|\vec{Z}^x_n-v_{n-1}\|.\tag{6.6}$$

Hence by (2.42), we get

$$\sup_{y\in\mathcal{S}}\left|\frac{\langle\vec{Z}^x_n,y\rangle}{\langle v_{n-1},y\rangle|}-1\right|\xrightarrow{\mathbb{P}_E}0.$$

The same argument applies when v_{n-1} is replaced by $v_{k,n-1}$. This ends the proof of (2.43).

(4) The a.s. convergence can be proved by the same argument as in Parts 1 and 3 above, using the a.s. convergence of $\bar{Z}^i_n-v_{0,n-1}$ established in [22, Theorem 2.14] under the additional moment condition (2.33).

□

7. Duality of the Kesten-Stigum theorem and proof of Theorem 2.8

In this section we prove Theorem 2.8, the duality of Theorem 2.6, for the convergence of $\frac{\langle Z_{n+k},y\rangle}{\langle Z_nM_{n,n+k-1},y\rangle}$ for fixed k as $n\to\infty$. We consider both the convergence in probability and the a.s. convergence.

Proof of Theorem 2.8. 1) We first prove (2.39), the uniform convergence in probability conditional on E. We fix $k\geqslant 1$. For $n\geqslant 0$, write Z_n in the form

$$Z_n=\sum_{r=1}^d Z_n(r)e_r=\sum_{r=1}^d\sum_{l=1}^{Z_n(r)} e_r. \tag{7.1}$$

By the definition of the branching process, we have for $n\geqslant 0$,

$$Z_{n+k}=\sum_{r=1}^d\sum_{l=1}^{Z_n(r)} N^r_{l,n,k}, \tag{7.2}$$

where for $j\in\{1,\cdots,d\}$ the j-th component of $N^r_{l,n,k}$ denotes the number of type-j offsprings at time $n+k$ of the l-th type r particle of generation n. By (7.1) and (7.2), for any $y\in\mathbb{R}^d_+\setminus\{0\}$,

$$\begin{aligned}\langle Z_{n+k},y\rangle-\langle Z_nM_{n,n+k-1},y\rangle&=\sum_{r=1}^d\sum_{l=1}^{Z_n(r)}\left(\langle N^r_{l,n,k},y\rangle-\langle e_rM_{n,n+k-1},y\rangle\right)\\&=\sum_{r=1}^d\langle e_rM_{n,n+k-1},y\rangle\sum_{l=1}^{Z_n(r)}\left(\frac{\langle N^r_{l,n,k},y\rangle}{\langle e_rM_{n,n+k-1},y\rangle}-1\right).\end{aligned} \tag{7.3}$$

By Proposition 5.1, dividing both sides by $\langle Z_nM_{n,n+k-1},y\rangle$, we find that, on $\{Z_n\neq 0\}$,

$$\frac{\langle Z_{n+k},y\rangle}{\langle Z_nM_{n,n+k-1},y\rangle}-1=\sum_{r=1}^d\frac{\langle e_rM_{n,n+k-1},y\rangle}{\langle Z_nM_{n,n+k-1},y\rangle}\sum_{l=1}^{Z_n(r)}\left(\frac{\langle N^r_{l,n,k},y\rangle}{\langle e_rM_{n,n+k-1},y\rangle}-1\right). \tag{7.4}$$

This implies that on $\{Z_n\neq 0\}$, for all $1\leqslant r\leqslant d$,

$$\left|\frac{\langle Z_{n+k},y\rangle}{\langle Z_nM_{n,n+k-1},y\rangle}-1\right|\leqslant\sum_{r=1}^d\frac{\langle e_rM_{n,n+k-1},y\rangle Z_n(r)}{\langle Z_nM_{n,n+k-1},y\rangle}R_{n,k}(r,y)\leqslant\max_{1\leqslant r\leqslant d}R_{n,k}(r,y), \tag{7.5}$$

where

$$R_{n,k}(r,y)=\left|\frac{1}{Z_n(r)}\sum_{l=1}^{Z_n(r)}\left(\frac{\langle N^r_{l,n,k},y\rangle}{\langle e_rM_{n,n+k-1},y\rangle}-1\right)\right|. \tag{7.6}$$

We will prove that $R_{n,k}(r,y)\to 0$ in probability conditionally on the explosion event E, uniformly in $y\in\mathcal{S}$, as $n\to\infty$: that is $\forall\varepsilon>0$,

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{y\in\mathcal{S}}R_{n,k}(r,y)>\varepsilon,E\right)=0. \tag{7.7}$$

Notice that by the independence of $\{N^r_{l,n,k} : l \geqslant 1\}$ and Z_n under $\mathbb{P}_\xi$, and the condition that the environment sequence (ξ_n) is i.i.d., we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{y\in\mathcal{S}} R_{n,k}(r,y) > \varepsilon, Z_n(r) > 0\right)\\
&= \mathbb{E}\left[\mathbb{P}_\xi\left(\sup_{y\in\mathcal{S}}\Big|\frac{1}{Z_n(r)}\sum_{l=1}^{Z_n(r)}\left(\frac{\langle N^r_{l,n,k}, y\rangle}{\langle e_r M_{n,n+k-1}, y\rangle} - 1\right)\Big| > \varepsilon, Z_n(r) > 0\right)\right]\\
&= \mathbb{E}\left[\sum_{j=1}^{\infty}\mathbb{P}_\xi\left(\frac{1}{j}\sup_{y\in\mathcal{S}}\Big|\sum_{l=1}^{j}\left(\frac{\langle N^r_{l,n,k}, y\rangle}{\langle e_r M_{n,n+k-1}, y\rangle} - 1\right)\Big| > \varepsilon\right)\mathbb{P}_\xi\Big(Z_n(r) = j\Big)\right]\\
&= \sum_{j=1}^{\infty}\mathbb{P}\left(\frac{1}{j}\sup_{y\in\mathcal{S}}\Big|\sum_{l=1}^{j}\left(\frac{\langle N^r_{l,0,k}, y\rangle}{\langle e_r M_{0,k-1}, y\rangle} - 1\right)\Big| > \varepsilon\right)\mathbb{P}\Big(Z_n(r) = j\Big).
\end{aligned} \tag{7.8}$$

By the uniform law of large numbers or the law of large numbers in a Banach space (see e.g. [49, p.189, Corollary 7.10]),

$$\mathbb{P}_\xi\left(\frac{1}{j}\sup_{y\in\mathcal{S}}\Big|\sum_{l=1}^{j}\left(\frac{\langle N^r_{l,0,k}, y\rangle}{\langle e_r M_{0,k-1}, y\rangle} - 1\right)\Big| > \varepsilon\right) \underset{j\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.}, \tag{7.9}$$

since $\min_{y\in\mathcal{S}}\langle e_r M_{0,k-1}, y\rangle = \langle e_r M_{0,k-1}, y_0\rangle$ for some $y_0 = y_0(\xi) \in \mathcal{S}$ and

$$\mathbb{E}_\xi\left(\sup_{y\in\mathcal{S}}\frac{\langle N^r_{l,0,k}, y\rangle}{\langle e_r M_{0,k-1}, y\rangle}\right) \leqslant \mathbb{E}_\xi\left(\frac{\sup_{y\in\mathcal{S}}\langle Z^r_k, y\rangle}{\langle e_r M_{0,k-1}, y_0\rangle}\right) \leqslant \frac{\langle e_r M_{0,k-1}, \mathbb{1}\rangle}{\langle e_r M_{0,k-1}, y_0\rangle} < \infty, \quad \mathbb{P}\text{-a.s.}$$

By the dominated convergence theorem, (7.9) implies that

$$\mathbb{P}\left(\frac{1}{j}\sup_{y\in\mathcal{S}}\Big|\sum_{l=1}^{j}\left(\frac{\langle N^r_{l,0,k}, y\rangle}{\langle e_r M_{0,k-1}, y\rangle} - 1\right)\Big| > \varepsilon\right) \underset{j\to\infty}{\longrightarrow} 0. \tag{7.10}$$

So for any $\eta > 0$, there is some integer $j_0 = j_0(\eta) > 0$ such that for all $j > j_0$,

$$\mathbb{P}\left(\frac{1}{j}\sup_{y\in\mathcal{S}}\Big|\sum_{l=1}^{j}\left(\frac{\langle N^r_{l,0,k}, y\rangle}{\langle e_r M_{0,k-1}, y\rangle} - 1\right)\Big| > \varepsilon\right) < \eta. \tag{7.11}$$

For short, we set

$$A_n := \left\{\sup_{y\in\mathcal{S}}\Big|\frac{1}{Z_n(r)}\sum_{l=1}^{Z_n(r)}\left(\frac{\langle N^r_{l,n,k}, y\rangle}{\langle e_r M_{n,n+k-1}, y\rangle} - 1\right)\Big| > \varepsilon\right\}.$$

Combining (7.8), (7.11) and Proposition 5.1, we obtain that for all $n \geqslant 0$,

$$\begin{aligned}
&\mathbb{P}\left(A_n, E, Z_n(r) > 0\right)\\
&\leqslant \sum_{j>j_0}\mathbb{P}\left(\frac{1}{j}\sup_{y\in\mathcal{S}}\Big|\sum_{l=1}^{j}\left(\frac{\langle N^r_{l,0,k}, y\rangle}{\langle e_r M_{0,k-1}, y\rangle} - 1\right)\Big| > \varepsilon\right)\mathbb{P}\Big(Z_n(r) = j\Big) + \sum_{1\leqslant j\leqslant j_0}\mathbb{P}\Big(Z_n(r) = j, E\Big)\\
&\leqslant \eta\sum_{j>j_0}\mathbb{P}\Big(Z_n(r) = j\Big) + \mathbb{P}\Big(1 \leqslant Z_n(r) \leqslant j_0, E\Big)\\
&\leqslant \eta + \mathbb{P}\Big(1 \leqslant Z_n(r) \leqslant j_0, E\Big).
\end{aligned} \tag{7.12}$$

Consequently, for all $n \geqslant 0$,

$$\begin{aligned}\mathbb{P}\Big(A_n \cap E\Big) &\leqslant \mathbb{P}\Big(A_n, E, Z_n(r) > j_0\Big) + \mathbb{P}\Big(E, 1 \leqslant Z_n(r) \leqslant j_0\Big) \\ &\leqslant \eta + 2\mathbb{P}\Big(E, 1 \leqslant Z_n(r) \leqslant j_0\Big). \end{aligned} \tag{7.13}$$

Recall that (cf. (5.6)) $\mathbb{P}\Big(E, 1 \leqslant Z_n(r) \leqslant j_0\Big) \to 0$. Therefore, from (7.13), we get

$$\limsup_{n\to\infty} \mathbb{P}\Big(A_n \cap E\Big) \leqslant \eta. \tag{7.14}$$

As $\eta > 0$ is arbitrary, this implies that

$$\lim_{n\to\infty} \mathbb{P}\Big(A_n \cap E\Big) = \lim_{n\to\infty} \mathbb{P}\left(\sup_{y\in\mathcal{S}} \left| \frac{1}{Z_n(r)} \sum_{l=1}^{Z_n(r)} \left(\frac{\langle N^r_{l,n,k}, y\rangle}{\langle e_r M_{n,n+k-1}, y\rangle} - 1 \right) \right| > \varepsilon, E \right) = 0. \tag{7.15}$$

So (7.7) is proved. Thus, by (7.5) we obtain (2.39).

2) We now prove (2.41), the a.s. uniform convergence on S, under the conditional condition (2.40). We first observe that condition (2.40) implies **H1**. Therefore, by applying Theorems 4.4 and 5.2, it follows that W^x is non-degenerate and $E = S$ a.s.

For the proof of (2.41), we proceed in two steps, by proving first the simple convergence, and then the uniform convergence, a.s. on E.

Step 1: prove the a.s. simple convergence in (2.41): that is, for any fixed $y \in \mathbb{R}^d_+ \setminus \{0\}$, and any fixed $k \geqslant 0$, as $n \to \infty$,

$$\left| \frac{\langle Z_{n+k}, y\rangle}{\langle Z_n M_{n,n+k-1}, y\rangle} - 1 \right| \underset{n\to\infty}{\longrightarrow} 0 \quad \mathbb{P}_E\text{-a.s.} \tag{7.16}$$

By homogeneity, we just need to consider $y \in \mathcal{S}$. Like in the proof for the convergence in probability, we still use (7.5), and it is sufficient to prove that the random variable $R_{n,k}(r, y)$ defined in (7.6) satisfies, for all $1 \leqslant r \leqslant d$,

$$R_{n,k}(r, y) = \left| \frac{1}{Z_n(r)} \sum_{l=1}^{Z_n(r)} \left(\frac{\langle N^r_{l,n,k}, y\rangle}{\langle e_r M_{n,n+k-1}, y\rangle} - 1 \right) \right| \underset{n\to\infty}{\longrightarrow} 0 \quad \mathbb{P}_E\text{-a.s.} \tag{7.17}$$

To prove (7.17), it is enough to establish the following equivalent statement: for all $1 \leqslant r \leqslant d$, $1 \leqslant k$ and $0 \leqslant b < k$, $\mathbb{P}$-a.s. on E,

$$\left| \frac{1}{Z_{kn+b}(r)} \sum_{l=1}^{Z_{kn+b}(r)} \left(\frac{\langle N^r_{l,kn+b,k}, y\rangle}{\langle e_r M_{kn+b,k(n+1)+b-1}, y\rangle} - 1 \right) \right| \underset{n\to\infty}{\longrightarrow} 0. \tag{7.18}$$

Let $(\tilde{\mathscr{F}}_n)$ be the filtration defined by

$$\tilde{\mathscr{F}}_0 = \{\emptyset\}, \quad \tilde{\mathscr{F}}_n = \sigma\left(\xi_s, N^r_{l,s}, 0 \leqslant s \leqslant n-1, 1 \leqslant r \leqslant d, l \geqslant 1\right) \text{ for } n \geqslant 1.$$

Applying the conditional Borel-Cantelli lemma [15, Theorem 5.3.2], we see that (7.18) holds $\mathbb{P}$-a.s. on the event E, if for all $1 \leqslant r \leqslant d$ and $C > 0$, $\mathbb{P}$-a.s. on E,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left| \sum_{l=1}^{Z_{kn+b}(r)} \left(\frac{\langle N^r_{l,kn+b,k}, y\rangle}{\langle e_r M_{kn+b,k(n+1)+b-1}, y\rangle} - 1 \right) \right| \geqslant C Z_{kn+b}(r) \;\Big|\; \tilde{\mathscr{F}}_{kn+b} \right) < \infty. \tag{7.19}$$

We can assume that condition (8.2) holds for some $1 < p \leqslant 2$ (as when it holds for some $p > 1$, then it also holds when p is replaced by any $\bar{p} \in (1, p)$). Since the environment sequence $(\xi_n)_{n\geqslant 0}$ is i.i.d., $\langle N^r_{l,kn+b,k}, y\rangle / \langle e_r M_{kn+b,k(n+1)+b-1}, y\rangle$ is independent of $\tilde{\mathscr{F}}_{kn+b}$ for all $1 \leqslant r \leqslant d$ and $n, l \geqslant 1, 1 \leqslant k \leqslant n$. Therefore, using Markov's inequality and [22, Lemma 10.1] (which is a consequence of the Marcinkiewicz-Zygmund inequality), the series in (7.19) can be bounded as follows:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{P}\Big(\Big| \sum_{l=1}^{Z_{kn+b}(r)} \Big(\frac{\langle N^r_{l,kn+b,k}, y\rangle}{\langle e_r M_{kn+b,k(n+1)+b-1}, y\rangle} - 1 \Big) \Big| \geqslant C Z_{kn+b}(r) \;\Big|\; \tilde{\mathscr{F}}_{kn+b} \Big) \\
\leqslant\ & \sum_{n=1}^{\infty} \frac{1}{C^p (Z_{kn+b}(r))^p} \mathbb{E}\Big(\Big| \sum_{l=1}^{Z_{kn+b}(r)} \Big(\frac{\langle N^r_{l,kn+b,k}, y\rangle}{\langle e_r M_{kn+b,k(n+1)+b-1}, y\rangle} - 1 \Big) \Big|^p \;\Big|\; \tilde{\mathscr{F}}_{kn+b} \Big) \\
\leqslant\ & \sum_{n=1}^{\infty} \Big(\frac{B_p^p\, Z_{kn+b}(r)}{C^p (Z_{kn+b}(r))^p} \mathbb{E} \Big| \frac{\langle Z^r_k, y\rangle}{\langle e_r M_{0,k-1}, y\rangle} - 1 \Big|^p \Big),
\end{aligned}$$

where B_p is a constant depending only on p. The last series converges provided that $\mathbb{P}$-a.s. on E,

$$\Big(\mathbb{E} \Big| \frac{\langle Z^r_k, y\rangle}{\langle e_r M_{0,k-1}, y\rangle} - 1 \Big|^p \Big) \sum_{n=1}^{\infty} (Z_{kn+b}(r))^{1-p} < \infty. \tag{7.20}$$

Therefore (7.19) holds if (7.20) is satisfied for all $1 \leqslant r \leqslant d$, $n \geqslant 1$.

It remains to prove (7.20), which is done below. Applying Corollary 2.10 with $y = e_r$, we obtain the strong law of large numbers for $Z^x_n(r)$: for all $1 \leqslant r \leqslant d$, $\mathbb{P}$-a.s. on E,

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(r) = \gamma > 0.$$

Therefore we deduce that, $\mathbb{P}$-a.s. on E,

$$\sum_{n=1}^{\infty} (Z_n(r))^{1-p} < \infty. \tag{7.21}$$

To finish the proof of (7.20) we just need to prove that, for all $1 \leqslant r \leqslant d$ and $1 \leqslant k \leqslant n$,

$$\mathbb{E} \Big| \frac{\langle Z^r_k, y\rangle}{\langle e_r M_{0,k-1}, y\rangle} \Big|^p < \infty. \tag{7.22}$$

We shall prove (7.22) by induction on k. For $k = 1$, it holds by the assumption (2.33) of Theorem 2.6. Suppose that (7.22) holds for some $k \geqslant 1$. We will prove that it still holds for $k + 1$. Using the decomposition (7.2) together with the inequality

$(a+b)^p \leqslant 2^{p-1}(a^p+b^p)$, $a,b \geqslant 0$, we have, for all $1 \leqslant r \leqslant d$ and $1 \leqslant k$,

$$\begin{aligned}
\mathbb{E}\left(\frac{\langle Z^r_{k+1}, y\rangle}{\langle e_r M_{0,k}, y\rangle}\right)^p &= \mathbb{E}\left(\sum_{j=1}^d \frac{\langle e_j M_{1,k}, y\rangle}{\langle e_r M_{0,k}, y\rangle} \sum_{l=1}^{Z^r_1(j)} \left|\frac{\langle N^j_{l,1,k}, y\rangle}{\langle e_j M_{1,k}, y\rangle} - 1 + 1\right|\right)^p \\
\leqslant\ & \mathbb{E}\left(\sum_{j=1}^d \frac{\langle e_j M_{1,k}, y\rangle}{\langle e_r M_{0,k}, y\rangle} \sum_{l=1}^{Z^r_1(j)} \left|\frac{\langle N^j_{l,1,k}, y\rangle}{\langle e_j M_{1,k}, y\rangle} - 1\right| + \sum_{j=1}^d \frac{\langle e_j M_{1,k}, y\rangle}{\langle e_r M_{0,k}, y\rangle} Z^r_1(j)\right)^p \\
\leqslant\ & 2^{p-1}\mathbb{E}\left(\sum_{j=1}^d \frac{\langle e_j M_{1,k}, y\rangle}{\langle e_r M_{0,k}, y\rangle} \sum_{l=1}^{Z^r_1(j)} \left|\frac{\langle N^j_{l,1,k}, y\rangle}{\langle e_j M_{1,k}, y\rangle} - 1\right|\right)^p \\
& + 2^{p-1}\mathbb{E}\left(\sum_{j=1}^d \frac{\langle e_j M_{1,k}, y\rangle}{\langle e_r M_{0,k}, y\rangle} Z^r_1(j)\right)^p. \qquad (7.23)
\end{aligned}$$

By the convexity of the function $x \mapsto x^p$ on $\mathbb{R}_+$ and the fact that $\sum_{j=1}^d \frac{\langle e_r M_0, e_j\rangle \langle e_j M_{1,n}, y\rangle}{\langle e_r M_{0,n}, y\rangle} = 1$, we get that, for all $1 \leqslant r, j \leqslant d$ and $1 \leqslant k$,

$$\begin{aligned}
\mathbb{E}\left(\sum_{j=1}^d \frac{\langle e_j M_{1,k}, y\rangle}{\langle e_r M_{0,k}, y\rangle} Z^r_1(j)\right)^p &= \mathbb{E}\left(\sum_{j=1}^d \frac{\langle e_r M_0, e_j\rangle \langle e_j M_{1,k}, y\rangle}{\langle e_r M_{0,k}, y\rangle} \frac{Z^r_1(j)}{\langle e_r M_0, e_j\rangle}\right)^p \\
&\leqslant \mathbb{E}\left(\sum_{j=1}^d \frac{\langle e_r M_0, e_j\rangle \langle e_j M_{1,k}, y\rangle}{\langle e_r M_{0,k}, y\rangle} \left(\frac{Z^r_1(j)}{\langle e_r M_0, e_j\rangle}\right)^p\right) \\
&\leqslant \sum_{j=1}^d \mathbb{E}\left(\frac{Z^r_1(j)}{M_0(r,j)}\right)^p. \qquad (7.24)
\end{aligned}$$

Using again the convexity of $x \mapsto x^p$ on $\mathbb{R}_+$, we next have for all $1 \leqslant r, j \leqslant d$ and $1 \leqslant k$,

$$\begin{aligned}
& \mathbb{E}\left(\sum_{j=1}^d \frac{\langle e_j M_{1,k}, y\rangle}{\langle e_r M_{0,k}, y\rangle} \sum_{l=1}^{Z^r_1(j)} \left|\frac{\langle N^j_{l,1,k}, y\rangle}{\langle e_j M_{1,k}, y\rangle} - 1\right|\right)^p \\
=\ & \mathbb{E}\left(\sum_{j=1}^d \frac{\langle e_r M_0, e_j\rangle \langle e_j M_{1,k}, y\rangle}{\langle e_r M_{0,k}, y\rangle} \frac{1}{\langle e_r M_0, e_j\rangle} \sum_{l=1}^{Z^r_1(j)} \left|\frac{\langle N^j_{l,1,k}, y\rangle}{\langle e_j M_{1,k}, y\rangle} - 1\right|\right)^p \\
\leqslant\ & \mathbb{E}\left(\sum_{j=1}^d \frac{\langle e_r M_0, e_j\rangle \langle e_j M_{1,k}, y\rangle}{\langle e_r M_{0,k}, y\rangle \langle e_r M_0, e_j\rangle^p} \mathbb{E}_\xi\left[\left(\sum_{l=1}^{Z^r_1(j)} \left|\frac{\langle N^j_{l,1,k}, y\rangle}{\langle e_j M_{1,k}, y\rangle} - 1\right|\right)^p \,\middle|\, Z^r_1(j)\right]\right) \\
\leqslant\ & \mathbb{E}\left[\sum_{j=1}^d \frac{\mathbb{E}_\xi Z^r_1(j)}{M_0(r,j)^p} \mathbb{E}_\xi\left|\frac{\langle N^j_{1,1,k}, y\rangle}{\langle e_j M_{1,k}, y\rangle} - 1\right|^p\right] \qquad (7.25) \\
\leqslant\ & \sum_{j=1}^d \mathbb{E} M_0(r,j)^{1-p} \mathbb{E}\left|\frac{\langle Z^j_k, y\rangle}{\langle e_j M_{0,k-1}, y\rangle} - 1\right|^p, \qquad (7.26)
\end{aligned}$$

where the last step holds since the environment sequence (ξ_n) is i.i.d.

Combining the relations (7.23)-(7.26), we obtain that for all $1 \leqslant r, j \leqslant d$ and $1 \leqslant k$,

$$\mathbb{E}\left(\frac{\langle Z_{k+1}^j, y\rangle}{\langle e_j M_{0,k}, y\rangle}\right)^p \leqslant 2^{p-1} B_p^p \sum_{j=1}^d \mathbb{E} M_0(r,j)^{1-p} \mathbb{E}\left|\frac{\langle Z_k^j, y\rangle}{\langle e_j M_{0,k-1}, y\rangle} - 1\right|^p + 2^{p-1} \sum_{j=1}^d \mathbb{E}\left(\frac{Z_1^r(j)}{M_0(r,j)}\right)^p. \tag{7.27}$$

From this and the hypothesis of induction, we see that (7.22) still holds for $k+1$. Therefore it holds for all $k \geqslant 1$.

From (7.22) and (7.21) we get (7.20), which implies consecutively (7.19) and (7.18). This ends the proof of (7.17), and thus of (7.16).

Step.2: prove the a.s. uniform convergence (2.41), with a decomposition argument. Applying the result proved in the first step for $y = e_j$, $j = 1, \cdots, d$, we obtain

$$\frac{\langle Z_{n+k}, e_j\rangle}{\langle Z_n M_{n,n+k-1}, e_j\rangle} = 1 + \varepsilon_{n,k}^1(j), \tag{7.28}$$

where $\varepsilon_{n,k}^1(j) \to 0$ $\mathbb{P}_E$-a.s., as $n \to \infty$. Thus

$$\begin{aligned}\frac{\langle Z_{n+k}, y\rangle}{\langle Z_n M_{n,n+k-1}, y\rangle} &= \frac{\sum_{j=1}^d \langle Z_{n+k}, y_j e_j\rangle}{\langle Z_n M_{n,n+k-1}, y\rangle} \\ &= \frac{\sum_{j=1}^d \langle Z_n M_{n,n+k-1}, y_j e_j\rangle (1 + \varepsilon_{n,k}^1(j))}{\langle Z_n M_{n,n+k-1}, y\rangle} \\ &= 1 + \varepsilon_{n,k}(y),\end{aligned} \tag{7.29}$$

where

$$\begin{aligned}\sup_{y \in \mathbb{R}_+^d \setminus \{0\}} |\varepsilon_{n,k}(y)| &= \left|\frac{\sum_{j=1}^d \langle Z_n M_{n,n+k-1}, y_j e_j\rangle \varepsilon_{n,k}^1(j)}{\langle Z_n M_{n,n+k-1}, y\rangle}\right| \\ &\leqslant \sup_{1 \leqslant j \leqslant d} |\varepsilon_{n,k}^1(j)| \to 0 \quad \mathbb{P}_E\text{-a.s.}\end{aligned} \tag{7.30}$$

This finishes the proof of the a.s. uniform convergence (2.41). □

8. Convergence of the normalized scalar product and proof of Theorem 2.6

In this section we establish the convergence of $W_n^x(y)$ uniformly in $y \in \mathbb{R}_+^d \setminus \{0\}$, first for the convergence in probability, and then for the almost sure convergence.

Recall that $E = \{\|Z_n\| \to \infty\}$ denotes the explosion event, and $\xrightarrow[n\to\infty]{\mathbb{P}_E}$ denotes the convergence in probability under the conditional probability $\mathbb{P}_E(\cdot) = \mathbb{P}(\cdot|E)$. Write $\xrightarrow[n\to\infty]{\mathbb{P}}$ for the convergence in probability under $\mathbb{P}$.

Theorem 8.1. *Assume conditions **A1**, **A3** and $\gamma > 0$. Assume additionally that the random environment sequence (ξ_n) is i.i.d. Then, for all $x \in \mathbb{N}^d \setminus \{0\}$,*

$$\sup_{y \in \mathbb{R}_+^d \setminus \{0\}} |W_n^x(y) - W^x| \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{8.1}$$

The convergence in probability can be improved to the a.s. convergence if additionally for some $p > 1$,

$$\max_{1\leqslant r,j\leqslant d} \mathbb{E}\left(\frac{Z_1^r(j)}{M_0(r,j)}\right)^p < \infty \quad \textit{and} \quad \mathbb{E}\|M_0\|^{1-p} < \infty. \tag{8.2}$$

Proof. The proof is based on the convergence of the direction (Theorem 2.9) and the Perron-Frobenius theorem (Theorem 2.1). In the following, we fix $x \in \mathbb{N}^d \setminus \{0\}$ and assume that $Z_0 = x$; for simplicity, we just write $W_n(y)$ and W instead of $W_n^x(y)$ and W^x.

1) We first prove (8.1) about the convergence in probability. By the definition (2.28) of $W_n(y)$ and the Perron-Frobenius theorem (Theorem 2.1), we have for all $y \in \mathbb{R}_+^d \setminus \{0\}$,

$$W_n(y) = \frac{1}{a_{0,n-1}\langle u_0, x\rangle}\frac{\langle Z_n, y\rangle}{\langle v_{n-1}, y\rangle}(1+\varepsilon_n(y)) \tag{8.3}$$

(in fact the expression $\varepsilon_n(y)$ depends also on x, but for simplicity we just write $\varepsilon_n(y)$ as x is fixed, and we will do the same in the following for similar expressions), where $\sup_{y\in\mathbb{R}_+^d\setminus\{0\}} |\varepsilon_n(y)| \to 0$ a.s. For $y = u_n$, it gives

$$W_n(u_n) = \frac{1}{a_{0,n-1}\langle u_0, x\rangle}\frac{\langle Z_n, u_n\rangle}{\langle v_{n-1}, u_n\rangle}(1+\varepsilon_n'), \tag{8.4}$$

where $\varepsilon_n' \to 0$ a.s. From (2.43) of Theorem 2.9 about the convergence of the direction, we know that on the explosion event E (in order that $\|Z_n\| \neq 0$), and for $y \in \mathbb{R}_+^d \setminus \{0\}$,

$$\frac{\langle Z_n, y\rangle}{\langle v_{n-1}, y\rangle\|Z_n\|} = 1 + \eta_n(y), \tag{8.5}$$

where $\eta_n(y)$ satisfies $\sup_{y\in\mathbb{R}_+^d\setminus\{0\}} |\eta_n(y)| \to 0$ in probability under $\mathbb{P}_E$. (Notice that $\eta_n(y)$ is in fact well defined on the survival event $S \supset E$, but we only need to consider E since we are conditioning on E.) Taking $y = u_n$, we see that on E,

$$\frac{\langle Z_n, u_n\rangle}{\langle v_{n-1}, u_n\rangle\|Z_n\|} = 1 + \eta_n(u_n), \tag{8.6}$$

where $\eta_n(u_n) \to 0$ in probability under $\mathbb{P}_E$. From (8.5) and (8.6), we get that on E and for $y \in \mathbb{R}_+^d \setminus \{0\}$,

$$\frac{\langle Z_n, y\rangle}{\langle v_{n-1}, y\rangle} = \frac{\langle Z_n, u_n\rangle}{\langle v_{n-1}, u_n\rangle}\big(1+\eta_n''(y)\big), \tag{8.7}$$

where $\eta_n''(y)$ satisfies $\sup_{y\in\mathbb{R}_+^d\setminus\{0\}} |\eta_n''(y)| \to 0$ in probability under $\mathbb{P}_E$. Therefore, from (8.3) and (8.4) we see that on E, we have a.s.

$$\frac{W_n(y)}{W_n(u_n)} = 1 + \varepsilon_n''(y), \tag{8.8}$$

for all $y \in \mathbb{R}_+^d \setminus \{0\}$, where $\varepsilon_n''(y)$ is well defined on E with $\sup_{y\in\mathbb{R}_+^d\setminus\{0\}} |\varepsilon_n''(y)| \xrightarrow[n\to\infty]{\mathbb{P}_E} 0$. Recall that from Lemma 4.2, we have $W_n(u_n) \to W$ a.s. So the above equality implies that a.s. on E, we have

$$W_n(y) = W + \varepsilon_n'''(y) \tag{8.9}$$

for all $y \in \mathbb{R}_+^d \setminus \{0\}$, where $\varepsilon_n'''(y)$ is well defined on E with $\sup_{y\in\mathbb{R}_+^d\setminus\{0\}} |\varepsilon_n'''(y)| \to 0$ in probability under $\mathbb{P}_E$. This gives that

$$\sup_{y\in\mathbb{R}_+^d\setminus\{0\}} |W_n^x(y) - W^x| \xrightarrow[n\to\infty]{\mathbb{P}_{\mathbb{E}}} 0. \tag{8.10}$$

Under condition **A3**, by [22, Lemma 7.1] or Lemma 3.6 (applied with $D_n = D$), we get that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\frac{1}{dD} \leqslant u_n(i) \leqslant 1 \text{ and } \frac{1}{dD} \leqslant v_n(i) \leqslant 1. \tag{8.11}$$

By (8.11) we have for all $y \in \mathbb{R}_+^d \setminus \{0\}$, $\mathbb{P}$-a.s.,

$$W_n(u_n) = \frac{\langle Z_n, u_n\rangle}{\langle Z_0 M_{0,n-1}, u_n\rangle} \geqslant \frac{1}{dD} \frac{\langle Z_n, \mathbb{1}\rangle}{\langle Z_0 M_{0,n-1}, u_n\rangle}. \tag{8.12}$$

By Theorem 2.1, we have, $\mathbb{P}$-a.s., as $n \to \infty$,

$$\frac{1}{dD} \frac{\langle Z_n, \mathbb{1}\rangle}{\langle Z_0 M_{0,n-1}, u_n\rangle} \sim \frac{1}{dD} \frac{\langle Z_n, \mathbb{1}\rangle}{\|M_{0,n-1}\|_{1,1}\langle Z_0, u_0\rangle\langle v_{n-1}, u_n\rangle}. \tag{8.13}$$

Moreover, in fact that $\langle v_{n-1}, u_n\rangle \leqslant 1$ for all $n \geqslant 1$, we get

$$\begin{aligned} \frac{1}{dD} \frac{\langle Z_n, \mathbb{1}\rangle}{\|M_{0,n-1}\|_{1,1}\langle Z_0, u_0\rangle\langle v_{n-1}, u_n\rangle} &\geqslant \frac{1}{dD\|y\|} \frac{\langle Z_n, y\rangle\langle v_{n-1}, y\rangle}{\|M_{0,n-1}\|_{1,1}\langle Z_0, u_0\rangle\langle v_{n-1}, y\rangle} \\ &\geqslant \frac{1}{(dD)^2} \frac{\langle Z_n, y\rangle}{\|M_{0,n-1}\|_{1,1}\langle Z_0, u_0\rangle\langle v_{n-1}, y\rangle}. \end{aligned} \tag{8.14}$$

Using again Theorem 2.1, we have, $\mathbb{P}$-a.s., as $n \to \infty$,

$$\frac{1}{(dD)^2} \frac{\langle Z_n, y\rangle}{\|M_{0,n-1}\|_{1,1}\langle Z_0, u_0\rangle\langle v_{n-1}, y\rangle} \sim \frac{1}{(dD)^2} \frac{\langle Z_n, y\rangle}{\langle Z_0 M_{0,n-1}, y\rangle} = \frac{1}{(dD)^2} W_n(y). \tag{8.15}$$

So from (8.12)-(8.15) (and the fact that if $x_n \geqslant y_n \sim z_n$ and $x_n \to 0$, then $z_n \to 0$), we get that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}_+^d\setminus\{0\}} W_n(y) = 0 \quad \text{a.s. on } \{W = 0\}. \tag{8.16}$$

On the other hand, we claim that

$$\{W > 0\} \subset E \quad a.s. \tag{8.17}$$

Indeed, we have

$$W_n(u_n) = \frac{\langle Z_n, u_n\rangle}{\langle Z_0 M_{0,n-1}, u_n\rangle} \leqslant \frac{\|Z_n\|}{\langle Z_0 M_{0,n-1}, u_n\rangle}. \tag{8.18}$$

From the result of Corollary 2.5 and (8.18), we get $\|Z_n\| \to \infty$ $\mathbb{P}$-a.s. on the event $\{W > 0\}$. This gives (8.17). From (8.16), (8.17), and Theorem 8.1 , we get that for

all $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{P}(\sup_{y\in\mathbb{R}^d_+\setminus\{0\}} |W_n(y) - W| > \varepsilon) \\
&= \mathbb{P}(\sup_{y\in\mathbb{R}^d_+\setminus\{0\}} |W_n(y) - W| > \varepsilon, W = 0) + \mathbb{P}(\sup_{y\in\mathbb{R}^d_+\setminus\{0\}} |W_n(y) - W| > \varepsilon, W > 0) \\
&\leqslant \mathbb{P}(\sup_{y\in\mathbb{R}^d_+\setminus\{0\}} W_n(y) > \varepsilon, W = 0) + \mathbb{P}(\sup_{y\in\mathbb{R}^d_+\setminus\{0\}} |W_n(y) - W| > \varepsilon, E) \\
&\to 0. \qquad (8.19)
\end{aligned}$$

This ends the proof of (8.1).

2) We then prove that under the additional condition (8.2),

$$\sup_{y\in\mathbb{R}^d_+\setminus\{0\}} |W^x_n(y) - W^x| \underset{n\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.} \qquad (8.20)$$

In fact, under the additional condition (8.2), with the same argument as in 1) but by using Theorem 2.9 for the a.s. convergence of the direction instead of the convergence in probability, we can see that the convergence in probability (8.10) can be improved to the a.s. convergence:

$$\sup_{y\in\mathbb{R}^d_+\setminus\{0\}} |W^x_n(y) - W^x| \underset{n\to\infty}{\longrightarrow} 0 \quad \mathbb{P}_E\text{-a.s.} \qquad (8.21)$$

Since $S = E = \{W > 0\}$ a.s. under the given conditions, (8.21) together with (8.16) implies (8.20). So the proof is finished. □

Proof of Theorem 2.6. Part 1 is contained in Lemma 4.2, Part 2 is contained in Theorem 4.4 and Proposition 4.8, while Part 3 is just Theorem 8.1. □

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