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# Stabilization and Optimal Control of Interconnected SDE - Scalar PDE System

Gabriel Velho<sup>1</sup>, Jean Auriol<sup>1</sup>, Riccardo Bonalli<sup>1</sup>, Islam Boussaada<sup>2</sup>

**Abstract**—In this paper, we design a controller for an interconnected system consisting of a linear Stochastic Differential Equation (SDE) actuated through a linear hyperbolic Partial Differential Equation (PDE). Our approach aims to minimize the variance of the state of the SDE component. We leverage a backstepping technique to transform the original PDE into an uncoupled stochastic PDE. As such, we reformulate our initial problem as the control of a delayed SDE with a non-deterministic drift. Under standard controllability assumptions, we design a controller steering the mean of the states to zero while keeping its covariance bounded. As final step, we address the optimal control of the delayed SDE employing Artstein’s transformation and Linear Quadratic stochastic control techniques.

## I. INTRODUCTION

The interest in interconnected systems of Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs) emerged when delays in ODEs were associated with transport equations, allowing in [1] for a re-interpretation of the classical Finite Spectrum Assignment [2]. Interconnections involving hyperbolic PDEs and ODEs can model the propagation of torsional waves in drilling systems [3], deepwater construction vessels [4], or heat exchangers systems. Lyapunov and backstepping methods then facilitated the design of stabilizing controllers for such interconnected systems [5]–[8].

In realistic scenarios, dynamical processes are frequently influenced by disturbances originating, e.g., from imprecise measurements, parameter uncertainties, external disruptions, etc. [9]. Such disturbances may considerably alter the dynamics. Therefore, effectively mitigating these uncertainties is crucial to establish the reliability and safety of such controlled systems [10]. Stochastic Differential Equations (SDEs) provide broad and accurate modelization of a large class of uncertain systems [11]. Stochastic control enables the effective design of stabilizing controllers for SDEs, which are also robust against random fluctuations. Stabilizing SDEs in expectation is a popular and effective control technique. However, when doing so, one must make sure the variance remains bounded to ensure reliability [10]. Robustness may also

be reliably achieved by seeking controllers that keep the state variance relatively small, see [10], [12].

In interconnected ODE-PDE systems, modeling uncertainties that appear in the ODE dynamics is often key to accuracy and robustness. Some methods have been proposed to stabilize PDE+ODE systems in the presence of bounded or model-based disturbances [7], [13]. It has been shown that SDEs may offer a broader framework for modeling such disturbances. Numerous methods exist for addressing the issue of SDEs with delays [14], [15]. While delays can be interpreted as transport PDEs, coupled systems with more complexity have yet to be thoroughly investigated. Our study aims to provide insights and methodologies for effectively controlling such interconnected SDE-PDE systems for the first time. Specifically, in this paper, we bridge the gap between deterministic PDE+ODE control and stochastic control by merging methodologies from both fields. Our main contribution is the introduction of a novel approach to reliably control a PDE+SDE system with bidirectional coupling. The proposed controllers are implementable and straightforward to compute. Concretely our contribution is twofold:

- We propose a feedback law achieving the stabilization of the PDE+SDE system, driving the mean of the states to zero while bounding the variance.
- Additionally, we introduce an optimal control approach to minimize the SDE variance over a finite time horizon, thus enhancing the robustness of the control strategy with respect to disturbances.

Our approach leverages the backstepping methodology to allow a reformulation of the coupled system as a system of delayed SDE with non-deterministic drift. We address our control objectives by employing Artstein’s transformation and Linear Quadratic (LQ) control techniques.

The paper is organized as follows. In Section II, we outline the problem formulation. Next, in Section III, we utilize the backstepping transformation to reduce the problem to controlling an input delayed SDE with a random drift term, while also establishing the well-posedness of the system. In Section IV, we propose a stabilizing feedback law for the SDE. Finally, in Section V, we present an optimal control approach to minimize the variance of the SDE.

## II. PROBLEM FORMULATION

### A. Notations

For a given  $n \in \mathbb{N} \setminus \{0\}$ , state variables take values in  $\mathbb{R}^n$ , while control variables take values in  $\mathbb{R}$ . We

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assume we are given a filtered probability space  $(\Omega, \mathcal{F} \triangleq (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$ , and that stochastic perturbations are due to a one-dimensional Wiener process  $W_t$ , which is adapted to the filtration  $\mathcal{F}$ . Let  $T > 0$  be some given time horizon. For any  $r \in \mathbb{N} \setminus \{0\}$ , we denote by  $L_{\mathcal{F}}^2([0, T], \mathbb{R}^r)$  the set of square integrable processes  $P : [0, T] \times \Omega \rightarrow \mathbb{R}^r$  that are  $\mathcal{F}$ -progressively measurable, whereas the subset  $C_{\mathcal{F}}^2([0, T], \mathbb{R}^r) \subseteq L_{\mathcal{F}}^2([0, T], \mathbb{R}^r)$  contains processes whose sample paths are continuous. The spaces of semi-definite and definite positive symmetric matrices in  $\mathbb{R}^n$  are denoted by  $\mathcal{S}_n^+$  and  $\mathcal{S}_n^{++}$ , respectively. If  $X \in L_{\mathcal{F}}^2([0, T], \mathbb{R}^n)$ , we denote by  $V_X(\cdot)$  its variance, that is  $V_X(t) \triangleq \mathbb{E}[(X(t) - \mathbb{E}[X(t)])^T (X(t) - \mathbb{E}[X(t)])] \in \mathbb{R}$ . Finally, we recall that if  $f_1$  and  $f_2$  are two deterministic functions in  $L^2([0, T], \mathbb{R})$ , Itô formula yields

$$\mathbb{E} \left[ \left( \int_0^t f_1(s) dW_s \right) \left( \int_0^t f_2(s) dW_s \right) \right] = \int_0^t f_1(s) f_2(s) ds. \quad (1)$$

### B. Control system

In this paper, we consider coupled PDE+SDE systems of the form

$$\begin{cases} dX(t) = (AX(t) + Bv(t, 0))dt + \sigma(t)dW_t \\ u_t(t, x) + \lambda u_x(t, x) = \eta^+(x)v(t, x) \\ v_t(t, x) - \mu v_x(t, x) = \eta^-(x)u(t, x) \\ u(t, 0) = qv(t, 0) + MX(t) \\ v(t, 1) = \rho u(t, 1) + V_{in}(t) \\ X(0) = X_0, u(0, x) = u_0(x), v(0, x) = v_0(x), \end{cases} \quad (2)$$

in the time-space domain  $[0, +\infty) \times [0, 1]$ . The state of the system is  $(X(\cdot), u(\cdot, x), v(\cdot, x)) \in \mathbb{R}^n \times (L^2([0, 1]))^2$ . The velocities  $\mu > 0$  and  $\lambda > 0$  are assumed to be constant, whereas the coupling terms  $\eta^+$  and  $\eta^-$  are continuous functions. The boundary coupling terms  $\rho$  and  $q$  verify  $|\rho q| < 1$  to avoid an infinite number of unstable poles, which is necessary to guarantee the existence of robustness margins for the closed-loop system [16]. We also assume  $q \neq 0$ . The matrices  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times 1}$ ,  $M \in \mathbb{R}^{1 \times n}$  are constant, and  $\sigma$  is a deterministic diffusion in  $L^\infty([0, T], \mathbb{R}^n)$ . Finally,  $(X_0, u_0, v_0) \in \mathbb{R}^n \times (L^2([0, 1]))^2$  and the control input  $V_{in}$  takes values in  $\mathbb{R}$ . The class of system (2) naturally appear when modeling a heat-exchanger connected to a temperature system subject to random perturbations (e.g., the temperature of a building disrupted by a random outside temperature or sunlight). We defer the proof of the closed-loop well-posedness of system (2) to Section III.

### C. Objective and approach overview

The system naturally contains multiple feedback loops or couplings that can potentially introduce instabilities. The objective of the paper is twofold:

- First, we aim to design a feedback control law to stabilize the mean of the system, driving it to zero while ensuring the variance remains bounded.
- Then, we provide a more noise-robust controller for the SDE state. This can be typically obtained through variance minimization [12].

To achieve the aforementioned design, we propose the following methodology:

- 1) First, we use an invertible backstepping transformation to map the coupled SDE-PDE system into a cascade SPDE-SDE system.
- 2) Secondly, using the method of characteristics, we prove the well-posedness of this latter system (and consequently, of the original system). We also show that the state  $X$  can be expressed as the solution of an SDE with input delay and random drifts.
- 3) Then, using the Artstein transform, we design an appropriate stabilizing control law.
- 4) Finally, we tackle the minimization of the variance using classical tools from stochastic LQ control.

In our approach, the control law  $V_{in}$  will write  $V_{in} = V_{BS} + V_{eff}$ . Here, the component  $V_{BS}$  corresponds to a backstepping controller that would stabilize the PDE subsystem in the absence of the SDE. The component  $V_{eff}$  corresponds instead to the controller that will be leveraged to solve the variance minimization problem.

## III. BACKSTEPPING TRANSFORMATION FOR SYSTEM SIMPLIFICATION.

In this section, we show that the original system (2) can be rewritten as a delayed SDE with random drifts.

### A. Backstepping transformation

Taking inspiration from [16], we consider the following backstepping change of variables:

$$\begin{pmatrix} \alpha(t, x) \\ \beta(t, x) \end{pmatrix} = \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} + \int_0^x K(x, y) \begin{pmatrix} u(t, y) \\ v(t, y) \end{pmatrix} dy + \gamma(x)X(t), \quad (3)$$

where,  $K(x, y) = \begin{pmatrix} K_{uu}(x, y) & K_{uv}(x, y) \\ K_{vu}(x, y) & K_{vv}(x, y) \end{pmatrix}$  and  $\gamma(x) = (\gamma_\alpha(x) \quad \gamma_\beta(x))^T$ . The kernels  $K_{uu}, K_{uv}, K_{vu}$ , and  $K_{vv}$  are continuous functions defined on the triangle domain  $\mathcal{T} \triangleq \{(x, y), x \in [0, 1], y \in [0, x]\}$ , whereas  $\gamma_\alpha$  and  $\gamma_\beta$  are in  $C^1([0, 1])$ . On their respective domains of definition,  $K$  and  $\gamma$  verify the following set of *kernel equations*:

$$\begin{cases} \Lambda K_x(x, y) + K_y(x, y)\Lambda + K(x, y)\eta(y) = 0, \\ \Lambda \gamma'(x) + \gamma(x)\bar{A} + \lambda K(x, 0)\bar{M} = 0, \\ \Lambda K(x, x) - K(x, x)\Lambda = -\eta(x), \quad \gamma_\alpha(0) = -M, \\ K(x, 0)(\lambda q \quad -\mu)^T + \gamma(x)B = 0, \quad \gamma_\beta(0) = 0, \end{cases} \quad (4)$$

where  $\Lambda = \text{diag}(\lambda, -\mu)$ ,  $\eta = \begin{pmatrix} 0 & \eta^+ \\ \eta^- & 0 \end{pmatrix}$ ,  $\bar{A} = \text{diag}(A, A)$ ,  $B = \text{diag}(B, B)$  and  $\bar{M} = \text{diag}(M, 0_{1 \times n})$ . The set of equations (21) is well-posed and admits a unique solution  $(K_{uu}, K_{uv}, K_{vu}, K_{vv}, \gamma_\alpha, \gamma_\beta)$  in  $(C^0(\mathcal{T}))^4 \times (C^1([0, 1]))^2$ , see, e.g., [16]. Differentiating equation (3) with respect to time and space (in the sense of distributions), and integrating by parts, we can show that the target states

$(\alpha, \beta, X)$  are solutions to the following *target system*:

$$\begin{cases} dX(t) = (AX(t) + B\beta(t, 0))dt + \sigma(t)dW_t, \\ d\alpha(t, x) + \lambda_{\alpha_x}(t, x)dt = \gamma_{\alpha}(x)\sigma(t)dW_t, \\ d\beta(t, x) - \mu\beta_x(t, x)dt = \gamma_{\beta}(x)\sigma(t)dW_t, \\ \alpha(t, 0) = Q\beta(t, 0), \quad \beta(t, 1) = V_{eff}(t), \end{cases} \quad (5)$$

where  $V_{BS}$  is defined as

$$\begin{aligned} V_{BS}(t) &= -\rho u(t, 1) - \gamma_{\beta}(1)X(t) \\ &- \int_0^1 K_{vu}(1, y)u(t, y)dy - \int_0^1 K_{vv}(1, y)v(t, y)dy. \end{aligned} \quad (6)$$

Unlike the existing results in the literature on stabilization of PDE+ODE systems [5], [16], we obtain additional terms in the target system. These are due to the additive noise in the SDE that is mapped back into the PDE, through the boundary coupling. Consequently, the states  $\alpha$  and  $\beta$  are solutions of an SPDE (whose solution should be understood in the weak sense [17]). As it will be seen in the following, this noise cannot be entirely compensated by the control input and will be mapped back into the SDE as additional random drift terms. Note that, in the absence of the SDE sub-system, the PDE would be finite-time stable. However, due to the cancellation of the term  $\rho u(t, 1)$ , the feedback operator is not strictly proper. This flaw may cause delay/robustness issues as emphasized in [16]. Partial cancellation of the reflection and filtering methods have been respectively proposed in [16] and [6] to guarantee the existence of robustness margins (with respect to delays and uncertainties). However, these techniques do not directly adjust to the current stochastic configuration. In particular, we should be able to show the robust mean square stability but quantifying the effects of these techniques on the variance seems challenging. Such a robustness analysis is out of the scope of this paper and left for future works.

### B. Well-posedness of the closed loop

*Lemma 1:* If  $V_{eff}$  is in  $C_{\mathcal{F}}^2([0, T])$ , then the closed-loop target system (5) is well-posed, that is the processes  $t \mapsto \alpha(t, \cdot)$  and  $t \mapsto \beta(t, \cdot)$  are in  $C_{\mathcal{F}}^2([0, T]; L^2(0, 1))$ , and the SDE state  $X$  is in  $C_{\mathcal{F}}^2([0, T]; \mathbb{R}^n)$ .

*Proof:* Using the method of characteristics, we have for all  $t > \frac{1}{\mu}$  and all  $x \in [0, 1]$ ,  $\beta(t, x) = V_{eff}\left(t - \frac{1-x}{\mu}\right) + \int_{t-\frac{1-x}{\mu}}^t \gamma_{\beta}(x + \mu(t-s))\sigma(s)dW_s$ . Since  $\gamma_{\beta}$  is continuously differentiable, we obtain [17, Theorem 6.10] that the stochastic convolution term  $t \mapsto \int_{t-\frac{1-x}{\mu}}^t \gamma_{\beta}(\cdot + \mu(t-s))\sigma(s)dW_s$ , is sample path continuous in  $L^2(0, 1)$ . Since  $V_{eff}$  is in  $C_{\mathcal{F}}^2([0, T])$ ,  $t \mapsto \beta(t, \cdot)$  is also sample path continuous in  $L^2(0, 1)$ . The regularity of  $\alpha$  can be proven in the same way. The solution of the SDE system is sample path continuous if the input  $t \mapsto \beta(t, 0)$  is in  $L_{\mathcal{F}}^2$  [9]. Using the Burkholder-Davis-Gundy inequality [18] we have that the stochastic process  $t \mapsto \int_{t-\frac{1}{\mu}}^t \gamma_{\beta}(\mu(t-s))\sigma(s)dW_s$  is in  $L_{\mathcal{F}}^2$ , which in turn means that  $\beta(t, 0)$  is in  $L_{\mathcal{F}}^2([0, T])$ , which concludes the proof. ■

Since the backstepping transformation is a Volterra transformation, it is boundedly invertible [1], which implies the well-posedness of the original system (2).

### C. A delayed SDE

Using the method of characteristics, we have that  $X(t)$  is the solution of the following input-delayed stochastic differential equation with random coefficients

$$\begin{cases} dX(t) = (AX(t) + BV_{eff}(t-h) + r(t))dt + \sigma(t)dW_t \\ X(0) = X_0, \quad V_{eff}(s) = \beta(0, 1 + \mu s) \quad \forall s \in [-h, 0] \end{cases} \quad (7)$$

where  $h \triangleq \frac{1}{\mu}$  and  $r(t) \triangleq B \int_{t-h}^t \gamma_{\beta}(\mu(t-s))\sigma(s)dW_s$ .

Therefore, we now only focus on controlling the delayed SDE (7) and minimizing the variance of the state. We follow a methodology similar to the one outlined in [15]. However, the inclusion of the extra random drift term  $r(t)$  requires to modify our approach to minimize the variance, as this term cannot be predicted.

## IV. STUDY OF THE DELAYED SDE : A STABILITY RESULT AND A MINIMAL BOUND OF THE VARIANCE

In this section, we extend the Artstein transform [2] to our stochastic setting to provide a predictor  $Y$  of the state  $X$  that follows a non-delayed SDE. From now on, we denote by  $T > h$  a possibly infinite final time.

### A. Artstein transform

Let  $X$  be the process solving equation (7). The Artstein transform of  $X$  is the process  $Y(t)$ , which is adapted to  $\mathcal{F}$ , that is defined by

$$Y(t) \triangleq X(t) + \int_{t-h}^t e^{A(t-s-h)} BV_{eff}(s)ds. \quad (8)$$

One readily verifies that  $Y$  satisfies the following non-delayed stochastic equation

$$\begin{cases} dY(t) = (A(t)Y(t) + \bar{B}V_{eff}(t) + r(t))dt + \sigma(t)dW_t \\ Y(0) = 0, \end{cases} \quad (9)$$

where  $\bar{B} \triangleq e^{-Ah}B$ . We can now apply known results from non-delayed stochastic control to steer (9) as desired. However, it is for now unclear how to compute the covariance of  $X$ , which we aim to estimate once the covariance of  $Y$  is available. We have the following lemma

*Lemma 2:* Let  $X$  be the process solving equation (7), and  $Y$  its Artstein transform defined in (8). Then, for all  $t \in [h, T]$ , we have

$$\begin{aligned} X(t) &= e^{Ah}Y(t-h) + \int_{t-h}^t e^{A(t-s)}r(s)ds \\ &\quad + \int_{t-h}^t e^{A(t-s)}\sigma(s)dW_s. \end{aligned} \quad (10)$$

*Proof:* Using the analytic formula for linear SDEs [9], we obtain  $X(t) = e^{Ah}X(t-h) + \int_{t-h}^t e^{A(t-s)}r(s)ds + \int_{t-h}^t e^{A(t-s)}BV_{eff}(s-h)ds + \int_{t-h}^t e^{A(t-s)}\sigma(s)dW_s$ . By expressing  $X(t-h)$  in terms of  $Y(t-h)$ , a change of variable in the regular integral of  $V_{eff}$  yields  $X(t) = e^{Ah}Y(t-h) +$

$\int_{t-h}^t e^{A(t-s)}r(s)ds - e^{Ah} \int_{t-2h}^{t-h} e^{A(t-s-2h)}BV_{eff}(s)ds + \int_{t-2h}^{t-h} e^{A(t-s-h)}BV_{eff}(s)ds + \int_{t-h}^t e^{A(t-s)}\sigma(s)dW_s$ , which gives the desired result. ■

Equation (10) states that  $X(t)$  can only be controlled through  $Y(t-h)$ , in that the additional noise term  $\int_{t-h}^t e^{A(t-s)}\sigma(s)dW_s$  cannot be controlled. The other noise term can only be partially mitigated by  $Y(t-h)$ .

### B. Stabilization of the delayed SDE through the Artstein transform.

If the pair  $(A, B)$  is controllable, Artstein's predictor enables the design of a feedback controller to stabilize the delayed SDE in the sense that it ensures exponential convergence to zero of its mean, while keeping the covariance bounded.

*Theorem 3:* Assume that the pair  $(A, B)$  is controllable. Define the feedback controller  $V_{eff}(t) = -KY(t)$ , where  $Y$  is given by equation (8), and  $K$  is such that  $H \triangleq A - BK$  is Hurwitz. Then, the control law  $V_{in}(t) = V_{BS}(t) + V_{eff}(t)$  drives the means of the states to zero while keeping their variances bounded. That is, there exist  $C > 0$ , (dependent on the parameters of the system and  $K$ ) and  $\nu > 0$  (dependent on the eigenvalues of  $A - BK$ ), such that, for any initial conditions  $(X_0, u_0, v_0)$ , for all  $x$  in  $[0, 1]$  and  $t > 0$ :

$$\|\mathbb{E}[X(t)]\| + \|\mathbb{E}[u(x, t)]\| + \|\mathbb{E}[v(x, t)]\| \leq Ce^{-\nu t} \times (\|\mathbb{E}[X_0]\| + \|\mathbb{E}[u_0]\| + \|\mathbb{E}[v_0]\|),$$

$$\mathbb{E}[\|X(t)\|^2] + \mathbb{E}[\|u(x, t)\|^2] + \mathbb{E}[\|v(x, t)\|^2] \leq C$$

The proof of the Theorem can be found in Appendix IX.

Despite its theoretical advantages, the Artstein's transform encounters practical limitations. Discretizing the integral during its computation can potentially render the closed-loop system unstable, as discussed in [19]. To address this issue, solutions such as filtering have been proposed to implement the controller safely [6], [19]. However, exploring these solutions further is beyond the scope of our study. Interestingly, setting  $\gamma_\beta(0) = K$  in the backstepping transformation, we obtain  $dX(t) = [(A - BK)X(t) + r(t)]dt + \sigma(t)dW_t$ , which implies the stability if  $(A - BK)$  is Hurwitz.

### C. Minimum variance bound of the SDE

We leverage equation (10) to write  $X(t)$  as the sum of a  $\mathcal{F}_{t-h}$ -measurable process and a stochastic integral depending only on values of  $dW_s$  for  $s \in [t-h, t]$ . The independence of both terms yields a direct relation between the variance of  $X(t)$  and of  $Y(t-h)$ , which is later used in the next section.

*Lemma 4:* Let  $X$  be the process solving equation (7), and let  $Y$  be its Artstein transform (8). For all  $t \in [h, T]$ , we have

$$V_X(t) = V_{\min}(t) + \mathbb{E} \left[ [Y(t-h) + G(t-h)]^T e^{A^T h} \times e^{Ah} [Y(t-h) + G(t-h)] \right], \quad (11)$$

where

$$V_{\min}(t) \triangleq \int_{t-h}^t \sigma(s)^T \left[ e^{A(t-s)} + N(t-s) \right]^T \times \left[ e^{A(t-s)} + N(t-s) \right] \sigma(s) ds,$$

and  $G(t) \triangleq \int_{t-h}^t \left[ \int_t^{s+h} e^{A(t-\tau)} \gamma_\beta(\mu(\tau-s)) d\tau \right] \sigma(s) dW_s$  and  $N(u) \triangleq \int_{-u}^0 e^{-A\tau} \gamma_\beta(\mu(\tau+u)) d\tau$ .

*Proof:* Fix  $t \in [h, T]$ . We start by separating the integral of  $r(s)$  in (10) into a  $\mathcal{F}_{t-h}$ -measurable process and a non-controllable process:

$$\int_{t-h}^t e^{A(t-\tau)} r(\tau) d\tau = \int_{t-h}^t \int_{\tau-h}^\tau e^{A(t-\tau)} \gamma_\beta(\mu(\tau-s)) \sigma(s) dW_s d\tau,$$

Since the integrand in the double integral is bounded, we can apply the stochastic Fubini theorem [20, Chapter 4, Theorem 45] to obtain

$$\int_{t-h}^t e^{A(t-\tau)} r(\tau) d\tau = \int_{t-2h}^{t-h} \left( \int_{t-h}^{s-h} e^{A(t-\tau)} \gamma_\beta(\mu(\tau-s)) d\tau \right) \times \sigma(s) dW_s + \int_{t-h}^t \left( \int_s^t e^{A(t-\tau)} \gamma_\beta(\mu(\tau-s)) d\tau \right) \sigma(s) dW_s$$

$$\int_{t-h}^t e^{A(t-\tau)} r(\tau) d\tau = e^{Ah} G(t-h) + \int_{t-h}^t N(t-s) \sigma(s) dW_s.$$

We can then rewrite the state  $X$  as the sum of a  $\mathcal{F}_{t-h}$  adapted process and a noise:

$$X(t) = e^{Ah} [Y(t-h) + G(t-h)] + \int_{t-h}^t \left[ e^{A(t-s)} + N(t-s) \right] \sigma(s) dW_s. \quad (12)$$

We have that  $Y(t-h)$  and  $G(t-h)$  are  $\sigma(W_s : 0 \leq s \leq t-h)$ -measurable, and that for  $r \in [0, t-h]$ ,  $W_r$  is independent from the stochastic integral in equation (12) (see details in [15, Lemma 3]). Therefore,  $Y(t-h) + G(t-h)$  is independent of the stochastic integral. Consequently, the variance of the sum of the integrals is equal to the sum of the variance of each integral. Using equation (1) for  $f_1(s) = f_2(s) = (e^{A(t-s)} + N(t-s))\sigma(s)$ , we obtain that the variance of the stochastic integral corresponds  $V_{\min}$ . This concludes the proof. ■

Similarly to [15], we note that the variance of  $X$  is bounded by a minimal variance caused by the delay induced by the PDE.

## V. MINIMIZATION OF THE VARIANCE THROUGH LQ CONTROL

In the previous section, we designed a feedback controller that stabilized in mean the interconnected PDE-SDE system while guaranteeing a bounded variance. However, such a control law may not guarantee that the variance remains small. In what follows, we provide an efficiently implementable control strategy that keeps the state covariance of the delayed SDE consistently low along the trajectory. For this, we propose to compute controllers via techniques from Linear Quadratic (LQ)

control [21]. In this section, we suppose the final time  $T$  is finite. The goal consists of minimizing the quadratic functional cost  $J_R$ , defined as:

$$J_R(V_{eff}, X) \triangleq \mathbb{E} \left[ \int_h^T X(t)^T Q(t) X(t) dt \right] + \mathbb{E} \left[ \int_0^{T-h} V_{eff}(t)^T R(t) V_{eff}(t) dt \right], \quad (13)$$

where  $Q \in L^\infty([0, T], \mathcal{S}_n^+)$  and  $R \in L^\infty([0, T], \mathcal{S}_n^{++})$ . This functional cost penalizes the weighted variance along the trajectory, as well as the control effort. These matrices are chosen based on the desired trade-off between state penalization and control effort penalization throughout time. Our problem thus states:

$$\min_{V_{eff} \in \mathcal{U}} J_R(V_{eff}, X), \quad X \text{ solves SDE (7)}. \quad (14)$$

Using equation (10), we replace the process  $X$  with the process  $Y+G$  in  $J_R$ . The benefit of this transformation is that optimal control techniques for non-delayed systems may be leveraged, e.g., [21].

#### A. Equivalent form of the system.

In this section, we use a change of variables to write the cost as a linear quadratic cost in terms of a non-delayed SDE. Let us introduce the state  $\bar{Y}$  defined by

$$\bar{Y}(t) \triangleq Y(t) + G(t), \quad (15)$$

where  $Y$  has been defined in equation (8). For  $t > h$ , let us denote  $V_{min,Q}(t)$  as

$$V_{min,Q}(t) \triangleq \int_{t-h}^t \sigma(s)^T \left[ e^{A(t-s)} + N(t-s) \right]^T Q(t) \left[ e^{A(t-s)} + N(t-s) \right] \sigma(s) ds.$$

*Lemma 5:* The cost function  $J_R$  rewrites as

$$J_R(V_{eff}, X) = \mathbb{E} \left[ \int_0^{T-h} \bar{Y}(t)^T \bar{Q}(t) \bar{Y}(t) dt \right] + \mathbb{E} \left[ \int_0^{T-h} V_{eff}(t)^T R(t) V_{eff}(t) dt \right] + \int_h^T V_{min,Q}(t) dt, \quad (16)$$

where  $\bar{Q}(t) \triangleq e^{A^T h} Q(t+h) e^{Ah}$  is symmetric positive.

*Proof:* We can rewrite the term  $\mathbb{E} [X(t)^T Q(t) X(t)]$  in equation (13) using  $\bar{Y}$ . Adjusting the computations proposed in Lemma 4, we directly obtain

$$\mathbb{E} [X(t)^T Q(t) X(t)] = \mathbb{E} [\bar{Y}(t-h)^T \bar{Q}(t-h) \bar{Y}(t-h)] + V_{min,Q}(t).$$

This concludes the proof.  $\blacksquare$

*Lemma 6:* The process  $\bar{Y}$  satisfies the following SDE

$$\begin{cases} d\bar{Y}(t) &= (A\bar{Y}(t) + \bar{B}V_{eff}(t) + \bar{r}(t)) dt \\ &+ (1 + g(0))\sigma(t)dW_t, \\ \bar{Y}(0) &= Y_0, \end{cases} \quad (17)$$

with  $\bar{r}(t) = \int_{t-h}^t \Gamma(t-s)\sigma(s)dW_s$ ,  $\Gamma(u) = B\gamma_\beta(\mu u) + g'(u) - Ag(u)$  and  $g(u) \triangleq \int_0^{h-u} e^{-A\tau}\gamma_\beta(\mu(\tau+u))d\tau$ .

*Proof:* Let us remark that the function  $g$  is continuously differentiable and that for all  $t > h$ ,  $G(t) = \int_{t-h}^t g(t-s)\sigma(s)dW_s$ . Ito's formula yields

$$dG(t) = g(0)\sigma(t)dW_t - g(h)\sigma(t-h)dW_{t-h} + \left( \int_{t-h}^t g'(t-s)\sigma(s)dW_s \right) dt$$

Since  $g(h) = 0$ , we obtain

$$d\bar{Y}(t) = dY(t) + dG(t) = (AY(t) + \bar{B}V_{eff}(t) + r(t) + AG(t) - AG(t)) dt + \left( \int_{t-h}^t g'(t-s)\sigma(s)dW_s \right) dt + (1 + g(0))\sigma(t)dW_t$$

which finally yields equation (17).  $\blacksquare$

We can now leverage the results from [21] to address the LQ problem expressed in terms of  $\bar{Y}$ . This approach yields an optimal control solution minimizing (13) in the form of a feedback in  $\bar{Y}$ .

#### B. Solving the optimal control problem.

When addressing a Linear Quadratic (LQ) stochastic control problem, the optimal control is characterized by a solution to a Riccati equation. Since, in our case, the SDE has a random coefficient ( $\bar{r}$ ), the corresponding Riccati equation is a Backward SDE (BSDE). Solving BSDEs is generally complex and requires sophisticated numerical algorithms. However, in our case, we are able to analytically compute an explicit solution for the considered BSDE, allowing a relatively straightforward computation of the optimal control.

*Proposition 7:* The optimal control  $V_{eff}^*$  minimizing (13) is given by

$$V_{eff}^*(t) = -R^{-1}\bar{B}^T (P(t)\bar{Y}(t) + \phi(t)), \quad (18)$$

where  $P$  is a symmetric matrix solution to the deterministic Riccati ODE

$$\dot{P}(t) = -A^T P(t) - P(t)A - \bar{Q}(t) + P(t)\bar{B}R^{-1}\bar{B}^T P(t),$$

with  $P(T-h) = 0$ , and where  $\phi$  is given by

$$\phi(t) = \int_{t-h}^t \left[ \int_t^{s+h} \Phi_\Pi(t, \tau) P(\tau) \Gamma(\tau-s) d\tau \right] \sigma(s) dW_s,$$

with  $\Phi_\Pi(\cdot, \cdot)$  the fundamental matrix associated to the matrix function  $\Pi(\cdot)$  defined as  $\Pi(t) \triangleq P(t)\bar{B}R^{-1}\bar{B}^T - A$ .

*Proof:* Since  $V_{min,Q}(t)$  is independent from the control variable, Lemma 5 shows that the minimization problem (14) is equivalent to the following:

$$\min_{V_{eff} \in \mathcal{U}} \bar{J}_R(V_{eff}, \bar{Y}), \quad \bar{Y} \text{ solves SDE (17)}. \quad (19)$$

where  $\bar{J}_R \triangleq \mathbb{E} \left[ \int_0^{T-h} \bar{Y}(t)^T \bar{Q}(t) \bar{Y}(t) dt \right] + \mathbb{E} \left[ \int_0^{T-h} V_{eff}(t)^T R(t) V_{eff}(t) dt \right]$ . To solve this LQ

problem, we may apply the results in [21] to obtain that the optimal control  $V_{eff}^*$  is given by (18), with  $P$  verifying the BSDE

$$\begin{cases} dP(t) = [-A^T P(t) - P(t)A - \bar{Q}(t) \\ + P(t)\bar{B}R^{-1}\bar{B}^T P(t)] dt + \Lambda(t)dW_t, \end{cases}$$

with  $P(T-h) = 0$ , and with  $\phi$  verifying the BSDE

$$\begin{cases} d\phi(t) = [\Pi(t)\phi(t) - P(t)\bar{r}(t)] dt + \xi(t)dW_t, \end{cases}$$

with  $\phi(T-h) = 0$ . The BSDE verified by  $P$  is only composed of deterministic coefficients, and therefore  $\Lambda = 0$ . As for the BSDE verified by  $\phi$ , it is linear affine and can, therefore, be solved explicitly. Using the method described in [11, Chapter 10.2], we obtain

$$\phi(t) = \mathbb{E} \left[ \int_t^{T-h} \Phi_{\Pi}(t, \tau) P(\tau) \bar{r}(\tau) d\tau \middle| \mathcal{F}_t \right]. \quad (20)$$

by writing  $\bar{r}$  as a stochastic integral, the stochastic Fubini theorem yields

$$\phi(t) = \mathbb{E} \left[ \int_{t-h}^{T-h} \left( \int_{\min(t, s)}^{\max(T-h, s+h)} \Psi(t, \tau, s) d\tau \right) \sigma(s) dW_s \middle| \mathcal{F}_t \right]$$

where  $\Psi(t, \tau, s) \triangleq \Phi_{\Pi}(t, \tau) P(\tau) \Gamma(\tau - s)$ . We conclude the proof using the classical result [18] that for any function  $f \in L^2_{\mathcal{F}}(0, T)$  and any  $0 \leq t_1 \leq t_2 \leq T$ ,  $\mathbb{E} \left[ \int_0^{t_2} f(s) dW_s \middle| \mathcal{F}_{t_1} \right] = \int_0^{t_1} f(s) dW_s$ . ■

In simulations, computing the stochastic integral is straightforward as we have access to the random realizations of the Brownian motion that is sampled. However, in practice, obtaining the Brownian motion trajectory may not always be feasible. We can address this challenge through Monte Carlo methods. A common approach in stochastic control is to precompute numerous optimal trajectories through simulation and store them. When employing the algorithm online, users can then reference these pre-generated optimal trajectories.

## VI. SIMULATIONS

In this section, we present simulation results applying our feedback stabilization method presented in Theorem 3 to an unstable one-dimensional SDE. This unstable system is academic, the parameters are chosen to demonstrate the efficiency of our method. The SDE parameters are chosen as follows :  $A = 0.6, B = 1, \sigma(t) = 0.6$  and  $X_0 = 2$ . The PDE parameters are :  $\mu = 2, \lambda = 1, \eta^+ = \eta^- = 0.3, M = \rho = 1$  and  $q = 0.25$ .

Results in Figure 1 demonstrate that, even though the delay caused by the PDE induces a short spike at the start, the controller effectively stabilizes the state while maintaining bounded deviation. Moreover, a higher gain in the controller induces a lower variance of the state and therefore a safer control.

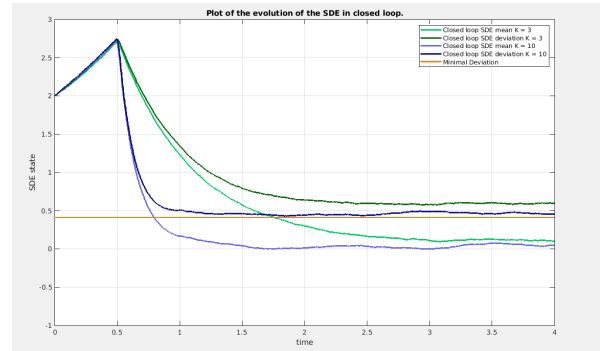


Fig. 1. Mean and deviation (square root of the variance) of the SDE state are depicted two systems stabilized with different gains. The values were computed through a Monte-Carlo approach.

## VII. CONCLUSION

In this paper, we have introduced two approaches for the control of bidirectionally coupled PDE+SDE systems. These two approaches utilize a combination of classical PDE+ODE tools and stochastic control techniques. One approach achieves stabilization in mean with bounded variance of the states, while the other minimizes the SDE variance to ensure additional robustness against noise. We plan to explore further avenues for research, including the generalization of the PDE to a system of  $n \times m$  linear first-order equations, as well as the extension of the SDE framework to incorporate multiplicative noise. We also plan to investigate the potential benefits of employing alternate backstepping techniques used successfully in PDE+ODE systems, such as the multi-step approach [22].

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## VIII. EQUATION DE NOYAU (TEMPORAIRE)

$$\begin{cases} \lambda(K_{uu})_x(x, y) + (K_{uu})_y(x, y)\lambda + K_{uv}(x, y)\eta^-(y) = 0 \\ \lambda(K_{uv})_x(x, y) - (K_{uv})_y(x, y)\mu + K_{uu}(x, y)\eta^+(y) = 0 \\ \lambda K_{uv}(x, x) + K_{uv}(x, x)\mu + \eta^+(x) = 0 \\ K_{uu}(x, 0)\lambda q - K_{uv}(x, 0)\mu + \gamma_\alpha(x)B = 0 \\ \lambda\gamma'_\alpha(x) + \gamma_\alpha(x)A + K_{uu}(x, 0)\lambda M = 0 \\ \gamma_\alpha(0) = -M, \end{cases} \quad (21)$$

as well as

$$\begin{cases} -\mu(K_{vu})_x(x, y) + (K_{vu})_y(x, y)\lambda + K_{vv}(x, y)\eta^-(y) = 0 \\ -\mu(K_{vv})_x(x, y) - (K_{vv})_y(x, y)\mu + K_{vu}(x, y)\eta^+(y) = 0 \\ -\mu K_{vv}(x, x) - K_{vv}(x, x)\lambda + \eta^-(x) = 0 \\ K_{vu}(x, 0)\lambda q - K_{vv}(x, 0)\mu + \gamma_\beta(x)B = 0 \\ -\mu\gamma'_\beta(x) + \gamma_\beta(x)A + K_{vu}(x, 0)\lambda M = 0 \\ \gamma_\beta(0) = 0. \end{cases} \quad (22)$$

## IX. APPENDIX - PROOF OF THEOREM 3

In what follows, if  $X$  is a stochastic process, we denote its expectation by  $m_X(t) \triangleq \mathbb{E}[X(t)]$ . If  $u$  is a function of time and space, we denote its expectation by  $m_u(t, x) = \mathbb{E}[u(x, t)]$ . We denote by  $C$  a possibly overloaded constant dependent on the parameters of the system.

*Proof:* Let  $(u, v, X)$  be the solution of the dynamics (2), and  $(\alpha, \beta)$  given by the backstepping transformation (3), and  $Y$  the Artstein transform of  $X$ . We consider a

controller  $V_{in} = V_{BS} + V_{eff}$  with the backstepping controller  $V_{BS}$  given by (6) and the effective SDE controller  $V_{eff}$  given by  $V_{eff}(t) = -KY(t)$ . The gain  $K$  is chosen such that  $H \triangleq A - \bar{B}K$  is Hurwitz, which is possible since if  $(A, B)$  is controllable, then  $(A, \bar{B})$  is controllable (see [15, Theorem 6]).

**First part of the proof: mean exponential stabilization.** With this controller, for  $t > 0$ ,  $Y$  follows the dynamic

$$dY(t) = (HY(t) + r(t))dt + \sigma(t)dW_t.$$

where  $r(t)$  is given by the stochastic integral as in equation (7). We note that it is therefore always of mean 0. The mean  $m_Y(t)$  is therefore given by

$$m_Y(t) = e^{Ht}m_Y(0).$$

At the initial time, the process  $Y$  is given by

$$\begin{aligned} Y(0) &= X(0) + \int_{-h}^0 e^{A(-s-h)} B V_{eff}(s) ds \\ &= X_0 + \int_{-h}^0 e^{A(-s-h)} B \beta(0, 1 + \mu s) ds. \end{aligned}$$

Through the definition of the backstepping transformation, since the kernels  $K$  are bounded, we have that

$$\|\beta(0, x)\| \leq \|v(0, x)\| + C(\|X_0\| + \|u_0\|_{L^2} + \|v_0\|_{L^2})$$

Therefore

$$\|m_Y(t)\| \leq C e^{-\nu t} (\|X_0\| + \|u_0\|_{L^2} + \|v_0\|_{L^2})$$

Where  $\nu$  is the highest eigenvalue of  $H$ . By taking the expectation of (10), we obtain

$$m_X(t) = e^{Ah} m_Y(t-h),$$

and by taking the expectation of the explicit value of  $\beta(t, x)$  (as in the proof of Lemma 1), we obtain

$$\begin{aligned} m_\beta(t, x) &= \mathbb{E} \left[ V_{eff} \left( t - \frac{1-x}{\mu} \right) \right. \\ &\quad \left. + \int_{t-\frac{1-x}{\mu}}^t \gamma_\beta(x + \mu(t-s)) \sigma(s) dW_s \right] \\ &= \mathbb{E} \left[ V_{eff} \left( t - \frac{1-x}{\mu} \right) \right] = -K m_Y \left( t - \frac{1-x}{\mu} \right). \end{aligned}$$

We can obtain a similar equation for  $\alpha$ . Therefore, the expectations of  $Y(t)$ ,  $X(t)$ ,  $\alpha(t, x)$  and  $\beta(t, x)$  are exponentially decreasing. We conclude by bounding in the exact same way the expectation of  $u$  and  $v$  through the linear inverse backstepping transformation

$$\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = \begin{pmatrix} \alpha(t, x) \\ \beta(t, x) \end{pmatrix} + \int_0^x L(x, y) \begin{pmatrix} \alpha(t, y) \\ \beta(t, y) \end{pmatrix} dy + \gamma_L(x)X(t),$$

where  $L$  is a bounded kernel.



**Second part of the proof: bound on the variance.**

We need first to bound the variance of  $Y(t)$ . We do this through its explicit formula:

$$Y(t) = e^{Ht}Y(0) + \int_0^t e^{H(t-s)}\sigma(s)dW_s + \int_0^t e^{H(t-s)}r(s)ds$$

therefore, by expanding the product and using Young inequality on the crossed terms we obtain

$$\begin{aligned} \mathbb{E}[Y(t)^T Y(t)] &\leq C \left( \mathbb{E} \left[ \|e^{Ht}Y(0)\|^2 \right] + \int_0^t \left\| e^{H(t-s)}\sigma(s) \right\|^2 ds \right. \\ &\quad \left. + \int_0^t e^{H^T(t-s)} \mathbb{E}[r(s)^T r(s)] e^{H(t-s)} ds \right). \end{aligned}$$

Thanks to the Itô formula for the quadratic variation of a stochastic integral, the expectation term in  $r$  can be bounded as follows

$$\mathbb{E}[r(s)^T r(s)] \leq h \|\sigma\|_\infty^2 \|\gamma\|_\infty^2 \leq C.$$

Since  $H$  is Hurwitz and all the other terms in the integrals are bounded, we have that

$$\mathbb{E}[Y(t)^T Y(t)] \leq C.$$

with  $C$  depending on the norms of the parameters of the system. From this bound, we can deduce in the same way that  $\mathbb{E}[X(t)^T X(t)] \leq C$  by using (10). The bound on the variance of  $\alpha(t, x)$  and  $\beta(t, x)$  is deduced from their explicit expression (always using Young inequality for the crossed terms, then the Ito formula for the stochastic integrals)

$$\begin{aligned} \mathbb{E}[\beta(t, x)^T \beta(t, x)] &\leq Ch \|\sigma\|_\infty^2 \|\gamma\|_\infty^2 \\ &\quad + C \|K\|^2 \mathbb{E} \left[ Y \left( t - \frac{1-x}{\mu} \right)^T Y \left( t - \frac{1-x}{\mu} \right) \right] \end{aligned}$$

Finally, the bound on  $u(t, x)$  and  $v(t, x)$  can be deduced from the inverse backstepping transformation and the fact that the kernels are bounded.

■