Dimensions of harmonic measures on non-autonomous Cantor sets

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Abstract

We consider Non Autonomous Conformal Iterative Function Systems (NACIFS) and their limit set. Our main concern is harmonic measure and its dimensions : Hausdorff and Packing. We prove that this two dimensions are continuous under perturbations and that they verify Bowen's and Manning's type formulas. In order to do so we prove general results about measures, and more generally about positive functionals, defined on a symbolic space, developing tools from thermodynamical formalism in a non-autonomous setting.

1 Introduction and statement of results.

Given a compact set in the plane defined dynamically by iteration of conformal contractions, this article focuses on the variations of Hausdorff dimension and packing dimension of the harmonic measure under perturbations of the conformal contractions. In a natural framework, which is precisely stated below, we demonstrate that these functions are continuous.

Let $\mathbb{K} \subset \mathbb{C}$ be a compact set and ω the harmonic measure of $\hat{\mathbb{C}}\backslash\mathbb{K}$ evaluated at ∞. Since the works of Makarov [Mak85], Jones, and Wolff [JW88], it is known that the Hausdorff dimension of ω is less than or equal to 1. This result was generalized by Bourgain to $\mathbb{R}^d$ in [Bou87], where he proved the existence of a constant $\ell(d) > 0$ such that the dimension of the harmonic measure of any open set in $\mathbb{R}^d$ is less than $d - \ell(d)$. According to Makarov, Jones, and Wolff we know that $\ell(2) = 1$.

If $\mathbb{K}$ has Hausdorff dimension in $]1, 2]$ the previous results tells us that harmonic measure only "sees" a small part of $\mathbb{K}$ of dimension less or equal to 1. Let for instance $\mathbb{K}$ be the Mandelbrot set. This is a compact and connected set, so that $\mathbb{C}\backslash\mathbb{K}$, is simply connected and we know by [Mak85] that the support of harmonic measure is a subset of the boundary of $\mathbb{K}$ of Hausdorff dimension equal to 1. On the other side Shishikura proved in [Shi98] that any intersection of this boundary with an open set has Hausdorff dimension 2.

When the compact set $\mathbb{K}$ is fractal and has Hausdorff dimension less than one it is still reasonable to believe that in many cases the dimension of harmonic measure remains stricly less than the Hausdorff dimension of $\mathbb{K}$. This has been proved to be true for several types of conformal Cantor sets [Car85], [LV95], [UZ00], [Bat96]. More recently Batakis and Zdunik proved in [BZ15] that this remains true for a class of non autonomous Cantor sets generated by similitudes. Note that in our knowledge the question whether this is true for all Conformal cantor sets is still opened. On the other side the first author exhibits in [Bat96] examples of Cantor sets for which harmonic measure has full dimension, and very recently David, Jeznach and Julia in [DJJ23] give an example of a Cantor set in the plane for which Harmonic measure is equivalent with Hausdorff measure.

In this paper, we develop tools to study entropies, Lyapunov exponents, pressures and dimensions for non-autonomous Cantor sets. Most results are proved in a symbolic space and for the set of positive functions defined on cylinders, see section 2, even if this is interesting on his own, our main focus in this article are Hausdorff and Packing dimension of the harmonic measure, and their continuity properties, for a class of non-autonomous conformal repellers in the plane.

In order to state our results precisely, we describe now our settings and we introduce some notations.

Following [RGU12], given a Jordan domain U, a non-autonomous conformal iterated function system is a sequence $\Psi = (\Psi_n)$ of finite collections of conformal contractions $\Psi_n = \{{}_n\psi_i : U \to U \ ; \ i = 1, ..., d_n\}$ verifying properties (1), (2) and (3) below for some $\eta > 0$ There exists $\eta > 0$ such that

1. *Conformality:* There is a fixed neighborhood $V \supset (1+\eta)U$ of U such that ${}_n\psi_i$ extends to a conformal diffeomorphism from V to V, for all $1 \leqslant i \leqslant d_n$ and all $n \in \mathbb{N}$.

2. *Open set condition and annulus condition:*

   - ${}_nV_i \cap {}_nV_j = \varnothing$ for all $1 \leqslant i \neq j \leqslant d_n$, where ${}_nV_i = {}_n\psi_i(V)$. [OSC]
   - Moreover, for any n and any $1 \leqslant i \leqslant d_n$ we have ${}_nV_i \subset (1-\eta)U$. [AC]

3. *Bounded Contraction:* For all $n \in \mathbb{N}$, all $1 \leqslant i \neq j \leqslant d_n$ and all $x \in {}_nV_i$

$$\eta < \|{}_n\psi_i'(x)\| < 1 - \eta \quad \text{[BC]}$$

Combining the open set and the bounded contraction conditions we get a uniform bound on d_n:

$$\text{there exists } N \text{ s.t. } d_n \leqslant N \text{ for all } n \in \mathbb{N}. \tag{1}$$

Notation.

- *For any $E \subset V$, any $n \in \mathbb{N}$ and any $1 \leqslant i \leqslant d_n$ let ${}_nE_i = {}_n\psi_i(E)$. The index on the left stands for the* **generation** *of the map while the one on the right stands for the chosen* ***branch*** *of the map.*

- *For $n \in \mathbb{N}$ we denote by $\mathcal{U}_n$ the collection $\{{}_nU_1, \cdots, {}_nU_{d_n}\}$. Let $\Phi_n = \sum_{i=1}^{d_n} {}_n\phi_i \mathbb{1}_{{}_nU_i}$ where ${}_n\phi_i$, defined on ${}_nU_i$, is the inverse conformal dilatation of ${}_n\psi_i$, ie. ${}_n\phi_i : {}_nU_i \to U$ and ${}_n\phi_i \circ {}_n\psi_i = id_U$.*

In this way, we have introduced piecewise conformal maps denoted again by Φ_n defined on the union of topological discs: $\Phi_n : \bigcup_{{}_nU_i \in \mathcal{U}_n} {}_nU_i \to U$ by the formula

$$\Phi_n(z) = {}_n\phi_i(z) \quad \text{for } z \in {}_nU_i.$$

Remark 1.1. *Note that the condition of bounded contraction is stronger than the one of [RGU12] (uniform contraction). This is due to the fact that dealing with harmonic measure implies estimates on capacity and hence on the size of the sub-components. Also the conformality condition has been strengthened to ensure existence of conformal annuli with bounded from below moduli. This is crucial in order to be able to use Harnack's principle (cf. [Car85]).*

Furthermore, without loss of generality we can suppose that U is the unit disk.

We consider the following non-autonomous limit set

Definition 1.2. *Given a non-autonomous iterated conformal functions system Ψ_n on a Jordan domain U we define the* limit set $\mathbb{X} \subset \mathbb{C}$ *by*

$$\mathbb{X} = \bigcap_{n=1}^{\infty} \left(\Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_2 \circ \Phi_1\right)^{-1}(U),$$

where Φ_n is the sequence of piecewise conformal dilatations introduced above.

For any n, the map $T_n := \Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_2 \circ \Phi_1$ is well defined on $\mathbb{X}$ and for any $x \in \mathbb{X}$ we have $T_n(x) \in U$. Moreover there exists $\prod_{i=1}^{n} d_i$ conformal inverse branches of T_n defined on V. Each of this branches sends U onto a conformal disk included in U, each point in $\mathbb{X}$ belongs to one and only one of these conformal disks. Remember that $\mathcal{U}_n$ is the collection of connected components of $T_n^{-1}(U)$.
There is an obvious way to code the situation.
For any integer n, let $\mathcal{A}_n := \{1 \cdots d_n\}$ and set $\mathbb{K} := \prod_{k=1}^{+\infty} \mathcal{A}_k$. There is a one-to-one correspondence from $\mathbb{K}$ to $\mathbb{X}$. More precisely if $\mathtt{a} = (a_n)$ is an element of $\mathbb{K}$, then the sequence of subsets of V defined by $U_0 = \bar{U}$ and $U_n = {}_n\psi_{a_n}(U_{n-1})$ is a decreasing sequence of compact sets with diameters converging to 0, thus converging to a point x, which is obviously an element of $\mathbb{X}$. Reciprocally, if $x \in \mathbb{X}$ then for any n we have : $\Phi_n \circ \cdots \circ \Phi_1(x) \in U$, and there exists $\mathtt{a} = (a_n) \in \mathbb{K}$ such that $x \in {}_1U_{a_1}$, $\Phi_1(x) \in {}_2U_{a_2}, \cdots, \Phi_n \circ \cdots \circ \Phi_1(x) \in {}_{n+1}U_{a_{n+1}}$. If $x \in \mathbb{X}$ is associated with $\mathtt{a} \in \mathbb{K}$, then for any n we have $x \in {}_1\psi_{a_1} \circ \cdots \circ {}_n\psi_{a_n}(U)$, which is the conformal disk mentioned above. In the sequel we will denote $X_n(x) := a_1 \cdots a_n$ the set $\mathbb{X} \cap {}_1\psi_{a_1} \circ \cdots \circ {}_n\psi_{a_n}(U)$, and $\Psi^{-1}_{a_1 \cdots a_n} := {}_1\psi_{a_1} \circ \cdots \circ {}_n\psi_{a_n}$.

Recall the definition of Hausdorff Dimension and Packing Dimension of a probability measure μ:

$$HD(\mu) = \inf_{\{Z:\mu(Z)=1\}} HD(Z) \quad \text{and} \quad PD(\mu) = \inf_{\{Z:\mu(Z)=1\}} PD(Z),$$

the infimum and the supremum being taken over all Borel subsets Z of the support of μ. Let ω be the harmonic measure on $\hat{\mathbb{C}}\backslash\mathbb{X}$ evaluated at ∞. By the celebrated results of N. Makarov [Mak85] and P. Jones-T. Wolff [JW88] the Hausdorff dimension of ω is not larger than one. On the other hand, it is clear that the Hausdorff dimension of ω is at most $HD(\mathbb{X})$.

For any continuous function defined on U let $\|f\| := \sup_U |f|$, with $|\cdot| : \mathbb{C} \to \mathbb{R}^+$ is the modulus. Let $\mathbb{X}$ and $\tilde{\mathbb{X}}$ be two non-autonomous limit sets associated to the conformal iterated systems Ψ and $\tilde{\Psi}$ respectively. We define

$$d(\Psi, \tilde{\Psi}) = \sup_n \max_{1 \leqslant i \leqslant d_n} \left(\left\| {}_n\psi_i - {}_n\tilde{\psi}_i \right\| \right)$$

Using general results on the symbolic space $\mathbb{K}_1$, and studied in the second and third part of this paper, we prove

Theorem 1.3. *Le $\mathbb{X}$ and $(\mathbb{X}_k)_{k\in\mathbb{N}}$ be non-autonomous limit sets associated with the conformal iterated systems Ψ and $(\Psi_k)_{k\in\mathbb{N}}$ respectively. Let ω and ω_k be the harmonic measures of their complementaries. Assume that $\lim_{k\to+\infty} d(\Psi,\Psi_k)=0$, then,*

$$\lim_{k\to+\infty} HD(\omega_k)=HD(\omega) \quad \textit{and} \quad \lim_{k\to+\infty} PD(\omega_k)=PD(\omega)\cdot$$

We also discuss validity of well known formulas to calculate these dimensions. In particular we show that an adapted form of Manning's formula applies.

Theorem 1.4. *Let ω be the harmonic measure of $\mathbb{C}\backslash\mathbb{X}$, where $\mathbb{X}$ is the limit set of a non-autonomous conformal iterated systems Ψ.*

We have

$$HD(\omega)=\liminf_n \frac{H_{\omega,n}}{\chi_{\omega,n}} \quad \textit{and} \quad PD(\omega)=\limsup_n \frac{H_{\omega,n}}{\chi_{\omega,n}},$$

where $H_{\omega,n}=-\dfrac{1}{n}\displaystyle\sum_{X\in\mathcal{U}_n}\omega(X)\log\omega(X)$ *and* $\chi_{\omega,n}=\dfrac{1}{n}\displaystyle\sum_{X\in\mathcal{U}_n}\omega(X)\log\|T_n'(X)\|$.

Moreover, ω-almost surely

$$HD(\omega)=\liminf_n \frac{h_{\omega,n}(x)}{\chi_n(x)} \quad \textit{and} \quad PD(\omega)=\limsup_n \frac{h_{\omega,n}(x)}{\chi_n(x)}$$

where for any $x\in\mathbb{X}_0$, $h_{\omega,n}(x)=-\dfrac{1}{n}\log\omega(X_n(x))$ and $\chi_n(x)=\dfrac{1}{n}\log|T_n'(x)|\cdot$

2 Symbolic point of view

This paragraph deals exclusively with symbolic Cantor sets. More precisely, for any integer n, let $\mathcal{A}_n$ be a finite alphabet with d_n symbols. Set $\mathbb{K}_n:=\displaystyle\prod_{k=n}^{+\infty}\mathcal{A}_k$. We are mainly interested in $\mathbb{K}_1$ that we denote $\mathbb{K}$.

Let $T:\mathbb{K}_n\to\mathbb{K}_{n+1}$ be the shift map sending a sequence $\mathtt{a}=(a_i)_{i\geqslant 0}$ to $T(\mathtt{a})=(a_i)_{i\geqslant 1}$. It should be emphasized that T goes from $\mathbb{K}_n$ to $\mathbb{K}_{n+1}$. This is a bit ambiguous, since the same letter designates different maps, but it lightens considerably the notations. We let $D_n^p:=\displaystyle\prod_{k=0}^{n-1} d_{k+p}$ be the degree of T^n on $\mathbb{K}_p$.

Given a sequence of symbols $\mathtt{a}=(a_i)_{i\in\mathbb{N}}\in\mathbb{K}$, we set

$$X_n(\mathtt{a}):=a_0\cdots a_{n-1}:=\{\mathtt{b}=(b_i)\in\mathbb{K}\,|\,a_i=b_i\ ,\ \text{for } 0\leqslant i\leqslant n-1\}.$$

The set $X_n(\mathtt{a})$ is the *cylinder* (word), of *length* (size) n, in which $\mathtt{a}$ belongs to. Given a cylinder X we denote by $|X|$ its length and, for any n, by $\mathcal{C}_n$ the set of all cylinders of length $n\geqslant 1$, in which, for practical reasons, we add $\varnothing$ with the convention $\varnothing X=X\varnothing=X$, and $|\varnothing|=0$. Note that $\mathcal{C}_n$ is a finite partition of $\mathbb{K}$ by D_n^1 cylinders.

The set $\mathbb{K}$ endowed with the product topology is compact. Let $\mathtt{d}$ be any metric on $\mathbb{K}$ that generates this topology and such that $\mathrm{diam}\mathbb{K}=1$. If $E\subset\mathbb{K}$, we denote by $|E|_{\mathtt{d}}:=\sup\{\mathtt{d}(x,y)\,|\,(x,y)\in E\times E\}$ its diameter.

Actually, since the planar Cantor sets that we will consider are conformal repellers, they are well described using the coding and the symbolic point of view. It will then be natural to choose a metric which ensures that a cylinder in $\mathbb{K}$, and its "image" in the plane, have approximately the same diameters.

2.1 Asymptotic Siblings Invariance

Because of the strong intra-scalar-similarity of the Cantor sets we are dealing with, one expects that natural geometric measures, or natural geometric objects, defined on them have nice scaling or invariance properties.

Note that in this section we do not need to assume any assumption on the finite numbers d_n.

In order to define precisely the properties we have in mind we need to introduce some definitions.

For any integer $p \geqslant 1$, let $\mathfrak{F}_p$ be the set of real positive maps defined on the set of cylinders $T^p(\mathcal{C}) := \bigcup_{n \geqslant p} T^p\mathcal{C}_n$. For $p = 0$ we denote by $\mathfrak{F} = \mathfrak{F}_0$ the set of maps defined on the set of cylinders $\mathcal{C} := \bigcup_{n \geqslant 0} \mathcal{C}_n$ with values in $]0, +\infty[$.

For $\nu = \nu_\varnothing \in \mathfrak{F}$ and XY a fixed element of $\mathcal{C}$ we note $\nu_X(Y) := \dfrac{\nu(XY)}{\nu(X)}$. Note that ν_X is an element of $\mathfrak{F}_{|X|}$.

A finite measure ν on $\mathbb{K}_p$ that gives positive mass to any open set, is naturally an element of $\mathfrak{F}_p$. Reciprocally we say that $\nu \in \mathfrak{F}_p$ is a (finite) measure if it projects as such on the compact set defined asymptotically by the cylinders, which is if and only if :

$$\forall n \in \mathbb{N} \quad \forall X \in T^p(\mathcal{C}) \quad \frac{1}{\nu(X)} \sum_{|Y|=n} \nu(XY) = 1.$$

Definition 2.1.

- *Given ν and ν' in $\mathfrak{F}_p$ we naturally define $\nu + \nu'$, $\lambda\nu$ for any $\lambda \in \mathbb{R}$, and $\nu.\nu'$.*
- *For any real function φ and any $\nu \in \mathfrak{F}_p$ we note $\varphi(\nu)$ the map from $T^p\mathcal{C}$ to $\mathbb{R}$ defined by $\varphi(\nu)(X) = \varphi(\nu(X))$. Clearly this only makes sense if φ is well-defined on $\nu(T^p\mathcal{C})$.*
- *Given ν and ν' in $\mathfrak{F}_p$ and $n \geqslant 1$ let : $\langle \nu, \nu' \rangle_n := \sum_{X \in T^p\mathcal{C}_{n+p}} \nu(X)\nu'(X)$.*
- *For any $\nu \in \mathfrak{F}_p$, we define its **mass** on $\mathcal{C}_n$ (with $n \geqslant p$)*

  $$\mathfrak{m}_n(\nu) := \langle \nu, 1 \rangle_n.$$

  Note that ν is a measure if $\mathfrak{m}_n(\nu_X) = 1$ for any n and any $X \in \mathcal{C}$.

- *We say that ν and ν' in $\mathfrak{F}_p$ are equivalent if there exists $C > 1$ such that for any X we have $\dfrac{1}{C} \leqslant \dfrac{\nu(X)}{\nu'(X)} \leqslant C$. We then note $\nu \sim \nu'$.*

In this paper we will mainly consider the following elements $\nu \in \mathfrak{F}_p$:

- $\nu(X) = \mu(X)$, where μ is a measure on $\mathbb{K}$,
- $\nu(X) = |X|_{\mathtt{d}}$,

Here are some elementary properties that will be used later on.

Properties.
For any $s \in \mathfrak{F}$, any $XYZ \in \mathcal{C}_n$ we have

$$s_X(YZ) = s_X(Y)s_{XY}(Z) \qquad \text{and} \qquad s_{X_Y}(Z) = s_{XY}(Z) \tag{2}$$

For any finite measure ν, any cylinder $X \in \mathcal{C}$ and any integers n and p we have

$$\langle \nu_X, \log s_X \rangle_{n+p} = \langle \nu_X, \log s_X \rangle_n + \sum_{Y \in T^{|X|}\mathcal{C}_{|X|+n}} \nu_X(Y) \langle \nu_{XY}, \log s_{XY} \rangle_p. \tag{3}$$

The following is an application of Jensen inequality.

Proposition 2.2. *Let ν and ν' be two elements of $\mathfrak{F}_p$. For any n we have*

$$\langle \nu, \log \frac{\nu'}{\nu} \rangle_n \leqslant \mathsf{m}_n(\nu) \log \frac{\mathsf{m}_n(\nu')}{\mathsf{m}_n(\nu)}, \tag{4}$$

with equality if and only if there exists k such that $\nu(X) = k\nu'(X)$ for any $X \in T^p\mathcal{C}_{n+p}$.

We introduce now our main assumption we will refer to as : Asymptotic Sibling Invariance. It is a kind of long range Markov property of a functional that asserts that along cylinders which share a big common part of history, the relative functional does not depend on its origin.

Definition 2.3. *Let ν be an element of $\mathfrak{F}$ and (β_n) a positive sequence converging to 0. We say that ν satisfies Asymptotic Siblings Invariance (ASI) with sequence (β_n), if for any integers n, k and p we have*

$$\forall (X, X') \in \mathcal{C}_n^2 \ \ \forall Y \in T^n\mathcal{C}_{n+k} \ \ \forall Z \in T^{n+k}\mathcal{C}_{n+k+p} \quad \left| \log \left(\frac{\nu_{XY}(Z)}{\nu_{X'Y}(Z)} \right) \right| \leqslant \beta_k \tag{5}$$

Note that we sometimes use the following version of the inequation :

$$\left| \frac{\nu_{XY}(Z)}{\nu_{X'Y}(Z)} - 1 \right| \leqslant \beta_k.$$

Notation. *To simplify notations with cylinders, we sometimes are a little bit imprecise and instead of writing XY with $X \in \mathcal{C}_n$ and $Y \in T^n\mathcal{C}_{n+k}$, we will write XY with $|X| = n$ and $|Y| = k$, the context reminding us that Y is not in $\mathcal{C}_k$ but in $T^n\mathcal{C}_{n+k}$.*

Definition 2.4. *Given two sequences of positif real numbers (γ_n) and (β_n), we say that they are compatible if (γ_n) is increasing, (β_n) converges towards 0 and there exists a sequence of integers, (p_n), such that $p_n \leqslant n$ and $\left(\frac{1}{n} \max(\gamma_{p_n}, \gamma_n \beta_{p_n}) \right)$ converges towards 0. We then denote $c(\gamma_n, \beta_n) := \max(\gamma_{p_n}, \gamma_n \beta_{p_n})$.*

As an example, one may consider the situation we will have to deal with studying the harmonic measure. In that case we will have $\gamma_n = O(n)$, it appears that such a sequence is compatible with any sequence (β_n) converging towards 0. As a matter of fact we have $\frac{1}{n} \max(\gamma_{p_n}, \gamma_n \beta_{p_n}) = \max(\frac{O(p_n)}{n}, \frac{O(n)}{n} \beta_{p_n})$, and any $p_n = o(n)$, for instance $p_n = \lfloor \sqrt{n} \rfloor$, fulfilled the desired growth condition.

Proposition 2.5. *Let (γ_n) and (β_n) be two compatible sequences.*
Assume that s and ν are two elements of $\mathfrak{F}$, that ν is a finite measure and that they both satisfied (ASI) with (β_n). Assume moreover that one of this condition is fulfilled :

$$\forall XY \in \mathcal{C} \quad |\log s_X(Y)| \leqslant \gamma_{|Y|} \tag{5.1}$$

or

$$\forall X \in \mathcal{C} \quad \forall n \in \mathbb{N} \quad s(X) \geqslant 1 \quad \text{and} \quad \langle \nu_X, \log(s_X)\rangle_n \leqslant \gamma_n. \tag{5.2}$$

Then for any $\tilde{s} \sim s$

- *There exists $C > 0$ such that :*

$$\alpha_n(\nu, \tilde{s}) := \sup_k \left\{ \max_{|X|=k} \langle \nu_X, \log \tilde{s}_X\rangle_n - \min_{|X|=k} \langle \nu_X, \log \tilde{s}_X\rangle_n \right\} \leqslant Cc(\gamma_n, \beta_n).$$

- *For any $(X, X') \in \mathcal{C}^2$ we have*

$$\lim_{n\to+\infty} \left(\frac{1}{n}\langle \nu_{X'}, \log \tilde{s}_{X'}\rangle_n - \frac{1}{n}\langle \nu_X, \log \tilde{s}_X\rangle_n \right) = 0.$$

Proof. The constant C will change from line to line but will stay independent of integers or cylinders, only depending on ν and s.
For any $\tilde{s} \sim s$ we have, for any $XY \in \mathcal{C}$: $-2\log C \leqslant \log \tilde{s}_X(Y) - \log s_X(Y) \leqslant 2\log C$.

From where we deduce, because ν is a finite measure

$$-2\log C \leqslant \langle \nu_X, \log \tilde{s}_X\rangle_n - \langle \nu_X, \log s_X\rangle_n \leqslant 2\log C.$$

This tells us that conclusions of the theorem are true for $\tilde{s}$ as soon as they are true for s, and in the sequel we focus on s.
Let

- for any $X \in \mathcal{C}$, $g_n(X) := \langle \nu_X, \log s_X\rangle_n$, with $g_n := g_n(\varnothing)$
- $\alpha_n := \alpha_n(\nu, s) = \sup_k \left\{ \max_{|X|=k} g_n(X) - \min_{|X|=k} g_n(X) \right\}$.

We first note that (5.1), and the fact that ν is a finite measure, imply that for any $X \in \mathcal{C}$ we have

$$|g_n(X)| = |\langle \nu_X, \log s_X\rangle_n| \leqslant \gamma_n, \tag{6}$$

inequality which is also true with (5.2) and leads to

$$\alpha_n \leqslant 2\gamma_n. \tag{7}$$

Note also that given any $X \in \mathcal{C}$, any integers n and p, we have from (3) :

$$g_{n+p}(X) = g_n(X) + \sum_{|Y|=n} \nu_X(Y) g_p(XY). \tag{8}$$

From where we deduce that for any $X' \in \mathcal{C}_{|X|}$ we have

$$\begin{aligned} g_{n+p}(X) - g_{n+p}(X') \quad = \quad & g_n(X) - g_n(X') + \sum_{|Y|=n} \left(\nu_X(Y) - \nu_{X'}(Y)\right) g_p(XY) + \\ & \sum_{|Y|=n} \nu_{X'}(Y) \left(g_p(XY) - g_p(X'Y)\right) \end{aligned} \tag{9}$$

Let $\mathcal{B}^+$ be the subset of $T^{|X|}\mathcal{C}_{|X|+n}$ for which we have $\nu_X(Y) - \nu_{X'}(Y) \geqslant 0$, and let $\mathcal{B}^-$ be its complementary. Then we have

$$\sum_{|Y|=n} (\nu_X(Y) - \nu_{X'}(Y))\, g_p(XY) \leqslant \max_{|Z|=|X|+n} g_p(Z) \sum_{Y\in\mathcal{B}^+} (\nu_X(Y) - \nu_{X'}(Y)) + \min_{|Z|=|X|+n} g_p(Z) \sum_{Y\in\mathcal{B}^-} (\nu_X(Y) - \nu_{X'}(Y)) \cdot$$

Since ν is a finite measure, we have $\sum_{|Y|=n} \nu_X(Y) = \sum_{|Y|=n} \nu_{X'}(Y) = 1$.

This implies that $\sum_{|Y|=n} (\nu_X(Y) - \nu_{X'}(Y)) = 0$.

So that $\sum_{Y\in\mathcal{B}^+} (\nu_X(Y) - \nu_{X'}(Y)) = -\sum_{Y\in\mathcal{B}^-} (\nu_X(Y) - \nu_{X'}(Y))$, which leads to

$$\begin{aligned}
\sum_{|Y|=n} (\nu_X(Y) - \nu_{X'}(Y))\, g_p(XY) &\leqslant \sum_{Y\in\mathcal{B}^+} (\nu_X(Y) - \nu_{X'}(Y)) \left(\max_{|Z|=|X|+n} g_p(Z) - \min_{|Z|=|X|+n} g_p(Z) \right) \\
&\leqslant \alpha_p \sum_{Y\in\mathcal{B}^+} (\nu_X(Y) - \nu_{X'}(Y)) \\
&\leqslant \alpha_p \sum_{Y\in\mathcal{B}^+} \nu_{X'}(Y) \left(\frac{\nu_X(Y)}{\nu_{X'}(Y)} - 1 \right) \\
&\leqslant \alpha_p\beta_0 \sum_{Y\in\mathcal{B}^+} \nu_{X'}(Y) \qquad \text{(because of (ASI))} \\
&\leqslant \alpha_p\beta_0 \qquad \text{(because } \nu \text{ is a finite measure)}
\end{aligned}$$

We thus have

$$\sum_{|Y|=n} (\nu_X(Y) - \nu_{X'}(Y))\, g_p(XY) \leqslant \lambda\alpha_p. \tag{10}$$

From (ASI) we deduce that :

$$\nu_{XY}(Z) \leqslant \left(1 + \beta_{|Y|}\right) \nu_{X'Y}(Z),$$

and also :

$$\log s_{XY}(Z) \leqslant \log s_{X'Y}(Z) + \beta_{|Y|}.$$

This implies :

– if (5.1) is fulfilled, that we have $|\log s_{X'Y}(Z)| \leqslant \gamma_{|Z|}$ so that

$$\begin{aligned}
\nu_{XY}(Z) \log s_{XY}(Z) &- \nu_{X'Y}(Z) \log s_{X'Y}(Z) \\
&= (\nu_{XY}(Z) - \nu_{X'Y}(Z)) \log s_{XY}(Z) + \nu_{X'Y}(Z) \log \tfrac{s_{XY}(Z)}{s_{X'Y}(Z)} \\
&\leqslant |\nu_{XY}(Z) - \nu_{X'Y}(Z)|\gamma_{|Z|} + \nu_{X'Y}(Z) \left|\log \tfrac{s_{XY}(Z)}{s_{X'Y}(Z)}\right|
\end{aligned}$$

and since ν and s fulfill (ASI) we get

$$\begin{aligned}
&\leqslant \beta_{|Y|}\nu_{X'Y}(Z)\gamma_{|Z|} + \nu_{X'Y}(Z)\beta_{|Y|} \\
&\leqslant \beta_{|Y|}(1 + \gamma_{|Z|})\nu_{X'Y}(Z).
\end{aligned}$$

Summing over all cylinders Z of length p, and using once again the fact that ν is a finite measure, we get

$$g_p(XY) - g_p(X'Y) \leqslant (1 + \gamma_p)\beta_{|Y|} \leqslant C\gamma_p\beta_{|Y|}. \tag{11}$$

– if (5.2) is fulfilled, that we have

$$\nu_{XY}(Z)\log s_{XY}(Z) - \nu_{X'Y}(Z)\log s_{X'Y}(Z) \quad \leqslant \quad \beta_{|Y|}\left(\nu_{X'Y}(Z)\log s_{X'Y}(Z) + C\nu_{X'Y}(Z)\right)$$

Summing over all cylinders Z of length p, and using (6), we also get (11).

Injecting (11) and (10), into (9) leads to :

$$g_{n+p}(X) - g_{n+p}(X') \leqslant g_n(X) - g_n(X') + C\gamma_p\beta_n + \lambda\alpha_p.$$

From where we get that :

$$\alpha_{n+p} \leqslant \alpha_n + C\gamma_p\beta_n + \lambda\alpha_p.$$

Which might be rewritten, for any $p \leqslant n$, in the following form

$$\alpha_n \leqslant \alpha_p + C\gamma_{n-p}\beta_p + \lambda\alpha_{n-p} \leqslant \alpha_p + C\gamma_n\beta_p + \lambda\alpha_{n-p}\,, \quad (\gamma_n) \text{ being increasing.}$$

Using (7) we get

$$\alpha_n \leqslant 2\gamma_p + C\gamma_n\beta_p + \lambda\alpha_{n-p} \leqslant C\max(\gamma_p, \gamma_n\beta_p) + \lambda\alpha_{n-p}.$$

Let q and $r \leqslant p$ be the quotient and the remainder of the euclidean division of n by p. By induction we get

$$\begin{array}{rcl} \alpha_n & \leqslant & C\max(\gamma_p, \gamma_n\beta_p)\displaystyle\sum_{k=0}^{q-1}\lambda^k + \lambda^q\alpha_r \\ & \leqslant & C\max(\gamma_p, \gamma_n\beta_p) + 2\lambda^q\gamma_r \\ & \leqslant & C\max(\gamma_p, \gamma_n\beta_p) + 2\lambda^q\gamma_p \\ & \leqslant & C\max(\gamma_p, \gamma_n\beta_p). \end{array}$$

Since (γ_n) and (β_n) are compatible, one may take $p = p_n$ to get

$$\alpha_n \leqslant Cc(\gamma_n, \beta_n),$$

which is the first part of the proposition and tells us that for any X and X' in $\mathcal{C}$, with $|X| = |X'|$, we have

$$\frac{1}{n}\left|g_{n(X)} - g_{n(X')}\right| \leqslant \frac{\alpha_n}{n} \leqslant C\frac{c(\gamma_n, \beta_n)}{n} \to_{n\to+\infty} 0.$$

We show now that we can get rid of the condition $|X| = |X'|$.

Let $X \in \mathcal{C}$, and let a be a cylinder of length 1. Apply (8) with $n = 1$ and $p = q$

$$g_{q+1}(X) = g_1(X) + \sum_{|b|=1}\nu_X(b)g_q(Xb) = g_q(Xa) + g_1(X) + \sum_{|b|=1}\nu_X(b)(g_q(Xb) - g_q(Xa)).$$

Apply now (8) with $n = q$ and $p = 1$:

$$g_{q+1}(X) = g_q(X) + \sum_{|Y|=q}\nu_X(Y)g_1(XY).$$

We thus have :

$$g_q(X) - g_q(Xa) = g_1(X) + \sum_{|b|=1}\nu_X(b)(g_q(Xb) - g_q(Xa)) - \sum_{|Y|=q}\nu_X(Y)g_1(XY).$$

By hypothesis on ν and s, there exists $C > 0$ such that for any cylinder $Z \in \mathcal{C}$ we have $g_1(Z) \leqslant C$. This leads to

$$|g_q(X) - g_q(Xa)| \leqslant C + \alpha_q,$$

and this easily leads, for any Y with $|Y| = p$, to

$$|g_q(X) - g_q(XY)| \leqslant (C + \alpha_q)p \leqslant C\alpha_q p\cdot \tag{12}$$

Let $X' \in \mathcal{C}_{|X|}$ and $XY \in \mathcal{C}$, we have

$$\left|g_q(X') - g_q(XY)\right| \leqslant \left|g_q(X') - g_q(X)\right| + |g_q(X) - g_q(XY)| \leqslant \alpha_q + Cp\alpha_q \leqslant C'p\alpha_q,$$

and then

$$\frac{1}{q}\left|g_q(X') - g_q(XY)\right| \leqslant C'p\frac{\alpha_q}{q}\cdot$$

Which finishes the proof. □

2.2 A continuity result

Let ν and ν' be two elements of $\mathfrak{F}$, we define

$$\mathcal{D}(\nu, \nu') = \sup_{p\geqslant 0} \max\left\{\left|\log\frac{\nu_X(a)}{\nu'_X(a)}\right|,\ Xa \in \mathcal{C}_{p+1} \text{ and } |a| = 1\right\}\cdot$$

Note that we obviously have $\mathcal{D}(\nu, \nu') = \mathcal{D}(\nu', \nu)$ and $\mathcal{D}(\nu, \nu') = 0$ if and only if $\nu = \nu'$. Moreover, it is also clear that we have

$$\mathcal{D}(\nu, \nu'') \leqslant \mathcal{D}(\nu, \nu') + \mathcal{D}(\nu', \nu'')\cdot$$

So that $\mathcal{D}$ could be seen as a metric on $\mathfrak{F}$ if $\mathcal{D}(\nu, \nu') < +\infty$ for any ν and ν', which is not the case. Nevertheless if $\mathcal{D}(\nu, \nu_k)$ converges to 0, when $k \to +\infty$, this tells us that (ν_k) is "converging" to ν.

Let $Y = y_1 \cdots y_n$, $Y_0 = \varnothing$ and, for any $1 \leqslant k \leqslant p$, $Y_k = y_1 \cdots y_k$. Since

$$\nu(Y) = \prod_{k=1}^{p} \nu_{Y_{i-1}}(Y_i),$$

we conclude that for any $X \in \mathcal{C}$ and any Ywe have

$$\left|\log\frac{\nu_X(Y)}{\nu'_X(Y)}\right| \leqslant |Y|\mathcal{D}(\nu, \nu'),$$

This also implies, for any $XY \in \mathcal{C}_{n+p}$, with $|Y| = n$, that we have

$$\left|\nu_X(Y) - \nu'_X(Y)\right| \leqslant \nu_X(Y)\left(e^{n\mathcal{D}(\nu,\nu')} - 1\right)\cdot$$

Let $g_n(X) := \langle \nu_X, \log s_X\rangle_n$ and $g'_n(X) := \langle \nu'_X, \log s'_X\rangle_n$. Assume that ν and ν' are both finite measures, and that $s_X \geqslant 1$, then we have

$$g_n(X) - g'_n(X) = \langle\left(1 - \frac{\nu'_X}{\nu_X}\right)\nu_X, \log s_X\rangle_n + \langle \nu'_X, \log\frac{s'_X}{s_X}\rangle_n,$$

which leads to

$$\left|g_n(X) - g'_n(X)\right| \leqslant \left(e^{n\mathcal{D}(\nu,\nu')} - 1\right)g_n(X) + n\mathcal{D}(s, s') \leqslant \left(e^{n\mathcal{D}(\nu,\nu')}n\mathcal{D}(\nu, \nu')g_n(X) + n\mathcal{D}(s, s')\right),$$

so that

$$\frac{1}{n}\left|g_n(X) - g'_n(X)\right| \leqslant e^{n\mathcal{D}(\nu,\nu')}\mathcal{D}(\nu, \nu')g_n(X) + \mathcal{D}(s, s')\cdot \tag{13}$$

Theorem 2.6. *Let ν and s be two elements of $\mathfrak{F}$, and let (ν_k) and (s_k) be two sequences of elements in $\mathfrak{F}$.*

For any $n \in \mathbb{N}$, set $g_n = \langle \nu_\varnothing, \log s_\varnothing \rangle_n$ and $g_{k,n} = \langle \nu_{k,\varnothing}, \log s_{k,\varnothing} \rangle_n$.

Assume that

- *ν and ν_n are finite measures.*
- *all those elements of $\mathfrak{F}$ verify (ASI), with a same sequence (β_n),*
- *there exists a sequence (γ_n), compatible with (β_n), such that for any $s' \in \{s,\, s_n\}_{n\in\mathbb{N}}$ and any $XY \in \mathcal{C}$*
$$\left|\log s'_X(Y)\right| \leqslant \gamma_{|Y|},$$
- $\lim\limits_{k\to+\infty} \mathcal{D}(\nu, \nu_k) = \lim\limits_{k\to+\infty} \mathcal{D}(s, s_k) = 0.$

Then we have
$$\lim_{k\to+\infty} \limsup_n \left| \frac{g_{n,k}}{n} - \frac{g_n}{n} \right| = 0,$$
And in particular we have
$$\lim_{k\to+\infty} \liminf_n \frac{g_{n,k}}{n} = \liminf_n \frac{1}{n} g_n \qquad \text{and} \qquad \lim_{k\to+\infty} \limsup_n \frac{g_{n,k}}{n} = \limsup_n \frac{1}{n} g_n.$$
Moreover, the conclusion are still valid for any $\tilde{s}$ and $(\tilde{s_n})$ such that $\tilde{s} \sim s$ and $\tilde{s}_n \sim s_n$ for any n.

Proof. By hypothesis we are in position to apply proposition 2.5 with ν and s, and ν_n and s_n and we will use the notation therein.

Let (A_n) be a sequence of cylinders, with $A_n \in \mathcal{C}_n$ for any n, and let $p \leqslant n$, with $q \in \mathbb{N}$ such that $qp < n \leqslant (q+1)p$. Using (8) we get
$$g_n(\varnothing) := g_n = g_{pq+r} = g_{pq} + \sum_{|Y|=pq} \nu(Y) g_r(Y),$$
and also
$$g_{pq} = g_{p(q-1)} + \sum_{|Y|=p(q-1)} \nu(Y) g_p(Y),$$
so that
$$g_{pq} = \sum_{i=0}^{q-1} \sum_{|Y|=ip} \nu(Y) g_p(Y).$$
From this we deduce that
$$g_n - \sum_{i=1}^{q-1} g_p(A_{ip}) = \sum_{i=0}^{q-1} \sum_{|Y|=ip} \nu(Y)(g_p(Y) - g_p(A_{ik})) + \sum_{|Y|=pq} \nu(Y) g_r(Y),$$
and thus
$$\left| \frac{g_n}{n} - \frac{1}{n} \sum_{i=1}^{q-1} g_p(A_{ip}) \right| \leqslant \frac{\alpha_p(\nu, s)}{p} + \frac{\gamma_p}{n} \leqslant C \frac{c(\gamma_p, \beta_p)}{p} + \frac{\gamma_p}{n}.$$

The same is clearly also true for ν_k and s_k so that we have

$$\begin{aligned}\left|\frac{g_n}{n}-\frac{g_{k,n}}{n}\right| &\leqslant \left|\frac{g_n}{n}-\frac{1}{n}\sum_{i=1}^{q-1}g_p(A_{ip})\right|+\left|\frac{g_{k,n}}{n}-\frac{1}{n}\sum_{i=1}^{q-1}g_{k,p}(A_{ip})\right|+\frac{1}{n}\left|\sum_{i=1}^{q-1}(g_p(A_{ip})-g_{k,p}(A_{ip}))\right| \\ &\leqslant 2\frac{\gamma_p}{n}+C\frac{c(\gamma_p,\beta_p)}{p}+\frac{1}{n}\left|\sum_{i=1}^{q-1}(g_p(A_{ip})-g_{k,p}(A_{ip}))\right|\end{aligned}$$

Using (13) and the fact that one may assume that $\mathcal{D}(\nu,\nu_k)\leqslant 1$, we get

$$\begin{aligned}\left|\frac{g_n}{n}-\frac{g_{k,n}}{n}\right| &\leqslant 2\frac{\gamma_p}{n}+C\frac{c(\gamma_p,\beta_p)}{p}+\frac{1}{n}\sum_{i=1}^{q-1}p\left(e^{p\mathcal{D}(\nu,\nu_k)}\mathcal{D}(\nu,\nu_k)g_p(A_{ip})+\mathcal{D}(s,s_k)\right) \\ &\leqslant 2\frac{\gamma_p}{n}+C\frac{c(\gamma_p,\beta_p)}{p}+e^p\gamma_p\mathcal{D}(\nu,\nu_k)+\mathcal{D}(s,s_k)\end{aligned}$$

Let $\varepsilon>0$.

Fix p big enough to ensure that $C\dfrac{c(\gamma_p,\beta_p)}{p}<\dfrac{\varepsilon}{3}$ and let $n_0\in\mathbb{N}$ such that $2\dfrac{\gamma_p}{n_0}<\dfrac{\varepsilon}{3}$.

Finally let k_0 such that for any $k\geqslant k_0$ we have $e^p\gamma_p\mathcal{D}(\nu,\nu_k)+\mathcal{D}(s,s_k)<\dfrac{\varepsilon}{3}$.

Then for any $k\geqslant k_0$ and any $n\geqslant n_0$ we have : $\left|\dfrac{g_n}{n}-\dfrac{g_{k,n}}{n}\right|<\varepsilon$, from where the result follows. □

2.3 Almost sure convergence

In that section we work with a probability measure μ. We are interested in almost sure asymptotic values of sequences of the form : $Z_n(x):=\log s(X_n(x))$, where $s\in\mathfrak{F}$, with $s\geqslant 1$, and for any $\mathtt{x}\in\mathbb{K}$, $X_n(\mathtt{x})$ is the unique element of $\mathcal{C}_n$ containing $\mathtt{x}$. We will use the following strong law of large numbers version, see for instance [HH80] :

Theorem 2.7. *Let $\{Y_n, n\geqslant 1\}$ be a sequence of random variables in a probability space. Let $\{\mathcal{F}_n, n\geqslant 1\}$ be an increasing sequence of σ-fields with Y_n measurable with respect to $\mathcal{F}_n$, for each n. Assume that (Y_n) is uniformly bounded. Then almost-surely*

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{n}\left(Y_i-\mathop{\mathbb{E}}_{\mu}[Y_i|\mathcal{F}_{i-1}]\right)=0\cdot$$

Theorem 2.8. *Let μ be a probability measure and let $s\in\mathfrak{F}$.*

Assume that μ and s fulfilled (ASI) with sequence (β_n), that there exists a sequence (γ_n), compatible with (β_n) and such that :

$$\forall XY\in\mathcal{C}\quad |\log s_X(Y)|\leqslant\gamma_{|Y|}$$

Then, for any $\tilde{s}\sim s$, there is a set $\mathcal{U}\subset\mathbb{K}$ of full μ-measure such that for any $\mathtt{a}\in U$ we have :

$$\lim_{n\to+\infty}\left(\frac{1}{n}\sum_{|Y|=n}\mu(Y)\log\tilde{s}(Y)-\frac{1}{n}\log\tilde{s}(X_n(\mathtt{a}))\right)=0\cdot$$

Proof. Given $n\in\mathbb{N}$, let $Z_n:=\frac{1}{n}\log s(X_n)$ and, for any $p\in\mathbb{N}^*$, $Y_{n,p}:=\dfrac{1}{p}\left(Z_{np}-Z_{(n-1)p}\right)$. Let also $\mathcal{F}_n$ be the σ-algebra generated by cylinders in $\mathcal{C}_n$. The idea, once again taking

from [Bat06], is to apply theorem 2.7 on $(Y_{n,p})_n$, with respect to $(\mathcal{F}_{np})_n$, and to prove that the conditional expectations are asymptotically, on n and p, independent of $\mathtt{x} \in \mathbb{K}$.

Fix $p > 0$ and $k > 0$ in $\mathbb{N}$. We have :

$$\mathop{\mathbb{E}}_{\mu}[Y_{k+1,p}|\mathcal{F}_{kp}] = \frac{1}{p}\sum_{|Y|=p}\frac{\mu(X_{kp}Y)}{\mu(X_{kp})}\log\frac{s(X_{kp}Y)}{s(X_{kp})} = \frac{1}{p}g_p(X_{kp}).$$

Recall that by (8), for any integer k, we have

$$g_{k+p}(X) = g_k(X) + \sum_{|Y|=k}\nu_X(Y)g_p(XY).$$

For any fixed cylinders Z and Y in $\mathcal{C}_k$ we have : $|g_p(XY) - g_p(XZ)| \leqslant \alpha_p$. From where we deduce that

$$|g_{k+p}(X) - g_k(X) - g_p(XZ)| = \left|\sum_{|Y|=k}\nu_X(Y)\left(g_p(XY) - g_p(XZ)\right)\right| \leqslant \alpha_p.$$

Applying this with $XZ = X_{kp} := X_pY_{(k-1)p}$, we end up with :

$$\left|g_{kp}(X_p) - g_{(k-1)p}(X_p) - g_p(X_{kp})\right| \leqslant \alpha_p.$$

Summing from 1 to q we get

$$\left|g_{qp}(X_p) - g_0(X_p) - \sum_{k=1}^{q} g_p(X_{kp})\right| \leqslant q\alpha_p.$$

This finally leads to

$$\left|\frac{1}{q}\sum_{k=1}^{q}\mathop{\mathbb{E}}_{\mu}[Y_{k+1,p}|\mathcal{F}_{kp}] - \frac{1}{qp}g_{qp}(X_p)\right| \leqslant \frac{\alpha_p}{p}. \tag{14}$$

Let $n > p$ be two integers, and let $q \in \mathbb{N}$ be such that : $qp < n \leqslant (q+1)p$. We have

$$Z_n = Z_{(q+1)p} + Z_n - Z_{(q+1)p} = p\sum_{k=1}^{q}Y_{k+1,p} + \log\frac{s(X_n)}{s(X_{(q+1)p})},$$

so that

$$\frac{1}{n}Z_n = \frac{pq}{n}\frac{1}{q}\sum_{k=1}^{q}Y_{k+1,p} - \frac{1}{n}\log s_{X_n}(T^nX_{(q+1)p}),$$

Using the hypothesis on s we get

$$\left|\frac{1}{n}Z_n - \frac{pq}{n}\frac{1}{q}\sum_{k=1}^{q}Y_{k+1,p}\right| \leqslant C\frac{\gamma_p}{n}. \tag{15}$$

Let

$$\begin{aligned}
① &= \frac{1}{n}Z_n - \frac{pq}{n}\frac{1}{q}\sum_{k=1}^{q}Y_{k+1,p}\\
②_q &= \frac{1}{q}\sum_{k=1}^{q}Y_{k+1,p} - \mathop{\mathbb{E}}_{\mu}[Y_{k+1,p}|\mathcal{F}_{kp}],\\
③ &= \mathop{\mathbb{E}}_{\mu}[Y_{k+1,p}|\mathcal{F}_{kp}] - \frac{1}{qp}g_{qp}(X_p),\\
④ &= \frac{1}{n}\left(g_{qp}(X_p) - g_n(X_p)\right),
\end{aligned}$$

We obviously have

$$\frac{1}{n}Z_n - \frac{1}{n}g_n(X_p) = ① + \frac{pq}{n}②_q + \frac{pq}{n}③ + ④$$

So that, using (14,15), we get

$$\left|\frac{1}{n}Z_n - \frac{1}{n}g_n(X_p)\right| \leqslant C\frac{\gamma_p}{n} + |②_q| + \frac{\alpha_p}{p} + |④|$$

Concerning ④, we use once again (8) with $r = n - pq \leqslant p$

$$g_n(X_p) = g_{pq}(X_p) + \sum_{|Y|=pq} \nu_{X_p}(Y) g_r(X_p Y)\cdot$$

From where we deduce that

$$|④| = \frac{1}{n}\left|\sum_{|Y|=pq} \nu_{X_p}(Y) g_r(X_p Y)\right| \leqslant C\frac{\gamma_p}{n},$$

so that we have

$$\left|\frac{1}{n}Z_n - \frac{1}{n}g_n(X_p)\right| \leqslant \frac{\alpha_p}{p} + C\frac{\gamma_p}{n} + |②_q|\cdot$$

Using (12) we get

$$\begin{aligned}\left|\frac{1}{n}Z_n - \frac{1}{n}g_n\right| &\leqslant \frac{1}{n}\left|g_n - g_n(X_p)\right| + \frac{\alpha_p}{p} + C\frac{\gamma_p}{n} + |②_q| \\ &\leqslant C\frac{\alpha_n}{n}p + \frac{\alpha_p}{p} + C\frac{\gamma_p}{n} + |②_q|\end{aligned} \tag{16}$$

Note that for any n we have by hypothesis that

$$|Y_{k,p}| = \left|\log\frac{s(X_{kp})}{s(X_{(k-1)p})}\right| = \left|\log s_{X_{(k-1)p}}(X_{kp})\right| \leqslant \gamma_p\cdot$$

We may thus apply theorem 2.7 with $(Y_{n,p})_n$, to conclude that p being fixed, $|②_q|$ μ-almost surely converges towards 0 as $q \to +\infty$, and we can easily conclude that there exists a set of full measure, denoted $\mathcal{U}$, such that it occurs for any p.

Let $\mathtt{x} \in \mathcal{U}$ and $\varepsilon > 0$. Choose p big enough to have $\frac{\alpha_p}{p} < \frac{\varepsilon}{2}$. Remembering that q goes to $+\infty$ with n, let $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$ we have

$$C\frac{\alpha_n}{n}p + C\frac{\gamma_p}{n} + |②_q| < \frac{\varepsilon}{2}\cdot$$

Then by (16) we have

$$\left|\frac{1}{n}Z_n - \frac{1}{n}g_n\right| < \varepsilon,$$

and we can conlcude that the sequence $\left(\frac{1}{n}Z_n(\cdot) - \frac{1}{n}g_n\right)$ μ-almost surely converges towards 0. Which is precisely

$$\lim_{n\to+\infty}\left(\frac{1}{n}\log s(X_n(\cdot)) - \frac{1}{n}\sum_{|Y|=n}\mu(Y)\log s(Y)\right) = 0\cdot$$

□

3 Pressures, Entropies, Lyapunov exponents and dimensions

In this section, using results from the previous one, we show how natural generalizations of notions coming from Thermodynamical Formalism are possible.

3.1 Lyapunov exponents and Entropies

Given a sequence (u_n) of real numbers, we let $\overline{u_n}$ be its limsup and $\underline{u_n}$ its liminf.

For any positive integer n and any probability measure, μ, on $\mathbb{K}$ we define

$$H_{\mu,n} := -\frac{1}{n}\sum_{|X|=n}\mu(X)\log\mu(X) \qquad \text{and} \qquad \mathcal{X}_{\mu,n} := -\frac{1}{n}\sum_{|X|=n}\mu(X)\log|X|_{\mathsf{d}},$$

where the sum is taken over all cylinders $X \in \mathcal{C}_n$, with the usual convention $0\ln(0) = 0$.

Let then $\overline{h_\mu} := \overline{H_{\mu,n}}$, $\underline{h_\mu} := \underline{H_{\mu,n}}$, $\overline{\mathcal{X}_\mu} := \overline{\mathcal{X}_{\mu,n}}$ and $\underline{\mathcal{X}_\mu} := \underline{\mathcal{X}_{\mu,n}}$.

Given any $\mathtt{a} \in \mathbb{K}$ and any positive number n, we define

$$\chi_n(\mathtt{a}) := -\frac{1}{n}\log|X_n(\mathtt{a})|_{\mathsf{d}} \quad \text{and} \quad h_{\mu,n}(\mathtt{a}) := -\frac{1}{n}\log\mu(X_n(\mathtt{a}))\cdot$$

Let as before note : $\overline{h_\mu}(\mathtt{a}) := \overline{h_{\mu,n}(\mathtt{a})}$, $\underline{h_\mu}(\mathtt{a}) := \underline{h_{\mu,n}}(\mathtt{a})$, $\overline{\mathcal{X}}(\mathtt{a}) := \overline{\mathcal{X}_n}(\mathtt{a})$ and $\underline{\mathcal{X}}(\mathtt{a}) := \underline{\mathcal{X}_n}(\mathtt{a})$. Remember that $\mathcal{F}_n$ is the σ-algebra generated by cylinders of length less or equal to n. We have

$$H_{\mu,n} = \mathop{\mathbb{E}}_{\mu}[h_{\mu,n}\,|\,\mathcal{F}_n] \qquad \text{and} \qquad \mathcal{X}_{\mu,n} = \mathop{\mathbb{E}}_{\mu}[\chi_n\,|\,\mathcal{F}_n]\cdot$$

Note also that with $s : X \mapsto \dfrac{1}{|X|_{\mathsf{d}}}$ we have :

$$H_{\mu,n} = \frac{1}{n}\langle\nu, \log\frac{1}{\nu}\rangle_n \qquad \text{and} \qquad \mathcal{X}_{\mu,n} = \frac{1}{n}\langle\nu, \log s\rangle_n\cdot$$

Applying proposition 2.2 to that setting, remembering that $D_n = d_0\cdots d_{n-1}$ is the degre of T^n, we get

Proposition 3.1. *For any n*

$$0 \leqslant H_{\mu,n} \leqslant \frac{1}{n}\log D_n \quad \textit{and} \quad H_{\mu,n} - \mathcal{X}_{\mu,n} \leqslant \frac{1}{n}\log \mathsf{m}_n(|.|_{\mathsf{d}})\cdot$$

Moreover, if X_m and X_M in $\mathcal{C}_n$ are such that : $\forall X \in \mathcal{C}_n$ we have $|X_m|_{\mathsf{d}} \leqslant |X|_{\mathsf{d}} \leqslant |X_M|_{\mathsf{d}}$ then

$$-\frac{1}{n}\log|X_M|_{\mathsf{d}} \leqslant \mathcal{X}_{\mu,n} \leqslant -\frac{1}{n}\log|X_m|_{\mathsf{d}}$$

Theorem 3.2. *Let μ be a probability measure that fulfilled (ASI) with sequence (β_n) and assume that there exists a sequence (γ_n), compatible with (β_n), and such that :*

$$\forall XY \in \mathcal{C} \quad |\log\mu_X(Y)| \leqslant \gamma_{|Y|}$$

Then there exists a set $\mathcal{U}$ of full μ-measure such that for any $\mathtt{a} \in \mathcal{U}$

$$\lim_{n\to+\infty}(h_{\mu,n}(\mathtt{a}) - H_{\mu,n}) = 0\cdot$$

and in particular we have μ-almost surely

$$\overline{h_\mu}(\cdot) = \overline{h_\mu} \qquad \textit{and} \qquad \underline{h_\mu}(\cdot) = \underline{h_\mu}\cdot$$

Assume moreover that $s : X \mapsto \dfrac{1}{|X|_{\mathsf{d}}}$ *fulfilled (ASI) and that there exist* $0 < \delta < 1$ *such that for any* $XY \in \mathcal{C}$ *we have*

$$\delta^{\gamma_{|Y|}} \leqslant \frac{|XY|_{\mathsf{d}}}{|X|_{\mathsf{d}}} \leqslant 1 \tag{17}$$

then there exists a set of full μ*-measure,* $\mathcal{V}$*, such that for any* $\mathsf{a} \in \mathcal{V}$

$$\lim_{n\to+\infty} (\mathcal{X}_n(\mathsf{a}) - \mathcal{X}_{\mu,n}) = 0,$$

and in particular we have μ*-almost surely*

$$\underline{\mathcal{X}} = \underline{\mathcal{X}}(\cdot) \leqslant \overline{\mathcal{X}}(\cdot) = \overline{\mathcal{X}}\cdot$$

Proof. The hypothesis allow us to apply theorem 2.8, with $\nu(X) = \mu(X)$ and $s(X) = \dfrac{1}{\mu(X)}$, to conclude that

$$\lim_{n\to+\infty} (h_{\mu,n}(\mathsf{a}) - H_{\mu,n}) = 0\cdot$$

Let now $s : X \mapsto \dfrac{1}{|X|_{\mathsf{d}}}$.

From the assumption $\delta^{\gamma_{|Y|}} \leqslant \dfrac{|XY|_{\mathsf{d}}}{|X|_{\mathsf{d}}} \leqslant 1$ we deduce that $1 \leqslant s_X(Y) \leqslant \delta^{-\gamma_{|Y|}}$, we easily conclude that for any $XY \in \mathcal{C}$, with $|Y| = n$ we have

$$|\log s_X(Y)| \leqslant -\gamma_n \log d\cdot$$

We may once again apply theorem 2.8 to conclude □

3.2 Pressure of a measure and dimensions

In that paragraph we assume uniform bounds for the diameters, which is a very natural hypothesis in our context : there exists $\delta \in]0,1[$ such that for any $n \in \mathbb{N}$, any cylinder XY with $|Y| = n$ we have

$$\delta^n \leqslant \frac{|XY|_{\mathsf{d}}}{|X|_{\mathsf{d}}} \leqslant (1-\delta)^n\cdot \tag{18}$$

In particular we thus have : $0 < -\log(1-\delta) \leqslant \underline{\mathcal{X}} \leqslant \overline{\mathcal{X}} \leqslant -\log\delta$

Definition 3.3. *For any real number* t *and any integer* n *let* $P_{\mu,n}(t) := H_{\mu,n} - t\mathcal{X}_{\mu,n}$.
We define $\underline{P_\mu}(t) = \underline{P_{\mu,n}}(t)$ *and* $\overline{P_\mu}(t) = \overline{P_{\mu,n}}(t)$.

Proposition 3.4. *Assume that* $|.|_{\mathsf{d}}$ *satisifes condition (18) and that* μ *and* $|.|_{\mathsf{d}}$ *both fulfill (ASI) with sequence* (β_n) *compatible with* $(\gamma_n = n)$*, then the function* $t \mapsto \underline{P_\mu}(t)$ *and* $t \mapsto \overline{P_\mu}(t)$ *are stricly decreasing and bi-Lipschitz*
We have :

1. $0 \leqslant \underline{P_\mu}(0) = \underline{h_\mu} \leqslant \overline{P_\mu}(0) = \overline{h_\mu}\cdot$
2. $\forall t \in \mathbb{R} \qquad \underline{h_\mu} - t\log(1-\delta) \leqslant \underline{P_\mu}(t) \leqslant \overline{P_\mu}(t) \leqslant \overline{h_\mu} - t\log\delta\cdot$
3. *Let* $HD(\mu)$ *be the Hausdorff dimension of* μ*, and* $PD(\mu)$ *its packing-dimension, then*

$$\underline{P_\mu}(HD(\mu)) = 0 = \overline{P_\mu}(PD(\mu))\cdot$$

4. For any $\mathtt{a}$ *in a set of full* μ*-measure we have :*

$$\liminf \frac{H_{\mu,n}}{\chi_{\mu,n}} = \liminf \frac{h_{\mu,n}(\mathtt{a})}{\chi_n(\mathtt{a})} = HD(\mu) \quad \textit{and} \quad \limsup \frac{H_{\mu,n}}{\chi_{\mu,n}} = \limsup \frac{h_{\mu,n}(\mathtt{a})}{\chi_n(\mathtt{a})} = PD(\mu).$$

Proof. From (18) we easily get for any $t < t+h$ and any $n \in \mathbb{N}$:

$$-h\log(1-\delta) + P_{\mu,n}(t+h) \leqslant P_{\mu,n}(t) \leqslant P_{\mu,n}(t+h) - h\log\delta.$$

From where we deduce, with P standing either for $\underline{\mathcal{P}_\mu}$ or $\overline{\mathcal{P}_\mu}$, that we have

$$0 < -h\log(1-\delta) \leqslant P(t) - P(t+h) \leqslant -h\log\delta.$$

Which tells us that $\mathcal{P}_\mu$ and $\overline{\mathcal{P}_\mu}$ are stricly decreasing and bi-Lipschitz.

Points (1) and (2) are obvious from definitions and hypothesis.

We turn to results about dimensions.

Hypothesis on μ and $|.|_{\mathtt{d}}$ allow us to apply theorem 2.8 with, for any $X \in \mathcal{C}$, $\nu(X) = \mu(X)$ and $s(X) = \dfrac{|X|_{\mathtt{d}}^t}{\mu(X)}$, as well as theorem 3.2. In particular, μ-almost surely we have

$$\begin{array}{lccc}
\text{(E)} & \lim\limits_{k\to+\infty} h_{\mu,n_k}(\cdot) = h & \iff & \lim\limits_{k\to+\infty} H_{\mu,n_k} = h, \\
\text{(L)} & \lim\limits_{k\to+\infty} \mathcal{X}_{n_k}(\cdot) = \chi & \iff & \lim\limits_{k\to+\infty} \mathcal{X}_{n_k} = \chi, \\
\text{(P)} & \lim\limits_{k\to+\infty} \dfrac{1}{n_k}\log\dfrac{|X_{n_k}(\cdot)|_{\mathtt{d}}^t}{\mu(X_{n_k}(\cdot))} = p & \iff & \lim_{k\to+\infty} H_{\mu,n_k} - t\mathcal{X}_{\mu,n_k} = p
\end{array}$$

Since $\underline{P_\mu}$ and $\overline{P_\mu}$ are stricly decreasing, continuous with positive and negative values, there exists a unique $(\underline{t},\bar{t}) \in \mathbb{R}^2$, with $0 \leqslant \underline{t} \leqslant \bar{t} \in \mathbb{R}$ such that : $\underline{P_\mu}(\underline{t}) = 0 = \overline{P_\mu}(\bar{t})$.

We focus on the result with liminf.

Take a point $\mathtt{a} \in \mathbb{K}$ such that (E), (L) and (P) are true.

Let (n_k) be a sequence of integers such that

$$\liminf \frac{h_{\mu,n}(\mathtt{a})}{\mathcal{X}_n(\mathtt{a})} = \lim_{k\to+\infty} \frac{h_{\mu,n_k}(\mathtt{a})}{\mathcal{X}_{n_k}(\mathtt{a})} = D(\mathtt{a}).$$

Extracting again if necessary one may suppose that

$$\lim_{k\to+\infty} h_{\mu,n_k}(\mathtt{a}) = h \text{ and } \lim_{k\to+\infty} \mathcal{X}_{\mu,n_k}(\mathtt{a}) = \chi.$$

By hypothesis on $|.|_{\mathtt{d}}$, we have $0 < \chi < +\infty$, thus $D(a) = \frac{h}{\chi}$ and we have

$$\underline{P_\mu}(D(a)) \leqslant h - D(a)\chi = 0.$$

From where we deduce that $D(a) \geqslant \underline{t}$.

Assume now that n_k is such that

$$\lim_{k\to+\infty} H_{\mu,n_k} - t\mathcal{X}_{\mu,n_k} = \underline{P_\mu}(t).$$

Pick a point $\mathtt{a}$ for which we can defined h and χ as above. We have $\dfrac{h}{\chi} \geqslant D(\mathtt{a})$ and also

$$\underline{P_\mu}(t) = h - t\chi = \chi\left(\frac{h}{\chi} - t\right) \geqslant \chi(D(a) - t).$$

We deduce that $\underline{P}_\mu(D(a)) \geqslant 0$ so that $D(a) \leqslant \underline{t}\cdot$

This tells us that μ-almost surely D is constant equal to $\underline{t}$ which, by classical results on Hausdorff dimension of measure, see for instance [Fal13], allows us to conclude that μ-almost surely

$$HD(\mu) = D(\cdot) = \underline{t}\cdot$$

□

3.3 Continuity of dimension

Theorem 3.5. *Let μ be a probability measure on $\mathbb{K}$ and (μ_k) be a sequence of probability measures.*

Let d be a metric on $\mathbb{K}$ and d_k a sequence of metrics. For any $X \in \mathcal{C}$ we let $\tilde{s}(X) = |X|_{\mathsf{d}}$ and $\tilde{s}_k(X) = |X|_{\mathsf{d}_k}$.

Assume that

- *The metrics $|\cdot|_{\mathsf{d}}$ and $|\cdot|_{\mathsf{d}_k}$, $k \in \mathbb{N}$, verify condition (18) with a common constant δ.*
- *μ, s, and for any k, s_k and μ_k, fulfilled (ASI) with a common sequence (β_n).*
- *there exists a sequence (γ_n), compatible with (β_n), such that for any $XY \in \mathcal{C}$ and any k,*

$$\log \frac{\mu(X)}{\mu(XY)} \leqslant \gamma_{|Y|} \quad \text{and} \quad \log \frac{\mu_k(X)}{\mu_k(XY)} \leqslant \gamma_{|Y|}\cdot$$

- *we have $\lim_{k\to+\infty} \mathcal{D}(\mu, \mu_k) = \lim_{k\to+\infty} \mathcal{D}(s, s_k) = 0$.*
- *(t_k) is a real sequence converging towards t.*

Then

$$\lim_{k\to+\infty} \limsup_n \left((H_{\mu,n} - t\chi_{\mu,n}) - (H_{\mu_k,n} - t_k\chi_{\mu_k,n})\right) = 0\cdot$$

In particular

$$\lim_{k\to+\infty} \underline{P}_{\mu_k}(t_k) = \underline{P}_\mu(t) \quad \text{and} \quad \lim_{k\to+\infty} \overline{P}_{\mu_k}(t_k) = \overline{P}_\mu(t),$$

$$\lim_{k\to+\infty} {}_kHD(\mu_k) = HD(\mu) \quad \text{and} \quad \lim_{k\to+\infty} {}_kPD(\mu_k) = PD(\mu)^{[1]}.$$

Proof. For any $X \in \mathcal{C}$ let

$$p(X) = \frac{|X|_{\mathsf{d}}^t}{\mu(X)} = \frac{s(X)^t}{\mu(X)} \qquad \text{and for any } k \qquad p_k(X) = \frac{|X|_{\mathsf{d}_k}^{t_k}}{\mu_k(X)} = \frac{s_k(X)^{t_k}}{\mu_k(X)}\cdot$$

Let $C = (1 + \sup_k\{|t|, |t_k|\})$. For any $XY \in \mathcal{C}$, we have

$$|\log p_X(Y)| \leqslant |t| \log s_X(Y) + \log \mu_X(Y) \leqslant C\gamma_{|Y|}\cdot$$

In the same way we check that $|\log p_{k_X}(Y)| \leqslant C\gamma_{|Y|}\cdot$ Thus one may take $\gamma'_n = C\gamma_n$.

[1] We would like to emphasize on the fact that the metrics changing with k, those dimensions are also changing, since Hausdorff and packing dimensions rely on the metric used.

We also have for any $XYZ \in \mathcal{C}$ and any $X' \in \mathcal{C}_{|X|}$

$$\begin{aligned}
\left|\frac{p_{XY}(Z)}{p_{X'Y}(Z)} - 1\right| &= \left|\frac{\mu_{X'Y}(Z)}{\mu_{XY}(Z)}\left(\frac{s_{XY}(Z)}{s_{X'Y}(Z)}\right)^t - 1\right| \\
&= \left|\left(\left(\frac{s_{XY}(Z)}{s_{X'Y}(Z)}\right)^t - 1\right)\frac{\mu_{X'Y}(Z)}{\mu_{XY}(Z)} + \frac{\mu_{X'Y}(Z)}{\mu_{XY}(Z)} - 1\right| \\
&\leqslant (1+\beta_0)\left|\left(\left(\frac{s_{XY}(Z)}{s_{X'Y}(Z)}\right)^t - 1\right)\right| + \left|\frac{\mu_{X'Y}(Z)}{\mu_{XY}(Z)} - 1\right| \\
&\leqslant 2(1+\beta_0)^{1+|t|}\beta_{|Y|} \\
&\leqslant C\beta_{|Y|}
\end{aligned}$$

We may thus take $\beta'_n = C\beta_n$. It is easy to prove that (γ'_n) and (β'_n) are compatible. We also have that $\mathcal{D}(p, p_k) \to 0$, so that we can apply theorem 2.6. Using notations therein we have :

$$\frac{1}{n}g_n = H_{\mu,n} - t\chi_{\mu,n} = P_{\mu,n}(t) \qquad \text{and} \qquad \frac{1}{n}g_{k,n} = H_{\mu_k,n} - t_k\chi_{\mu_k,n} = P_{\mu_k,n}(t_k),$$

and theorem 2.6 asserts that

$$\lim_{k\to+\infty} \limsup_n \left(P_{\mu,n}(t) - P_{\mu_k,n}(t_k)\right) = 0.$$

From this we conclude that

$$\lim_{k\to+\infty} \underline{P}_{\mu_k}(t_k) = \underline{P}_\mu(t) \quad \text{and} \quad \lim_{k\to+\infty} \overline{P}_{\mu_k}(t_k) = \overline{P}_\mu(t).$$

With $t_k = {}_kHD(\mu_k)$, and δ any limit value of (t_k) we conclude, using proposition 3.4, that

$$0 = \lim_{k\to+\infty} \underline{P}_{\mu_k}({}_kHD(\mu_k)) = \underline{P}_\mu(\delta),$$

which, by proposition 3.4, tells us that $\delta = HD(\mu)$, so that $({}_kHD(\mu_k))$ converges towards $HD(\mu)$. □

4 Harmonic measure of non-autonomous conformal repellers

We apply in this last section, results of the previous ones to study harmonic measure of the complementary of non-autonomous conformal repellers.

In the first paragraph we study geometric aspects of a given non-autonomous conformal iterated function system Ψ, proving that diameter and harmonic measure fulfilled the desired conditions.

Then we deal with sequence of such iterated function systems proving continuity of Hausdorff and Packing dimensions of harmonic measures.

Let $\Psi = (\Psi_n) = \{{}_n\psi_i : U = \mathbb{D} \to U = \mathbb{D} \ ; \ i = 1, ..., d_n\}$ be a non-autonomous conformal iterated function system satisfying our assumptions with a constant η controlling contraction and modulus of the annulus, and a fixed neighborhood of $\mathbb{D}$, V. We note $\mathbb{X}$ its limit set. In this last section, we adopt the following notations:

- For any symbolic cylinder $X = a_1 \cdots a_n$ we note ψ_X the contraction ${}_1\psi_{a_1} \circ \cdots \circ {}_n\psi_{a_n}$, and $\phi_X = {}_n\varphi_{a_n} \circ \cdots \circ {}_1\varphi_{a_1}$ its inverse.

- If X is a symbolic cylinder, we also denote its geometric realization by X. That is, $X = \psi_X(\mathbb{D}) \cap \mathbb{X}$.
- If $E \subset V$ and X is a cylinder, then $E_X := \psi_X(E)$. In particular, we denote $\mathbb{D}_X = \Psi_X(\mathbb{D})$ and $\mathbb{D}_{\gamma X} = \Psi_X(\mathbb{D}_\gamma)$, where $\mathbb{D}_\gamma = \{|z| \leqslant \gamma\}$.
- Given a Green domain $\Omega \subset \mathbb{C}$, a point $x \in \Omega$ and a Borel set F, we will denote $\omega(x, F, \Omega)$ the harmonic measure assigned to the domain Ω of F at x.
- In particular we will note $\omega := \omega(\infty, \cdot, \hat{\mathbb{C}}\backslash\mathbb{X})$, and for any k, $\omega_k := \omega(\infty, \cdot, \hat{\mathbb{C}}\backslash\mathbb{X}_k)$, where $\mathbb{X}_k = T^k(\mathbb{X})$.

4.1 Controlling diameters of cylinders

As one may expect we will use Kœbe distortion theorem, and more precisely its quantified versions, to cope with diameters.

Proposition 4.1. *For any $0 < \delta < 1$ fixed, there exists $C > 1$, such that for any conformal map, $f : \mathbb{D} \to \mathbb{C}$, any compact set $\mathbb{K} \subset \{|z| \leqslant 1 - \delta\}$ and any $(x, y) \in \mathbb{K} \times \mathbb{K}$*

$$(1 - C \operatorname{diam}(\mathbb{K})) \leqslant \frac{|f'(x)|}{|f'(y)|} \leqslant (1 + C \operatorname{diam}(\mathbb{K})) \tag{19}$$

and

$$(1 - C \operatorname{diam}(\mathbb{K}))|f'(x)| \leqslant \frac{\operatorname{diam} f(\mathbb{K})}{\operatorname{diam} \mathbb{K}} \leqslant |f'(x)|(1 + C \operatorname{diam}(\mathbb{K})) \tag{20}$$

Proof. First note that relation (19) and (20) are relevant only if $\operatorname{diam}(\mathbb{K})$ is small. Otherwise it is nothing else than the standard Koebe distortion theorem. And we assume from now on that $\operatorname{diam}(\mathbb{K}) \leqslant \frac{\delta}{2}$.

Let x and y be two points in $\mathbb{K}$. We apply corollary 1.5 in [Pom92], with $z = \dfrac{x - y}{1 - \bar{y}x}$, we have

$$\frac{|f'(x)|}{|f'(y)|} \leqslant \left(\frac{1 + |z|}{1 - |z|}\right)^3,$$

Since $\mathbb{K} \subset \{|z| \leqslant 1 - \delta\}$ we have $|z| \leqslant \frac{1}{\delta}|x - y| \leqslant \frac{1}{\delta} \operatorname{diam} \mathbb{K}$. We also have $|z| < 1$ and we can deduce that we have :

$$\frac{|f'(x)|}{|f'(y)|} \leqslant 1 + C \operatorname{diam}(\mathbb{K}). \tag{21}$$

By exchanging x and y we also get the opposite inequality.

Let $Z = 1 - x\bar{y}$. Theorem 1 in [Fan78] asserts that we have

$$\frac{|Z|}{1 - |x|^2} \frac{|Z| - |x - y|}{|Z| + |x - y|} \leqslant \frac{|f(x) - f(y)|}{|f'(x)||x - y|} \leqslant \frac{|Z|}{1 - |x|^2} \frac{|Z| + |x - y|}{|Z| - |x - y|} \tag{22}$$

Since $Z = 1 - |x|^2 - x(\overline{y - x})$ we have

$$1 - |x|^2 - |x - y| \leqslant |Z| \leqslant 1 - |x|^2 + |x - y|$$

Which leads to

$$1 - \frac{|x - y|}{1 - |x|^2} \leqslant \frac{|Z|}{1 - |x|^2} \leqslant 1 + \frac{|x - y|}{1 - |x|^2}.$$

But $|x| \leqslant 1-\delta$, so $\delta \leqslant 1-|x|^2 \leqslant 1$, and $|x-y| \leqslant \operatorname{diam}\mathbb{K}$ which gives

$$1 - \frac{\operatorname{diam}\mathbb{K}}{\delta} \leqslant \frac{|Z|}{1-|x|^2} \leqslant 1 + \frac{\operatorname{diam}\mathbb{K}}{\delta} \tag{23}$$

We have assumed that $\operatorname{diam}\mathbb{K} \leqslant \frac{\delta}{2}$ thus

$$\frac{|Z|+|x-y|}{|Z|-|x-y|} = 1 + 2\frac{|x-y|}{|Z|-|x-y|} \leqslant 1 + 2\frac{|x-y|}{1-|x|^2-2|x-y|} \leqslant 1 + \frac{2}{\delta(\delta-1)}\operatorname{diam}\mathbb{K},$$

and we prove the same way that

$$1 - \frac{2}{\delta}\operatorname{diam}\mathbb{K} \leqslant \frac{|Z|-|x-y|}{|Z|+|x-y|}.$$

Using those two last inequalities and (23) in (22) we end up with

$$\left(1 - \frac{2}{\delta}\operatorname{diam}\mathbb{K}\right)^2 \leqslant \frac{|f(x)-f(y)|}{|f'(x)||x-y|} \leqslant \left(1 + \frac{2}{\delta(\delta-1)}\operatorname{diam}\mathbb{K}\right)^2.$$

Using (21), with an adjustement of the constant, we may change $|f'(x)|$ with $|f'(z)|$, for any $z \in \mathbb{K}$, so that for any x, y and z in $\mathbb{K}$ we have

$$1 - C\operatorname{diam}\mathbb{K} \leqslant \frac{|f(x)-f(y)|}{|f'(z)||x-y|} \leqslant 1 + C\operatorname{diam}\mathbb{K}.$$

We easily conclude that for any $z \in \mathbb{K}$

$$(1 - C\operatorname{diam}\mathbb{K})\,|f'(z)| \leqslant \frac{\operatorname{diam} f(\mathbb{K})}{\operatorname{diam}\mathbb{K}} \leqslant (1 + C\operatorname{diam}\mathbb{K})\,|f'(z)|.$$

□

We use this result to prove that asserts that relevant functionals verify (ASI).

For any $X \in \mathcal{C}$ let $s(X) := \operatorname{diam}(X)$, where diam is the euclidean diameter of the geometric cylinder X, and let $\tilde{s} := |(\psi_X)'(0)|$.

Corollary 4.2. *There exists $C > 1$ and $q \in]0,1[$ such that for any cylinder $XYZ \in \mathcal{C}$ we have*

$$\begin{aligned} 1 - Cq^{|Y|} \quad &\leqslant \quad \tilde{s}_{XY}(Z)\left|\frac{(\psi_Y)'(0)}{(\psi_{YZ})'(0)}\right| \leqslant 1 + Cq^{|Y|} \\ &\textit{and} \\ 1 - Cq^{|Y|} \quad &\leqslant \quad s_{XY}(Z)\left|\frac{\operatorname{diam}(Y)}{\operatorname{diam}(YZ)}\right| \leqslant 1 + Cq^{|Y|} \end{aligned} \tag{24}$$

We have $s \sim \tilde{s}$ and they both verify (ASI) with $\beta_n = Cq^n$.

Proof. If we apply classical Koebe distortion estimates with $f = \psi_X$, $\mathbb{K} = \mathbb{X}_k$ and $x = 0$ we get

$$\frac{1}{C} \leqslant \frac{\operatorname{diam}(X)}{|(\psi_X)'(0)|} \leqslant C.$$

Thus s and $\tilde{s}$ are equivalent.

We have

$$\left|\frac{(\psi_{XYZ})'(0)}{(\psi_{XY})'(0)}\right| = \left|\frac{(\psi_X)'(\psi_{YZ}(0))}{(\psi_X)'(\psi_Y(0))}\right|\left|\frac{(\psi_{YZ})'(0)}{(\psi_Y)'(0)}\right|.$$

Since $\psi_{YZ}(0)$ and $\psi_Y(0)$ both lie in Y, by (19) we have

$$(1 - C\operatorname{diam}(Y)) \leqslant \left|\frac{(\psi_X)'(\psi_{YZ}(0))}{(\psi_X)'(\psi_Y(0))}\right| \leqslant (1 + C\operatorname{diam}(Y)),$$

from where we deduce, using also the fact that by hypothesis we have $\operatorname{diam}(Y) \leqslant Cq^{|Y|}$, that

$$(1 - Cq^{|Y|})\left|\frac{(\psi_{YZ})'(0)}{(\psi_Y)'(0)}\right| \leqslant \left|\frac{(\psi_{XYZ})'(0)}{(\psi_{XY})'(0)}\right| \leqslant (1 + Cq^{|Y|})\left|\frac{(\psi_{YZ})'(0)}{(\psi_Y)'(0)}\right|.$$

This can be rewritten

$$1 - Cq^{|Y|} \leqslant \tilde{s}_{XY}(Z)\left|\frac{(\psi_Y)'(0)}{(\psi_{YZ})'(0)}\right| \leqslant 1 + Cq^{|Y|}.$$

The same is also true with X' instead of X and this leads to

$$\left|\frac{\tilde{s}_{XY}(Z)}{\tilde{s}_{X'Y}(Z)} - 1\right| \leqslant Cq^{|Y|}.$$

One can apply relation (20) with $f = \psi_X$, $\mathbb{K} = YZ$, and also $\mathbb{K} = Y$, and $x \in YZ$, to get

$$\frac{1 - C\operatorname{diam}(YZ)}{1 + C\operatorname{diam}(Y)} \leqslant \frac{\operatorname{diam}(XYZ)}{\operatorname{diam}(XY)}\frac{\operatorname{diam}(Y)}{\operatorname{diam}(YZ)} \leqslant \frac{1 + C\operatorname{diam}(YZ)}{1 - C\operatorname{diam}(Y)}.$$

We have $\operatorname{diam}(YZ) < \operatorname{diam}(Y)$, and by hypothesis we know that $\operatorname{diam}(Y) \leqslant Cq^{|Y|}$. Adjusting the constant C, we easily conclude that we have

$$1 - Cq^{|Y|} \leqslant \frac{\operatorname{diam}(XYZ)}{\operatorname{diam}(XY)}\frac{\operatorname{diam}(Y)}{\operatorname{diam}(YZ)} \leqslant 1 + Cq^{|Y|}.$$

Which is exactly

$$1 - Cq^{|Y|} \leqslant s_{XY}(Z)\frac{\operatorname{diam}(Y)}{\operatorname{diam}(YZ)} \leqslant 1 + Cq^{|Y|}.$$

As a matter of fact this implies that for any other geometric cylinder $X' \subset \mathbb{X}$, with $|X'| = |X|$, that we have

$$\left|\frac{s_{XY}(Z)}{s_{X'Y}(Z)} - 1\right| \leqslant Cq^{|Y|}.$$

□

4.2 Controlling diameters under perturbation

We say that two NACIFS Ψ and $\tilde{\Psi}$ are compatible if for any n,

- the degree of Φ_n and the degree of $\tilde{\Phi}_n$ are the same,
- for any i and j, ${}_n\psi_i(V) \cap {}_n\tilde{\psi}_j(V) \neq \varnothing$ if and only if $i = j$.

For two compatible NACIFS Ψ and $\tilde{\Psi}$ we define

$$d(\Psi, \tilde{\Psi}) = \sup_n \max_{1 \leqslant i \leqslant d_n} \{\|{}_n\psi_i - {}_n\tilde{\psi}_i\|_\infty\},$$

with $\|f\|_\infty = \sup_{z \in \bar{\mathbb{D}}_\gamma} |f(z)|$.

Proposition 4.3. *Assume that Ψ and $\tilde{\Psi}$ are two compatible NACIFS and that $0 < \eta < 1$ is the constant controling their annulus and their contractions. Let $\alpha = \dfrac{\log(1-\eta)}{\log(\eta(1-\eta))}$. For any symbolic cylinder X let $s(X) = |\psi'_X(0)|$ and $\tilde{s}(X) = |\tilde{\psi}'_X(0)|$. There exists $C > 0$ such that*

$$\mathcal{D}(s, \tilde{s}) \leqslant C d(\Psi, \tilde{\Psi})^{\alpha}.$$

Proof. Let $A_n = sup_{|X|=n} \|\psi_X - \tilde{\psi}_X\|_\infty$. Note that we have $A_1 = d(\Psi, \tilde{\Psi})$.

Let Xa be a cylinder with $|X| = n$ and $|a| = 1$. For any $z \in \mathbb{D}_\gamma$ we have

$$\begin{aligned} \left|\psi_{Xa}(z) - \tilde{\psi}_{Xa}(z)\right| &\leqslant \left|\psi_X(\psi_a(z)) - \psi_X(\tilde{\psi}_a(z))\right| + \left|\psi_X(\tilde{\psi}_a(z)) - \tilde{\psi}_X(\tilde{\psi}_a(z))\right| \\ &\leqslant (1-\eta)^n |\psi_a(z) - \tilde{\psi}_a(z)| + A_n \\ &\leqslant (1-\eta)^n A_1 + A_n, \end{aligned}$$

from where we deduce that

$$A_n \leqslant \frac{1}{\eta} A_1 = \frac{1}{\eta} d(\Psi, \tilde{\Psi}) \tag{25}$$

Using Cauchy representation theorem, for any cylinder X with $|X| = k$ we have

$$\left|\psi'_X(0) - \tilde{\psi}'_X(0)\right| = \frac{1}{2\pi} \left| \int_{\partial \mathbb{D}} \frac{\psi_X(z) - \tilde{\psi}_X(z)}{z^2} dz \right| \leqslant A_k.$$

Since by assumption $\eta^k \leqslant \left|\tilde{\psi}'_X(0)\right|$, we deduce that

$$\left| \frac{\psi'_X(0)}{\tilde{\psi}'_X(0)} \right| \leqslant 1 + \frac{1}{\eta^{k+1}} d(\Psi, \tilde{\Psi}). \tag{26}$$

Inequality which is true for any cylinder X such that $|X| \leqslant k$, and stays true if we exchange the rôle played by ψ and $\tilde{\psi}$.

Let now $X = YX_k$ be any cylinder of length n, with $X_k = \varnothing$ if $n \leqslant k$, and $|X_k| = k$ otherwise.

By chain rule we have

$$\psi'_{YX_k}(0) = \psi'_Y(\psi_{X_k}(0)) \psi'_{X_k}(0).$$

So that for any cylinder a of length 1 we have

$$\frac{s_X(a)}{\tilde{s}_X(a)} = \left| \frac{\psi'_Y(\psi_{X_k a}(0))}{\psi'_Y(\psi_{X_k}(0))} \right| \left| \frac{\tilde{\psi}'_Y(\tilde{\psi}_{X_k}(0))}{\tilde{\psi}'_Y(\tilde{\psi}_{X_k a}(0))} \right| \left| \frac{\psi'_{X_k a}(0)}{\tilde{\psi}'_{X_k a}(0)} \right| \left| \frac{\tilde{\psi}'_{X_k}(0)}{\psi'_{X_k}(0)} \right|$$

From (19) in proposition 4.1 there exists $C > 0$ such that

$$\left| \frac{\psi'_Y(\psi_{X_k a}(0))}{\psi'_Y(\psi_{X_k}(0))} \right| \leqslant 1 + C \mathrm{diam} X_k \quad \text{and} \quad \left| \frac{\tilde{\psi}'_Y(\tilde{\psi}_{X_k}(0))}{\tilde{\psi}'_Y(\tilde{\psi}_{X_k a}(0))} \right| \leqslant 1 + C \mathrm{diam} \tilde{X}_k.$$

And from (26)

$$\left| \frac{\psi'_{X_k a}(0)}{\tilde{\psi}'_{X_k a}(0)} \right| \left| \frac{\tilde{\psi}'_{X_k}(0)}{\psi'_{X_k}(0)} \right| \leqslant \left(1 + \frac{1}{\eta^{k+1}} d(\Psi, \tilde{\Psi}) \right)^2.$$

Using the fact that $\mathrm{diam} X_k$ and $\mathrm{diam}\tilde{X}_k$ are both smaller than $(1-\eta)^k$, and adjusting the constant C we end up with

$$\frac{s_X(a)}{\tilde{s}_X(a)} \leqslant 1 + C\left((1-\eta)^k + \frac{1}{\eta^k} d(\Psi, \tilde{\Psi})\right) = 1 + \frac{C}{\eta^k}\left((\eta(1-\eta))^k + d(\Psi, \tilde{\Psi})\right). \tag{27}$$

Let k be the integer part of $\dfrac{\log(d(\Psi, \tilde{\Psi}))}{\log(\eta(1-\eta))} + 1$, then $k \geqslant \dfrac{\log(d(\Psi, \tilde{\Psi}))}{\log(\eta(1-\eta))}$ and $(\eta(1-\eta))^k \leqslant d(\Psi, \tilde{\Psi})$, so that we have

$$\frac{s_X(a)}{\tilde{s}_X(a)} \leqslant 1 + 2C\frac{d(\Psi, \tilde{\Psi})}{\eta^k}.$$

On the other side we have $k \leqslant \dfrac{\log(d(\Psi, \tilde{\Psi}))}{\log(\eta(1-\eta))} + 1$ so that

$$\frac{1}{\eta^k} \leqslant \frac{1}{\eta} \exp\left(\frac{\log(d(\Psi, \tilde{\Psi}))}{\log(\eta(1-\eta))} \log\frac{1}{\eta}\right) = \frac{1}{\eta}\left(\frac{1}{d(\Psi, \tilde{\Psi})}\right)^{\frac{\log\eta}{\log(\eta(1-\eta))}}.$$

Which leads, with $\alpha = \frac{\log(1-\eta)}{\log(\eta(1-\eta))}$, to

$$\frac{s_X(a)}{\tilde{s}_X(a)} \leqslant 1 + Cd(\Psi, \tilde{\Psi})^\alpha,$$

and gives

$$\left|\log\left(\frac{s_X(a)}{\tilde{s}_X(a)}\right)\right| \leqslant Cd(\Psi, \tilde{\Psi})^\alpha.$$

This being true for any cylinder X we conclude that

$$\mathcal{D}(s, \tilde{s}) \leqslant Cd(\Psi, \tilde{\Psi})^\alpha.$$

□

4.3 Controlling harmonic measure

In this section we extend well known properties of harmonic measure of usual IFS to NACIFS. We have tried to give all the ideas without drowning the reader with the total amount of technicalities.

Let Ω be a domain in $\mathbb{R}^2$ and $F \subset \partial\Omega$. For $x \in \Omega$ we denote $\omega(x, F, \Omega)$ the harmonic measure of F in Ω evaluated at x. If Ω^c is bounded, we denote $\omega(F, \Omega)$ the harmonic measure of F in Ω evaluated at infinity.

We will frequently make use of the following (see for instance [Bre69]): If $\Omega \subset \Omega'$ are two domains, $F \subset \partial\Omega \cap \partial\Omega'$ and $x \in \Omega$, then

$$\omega(x, F, \Omega) = \omega(x, F, \Omega') - \int_{z \in \partial\Omega \cap \Omega'} \omega(z, F, \Omega')\omega(x, dz, \Omega)$$

For α, β positive real functions we will write $\alpha \overset{c}{\sim} \beta$ if $\frac{1}{c}\alpha \leqslant \beta \leqslant c\alpha$ with $c > 0$ constant. Finally, with no loss of generality we will take $U = \mathbb{D}$ the unit disc.

Remember that for any symbolic cylinder $X = a_1 \cdots a_n$ we note $\Psi_X = {}_1\psi_{a_1} \circ \cdots \circ {}_n\psi_{a_n}$, and that we also denote X the geometric cylinder $\mathbb{X} \cap \Psi_X(\mathbb{D})$.

In the following we will deal with two NACIFS Ψ and $\tilde{\Psi}$ and we will denote $\tilde{X} = \tilde{\mathbb{X}} \cap \tilde{\Psi}_X(\mathbb{D})$ the geometric cylinder for $\tilde{\Psi}$.

In order to apply the previous machinery we need to prove the following result

Theorem 4.4. *Let Ψ and $\tilde{\Psi}$ be two NACIFS, $\mathbb{X}$, $\tilde{\mathbb{X}}$ the corresponding limit (Cantor) sets and ω, $\tilde{\omega}$ the harmonic measures on these sets.*

We have :

$$\lim_{d(\Psi,\tilde{\Psi})\to 0} \mathcal{D}(\omega,\tilde{\omega}) = 0\cdot \tag{28}$$

The proof follows the ideas of [Bat06] and heavily relies on a repeated use of the following well known results.

Proposition 4.5. *([MV86],[Car85]) Let Ω be a domain containing ∞ and let $A_1 \subset B_1 \subset A_2 \subset B_2 \subset ... \subset A_n \subset B_n \subset \Omega$ be conformal discs such that the annuli $B_i\backslash A_i$ are contained in Ω, for $1 \leqslant i \leqslant n$. If the modules of the annuli are uniformly bounded away from zero and if $\infty \in \Omega\backslash B_n$ then, for all pairs of positive harmonic functions u, v vanishing on $\partial\Omega\backslash A_1$ and for all $x \in \Omega\backslash B_n$ we have*

$$\left|\frac{u(x)}{v(x)} : \frac{u(\infty)}{v(\infty)} - 1\right| \leqslant Kq^n \tag{29}$$

where $q < 1$ and K are two constants that depend only on the lower bound of the modules of the annuli.

Lemma 4.6. *Let Ψ be a NACIFS, $\mathbb{X}$ the corresponding limit (Cantor) set and ω the harmonic measures on this sets. There exists $c > 0$ depending only on $\gamma > 1$, small enough so that $\gamma\mathbb{D} \subset V$, such that $\omega(x, \mathbb{X}, \mathbb{D}_\gamma\backslash\mathbb{X}) \geqslant c$ whenever $|x| = 1$.*

Proof. The proof implies standard arguments concerning capacity and harmonic measure on Cantor type sets: we hence leave some details to the reader (see also [Bat96]).

Let μ be the uniform probability measure on $\mathbb{X}$, that is every cylinder of the n^{th} generation is charged by $1/(d_1 \cdot \ldots \cdot d_n)$ mass. For $x \neq y$ in $\mathbb{D}$ the Green function satisfies $\ln\left(\frac{C_1}{|x-y|}\right) \leqslant G(x,y) \leqslant \ln\left(\frac{C_2}{|x-y|}\right)$, where C_1, C_2 are constants depending on η.

Therefore, for $y \in \mathbb{X}$,

$$\sum_{k=1}^{\infty} (d_1 \cdot \ldots \cdot d_k)^{-1} \ln(C_1/(1-\eta)^k) \leqslant G\mu(y) \leqslant \sum_{k=1}^{\infty} (d_1 \cdot \ldots \cdot d_k)^{-1} \ln(C_2/\eta^k),$$

which implies $G\mu \overset{c}{\sim} 1$ on $\mathbb{X}$. It follows on the maximum principle that $G\mu \overset{c}{\sim} \omega_{\mathbb{D}\backslash\mathbb{X}}(\mathbb{X})$ in $\mathbb{D}$. On the other hand, clearly $G\mu(x) \geqslant c > 0$ for $x \in \mathbb{D}$ and hence the result. □

The following lemma is a first application of this result. It will subsequently be used several times to obtain "localization" results for harmonic measure.

Lemma 4.7. *Let Ψ be a NACIFS and $\mathbb{X}$ its limit Cantor set. Assume that $\gamma > 1$ is small enough so that $\mathbb{D}_\gamma := \{|z| < \gamma\} \subset V$. There exists $C > 0$ depending only on γ, so that for any cylinder $X \in \mathcal{C}$, and any $x \in \partial\mathbb{D}_X$ we have*

$$1 \leqslant \frac{\omega(x, X, \hat{\mathbb{C}}\backslash\mathbb{X})}{\omega(x, X, \mathbb{D}_{\gamma X}\backslash X)} \leqslant C\cdot$$

Proof. Let $\Gamma := \{|x| = \gamma\}$, $u : x \mapsto \omega(x, X, \hat{\mathbb{C}}\backslash\mathbb{X})$, $\tilde{u} : x \mapsto \omega(x, X, \mathbb{D}_{\gamma X}\backslash X)$ and $h = u - \tilde{u}$. The function h is harmonic on $\mathbb{D}_{\gamma X}\backslash X$, with $h = 0$ on X and $h = u$ on $\partial\mathbb{D}_{\gamma X}$ so that

$$h(x) = \int_{\partial\mathbb{D}_{\gamma X}} u(z)\omega(x, dz, \mathbb{D}_{\gamma X}\backslash\mathbb{X})\cdot$$

Let $x_\gamma \in \partial\mathbb{D}_X$ be such that for any $x \in \partial\mathbb{D}_X$, $u(x) \leqslant u(x_\gamma)$, and $x_m \in \partial\mathbb{D}_{\gamma X}$ be such that for any $x \in \partial\mathbb{D}_{\gamma X}$ we have $u(x) \leqslant u(x_m)$.

The function $x \mapsto u(x) - u(x_\gamma)$ is harmonic outside $\mathbb{D}_X \cup \mathbb{X}$, non-positive on $\partial\mathbb{D}_X \cup \mathbb{X}\backslash X$, it is thus negative on $\mathbb{C}\backslash\mathbb{D}_X$ and in particular we have $u(x_m) < u(x_\gamma)$.

We deduce the following

$$\tilde{u}(x_\gamma) = u(x_\gamma) - \int_{\partial\mathbb{D}_{\gamma X}} u(z)\omega(x_\gamma, dz, \mathbb{D}_{\gamma X}\backslash\mathbb{X}) \geqslant u(x_\gamma)\left(1 - \omega(x_\gamma, \partial\mathbb{D}_{\gamma X}, \mathbb{D}_X\backslash\mathbb{X})\right) = u(x_\gamma)\omega(x_\gamma, X, \mathbb{D}_{\gamma X}\backslash X).$$

By conformal invariance, with $n = |X|$, we have

$$\omega(x_\gamma, X, \mathbb{D}_{\gamma X}\backslash\mathbb{X}) = \omega(\Phi_X(x_\gamma), \mathbb{X}_n, \mathbb{D}_\gamma\backslash\mathbb{X}_n),$$

with by definition $|\Phi_X(x_\gamma)| = 1$. Lemma 4.6 leads to $\tilde{u}(x_\gamma) \geqslant u(x_\eta)c$, since $\mathbb{X}_n$ is the limit Cantor set associated to an NACIFS with the same constant than $\mathbb{X}$.

We conclude that we have : $1 \leqslant \dfrac{u(x_\gamma)}{\tilde{u}(x_\gamma)} \leqslant C$.

By Harnack inequality on $\partial\mathbb{D}_X$ we conclude that $1 \leqslant \dfrac{u(x)}{\tilde{u}(x)} \leqslant C$ is true for any $x \in \partial\mathbb{D}_X$, with a different C, still uniquely depending on γ. □

We also need to establish the following lemma that restrains the number of generations necessary to establish an estimate on distribution of harmonic measure.

Lemma 4.8. *Let Ψ be a NACIFS, $\mathbb{X}$ its limit Cantor set and for any $\ell \geqslant 1$ let $\mathbb{X}_\ell = T^\ell(\mathbb{X})$.*

There exist $q < 1$ and $K > 0$ depending only on η such that for any $XYZ \in \mathcal{C}$ with $|X| = k$, $|Y| = n$ we have

$$\left|\log\left(\frac{\omega(XYZ)}{\omega(XY)}\frac{\omega_k(Y)}{\omega_k(YZ)}\right)\right| < Kq^n \tag{30}$$

The proof of this lemma follows the lines of the one in [Bat06], therefore we will only give the highlights and main ideas of proof to facilitate the reading.

Proof. Let $u : x \mapsto \frac{\omega(x, XYZ, \hat{\mathbb{C}}\backslash\mathbb{X})}{\omega(XYZ)}$ and $v : x \mapsto \frac{\omega(x, XY, \hat{\mathbb{C}}\backslash\mathbb{X})}{\omega(XY)}$. Note that $u(\infty) = v(\infty) = 1$.

Assume that $Y = y_1 \cdots y_n$ and for any $\ell \in [1, n]$ let $X_0 = X$, $X_\ell = Xy_1 \cdots y_\ell$, $A_\ell = \mathbb{D}_{X_{n-\ell}}$ and $B_\ell = \mathbb{D}_{\gamma X_{n-\ell}}$.

We have $A_0 \subset B_0 \subset \cdots \subset A_{n-1} \subset B_{n-1} \subset \hat{\mathbb{C}}\backslash\mathbb{X}$, and each annulus $B_i\backslash A_i$ is conformally equal to $\mathbb{D}_\gamma\backslash\mathbb{D}$.

The function u and v are both harmonic in $\hat{\mathbb{C}}\backslash\mathbb{X}$, and they vanished on $\mathbb{X}\backslash XY$.

We are thus in position to apply proposition 4.5 to conclude that for any $x \notin \mathbb{X} \cup \mathbb{D}_{\gamma X y_1}$, and in particular for any $x \in \partial\mathbb{D}_X \cup \partial\mathbb{D}_{\gamma X}$

$$\left|\log\left(\frac{\omega\left(x, XYZ, \hat{\mathbb{C}}\backslash\mathbb{X}\right)}{\omega\left(x, XY, \hat{\mathbb{C}}\backslash\mathbb{X}\right)}\frac{\omega(XY)}{\omega(XYZ)}\right)\right| = \left|\log\left(\frac{u(x)}{v(x)}\frac{v(\infty)}{u(\infty)}\right)\right| \leqslant Cq^n. \tag{31}$$

Let now $\tilde{u} : x \mapsto \frac{\omega(x, XYZ, \mathbb{D}_{\gamma X}\backslash X)}{\omega(XYZ)}$ and $\tilde{v} : x \mapsto \frac{\omega(x, XY, \mathbb{D}_{\gamma X}\backslash X)}{\omega(XY)}$. We have for any $x \in \mathbb{D}_{\gamma X}\backslash X$

$$\tilde{u}(x) = u(x) - \int_{\partial\mathbb{D}_{\gamma X}} u(z)\omega(x, dz, \mathbb{D}_{\gamma X}\backslash X),$$

and also

$$\tilde{v}(x) = v(x) - \int_{\partial\mathbb{D}_{\gamma X}} v(z)\omega(x, dz, \mathbb{D}_{\gamma X}\backslash X).$$

We thus have

$$\tilde{u}(x) - \tilde{v}(x) = u(x) - v(x) - \int_{\partial \mathbb{D}_{\gamma X}} (u(z) - v(z)) \omega(x, dz, \mathbb{D}_{\gamma X} \backslash X).$$

Which may be written :

$$\frac{\tilde{u}(x)}{\tilde{v}(x)} - 1 = \frac{v(x)}{\tilde{v}(x)} \left(\left(\frac{u(x)}{v(x)} - 1 \right) - \int_{\partial \mathbb{D}_{\gamma X}} \frac{v(z)}{v(x)} \left(\frac{u(z)}{v(z)} - 1 \right) \omega(x, dz, \mathbb{D}_{\gamma X} \backslash X) \right).$$

Since (31) is true for any $x \in \partial \mathbb{D}_X \cup \partial \mathbb{D}_{\gamma X}$, and by lemma 4.7, we conclude that there exist $C > 0$ and $q < 1$ such that for any $x \in \partial \mathbb{D}_X$

$$\left| \frac{\tilde{u}(x)}{\tilde{v}(x)} - 1 \right| \leqslant C q^n \left(1 + \int_{\partial \mathbb{D}_{\gamma X}} \frac{v(z)}{v(x)} \omega(x, dz, \mathbb{D}_{\gamma X} \backslash X) \right) = C q^n \left(1 + \frac{v(x) - \tilde{v}(x)}{v(x)} \right) \leqslant 2 C q^n.$$

From this we conclude that for any $x \in \partial \mathbb{D}_X$ we have

$$\left| \log \left(\frac{\omega(x, XYZ, \mathbb{D}_{\gamma X} \backslash X)}{\omega(x, XY, \mathbb{D}_{\gamma X} \backslash X)} \frac{\omega(XY)}{\omega(XYZ)} \right) \right| = \left| \log \left(\frac{\tilde{u}(x)}{\tilde{v}(x)} \right) \right| \leqslant C q^n. \tag{32}$$

By conformal invariance of harmonic measure, we conclude that for any $x \in \partial \mathbb{D}$

$$\left| \log \left(\frac{\omega(x, YZ, \mathbb{D}_\gamma \backslash \mathbb{X}_k)}{\omega(x, Y, \mathbb{D}_\gamma \backslash \mathbb{X}_k)} \frac{\omega(XY)}{\omega(XYZ)} \right) \right| \leqslant C q^n.$$

Applying (31) with $\mathbb{X}_k$ and ω_k, this is possible since the NACIFS giving birth to $\mathbb{X}_k$ is comparable to the one giving birth to $\mathbb{X}$, we get for any $x \in \partial \mathbb{D}$

$$\left| \log \left(\frac{\omega(x, YZ, \mathbb{D}_\gamma \backslash \mathbb{X}_k)}{\omega(x, Y, \mathbb{D}_\gamma \backslash \mathbb{X}_k)} \frac{\omega_k(Y)}{\omega_k(YZ)} \right) \right| \leqslant C q^n. \tag{33}$$

Combining (31,32,33) we end up with

$$\left| \log \left(\frac{\omega(XYZ)}{\omega(XY)} \frac{\omega_k(Y)}{\omega_k(YZ)} \right) \right| \leqslant C q^n. \tag{34}$$

□

From this result we easily deduce that harmonic measure verifies the two assumptions needed to apply main results of the previous section : (ASI) and uniform control on the measure of cylinders.

Corollary 4.9. *Let Ψ be a NACIF, $\mathbb{X}$ its limit set and ω the harmonic measure on $\mathbb{C} \backslash \mathbb{X}$ evaluated at ∞. Under our geometric assumptions on the NACIF there exists $q \in]0, 1[$ and $C > 0$ such that for any geometric cylinders $XYZ \in \mathcal{C}$ and $X'YZ \in \mathcal{C}$, with $|X| = |X'|$ and $|Y| = n$*

$$\left| \log \left(\frac{\omega_{XY}(Z)}{\omega_{X'Y}(Z)} \right) \right| \leqslant C q^n \tag{35}$$

$$|\log(\omega_X(Y))| \leqslant Cn \tag{36}$$

Proof. Relation (34) asserts that for any $XYZ \in \mathcal{C}$ with $|X| = k$ and $|Y| = n$:

$$\left| \log \left(\omega_{XY}(Z) \frac{\omega_k(Y)}{\omega_k(YZ)} \right) \right| \leqslant C q^n,$$

from where we get, for any $X' \in \mathcal{C}$ with $|X'| = |X|$

$$\begin{aligned}\left|\log\left(\frac{\omega_{XY}(Z)}{\omega_{X'Y}(Z)}\right)\right| &= \left|\log\left(\omega_{XY}(Z)\frac{\omega_k(Y)}{\omega_k(YZ)}\right) - \log\left(\omega_{X'Y}(Z)\frac{\omega_k(Y)}{\omega_k(YZ)}\right)\right| \\ &\leqslant \left|\log\left(\omega_{XY}(Z)\frac{\omega_k(Y)}{\omega_k(YZ)}\right)\right| + \left|\log\left(\omega_{X'Y}(Z)\frac{\omega_k(Y)}{\omega_k(YZ)}\right)\right| \\ &\leqslant Cq^n\end{aligned}$$

From (34) we deduce that there exists $C > 1$ such that

$$\frac{1}{C}\omega_k(Z) \leqslant \frac{\omega(XZ)}{\omega(X)} \leqslant 1.$$

Assume that $Z = z_0 \cdots z_{n-1}$ and for any $i \in [\![0, n-1]\!]$ let $Z_i = z_0 \cdots z_i$ then we have :

$$\omega_k(Z) = \omega_k(z_0)\prod_{i=1}^{n-1}\frac{\omega_k(Z_i)}{\omega_k(Z_{i-1})},$$

since for any $i \in [\![1, n-1]\!]$ relation (34) tells us that : $\frac{1}{C}\omega_{k+i}(z_i) \leqslant \frac{\omega_k(Z_i)}{\omega_k(Z_{i-1})} \leqslant 1$, we conclude that

$$\frac{1}{C^n}\prod_{i=1}^{n-1}\omega_{k+i}(z_i) \leqslant \frac{\omega(XZ)}{\omega(X)} \leqslant 1.$$

From Lemma 4.6 we can deduce that given a NACIF Ψ there exists a constant $\alpha > 0$, depending only the geometric constants that constrain the geometry of the NACIF, such that for any $\ell \geqslant 0$ and any $a \in \mathcal{C}_\ell$ with $|a| = 1$, we have $\omega_\ell(a) \geqslant \alpha$, we conclude that

$$\left(\frac{\alpha}{C}\right)^n \leqslant \frac{\omega(XZ)}{\omega(X)} \leqslant 1,$$

and the result follows.

□

We turn to a comparison of harmonic measures of the limit sets X and $\tilde{X}$.

Lemma 4.10. *Let $\gamma > 1$ such that $\mathbb{D}_\gamma \subset V$ appearing in the definition of the NACIF Ψ. There exist $K > 0$ and $\alpha > 0$ such that for any $x \in \mathbb{D}\backslash\mathbb{X}$:*

$$\omega(x, \mathbb{X}, \mathbb{D}_\gamma\backslash\mathbb{X}) \geqslant 1 - K\,\mathrm{dist}(x, \mathbb{X})^\alpha.$$

Proof. Given any $x \in \mathbb{D}\backslash\mathbb{X}$ let $n(x) = \max\{k \in \mathbb{N} \,|\, \exists X \in \mathcal{C}_n\,,\, x \in \mathbb{D}_X\}$. Note that $n(x)$ is such that, there exists $X \in \mathcal{C}_n$ with $x \in \mathbb{D}_X$ and $x \notin \mathbb{D}_{Xa}$ for any cylinder $Xa \in \mathcal{C}_{n+1}$.

Using open set and bounded contraction hypothesis, we deduce from Kœbe distortion theorem that there exist three constants $K' > 0$ and $0 < \beta' < \beta$ such that

$$\frac{1}{K'}\,\mathrm{dist}(x, \mathbb{X})^\beta \leqslant e^{-n} \leqslant K'\,\mathrm{dist}(x, \mathbb{X})^{\beta'} \tag{37}$$

It is important to note that K', β' and β only depend on the constant η used to control the geometry of the NACIF.

Let $x \in \mathbb{D}_X$, with $X = a_1 \cdots a_n$ and $|X| = n(x) := n$. Let $X_0 = \mathbb{X}$ and, for any $1 \leqslant k \leqslant n$, $X_k = a_1 \cdots a_k$ and set $u_k : y \mapsto \omega(y, X_k, \mathbb{D}_{\gamma X_k}\backslash X_k)$. The function u_k is harmonic on $\mathbb{D}_{\gamma X_k}\backslash X_k$, and equal 1 on X_k. Our goal is to prove that

$$u_0(x) = \omega(x, \mathbb{X}, \mathbb{D}_\gamma\backslash\mathbb{X}) \geqslant 1 - K\,\mathrm{dist}(x, \mathbb{X})^\alpha.$$

First we note, using Lemma 4.6, conformal invariance and maximum principle that there exists $c > 0$ such that for any $x \in \mathbb{D}$

$$u_k(x) \geqslant c. \tag{38}$$

For any $1 \leqslant k \leqslant n$ we have

$$u_{k-1}(x) - u_k(x) = \int_{\partial \mathbb{D}_{\gamma X_k}} u_{k-1}(z)\omega(x, dz, \mathbb{D}_{\gamma X_k} \backslash X_k).$$

Because of our geometric assumptions on the NACIF and Harnack inequalities, there exists a constant $\lambda > 0$ such that for any $Y \in \mathcal{C}$, any continuous function $h > 0$, defined on $\mathbb{D}_Y$ and harmonic on $\mathbb{D}_Y \backslash Y$, and any y and y' in $\mathbb{D}_Y \backslash Y$: $\lambda h(y) \leqslant h(y')$.

From this we deduce that

$$u_{k-1}(x) - u_k(x) \geqslant \lambda u_{k-1}(x)\omega(x, \partial \mathbb{D}_{\gamma X_k}, D_{\gamma X_k} \backslash X_k) = \lambda u_{k-1}(x)(1 - u_k(x)).$$

Let $v_k = 1 - u_k$, we drop x to lighten notations.
Using (38) we get : $\lambda c v_k \leqslant \lambda u_k v_k \leqslant v_k - v_{k-1}$ and

$$c\lambda \leqslant \frac{v_k - v_{k-1}}{v_k} \leqslant \int_{v_{k-1}}^{v_k} \frac{1}{t} dt = \log(v_k) - \log(v_{k-1}).$$

Summing for $1 \leqslant k \leqslant n$

$$c\lambda n \leqslant \log v_n - \log v_0,$$

so that

$$1 - u_0 = v_0 \leqslant e^{-c\lambda n}.$$

We use (37) to conclude that : $u_0(x) = \omega(x, \mathbb{X}, \mathbb{D}_\gamma \backslash \mathbb{X}) \geqslant 1 - \left(\frac{1}{K'} \operatorname{dist}(x, \mathbb{X})^\beta\right)^{c\lambda}$. This is the desired result with $K = \frac{1}{K'^{c\lambda}}$ and $\alpha = c\lambda\beta$. □

This corollary happens to be crucial in order to prove continuity of dimensions.

Corollary 4.11. *Let $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that for any $XY \in \mathcal{C}$ with $|Y| = n$ and any $x \in \mathbb{D}_{XY}$*

$$\omega(x, X, \mathbb{C} \backslash \mathbb{X}) \geqslant 1 - \varepsilon.$$

Proof. Let $\varepsilon > 0$ and $\eta > 0$ given by Lemma 4.10. Let n big enough so that for any $k \in \mathbb{N}$ and any $Y \in \mathcal{C}$ with $|Y| = n$ we have $\operatorname{diam} \mathbb{D}_{\gamma Y} < \eta$. Let $X \in \mathcal{C}$ and $x \in \partial D_{XY}$, with $|Y| = n$. We have

$$\omega(x, X, \mathbb{C} \backslash \mathbb{X}) \geqslant \omega(x, X, \mathbb{D}_{\gamma X} \backslash \mathbb{X}).$$

By conformal invariance, with $k = |X|$, we have

$$\omega(x, X, \mathbb{D}_{\gamma X} \backslash \mathbb{X}) = \omega_k(\varphi_X(x), \mathbb{X}_k, \mathbb{D}_\gamma \backslash \mathbb{X}_k).$$

Since $x \in \partial \mathbb{D}_{XY}$ we have $\varphi_X(x) \in \partial \mathbb{D}_Y$, so that $\operatorname{dist}(\varphi_X(x), \mathbb{X}_k) \leqslant \operatorname{diam} \mathbb{D}_{\gamma Y} < \eta$ and Lemma 4.10 gives : $\omega_k(\varphi_X(x), \mathbb{X}_k, \mathbb{D}_\gamma \backslash \mathbb{X}_k) > 1 - \varepsilon$.

We have proved that for any $x \in \partial D_{XY}$ we have

$$\omega(x, X, \mathbb{C} \backslash \mathbb{X}) \geqslant \omega(x, X, \mathbb{D}_{\gamma XY} \backslash \mathbb{X}) = \omega_k(\varphi_X(x), \mathbb{X}_k, \mathbb{D}_\gamma \backslash \mathbb{X}_k) \geqslant 1 - \varepsilon.$$

This inequality extends to $\mathbb{D}_{XY}$ by maximum principle.

□

Proof of theorem 4.4. Let $\varepsilon > 0$ and $N_1 \in \mathbb{N}$ big enough so that Cq^{N_1} in Corollary 4.9 is strictly less than ε.

From Corollary 4.9 we know that : $cq^{N_1} \leqslant \omega_k(Z)$ for any $k \in \mathbb{N}$, any $Z \in T^k(\mathcal{C})$ with $|Z| \leqslant N_1$. Let $\varepsilon' = cq^{N_1}\varepsilon$. For any $k \in \mathbb{N}$, any $Z \in T^k(\mathcal{C})$ with $|Z| \leqslant N_1$ we have

$$\frac{\varepsilon'}{\omega(X)} \leqslant \varepsilon. \tag{39}$$

Let N_2 be given by Corollary 4.11 with $\varepsilon' > 0$ so that for any $XY \in \mathcal{C}$ with $|Y| = N_2$ and any $x \in \partial\mathbb{D}_{XY}$ we have : $\omega(x, X, \mathbb{C}\backslash\mathbb{X}) \geqslant 1 - \varepsilon'$.

Let now $d(\Psi, \tilde{\Psi})$ be as small as necessary to ensure that for any $k \in \mathbb{N}$, any $Y \in T^k(\mathcal{C})$ with $|Y| \leqslant N_1 + N_2$ we have $\tilde{Y} \subset \mathbb{D}_Y$. Where $\tilde{Y}$ is the geometric cylinder associated to $\tilde{\Psi}$ with coding Y.

Let $Xa \in \mathcal{C}$ with $|a| = 1$ and $|X| = k$. We have

$$\log\left(\frac{\omega_X(a)}{\tilde{\omega}_X(a)}\right) = \log\left(\frac{\omega_k(a)}{\tilde{\omega}_k(a)}\right) + \log\left(\frac{\omega_X(a)}{\tilde{\omega}_X(a)}\frac{\tilde{\omega}_k(a)}{\omega_k(a)}\right).$$

If $|X| = k > N_1$ we have by Corollary 4.9

$$\log\left(\frac{\omega_X(a)}{\tilde{\omega}_X(a)}\right) \leqslant \log\left(\frac{\omega_k(a)}{\tilde{\omega}_k(a)}\right) + \varepsilon \tag{40}$$

We are thus led to prove that $\log\left(\frac{\omega_X(a)}{\tilde{\omega}_X(a)}\right)$ is small for any cylinder X with $|X| \leqslant N_1$.

Actually we are going to prove that $\log\left(\frac{\omega_k(X)}{\tilde{\omega}_k(X)}\right)$ is small for any cylinder X with $|X| \leqslant N_1$ and any k.

In order to lighten notations, we drop the letter k. Remember that our estimates about harmonic measure are only depending on the geometric constants in our hypothesis : Bounded contraction and Annulus condition.

Let $|X| = k \leqslant N_1$, $F = \bigcup_{|Z|=N_1+N_2} \mathbb{D}_{\gamma Z}$ and $\Omega = \mathbb{C}\backslash F$. Our choices imply that for any Z with $|Z| = N_1 + N_2$ we have $\tilde{Z} \subset \mathbb{D}_Z$ so that $\tilde{\mathbb{X}} \subset F$.

The functions $u : x \mapsto \omega(x, X, \mathbb{C}\backslash\mathbb{X})$ and $v : x \mapsto \omega(x, \tilde{X}, \mathbb{C}\backslash\tilde{\mathbb{X}})$ are harmonic on Ω.

Let $|Z| = N_1 + N_2$ with $Z \subset X$, then from Corollary 4.11 we deduce, for any $x \in \partial_{\gamma Z}$, that $u(x) \geqslant 1 - \varepsilon$ as well as $v(x) \geqslant 1 - \varepsilon'$, which implies that : $|u(x) - v(x)| \leqslant \varepsilon'$ for any $x \in \partial_{\gamma Z}$.

Let now $Z = YW$, with $|Y| = |X|$, $|Z| = N_1 + N_2$ and $Z \not\subset X$. By Corollary 4.11 we have $\omega(x, Y, \mathbb{C}\backslash\mathbb{X}) > 1 - \varepsilon'$ and also $\omega(x, \tilde{Y}, \mathbb{C}\backslash\tilde{\mathbb{X}}) > 1 - \varepsilon'$ for any $x \in \partial\mathbb{D}_{\gamma Z}$.

Since

$$1 = \sum_{|V|=|X|} \omega(x, V, \mathbb{C}\backslash\mathbb{X}) = \sum_{|V|=|X|} \omega(x, \tilde{V}, \mathbb{C}\backslash\tilde{\mathbb{X}}),$$

we deduce that $u(x) \leqslant 1 - \omega(x, Y, \mathbb{C}\backslash\mathbb{X}) \leqslant \varepsilon'$ and $v(x) \leqslant 1 - \omega(x, \tilde{Y}, \mathbb{C}\backslash\tilde{\mathbb{X}}) \leqslant \varepsilon'$. We thus have, for any $x \in \partial\mathbb{D}_{\gamma Z}$, $|u(x) - v(x)| \leqslant 2\varepsilon'$.

The functions u and v are both harmonic on Ω and for any $x \in \partial\Omega$ we have $|u(x) - v(x)| \leqslant 2\varepsilon'$. By maximum principle we have $|u(x) - v(x)| \leqslant 2\varepsilon'$ for any $x \in \Omega$ and in particular for $x = \infty$, so that $|\omega(X) - \tilde{\omega}(X)| \leqslant 2\varepsilon'$.

We deduce from (39) that for any cylinder X with $|X| \leqslant N_1$ we have :

$$\left|\frac{\tilde{\omega}(X)}{\omega(X)} - 1\right| \leqslant 2\varepsilon' \frac{1}{\omega(X)} \leqslant 2\varepsilon.$$

From where it is easy to deduce that we have for such cylinders :

$$\log\left(\frac{\omega_X(a)}{\tilde{\omega}_X(a)}\right) = \log\left(\frac{\omega(Xa)}{\tilde{\omega}(Xa)}\right) + \log\left(\frac{\tilde{\omega}(X)}{\omega(X)}\right) \leqslant 4\varepsilon$$

With inequality (40) we can conclude that for any cylinder X of any length we have

$$\log\left(\frac{\omega_X(a)}{\tilde{\omega}_X(a)}\right) \leqslant 5\varepsilon.$$

Which tells us that $\mathcal{D}(\omega, \tilde{\omega}) \leqslant 5\varepsilon$ and concludes the proof.

□

4.4 Back to the complex plane

We are now in position to give a proof of the main results of this paper about harmonic measure on the plane: Theorems 1.3 and 1.4

Let $\mathbb{X}$ be a non-autonomous Cantor set associated with the conformal iterated systems Ψ and let ω be the harmonic measures of its complementary in $\mathbb{C}$.

We defined on the symbolic space $\mathbb{K}$ the metric d. For any $\mathtt{a} = (a_n) \in \mathbb{K}$ and $\mathtt{b} = (b_n) \in \mathbb{K}$ let $d(\mathtt{a}, \mathtt{b}) = |x(\mathtt{a}) - x(\mathtt{b})|$, which is the euclidian distance in $\mathbb{C}$ of the points encoded by $\mathtt{a}$ and $\mathtt{b}$. In that way, symbolic cylinders and geometric cylinders have the same diameters.

From the discussions in paragraph 4.1 we easily get :

$$\frac{|XY|_{\mathtt{d}}}{|X|_{\mathtt{d}}} \sim \frac{|\Psi'_{XY}(0)|}{|\Psi'_X(0)|} = \frac{|\Psi'_X(\Psi_Y(0))|}{|\Psi'_X(0)|}|\Psi'_Y(0)| \sim |\Psi'_Y(0)|.$$

Hypothesis Bounded Contraction implies that there exists a constant $\eta \in]0,1[$ such that : $(1-\eta)^{|Y|} \leqslant |\Psi'_Y(0)| \leqslant \eta^{|Y|}$.

And we can conclude that d satisfies condition (18).

By Corollary 4.2 we also know that $s(X) := |X|_{\mathtt{d}}$ satisfies (ASI) with sequence $(\beta_n = Cq^n)$ with $0 < q < 1$.

Corollary 4.9 expresses the fact that ω also fulfills (ASI) with sequence $(\beta_n = Cq^n)$, with $0 < q < 1$, with an adjustment of the constant q, as well as condition (5.1) with $\gamma_n \sim Cn$.

We are thus in position to apply Theorem 3.4 with $\mu = \omega$ to conclude that Theorem 1.4 is true.

Let now $(\mathbb{X}^k)_{k\in\mathbb{N}}$ be a sequence of non-autonomous Cantor sets associated with the conformal iterated systems $(\Psi^k)_{k\in\mathbb{N}}$, and Let $(\omega^k)_{k\in\mathbb{N}}$ be the harmonic measures of their complementaries. Assume that $\lim\limits_k d(\Psi, \Psi^k) = 0$.

We define for any k a metric d_k such that geometric cylinders for Ψ^k and symbolic cylinders for d_k have the same diameter.

We know that d_k and ω^k fulfill the same properties as d and ω, with the same constants, because of our hypothesis : Annulus Condition and Bounded Contraction, and the fact that (Ψ^k) is converging towards Ψ.

Let $s_k(X) = |X|_{\mathtt{d}_k}$. Since $\lim\limits_k d(\Psi, \Psi^k) = 0$, by Proposition 4.3 we know that $\lim\limits_k \mathcal{D}(s, s_k) = 0$, and by Theorem 4.4 that $\lim\limits_k \mathcal{D}(\omega, \omega^k) = 0$ we may thus apply Theorem 3.5 to conclude for the continuity of the dimensions of harmonic measure.

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