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ALGEBRAIC ISOMORPHISMS OF QUANTIZED HOMOGENEOUS SPACES

ROBERT YUNCKEN

ABSTRACT. We describe a proof of the following folklore theorem: If $\mathcal{X} = G/K$ is the homogeneous space of a simply connected compact semisimple Lie group with Poisson-Lie stabilizers, then the q -deformed algebras of regular functions $\mathbb{C}[\mathcal{X}_q]$ with $0 < q \leq 1$ are mutually non-isomorphic as $*$ -algebras.

1. INTRODUCTION

Fix G a compact simply connected semisimple Lie group and K a proper closed Poisson-Lie subgroup. The homogeneous space $\mathcal{X} = G/K$ admits a family of q -deformations, $0 < q < \infty$, realized via their $*$ -algebras of regular functions $\mathbb{C}[\mathcal{X}_q]$.

The following theorem is folklore.

Theorem 1.1. $\mathbb{C}[\mathcal{X}_p] \cong \mathbb{C}[\mathcal{X}_q]$ if and only if $p = q^{\pm 1}$.

In this expository note, we will give a proof of this fact. The ideas are mostly not original. A proof for the quantum spheres $\mathcal{X}_q = S_q^{2n+1}$ was given by D'Andrea [DAn24]. Krämer [Krä08] proved a similar result for the nonstandard Podleś spheres.

In the present context, the algebras $\mathbb{C}[G_q]$ are due to Soibelman [Soi91], see also [Wor87]. The algebras $\mathbb{C}[\mathcal{X}_q]$ were introduced over the following years, but a systematic study of their structure was undertaken by Stokman and Dijkhuizen [SD99; Sto03]. This was further developed by Neshveyev and Tuset in [NT12]. The essential points necessary for proving Theorem 1.1 are already contained in the papers just listed.

Gisselsson [Gis20] proved that the enveloping C^* -algebras $C^*(\mathcal{X}_q)$ are all isomorphic for $q \in (0, \infty) \setminus \{1\}$. In other words, the quantum spaces \mathcal{X}_q ($0 < q < 1$) are isomorphic as noncommutative topological spaces but not as noncommutative algebraic varieties.

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2. PRELIMINARIES

This article is not completely self-contained. We will make heavy use of the notation and results of [NT12], especially Sections 1, 2 and the beginning of Section 3.

The isomorphism $\mathbb{C}[\mathcal{X}_q] \cong \mathbb{C}[\mathcal{X}_{q^{-1}}]$ is well-known, cf. [NT13, Lemma 2.4.2]. Also, $\mathbb{C}[\mathcal{X}_q]$ is commutative if and only if $q = 1$. We therefore work throughout with $0 < p, q < 1$.

3. $SU_q(2)$ AND THE QUANTUM DISK

The algebra $\mathbb{C}[SU_q(2)]$ is the universal $*$ -algebra generated by two elements α and γ satisfying

$$\begin{aligned} \alpha\gamma &= q\gamma\alpha, & \alpha\gamma^* &= q\gamma^*\alpha, & \gamma\gamma^* &= \gamma^*\gamma \\ \alpha^*\alpha + \gamma^*\gamma &= 1 = \alpha\alpha^* + q^2\gamma^*\gamma. \end{aligned} \quad (3.1)$$

The $*$ -subalgebra $\mathbb{C}[\gamma, \gamma^*]$ is a polynomial algebra in 2 commuting conjugate variables. We use the notation

$$\alpha^{(i)} = \begin{cases} \alpha^i, & \text{if } i \geq 0, \\ (\alpha^*)^{-i}, & \text{if } i < 0. \end{cases}$$

Putting $\mathcal{A}_i = \alpha^{(i)}\mathbb{C}[\gamma, \gamma^*] = \mathbb{C}[\gamma, \gamma^*]\alpha^{(i)}$, we get an algebra grading

$$\mathbb{C}[SU_q(2)] = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i. \quad (3.2)$$

The standard irreducible representation ρ_q of $\mathbb{C}[SU_q(2)]$ on $\mathcal{B}(\ell^2(\mathbb{Z}_+))$ is given in the basis $(e_n)_{n \in \mathbb{Z}_+}$ by

$$\rho_q(\alpha) : e_n \mapsto \sqrt{1 - q^{2n}} e_{n-1}, \quad \rho_q(\gamma) : e_n \mapsto -q^n e_n. \quad (3.3)$$

Lemma 3.1. *The kernel of ρ_q is the ideal generated by $\gamma - \gamma^*$.*

Proof. Let $a \in \ker(\rho_q)$. Write $a = \sum_i c_i \alpha^{(i)} \in \bigoplus_i \mathcal{A}_i$ where $c_i \in \mathbb{C}[\gamma, \gamma^*]$. Since ρ_q maps \mathcal{A}_i to weighted shifts of degree i , we have $\rho_q(c_i \alpha^{(i)}) = 0$ for each i .

Let $i \geq 0$. We have $\rho_q(c_i \alpha^{(i)} \alpha^{(i)*}) = 0$. Since $\rho_q(\alpha^{(i)} \alpha^{(i)*})$ is diagonal with no zero entries, we get $\rho_q(c_i) = 0$. The spectrum of $\rho_q(\gamma) = \rho_q(\gamma^*)$ is $-q^{\mathbb{Z}_+}$ which is Zariski dense in \mathbb{R} . So if we view c_i as a polynomial in $\mathbb{C}[\gamma, \gamma^*] \cong \mathbb{C}[Z, \bar{Z}]$, it is zero on the real line and hence divisible by $\gamma - \gamma^*$.

For $i < 0$, we can take the adjoint to reduce to the previous case. \square

We define $\mathbb{C}[\overline{\mathbb{D}}_q] = \mathbb{C}[SU_q(2)] / \langle \gamma - \gamma^* \rangle \cong \rho_q(\mathbb{C}[SU_q(2)])$ and let $F_q : \mathbb{C}[SU_q(2)] \rightarrow \mathbb{C}[\overline{\mathbb{D}}_q]$ denote the quotient map. Then $\mathbb{C}[\overline{\mathbb{D}}_q]$ has generators

$$y = F_q(\gamma), \quad z = F_q(\alpha)^* \quad (3.4)$$

and relations

$$yz = qzy, \quad y = y^*, \quad zz^* + y^2 = 1 = z^*z + q^2y^2. \quad (3.5)$$

In Poisson language, F_q is the restriction of polynomials on $SU_q(2)$ to a certain quantized symplectic leaf.

Remark 3.2. The $*$ -algebra $\mathbb{C}[\overline{\mathbb{D}}_q]$ embeds into the C^* -algebra of the quantum disk $C(\overline{\mathbb{D}}_q)$ of [NT12, §3] via the homomorphism

$$z \mapsto Z_q, \quad y \mapsto -(1 - Z_q Z_q^*)^{\frac{1}{2}}. \quad (3.6)$$

This is not an algebraic map, which is why we add the generator y .

4. HOMOGENEOUS SPACES FOR $SU_q(2)$

Write T for the diagonal torus in $SU(2)$. For $n \in \{1, 2, \dots\}$, let T_n denote the closed subgroup of T generated by the n th roots of unity. We also use the convention $T_\infty = T$. The associated quantum homogeneous spaces [NT12, §2] are given by the quantized function algebras

$$\mathbb{C}[SU_q(2)/T_n] = \text{span}\{\alpha^{(i)} \gamma^j \gamma^{*k} \mid i + j - k \in n\mathbb{Z}\} \subseteq \mathbb{C}[SU_q(2)], \quad (4.1)$$

where we use the convention $\infty\mathbb{Z} = \{0\}$. Explicitly $SU_q(2)/T_n$ is $SU_q(2)$ when $n = 1$, $SO_q(3)$ when $n = 2$ [Pod95], the quantum lens space $L_q^3(n, 1)$ when $3 \leq n < \infty$ [HS03] and the standard Podleś sphere when $n = \infty$ [Pod87].

Let us also write

$$\mathbb{C}[\overline{\mathbb{D}}_q/T_n] = F_q(\mathbb{C}[SU_q(2)/T_n]) \cong \rho_q(\mathbb{C}[SU_q(2)/T_n]).$$

This is the quantization of a 2-dimensional symplectic leaf of $SU_q(2)/T_n$.

5. COMMUTATOR SPECTRUM

Definition 5.1. Define the *commutator spectrum* of an algebra \mathcal{A} to be

$$\text{CommSp}(\mathcal{A}) = \{\omega \in \mathbb{C}^\times \mid \exists a, b \in \mathcal{A} \setminus \{0\} \text{ with } ab = \omega ba\}.$$

Lemma 5.2. *The commutator spectrum of $\mathbb{C}[\overline{\mathbb{D}}_q/T_n]$ is*

$$\mathcal{Q}(\mathbb{C}[\overline{\mathbb{D}}_q/T_n]) = \begin{cases} q^{\mathbb{Z}} & \text{if } n \text{ is odd} \\ q^{2\mathbb{Z}} & \text{if } n \text{ is even or } \infty. \end{cases}$$

Proof. Consider first the quantum plane algebra $\mathbb{C}_q[y, z]$ generated by y, z with $yz = qzy$. We do not impose a $*$ -structure for the moment. We have

$$(y^j z^k)(y^{j'} z^{k'}) = q^{jk' - j'k} (y^{j'} z^{k'}) (y^j z^k). \quad (5.1)$$

This shows that $q^{\mathbb{Z}} \subseteq \text{CommSp}[\mathbb{C}_q[y, z]]$. Moreover, by considering the leading terms for the lexicographical ordering on powers of y and z , one can deduce that $\text{CommSp}(\mathbb{C}_q[y, z]) = q^{\mathbb{Z}}$.

Now suppose $\omega \in \text{CommSp}(\mathbb{C}[\overline{\mathbb{D}}_q/T_n])$, so we have $a, b \in \mathbb{C}[\overline{\mathbb{D}}_q/T_n]$ nonzero with $ab = \omega ba$. By considering the highest order terms in the grading (3.2), we can assume without loss of generality that $a \in F_q(\mathcal{A}_i)$, $b \in F_q(\mathcal{A}_{i'})$ for some i, i' . If both $i, i' \leq 0$ then $a, b \in \mathbb{C}_q[y, z]$ so $\omega \in q^{\mathbb{Z}}$. If both $i, i' > 0$ we can reduce to the previous case by taking adjoints. Finally, if $i < 0$ and $i' > 0$ then we can reduce to the previous case by replacing b by $a^m b$ for some sufficiently large m .

This proves that $\text{CommSp}(\mathbb{C}[\overline{\mathbb{D}}_q/T_n]) \subseteq q^{\mathbb{Z}}$, and also that it is realized on monomials $a = y^j z^k$, $b = y^{j'} z^{k'} \in \mathbb{C}[\overline{\mathbb{D}}_q/T_n]$. Note also that it is closed under taking integer powers, by considering a and b^m .

If n is even or ∞ , then Equation (4.1) forces $j \equiv k$ [2] and $j' \equiv k'$ [2]. Therefore, by Equation (5.1), $\text{CommSp}(\mathbb{C}[\overline{\mathbb{D}}_q/T_n]) \subseteq q^{2\mathbb{Z}}$. Putting $a = F_q(\gamma^* \alpha) = yz^*$, $b = F_q(\gamma^* \gamma) = y^2$ gives $\omega = q^2$.

If n is odd, we can put $a = F_q(\gamma^{n+1} \alpha^*) = y^{n+1} z$, $b = F_q(\gamma^{\frac{n+1}{2}} \gamma^{*\frac{n-1}{2}} \alpha^*) = y^n z$ and we get $\omega = q$. \square

6. QUANTIZED HOMOGENEOUS SPACES

Let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be the simple roots of G and $\{\omega_1, \dots, \omega_r\}$ the associated fundamental weights. We write (\cdot, \cdot) for the G -invariant bilinear form on \mathfrak{h}^* with $(\alpha, \alpha) = 2$ for all short roots α , and we use the standard notation $q_i = q^{(\alpha_i, \alpha_i)/2}$. Thus $q_i = q, q^2$ or q^3 depending on the length of α_i . For every $\alpha_i \in \Pi$, there is a restriction map $\sigma_i : \mathbb{C}[G_q] \rightarrow \mathbb{C}[\text{SU}_{q_i}(2)]$.

The Poisson-Lie subgroups of G are determined by pairs (S, L) where S is a subset of the simple roots for G and L is a subgroup of the lattice $P(S^c) = \mathbb{Z}\{\omega_i \mid \alpha_i \in S^c\}$. We refer to [NT12, Proposition 1.1] for the details. Fix such a subgroup $K^{S,L}$ and consider $\mathcal{X} = G/K^{S,L}$ and its quantized algebra of function $\mathbb{C}[\mathcal{X}_q]$.

Lemma 6.1. *For every $\alpha_i \in S^c$ we have $\sigma_i(\mathbb{C}[\mathcal{X}_q]) = \mathbb{C}[\text{SU}_{q_i}(2)/T_{n_i}]$ for some $n_i \in \{1, 2, \dots, \infty\}$.*

Proof. This follows from [NT12, Corollary 2.4]. Explicitly, n_i is the generator of the subgroup of \mathbb{Z} obtained by restricting the weights $\mu \in L$ to weights for $\mathcal{U}_{q_i}(\mathfrak{sl}_2)$, i.e., $n_i \mathbb{Z} = \{(\mu, \alpha_i) \mid \mu \in L\}$. \square

Let T be the maximal torus of G , and $T_L \subseteq T$ the subgroup annihilated by $L \subseteq P = \widehat{T}$. Let $W^S \subset W$ be the set of $w \in W$ such that $w(\alpha_i) > 0$ for all $\alpha_i \in S$.

The irreducible representations of $\mathbb{C}[\mathcal{X}_q]$ are indexed by pairs $(w, t) \in W^S \times T/T_L$. These representations $\pi_{w,t}$ are described explicitly in the preamble to Theorem 2.2 of [NT12]. See also [SD99, Theorem 5.9]. We recall only that in the case where $w = s_i$ is the simple reflection in $\alpha_i \in S^c$, the associated representation is

$$\pi_{s_i, 1} = \rho_q \circ \sigma_i : \mathbb{C}[\mathcal{X}_q] \rightarrow \mathbb{C}[\mathbb{D}_{q_i}/T_{n_i}] \rightarrow \mathcal{B}(\ell^2(\mathbb{Z}_+)). \quad (6.1)$$

7. QUANTUM 2-CELL REPRESENTATIONS

Definition 7.1. An irreducible representation π of $\mathbb{C}[\mathcal{X}_q]$ will be called a *2-cell representation* if the norm closure of the image is an extension of an abelian C^* -algebra by the compacts—that is, if it fits into an exact sequence of C^* -algebras

$$0 \rightarrow \mathcal{K} \rightarrow \pi(\mathbb{C}[\mathcal{X}_q]) \rightarrow A \rightarrow 0$$

with A abelian and \mathcal{K} being the compacts on a separable infinite dimensional Hilbert space.

Example 7.2. The standard representation ρ_q of $\mathbb{C}[\mathrm{SU}_q(2)/T_n]$ is a 2-cell representation for any $n \in \{1, 2, \dots, \infty\}$. To see this, let I denote the $*$ -ideal of $\mathbb{C}[\mathrm{SU}_q(2)]$ generated by γ and put $I_n = I \cap \mathbb{C}[\mathrm{SU}_q(2)/T_n]$. Then $\overline{\rho_q(I_n)} = \mathcal{K}(\ell^2(\mathbb{Z}_+))$ since it contains $\rho_q(\alpha^* \gamma)$ which is a compact weighted shift with distinct nonzero weights. And putting $\gamma = 0$ in the relations (3.1) shows that $\mathbb{C}[\mathrm{SU}_q(2)/T_n]/I_n$ is abelian.

Lemma 7.3. *The representation $\pi_{w,t}$ is a 2-cell representation if and only if w is a simple reflection, i.e. $w = s_i$ for some $\alpha_i \in S^c$.*

Proof. The representation $\pi_{w,t}$ is the representation $\pi_{w,1}$ twisted by a character, so the image of $\pi_{t,w}(\mathbb{C}[\mathcal{X}_q])$ is independent of t . Therefore, we may take $t = 1$.

If $w = 1$, then $\pi_{1,1}$ is the counit.

If $w = s_i$ for $i \in S^c$, then by Lemma 6.1 $\pi_{w,1}(\mathbb{C}[\mathcal{X}_q]) = \rho_{q_i}(\mathbb{C}[\mathrm{SU}_{q_i}(2)/T_{n_i}])$, so π_w is a 2-cell representation by Example 7.2.

If $w \in W^S$ has Bruhat length two or more, then [NT12, Theorem 4.1] shows that $\pi_{w,1}(\mathbb{C}[\mathcal{X}_q])$ has a nontrivial nonabelian quotient, so $\pi_{w,1}$ is not a 2-cell representation. □

8. PROOF OF THE MAIN THEOREM

Proof of Theorem 1.1. By Lemmas 5.2, 6.1 and 7.3, the greatest value, less than one, of the commutator spectrum of the image of any 2-cell representation of $\mathbb{C}[\mathcal{X}_q]$ is q^m , for some $m \in \mathbb{N}$ independent of q . This is an invariant of $\mathbb{C}[\mathcal{X}_q]$. □

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