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# Optimal investment and consumption under forward utilities with relative performance concerns\*

Guillaume Broux-Quemerais<sup>†</sup>      Anis Matoussi <sup>†</sup>      Chao Zhou<sup>‡</sup>

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## Abstract

We study a  $n$ -player and mean-field portfolio optimization problem under relative performance concerns with non-zero volatility, for wealth and consumption. The consistency assumption defining forward relative performance processes leads to a sufficient characterization of such processes with mean of a HJB-SPDE which highlights the link between wealth and consumption utility, and also characterizes the optimal strategies. In particular, forward relative performance processes with a wealth utility of CRRA type and separable time and space dependence necessarily have a consumption utility of the same form, with the same risk aversion parameter. This characterization gives a better understanding of the drift condition ensuring time consistency. In this setting, we establish closed form of the Nash equilibrium for both the  $n$ -player and mean field problems. We also provide some numerical examples.

**Keywords:** Forward utilities, relative performance, Mean-Field Games, stochastic control, investment-consumption optimization, stochastic HJB equation, stochastic PDEs.

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## Introduction

We study a  $n$ -player and mean-field investment-consumption optimization problem under relative performance concern. More precisely, we focus on an asset specialization setting with common noise under the forward performance process framework. The concept of asset specialization in finance refers to a situation where investment managers focus on specific assets. This can be motivated by a variety of reasons like familiarity with a certain sector, trading costs and constraints, liquidity costs ([30]) and ambiguity aversion ([26]). Each agent wishes to choose her portfolio-consumption strategy while having concerns towards the average wealth and consumption of other investors. This optimization problem has been studied through the classical expected utility maximization in [22], [21] and through forward relative performance process with zero volatility in [1], [6] and [7].

The study of optimization problems within a large population naturally leads to the situation of an infinite number of players, formalized through the mean field game theory introduced in [16], [23]. The idea is to model the asymptotic behavior of a large number of agents interacting with each other and subject to financial or energetic constraints. The mean-field formulation consists in considering a control problem for one representative agent faced with a continuum of competitors. In the pioneer works [23], the randomness of the population dynamics is assumed to be independent for each individual. The case of correlated randomness have been later developed in [3], [2].

In a competitive setting i.e. in a market in which multiple investors act, utility functions include a relative performance metric, to evaluate the impact of a strategy relatively to others. The first work in this direction [11] studies an optimal investment problem, through a classical expected utility maximization with a relative performance metric compatible with exponential utility. The authors prove the existence of a Nash equilibrium when there is no investment constraint, and when there is investment constraint for exponential utility in the Black and Scholes model. Later, relative performance in competition have been studied in [22], where the authors study an asset specialization problem through a classical expected relative performance maximization for CARA and CRRA risk preferences. The Nash equilibrium is determined in the  $n$ -player and mean-field game with HJB arguments. This work has been generalized to an optimal investment-consumption problem in [21]. The authors in [15] extended the study of investment consumption problems to more general Itô-diffusion environment. A one-to-one correspondence between the Nash equilibrium for the mean field asset specialization problem with random parameter under classical maximization of the expected relative power utility and some mean field FBSDE with quadratic growth is given in [12]. Studying this equation, the authors prove the uniqueness of the equilibrium and then solve the FBSDE for a small competition parameter  $\theta$ . Optimal investment consumption problems with relative concerns have recently been studied in [5], under the framework of recursive utilities of type Epstein-Zin.

Consistent forward utility have been introduced and developed in [27] to overcome some limits of the classical expected utility. The latest is a deterministic function depending on its value at the horizon time  $T$ . The value function being constructed backward in time with the dynamic programming principle, the classical utility theory is not adapted to the updating of risk preferences or the time horizon. Forward utilities enable the dynamic adjustment of the decision criteria, starting from preferences which are known at an initial time, rather than imposing a potentially distant and arbitrary time horizon. The preferences of an agent are thus described by a (random) dynamic utility  $(U(t, \cdot))$ .

The decision criterion maintains time consistency within the given investment or decision-making context, in the sense that if  $X_t^\pi$  is the observable process  $X_t^\pi$  (typically the wealth) resulting from the admissible decision/strategy  $\pi$ , then the preference process  $U(t, X_t^\pi)$  is a supermartingale, and there exists an optimal strategy such that the preference process is a martingale.

Since their introduction, there has been numerous theoretical developments in the field. In a general setting, [29] established a sufficient condition for time-consistency when the dynamic utility is an Itô random field. The consistent dynamic utility verifies a nonlinear SPDE of HJB type. This work has been extended to consistent utility of investment and consumption in [9], where dynamics of wealth and consumption utilities are linked by the consistency SPDE. Consistent dynamic utilities have found diverse applications over recent years, including but not limited to option valuation, insurance, mean field games ([22], [6]), long term interest rate modeling ([10]), risk measures [4] or more recently pension design ([14],[28]).

The analogue of forward utilities in a competitive setting are forward relative performance processes. One of the first work on forward performance processes in a competitive environment is [1]. The authors focus on optimal investment problems in general Itô-diffusion markets, for two investors with homothetic locally riskless forward utilities. The study of the asset diversification problem under more general monotone forward preferences has been extended to the mean-field and  $n$ -player settings in [32]. Similar investment-consumption optimization problems under forward relative criteria in a many player and mean field settings have also been considered in [6] and [7]. The authors consider separable forward utilities in power form defined for all agents  $i = 1, \dots, n$  as  $U^i(t, x) = f^i(t)u^{(\alpha_i)}(x)$  and  $V^i(t, x) = g^i(t)u^{(\alpha_i)}(x)$ , for deterministic functions  $f^i$  and  $g^i$ , and  $u^{(\alpha_i)}(x) = \frac{1}{1-\alpha_i}x^{1-\alpha_i}$  is the standard power utility function, with risk aversion coefficient  $\alpha_i < 1$ . In [7], a constant Nash equilibrium is determined under the assumption that for every agent,  $g^i(t) = f^i(t)^{1-\kappa}$ , where  $\kappa \in \mathbb{R}$  is the market-risk relative preference parameter is common to all the agent. However, for a best response strategy, this personal preference parameter may be distinct for all agent.

Our objective is to study the  $n$ -player and mean-field investment-consumption optimization problem in competition under forward relative performance processes with non-zero volatility. We will be interested in relative performance process  $Q^i$  of the form

$$Q^i(t, x) = U^i(t, x) + \int_0^t V^i(s, \widehat{c}_s^i x) ds, \quad (0.1)$$

where  $U^i$  and  $V^i$  are forward utilities from wealth and consumption. Assuming the wealth utility to be an Itô random field, with characteristics  $(\beta, \gamma)$ , Itô-Ventzel formula allows to get the dynamics of the compound random field  $Q$  along the wealth process, as to exploit the consistency condition. Following the initial work [9], this allows us to point out a condition on the drift  $\beta$ , which translates as an SPDE that is sufficient for  $U^i$  to solve to ensure time consistency of  $Q^i$ . Wealth and consumption utilities are linked by the consistency SPDE characterizing forward performance processes. This condition takes a convenient form for separable utilities of CRRA type

$$U^{(\alpha_i)}(t, x) = Z_t^i u^{(\alpha_i)}(x), \quad V^{(\alpha_i)}(t, c) = \phi_t^i u^{(\alpha_i)}(c), \quad i = 1, \dots, n \quad (0.2)$$

where  $(Z_t^i)_{t \geq 0}$  and  $(\phi_t^i)_{t \geq 0}$  are continuous stochastic processes with log-normal dynamics. We investigate the existence of a Nash equilibrium for both the  $n$ -player and mean field settings.

We define the notion of Nash equilibrium in the framework of forward relative performance processes. A forward Nash equilibrium consists in  $n$ -triples  $(Q^i, \pi^{i,*}, c^{i,*})$  so that  $U^i$  and  $V^i$  are forward utilities and for all agent  $i = 1, \dots, n$  using optimal strategy  $\pi_t^{i,*}$  from their individual optimization problem,  $Q^i$  satisfies the time consistency property. This definition is coherent with the classical notion of Nash equilibrium in the sense that time consistency implies that no manager can increase her expected utility by being the only one changing her strategy. The case of separable utilities of CRRA type is tractable, and we exhibit the Nash equilibrium in this framework. However, optimal strategies in this context present complex dependencies due to relative concerns, which motivates the study of the mean-field situation.

The mean-field game formulation relies on a random type vector representation introduced in [22]. We generalize the example of CRRA separable utilities to the mean field setting. Consider  $U$  and  $V$  two dynamic utilities of wealth and consumption of the form (0.2) with risk aversion  $\alpha$ , whose time variations are driven by log-normal processes  $Z$  and  $\phi$ , with local characteristics  $(b^Z, \delta_W^Z, \delta_B^Z)$  and  $(b^\phi, \delta_W^\phi, \delta_B^\phi)$ . In order to allow different preferences across the population, additionally to competition and risk aversion concerns, as well as parameters relative to stock prices as presented in Section ??, the random vector also includes local characteristics of processes  $Z$  and  $\phi$ , namely

$$\zeta = \left( \delta_W^Z, \delta_B^Z, b^\phi, \delta_W^\phi, \delta_B^\phi, \xi, \alpha, \theta, m, \nu, \sigma \right).$$

In this setting, the forward performance criterion  $Q$  of type (0.1) built with  $U$  and  $V$ , is evaluated along relative wealth and consumption processes

$$\hat{X}_t = \frac{X_t}{(\bar{X}_t)^\theta}, \quad \hat{c}_t = \frac{c_t}{(\bar{C}_t)^\theta},$$

where  $\bar{X}_t$  and  $\bar{C}_t$  are geometric average wealth and consumption process of the continuum of agents. The strategy for solving the mean-field optimization problem given by the martingale optimality condition on  $Q(t, \hat{X}_t)$  is the following. First fix a probability measure  $(\bar{X}_t)_{t \geq 0}$  representing the geometric average wealth process of the population, and solve the optimization problem given by the consistency condition on the metric relative to  $\bar{X}_t$ . Then, note that conditionally on the common noise  $B$ , all agents face i.i.d copies of the same optimization problem. Given this optimal control, determine the law of  $\bar{X}_t^*$  conditionally on the common noise  $B$ . Since  $\bar{X}_t$  models the geometric average wealth of the continuum of agents, the desired law is the one of  $\exp(\mathbb{E}[\log X_t^* | \mathcal{F}_t^B])$ . Finally, find a fix point satisfying

$$\bar{X}_t = \exp(\mathbb{E}[\log X_t^* | \mathcal{F}_t^B]). \tag{0.3}$$

This equation reflects that the optimal strategy conditionally on the common noise must be typical of the population. The conditioning on  $B$  appears because the effect of independent noises  $W^i$  on the empirical measure of  $\bar{X}_t$  averages out as  $n$  goes to infinity, whereas the effect of the common noise does not. Relying on this mean-field characterization, which also stands for the equilibrium consumption, we exploit the time consistency of  $Q(\frac{X^{\pi,c}}{(\bar{X}^{\pi,c})^\theta}, t)$  using Itô-Ventzel's formula and the martingale optimality condition. The case of separable utilities of CRRA type is tractable, and we give explicit forms of optimal strategies in this context.

Our characterization of forward relative performance process is coherent with [9] in the non-competitive setting. The optimal investment strategy also coincides with the one from [6], with additional terms related to the wealth utility non-zero volatility. The consumption equilibrium differs from [7] because of our use of Itô-Ventzel's formula to study the consistency property of forward relative performance processes. More precisely, the optimal consumption rate is stochastic, characterized by the relative importance the agents gives to wealth utility compared to consumption utility, represented by the ratio  $\frac{\phi_t}{Z_t}$ . We discuss the behavior of the MF equilibrium strategies in some examples, namely when this ratio is constant or a power of  $Z_t$ .

The paper is organized as follows. In Section 1, we introduce forward relative performance process and establish the SPDE which gives a sufficient condition on  $U^i$  so that the performance process  $Q^i$  is consistent. We give the optimal strategy for agent  $i$  using the martingale optimality condition. We specify the consistency condition for separable utilities of CRRA type. In Section 2, we define and investigate the existence of a Nash equilibrium in the  $n$ -player situation. The mean-field optimization problem against a continuum of agent is studied in Section 3. We also introduce two examples of study of the mean-field consumption equilibrium strategy. Finally, we illustrate our results with some numerical examples in Section 4. Regular random field space for the study of differentiability of Itô random fields are recalled in Annex A.

**Notations:**

All stochastic processes in the sequel are defined on a standard probability space  $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$  supporting  $n + 1$  independent Brownian motions  $B, W^1, \dots, W^n$ , where  $(\mathcal{F}_t)_{t \geq 0}$  is the natural augmented filtration they generate. For  $x, y \in \mathbb{R}^d$ , we denote by  $x^\top$  the transpose of vector  $x$ ,  $x \cdot y = xy^\top$  the scalar product between  $x$  and  $y$ , and  $\|\cdot\|$  the usual norm  $\|x\| = \text{Tr}(xx^\top)^{\frac{1}{2}}$ . We also define the space for  $p \in \mathbb{N}^*$  and  $I$  a subset of  $\mathbb{R}^d$

$$\mathcal{L}_{loc}^p(I) = \left\{ \varphi I - \text{valued prog. measurable process s.t. for all } T > 0, \int_0^T \|\varphi_s\|^p ds < \infty \right\}.$$

# 1 Forward relative utility in a many player setting

In this paper, we study portfolio management problem in an asset specialization setting. More precisely, we consider a finite population of agents, each one investing only in one idiosyncratic asset. The price randomness is modeled through an individual noise and a noise common to the entire population. The agents preferences are represented with forward utilities on relative wealth and consumption metrics, to reflect their will to compete against each other. We investigate the characterization of forward performance processes with relative concerns, and the associated optimal policies. We start by defining the relative performance metric and the associated dynamics for the relative wealth process. The martingale property ensuring time consistency allows to derive the consistency SPDE and the associated optimal strategies. The Itô-Ventzel's formula and regular random field spaces for the study of the differentiability of Itô random fields are recalled in Annex A.

## 1.1 Asset specialization and relative performance

Consider a market consisting in one riskless asset with interest rate  $r > 0$  and  $n$  risky securities, in which the price  $(S_t^i)$  of stock  $i$  traded exclusively by the  $i^{th}$  agent solves

$$\frac{dS_t^i}{S_t^i} = m_i dt + \nu_i dW_t^i + \sigma_i dB_t, \quad S_0^i = s_0^i > 0, \quad (1.1)$$

with constant real parameters  $m_i \in \mathbb{R}$ ,  $\sigma_i > 0$  et  $\nu_i > 0$ .  $B$  is called the common noise since it induces a correlation between the stocks while the  $W_t^i$  are idiosyncratic noises, specific to each agent  $i$ .

**Agent's wealth** - For  $i = 1, \dots, n$ , the  $i^{th}$  agent uses a self-financing strategy  $(\pi_t^i)_{t \geq 0}$  representing the proportion of wealth invested in the  $i^{th}$  stock. Denote  $(c_t^i)_{t \geq 0}$  the rate of consumption per unit of wealth. The  $i^{th}$  agent's wealth dynamics writes as

$$dX_t^i = rX_t^i dt + \pi_t^i X_t^i (\mu_i dt + \nu_i dW_t^i + \sigma_i dB_t) - c_t^i X_t^i dt, \quad \text{with } \mu_i = m_i - r. \quad (1.2)$$

An investment-consumption strategy is admissible if  $(\pi^i, c^i)$  is a progressively measurable  $\mathbb{R} \times (0, \infty)$  process such that  $\mathbb{E} \left[ \int_0^t (|\pi_s^i|^2 + |c_s^i|^2) ds \right] < \infty$  for any  $t > 0$ . We denote  $\mathcal{A}_i$  the set of admissible strategies for agent  $i$ .

**Remark 1.1.** Our work focus on the asset specialization model defined above (see [22], [21], [7]). However, we are also interested in forward relative performance processes in a more general incomplete Itô market model as in [9]. The incompleteness can be expressed by restrictions on admissible portfolios  $\pi_t^i$ , required to live in a progressive vector space  $\mathcal{R}_t^i$ , see [19]. Denote  $\Sigma_i = (\nu_i \sigma_i)$  the volatility vector and  $\bar{W}_t^i$  the two-dimensional Brownian motion  $(W_t^i B_t)$ . Introducing the market price of risk vector  $\eta_i = \Sigma_i^\top (\Sigma_i \Sigma_i^\top)^{-1} \mu_i$ , the wealth process (1.2) takes the form

$$dX_t^i = X_t^i \left[ (r - c_t^i) dt + \pi_t^i \Sigma_i \cdot (d\bar{W}_t^i + \eta_i dt) \right]. \quad (1.3)$$

The existence of a risk premium is a weak form of absence of arbitrage opportunity, in the sense that  $\Sigma_i \Sigma_i^\top$  is non-degenerate. Note that the risk premium impact on wealth dynamics occurs through the scalar product  $\pi_t \Sigma_i \cdot \eta_i$ . If the incompleteness is modeled by restrictions on the portfolio rescaled by

the volatility, of type  $\pi_t^i \Sigma_i \in \mathcal{R}_t^i$ , the market price of risk action is limited to its projection over the constraint set. In other words,  $\pi_t \Sigma_i \cdot \eta_i = \pi_t \Sigma_i \cdot \eta_i^{\mathcal{R}_t^i}$  where  $\eta_i^{\mathcal{R}_t^i}$  denotes the projection of  $\eta_i$  on the space  $\mathcal{R}_t^i$  and is referred to as the minimal risk premium.

**Agent's interaction and relative performance** - Each agent measures the performance of her own strategy taking into account the policies of others. One can thus define the relative wealth process and the relative consumption metric as

$$\widehat{X}^i = \frac{X^i}{\left(\widetilde{X}^{(-i)}\right)^{\theta_i}}, \quad \text{where} \quad \widetilde{X}^{(-i)} = \left(\prod_{k \neq i}^n X^k\right)^{\frac{1}{n-1}}, \quad (1.4)$$

$$\widehat{c}^i = \frac{c^i}{\left(\widetilde{c}^{(-i)}\right)^{\theta_i}}, \quad \text{where} \quad \widetilde{c}^{(-i)} = \left(\prod_{k \neq i}^n c^k\right)^{\frac{1}{n-1}}, \quad (1.5)$$

where  $\theta_i \in [0, 1]$  is the relative concern parameter. The closer  $\theta_i$  is to one the more agent  $i$  is concerned with the geometric average wealth and consumption of his competitors. An application of Itô's formula leads to the dynamics of the geometric average of other agents

$$\begin{aligned} \frac{d\widetilde{X}_t^{(-i)}}{\widetilde{X}_t^{(-i)}} &= \left( r + \overline{\mu\pi}_t^{(-i)} - \frac{1}{2} \left( \overline{\Sigma\pi}_t^2^{(-i)} - \left( \overline{\sigma\pi}_t^{(-i)} \right)^2 - \frac{1}{n-1} \overline{(\nu\pi)_t^2}^{(-i)} \right) - \overline{c}_t^{(-i)} \right) dt \\ &\quad + \frac{1}{n-1} \sum_{k \neq i}^n \nu_k \pi_t^k dW_t^k + \overline{\sigma\pi}_t^{(-i)} dB_t, \end{aligned} \quad (1.6)$$

where we denote quantities relative to sum over all agents except  $i$

$$\begin{aligned} \overline{\mu\pi}_t^{(-i)} &= \frac{1}{n-1} \sum_{k \neq i}^n \mu_k \pi_t^k, \quad \overline{(\nu\pi)_t^2}^{(-i)} = \frac{1}{n-1} \sum_{k \neq i}^n (\nu_k \pi_t^k)^2, \quad \overline{\sigma\pi}_t^{(-i)} = \frac{1}{n-1} \sum_{k \neq i}^n \sigma_k \pi_t^k, \\ \overline{\Sigma\pi}_t^2^{(-i)} &= \frac{1}{n-1} \sum_{k \neq i}^n \Sigma_k (\pi_t^k)^2, \quad \overline{c}_t^{(-i)} = \frac{1}{n-1} \sum_{k \neq i}^n c_t^k, \quad \Sigma_k = \sigma_k^2 + \nu_k^2. \end{aligned}$$

We then get the relative wealth process dynamics

$$\begin{aligned} \frac{d\widehat{X}_t^i}{\widehat{X}_t^i} &= \xi_i dt - \left( c_t^i - \theta_i \overline{c}_t^{(-i)} \right) dt + \left( \nu_i \pi_t^i dW_t^i - \theta_i \frac{1}{n-1} \sum_{k \neq i}^n \nu_k \pi_t^k dW_t^k \right) \\ &\quad + \left( \sigma_i \pi_t^i - \theta_i \overline{\sigma\pi}_t^{(-i)} \right) dB_t, \end{aligned} \quad (1.7)$$

where

$$\begin{aligned} \xi_i &= r(1 - \theta) + \mu_i \pi_t^i - \theta_i \overline{\mu\pi}_t^{(-i)} + \frac{\theta_i}{2} \overline{\Sigma\pi}_t^2^{(-i)} - \frac{\theta_i^2}{2} \left( \left( \overline{\sigma\pi}_t^{(-i)} \right)^2 + \frac{1}{n-1} \overline{(\nu\pi)_t^2}^{(-i)} \right) \\ &\quad - \theta_i \sigma_i \pi_t^i \overline{\sigma\pi}_t^{(-i)}. \end{aligned} \quad (1.8)$$



## 1.2 Forward performance process with utilities taken as Itô random fields

Each manager  $i = 1, \dots, n$  measures the output of her relative performance metric using a forward relative performance process, modeled as an  $\mathcal{F}_t$  progressively measurable random field  $Q^i : \Omega \times (0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ . This process captures respectively the utility from wealth and consumption with adapted continuous random fields  $U(t, x)$  and  $V(t, c)$  on  $(\mathbb{R}^+)^2$ , such that  $\mathbb{P}$ -almost surely, for any  $t \geq 0$ , the maps  $x \mapsto U(t, x)$  and  $x \mapsto V(t, x)$  are standard utility functions. Those processes will be called dynamic utilities.

**Assumption 1.1.** Assume that  $U^i$  and  $V^i$  are adapted continuous random fields, such that:

1. For any  $t \geq 0$ , the maps  $U^i$  and  $V^i$  are non-negative, strictly increasing and strictly concave in  $x$ .
2. The partial derivatives  $U_x^i(t, x), U_{xx}^i(t, x)$  and  $V_x^i(t, x), V_{xx}^i(t, x)$  exist for all  $t \geq 0, x > 0, \mathbb{P} - a.s.$

**Definition 1.1.** Consider an agent  $i \in \{1, \dots, n\}$  and assume that each manager  $j \neq i$  follow an admissible strategy  $(\pi^j, c^j)$ . For  $(\pi^i, c^i)$  an admissible strategy, define:

$$Q^i(t, x) = U^i(t, x) + \int_0^t V^i(s, \widehat{c}_s^i x) ds, \quad (1.9)$$

where  $U^i$  and  $V^i$  are progressively measurable random fields satisfying Assumption 1.1.  $Q^i$  is a forward relative performance process for manager  $i$  if, for all  $t \geq 0$ :

- *Time consistency* - For any admissible  $(\pi^i, c^i)$ ,  $Q^i(t, \widehat{X}_t^i)$  is a (local) supermartingale, and there exists an admissible strategy  $(\pi^{i,*}, c^{i,*})$  such that  $Q^i(t, \widehat{X}_t^{i,*})$  is a (local) martingale.

Assuming Itô dynamics for utilities from wealth and consumption, the consistency property induces a condition on the drift of  $Q^i$ . Under enough regularity conditions, this leads to an SPDE characterizing the forward performance process. To study the drift variations, we need to introduce the notion of Fenchel-Legendre transform, which is well defined under the regularity Assumption 1.1 on  $V^i$ .

**Definition 1.2.** Let  $V : \Omega \times (0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  be a random field, strictly concave in  $x$ . The Fenchel-Legendre transform of  $V$ , denoted  $\widetilde{V}$  is the random field such that:

$$\widetilde{V}(t, x') = \sup_{x > 0} \{V(t, x) - x'x\}, \quad x' > 0, t \geq 0. \quad (1.10)$$

Let's assume that the forward utility  $U^i$  is an Itô random field with local characteristics  $(\beta^i, (\gamma_W^i, \gamma_B^i))$ , whose dynamic is given by

$$dU^i(t, x) = \beta^i(t, x)dt + \gamma_W^i(t, x)dW_t^i + \gamma_B^i(t, x)dB_t, \quad \mathbb{P}.a.s \quad (1.11)$$

We also define the quantity

$$\begin{aligned} \varphi(t, x) = & U_x^i(t, x)x \left( r(1 - \theta) - \theta_i \overline{\mu \pi}_t^{(-i)} + \frac{\theta_i}{2} \overline{\Sigma \pi}_t^2^{(-i)} + \frac{\theta_i^2}{2} \left( (\overline{\sigma \pi}_t^{(-i)})^2 + \frac{1}{n-1} \overline{(\nu \pi)_t^2}^{(-i)} \right) + \theta_i \overline{c}_t^{(-i)} \right) \\ & + \frac{1}{2} U_{xx}^i(t, x)(x)^2 \left( \frac{\theta_i^2}{n-1} \overline{(\nu \pi)_t^2}^{(-i)} + (\theta_i \overline{\sigma \pi}_t^{(-i)})^2 \right) - \gamma_{B_x}^i(t, \widehat{X}_t^i) \theta_i \overline{\sigma \pi}_t^{(-i)} x. \end{aligned} \quad (1.12)$$

Applying Itô-Ventzel's formula to the forward performance process  $Q^i$  along the relative wealth process  $\widehat{X}_t^i$  allows to study the martingale condition of optimality from the Definition 1.1. This leads to a sufficient condition on the drift of the wealth utility  $U$  as well as the characterization of the optimal strategy in this context.

**Proposition 1.1.** *Consider an agent  $i \in \{1, \dots, n\}$  and assume that each manager  $j \neq i$  follow an admissible strategy  $(\pi^j, c^j) \in \mathcal{A}_j$ . Under the assumption that  $U^i$  solves the SPDE*

$$dU^i(t, x) = \left( -\varphi(t, x) + \frac{1}{2}U_{xx}^i(t, x)x^2(\nu_i^2 + \sigma_i^2) \left( \pi_t^{i,*} \right)^2 - \widetilde{V}^i(t, (\widetilde{c}_t^{(-i)})^{\theta_i} U_x^i(t, \widehat{X}_t^i)) \right) dt + \gamma_{W_t}^i(t, x) dW_t^i + \gamma_{B_t}^i(t, x) dB_t, \quad (1.13)$$

then  $Q^i$  defined by (1.9) is a forward relative performance process for manager  $i$  in the sens of Definition 1.1. The related optimal policy  $(\pi^{i,*}, c^{i,*})$  is given by

$$\pi_t^{i,*} = \frac{1}{\nu_i^2 + \sigma_i^2} \left( \sigma_i \theta_i \overline{\sigma \pi_t^{(-i)}} - \frac{1}{U_{xx}^i(t, \widehat{X}_t^i) \widehat{X}_t^i} \left( \gamma_{W_x}^i(t, \widehat{X}_t^i) \nu_i + \gamma_{B_x}^i(t, \widehat{X}_t^i) \sigma_i + (\mu - \theta_i \sigma_i \overline{\sigma \pi_t^{(-i)}}) U_x^i(t, \widehat{X}_t^i) \right) \right) \quad (1.14)$$

$$c_t^{i,*} = \frac{(V_x^i)^{-1} \left( U_x^i(t, \widehat{X}_t^{i,*}) (\widetilde{c}_t^{(-i)})^{\theta_i}, t \right) (\widetilde{c}_t^{(-i)})^{\theta_i}}{\widehat{X}_t^{i,*}}. \quad (1.15)$$

Taking a competition parameter  $\theta^i = 0$ , we retrieve the same SPDE characterization of forward performance processes as well as the same optimal strategies as in [9]. Also note that taking a null volatility  $\gamma^i(t, x) = (0, 0)$ , we recover the same optimal portfolio as well as the same expression for the optimal consumption process (1.15) as in [7]. However, we characterize the utility from consumption  $V^i$  as a function of the utility from wealth  $U^i$  through the SPDE (1.13), whereas [7] relies on a PDE with random coefficient. This is due to the use of Itô-Ventzel's formula instead of Itô's to exhibit the dynamics of  $Q^i(t, \widehat{X}_t^i)$ , involving the drift  $\beta^i$  of the forward utility  $U^i$  instead of its time derivative.

*Proof.* The proof of this result is in three steps. First, we apply Itô-Ventzel's formula to get the dynamics of  $dQ^i(t, \widehat{X}_t^i)$ . Then, exploiting the martingale condition of Definition 1.1, we derive the optimal strategy processes using the first order condition on the drift of  $dQ^i(t, \widehat{X}_t^i)$ . We conclude on the optimality and injecting the constraint on the drift in the Itô form of  $U$  leads to the above SPDE.

**Time Consistency** - Applying Itô-Ventzel's formula to get the dynamics of  $dQ^i(t, \widehat{X}_t^i)$

$$\begin{aligned} dQ^i(t, \widehat{X}_t^i) &= \left( \beta^i(t, \widehat{X}_t^i) + V^i(t, \widehat{c}_t^i \widehat{X}_t^i) \right) dt + \gamma_{W_t}^i(t, \widehat{X}_t^i) dW_t^i + \gamma_{B_t}^i(t, \widehat{X}_t^i) dB_t \\ &+ U_x^i(t, \widehat{X}_t^i) d\widehat{X}_t^i + \frac{1}{2} U_{xx}^i(t, \widehat{X}_t^i) d\langle \widehat{X}_t^i, \widehat{X}_t^i \rangle + \left\langle \gamma_{W_x}^i(t, \widehat{X}_t^i) dW_t^i + \gamma_{B_x}^i(t, \widehat{X}_t^i) dB_t, d\widehat{X}_t^i \right\rangle \\ &= \left( \beta^i(t, \widehat{X}_t^i) + V^i(t, \widehat{c}_t^i \widehat{X}_t^i) + U_x^i(t, \widehat{X}_t^i) \widehat{X}_t^i \left( \xi_i - (\widehat{c}_t^i - \theta_i \overline{c}_t^{(-i)}) \right) \right) dt + \gamma_{B_t}^i(t, \widehat{X}_t^i) dB_t \\ &+ \gamma^i(t, \widehat{X}_t^i) dW_t^i + U_x^i \widehat{X}_t^i \left( (t, \widehat{X}_t^i) \left( \sigma_t \pi_t^i - \theta_i \overline{\sigma \pi_t^{(-i)}} \right) dB_t + \left( \nu_i \pi_t^i dW_t^i - \theta_i \frac{1}{n-1} \sum_{k \neq i}^n \nu_k \pi_t^k dW_t^k \right) \right) \\ &+ \frac{1}{2} U_{xx}^i(t, \widehat{X}_t^i) (\widehat{X}_t^i)^2 \left( (\nu_i \pi_t^i)^2 + \frac{\theta_i^2}{n-1} (\overline{\nu \pi_t})^2^{(-i)} + (\sigma_i \pi_t^i - \theta_i \overline{\sigma \pi_t^{(-i)}})^2 \right) dt \\ &+ \gamma_{W_x}^i(t, \widehat{X}_t^i) \nu_i \pi_t^i \widehat{X}_t^i dt + \gamma_{B_x}^i(t, \widehat{X}_t^i) \left( \sigma_i \pi_t^i - \theta_i \overline{\sigma \pi_t^{(-i)}} \right) \widehat{X}_t^i dt \end{aligned}$$

In particular, the drift of the above random field takes the form

$$\begin{aligned}
\text{drift } Q^i(t, \widehat{X}_t^i) &= \beta^i(t, \widehat{X}_t^i) + V^i(t, \widehat{c}_t^i \widehat{X}_t^i) + \gamma_{B_x}^i(t, \widehat{X}_t^i) \left( \sigma_i \pi_t^i - \theta_i \overline{\sigma \pi}_t^{(-i)} \right) \widehat{X}_t^i + U_x^i(t, \widehat{X}_t^i) (r(1 - \theta) + \mu_i \pi_t^i \\
&\quad - \theta_i \overline{\mu \pi}_t^{(-i)} + \frac{\theta_i \overline{\Sigma \pi}_t^{(-i)}}{2} + \frac{\theta_i^2}{2} \left( (\overline{\sigma \pi}_t^{(-i)})^2 + \frac{1}{n-1} (\overline{\nu \pi}_t)^{2(-i)} \right) - \theta_i \sigma_i \pi_t^i \overline{\sigma \pi}_t^{(-i)} - (c_t^i - \theta_i \overline{c}_t^{(-i)}) \\
&\quad + \frac{1}{2} U_{xx}^i(t, \widehat{X}_t^i) (\widehat{X}_t^i)^2 \left( (\nu_i \pi_t^i)^2 + \frac{\theta_i^2}{n-1} (\overline{\nu \pi}_t)^{2(-i)} + (\sigma_i \pi_t^i - \theta_i \overline{\sigma \pi}_t^{(-i)})^2 \right) + \gamma_{W_x}^i(t, \widehat{X}_t^i) \nu_i \pi_t^i \widehat{X}_t^i
\end{aligned}$$

**Optimal strategies** - We find the optimal strategy exploiting the first order condition on the drift of  $Q^i(t, \widehat{X}_t^i)$ . In other words, the drift of this compound random field should be non positive for any admissible strategy and vanish at the optimum. In other words, drift  $Q^i(t, \widehat{X}_t^i)$  reaches its maximum at the optimal strategy, and its derivatives with respect to variables  $\pi$  and  $c$  vanish. The optimal portfolio process is close to the one in [7] with an additional term involving  $\gamma_x^i$ , and the optimal consumption process is the same, namely

$$\begin{aligned}
\pi_t^{i,*} &= \frac{1}{\nu_i^2 + \sigma_i^2} \left( \sigma_i \theta_i \overline{\sigma \pi}_t^{(-i)} - \frac{1}{U_{xx}^i(t, \widehat{X}_t^i) \widehat{X}_t^i} \left( \gamma_{W_x}^i(t, \widehat{X}_t^i) \nu_i + \gamma_{B_x}^i(t, \widehat{X}_t^i) \sigma_i + (\mu_i - \theta_i \sigma_i \overline{\sigma \pi}_t^{(-i)}) U_x^i(t, \widehat{X}_t^i) \right) \right) \\
c_t^{i,*} &= \frac{(V_x^i)^{-1} \left( U_x^i(t, \widehat{X}_t^{i,*}) (\widehat{c}_t^{(-i)})^{\theta_i}, t \right) (\widehat{c}_t^{(-i)})^{\theta_i}}{\widehat{X}_t^{i,*}}.
\end{aligned}$$

There only remains to check that the process  $Q^i$  effectively satisfies the time consistency property of Definition 1.1. Using the function  $\varphi$  introduced in (1.12) to shorten the notation, the drift of  $Q^i$  rewrites as

$$\begin{aligned}
\text{drift } Q^i(t, \widehat{X}_t^i) &= \beta^i(t, \widehat{X}_t^i) + \varphi(t, \widehat{X}_t^i) + \widetilde{V}^i(t, (\widetilde{c}_t^{(-i)})^{\theta_i} U_x^i(t, \widehat{X}_t^i)) + \frac{1}{2} U_{xx}^i(t, \widehat{X}_t^i) (\widehat{X}_t^i)^2 ((\nu_i^2 + \sigma_i^2) (\pi_t^i)^2 \\
&\quad - 2\theta_i \sigma_i \overline{\sigma \pi}_t^{(-i)} \cdot \pi_t^i + 2(\mu - \theta_i \sigma_i \overline{\sigma \pi}_t^{(-i)}) \pi_t^i \frac{U_x^i(t, \widehat{X}_t^i)}{U_{xx}^i(t, \widehat{X}_t^i) \widehat{X}_t^i} \\
&\quad + \frac{1}{U_{xx}^i(t, \widehat{X}_t^i) \widehat{X}_t^i} \left( 2\gamma_{W_x}^i(t, \widehat{X}_t^i) \nu_i \pi_t^i + 2\gamma_{B_x}^i(t, \widehat{X}_t^i) \sigma_i \pi_t^i \right) \\
&\quad + V^i(t, \widehat{c}_t^i \widehat{X}_t^i) - c_t^i U_x(t, \widehat{X}_t^i) \widehat{X}_t^i - \widetilde{V}^i(t, (\widetilde{c}_t^{(-i)})^{\theta_i} U_x^i(t, \widehat{X}_t^i)) \\
&= \beta^i(t, \widehat{X}_t^i) + \varphi(t, \widehat{X}_t^i) + \widetilde{V}^i(t, (\widetilde{c}_t^{(-i)})^{\theta_i} U_x^i(t, \widehat{X}_t^i)) + V^i(t, \widehat{c}_t^i \widehat{X}_t^i) - c_t^i U_x(t, \widehat{X}_t^i) \widehat{X}_t^i \\
&\quad - \widetilde{V}^i(t, (\widetilde{c}_t^{(-i)})^{\theta_i} U_x^i(t, \widehat{X}_t^i)) + \frac{1}{2} U_{xx}^i(t, \widehat{X}_t^i) (\widehat{X}_t^i)^2 (\nu_i^2 + \sigma_i^2) \left( (\pi_t^i)^2 - 2\pi_t^i \cdot \pi_t^{i,*} \right)
\end{aligned}$$

**Necessary condition on  $\beta$  so that  $Q$  is consistent** - From the first order condition, the above drift is minimal for the optimal portfolio  $\pi_t^{i,*}$ , and this minimum equals

$$\begin{aligned}
\text{drift } Q^i(t, \widehat{X}_t^{i,*}) &= \beta^i(t, \widehat{X}_t^{i,*}) + \varphi(t, \widehat{X}_t^{i,*}) + \widetilde{V}^i(t, (\widetilde{c}_t^{(-i)})^{\theta_i} U_x^i(t, \widehat{X}_t^{i,*})) + V^i(t, \widehat{c}_t^i \widehat{X}_t^{i,*}) \\
&\quad - c_t^i U_x(t, \widehat{X}_t^{i,*}) \widehat{X}_t^{i,*} - \widetilde{V}^i(t, (\widetilde{c}_t^{(-i)})^{\theta_i} U_x^i(t, \widehat{X}_t^{i,*})) - \frac{1}{2} U_{xx}^i(t, \widehat{X}_t^{i,*}) (\widehat{X}_t^{i,*})^2 (\nu_i^2 + \sigma_i^2) \left( \pi_t^{i,*} \right)^2. \quad (1.16)
\end{aligned}$$

Moreover, by definition of the Fenchel-Legendre transform  $\widetilde{V}^i$  and using (1.15), one can note that

$$\widetilde{V}^i(t, (\widetilde{c}_t^{(-i)})^{\theta_i} U_x^i(t, \widehat{X}_t^{i,*})) = V^i(t, \widehat{c}_t^{i,*} \widehat{X}_t^{i,*}) - c_t^{i,*} U_x(t, \widehat{X}_t^{i,*}) \widehat{X}_t^{i,*}. \quad (1.17)$$

Imposing that the quantity (1.16) equals zero ensures time consistency of the forward relative performance process  $Q^i$ . From the concavity of  $U^i$ , we get that the drift of  $Q^i(t, \widehat{X}_t^i)$  is non positive for any admissible portfolio and vanishes for the optimal strategy  $(\pi_t^{i,*}, c_t^{i,*})$ . This last condition in particular implies a constraint on the drift  $\beta^i$  of the forward utility  $U^i$ , which takes the form

$$\beta^i(t, x) = -\varphi(t, x) + \frac{1}{2}U_{xx}^i(t, x)x^2(\nu_i^2 + \sigma_i^2)\left(\pi_t^{i,*}\right)^2 - \widetilde{V}^i(t, (\widetilde{c}_t^{(-i)})^{\theta_i}U_x^i(t, \widehat{X}_t^i)). \quad (1.18)$$

Injecting this relation in the Itô decomposition of  $U^i$  leads to the SPDE

$$dU^i(t, x) = \left( -\varphi(t, x) + \frac{1}{2}U_{xx}^i(t, x)x^2(\nu_i^2 + \sigma_i^2)\left(\pi_t^{i,*}\right)^2 - \widetilde{V}^i(t, (\widetilde{c}_t^{(-i)})^{\theta_i}U_x^i(t, \widehat{X}_t^i)) \right) dt + \gamma_W^i(t, x)dW_t^i + \gamma_B^i(t, x)dB_t. \quad (1.19)$$

□

### 1.3 CRRA forward relative performance process

In the following, we focus on forward relative performance processes of CRRA type, with separable time and space dependence. The drift constraint (1.18) can be specified when the space dependence of wealth utility is of power type.

Consider an agent  $i$  with utility from wealth  $U^{(\alpha_i)}(t, x) = Z_t^i u^{(\alpha_i)}(x)$ , whose dynamics are driven by those of some Itô process  $Z_t^i$  and where  $u^{(\alpha_i)}(x) = \frac{x^{1-\alpha_i}}{1-\alpha_i}$  denotes the standard power utility function with risk aversion parameter  $\alpha_i \in [0, 1]$ . Moreover, we assume the process  $Z_t^i$  to have log-normal dynamics that we denote

$$dZ_t^i = Z_t^i \left( b^{Z^i}(t)dt + \delta^{W, Z^i}(t)dW_t^i + \delta^{B, Z^i}(t)dB_t \right), \quad Z_0^i = 1. \quad (1.20)$$

The local characteristics of the forward utility  $U^{(\alpha_i)}$  seen as an Itô random field of type (1.11) are then given by

$$\beta^{(\alpha_i)}(t, x) = b^{Z^i}(t)U^{(\alpha_i)}(t, x) \quad \text{and} \quad \gamma^{(\alpha_i)}(t, x) = \left( \delta^{W, Z^i}(t) \delta^{B, Z^i}(t) \right) U^{(\alpha_i)}(t, x). \quad (1.21)$$

The deterministic power utility function  $u^{(\alpha_i)}$  satisfies the following useful properties:

$$\begin{aligned} x u_x^{(\alpha_i)}(x) &= x^{1-\alpha_i} = (1 - \alpha_i) u^{(\alpha_i)}(x) \\ x^2 u_{xx}^{(\alpha_i)}(x) &= -\alpha_i x u_x^{(\alpha_i)}(x) = -\alpha_i (1 - \alpha_i) u^{(\alpha_i)}(x). \end{aligned}$$

Let  $V^i$  be a forward utility from consumption. The time consistency of the relative performance process  $Q^i$  defined by (1.9), generated by  $U^{(\alpha_i)}$  and  $V^i$  implies that  $V^i$  is also a forward utility of CRRA type with the same risk aversion parameter  $\alpha_i$  and separable time and space dependence, see [9]. We will thus consider a utility from consumption of the form  $V^{(\alpha_i)}(t, c) = \phi_t^i u^{(\alpha_i)}(c)$ , where  $(\phi_t^i)_{t \geq 0}$  follows

$$d\phi_t^i = \phi_t^i \left( b^{\phi^i}(t)dt + \delta^{W, \phi^i}(t)dW_t^i + \delta^{B, \phi^i}(t)dB_t \right), \quad \phi_0^i = 1. \quad (1.22)$$

For ease of notation, we introduce for  $i = 1, \dots, n$

$$f^i(t) = r(1 - \theta_i) - \theta_i \overline{\mu \pi_t^{(-i)}} + \frac{\theta_i}{2} \overline{\Sigma \pi_t^2}^{(-i)} + \frac{\theta_i^2}{2} \left( (\overline{\sigma \pi_t}^{(-i)})^2 + \frac{1}{n-1} \overline{(\nu \pi_t)^2}^{(-i)} \right) \quad (1.23)$$

$$g^i(t) = \frac{\theta_i^2}{n-1} \overline{(\nu \pi_t)^2}^{(-i)} + (\theta_i \overline{\sigma \pi_t}^{(-i)})^2. \quad (1.24)$$

**Proposition 1.2.** Consider a couple of CRRA utility random fields with separable time and space dependence  $U^{(\alpha_i)}(t, x) = Z_t^i u^{(\alpha_i)}(x)$  and  $V^{(\alpha_i)}(t, c) = \phi_t^i u^{(\alpha_i)}(c)$ , and let the conditions of Proposition 1.1 hold. Then the drift constraint (1.18) takes the form

$$b^{Z^i}(t) = (1 - \alpha_i) \delta_B^{Z^i}(t) \theta_i \overline{\sigma \pi}_t^{(-i)} - (1 - \alpha_i) f^i(t) + \frac{1}{2} \alpha_i (1 - \alpha_i) \left( g^i(t) - (\nu_i^2 + \sigma_i^2) \pi_t^{i,*} \right) - (1 - \alpha_i) \theta_i \overline{c}_t^{(-i)} + \frac{\alpha_i}{(\tilde{c}_t^{(-i)})^{\frac{\theta_i(1-\alpha_i)}{\alpha_i}}} \left( \frac{\phi_t^i}{Z_t^i} \right)^{\frac{1}{\alpha_i}}. \quad (1.25)$$

*Proof.* **Characterization of the utility from consumption** - Wealth and consumption utilities are related through the SPDE (1.13), which requires

$$\beta^i(t, x) = -\varphi(t, x) + \frac{1}{2} U_{xx}^i(t, x) x^2 (\nu_i^2 + \sigma_i^2) \left( \pi_t^{i,*} \right)^2 - \tilde{V}^i(t, (\tilde{c}_t^{(-i)})^{\theta_i} U_x^i(t, x)),$$

where

$$\varphi(t, \widehat{X}_t^i) = U_x(t, \widehat{X}_t^i) \widehat{X}_t^i (f^i(t) + \theta_i \overline{c}_t^{(-i)}) + \frac{1}{2} U_{xx}(t, \widehat{X}_t^i) (\widehat{X}_t^i)^2 g^i(t) - \gamma_{B_x}^i(t, \widehat{X}_t^i) \theta_i \overline{\sigma \pi}_t^{(-i)} \widehat{X}_t^i,$$

and  $f^i$  and  $g^i$  are defined as (1.23) and (1.24) respectively. Replacing (1.21) and using the identities of the power form utility, we obtain

$$\begin{aligned} b^{Z^i}(t) U^{(\alpha_i)}(t, x) &= -(1 - \alpha_i) U^{(\alpha_i)}(t, x) (f^i(t) + \theta_i \overline{c}_t^{(-i)}) + \frac{1}{2} \alpha_i (1 - \alpha_i) U^{(\alpha_i)}(t, x) g^i(t) \\ &\quad - \frac{1}{2} \alpha_i (1 - \alpha_i) U^{(\alpha_i)}(t, x) (\nu_i^2 + \sigma_i^2) (\pi_t^{i,*})^2 - \tilde{V}^i(t, (\tilde{c}_t^{(-i)})^{\theta_i} U_x^{\alpha_i}(t, x)) \\ &\quad + \delta_B^{Z^i}(t) \theta_i \overline{\sigma \pi}_t^{(-i)} (1 - \alpha_i) U^{(\alpha_i)}(t, x). \end{aligned} \quad (1.26)$$

This constraint implies

$$\tilde{V}^i(t, (\tilde{c}_t^{(-i)})^{\theta_i} U_x^{(\alpha_i)}(t, x)) = (\overline{v}_t^i - (1 - \alpha_i) \theta_i \overline{c}_t^{(-i)}) U^{(\alpha_i)}(t, x), \quad (1.27)$$

where  $\overline{v}_t^i = -\left[ b^{Z^i}(t) - (1 - \alpha_i) \delta_B^{Z^i}(t) \theta_i \overline{\sigma \pi}_t^{(-i)} \right] + (1 - \alpha_i) f^i(t) - \frac{1}{2} \alpha_i (1 - \alpha_i) (g^i(t) - (\nu_i^2 + \sigma_i^2) \pi_t^{i,*})$ . Denote  $\overline{w}_t^i = \overline{v}_t^i - (1 - \alpha_i) \theta_i \overline{c}_t^{(-i)}$ . Differentiating (1.27) with respect to the space variable leads

$$\tilde{V}_x^i \left( (\tilde{c}_t^{(-i)})^{\theta_i} U_x^{(\alpha_i)}(t, x) \right) = \frac{\overline{w}_t^i}{(\tilde{c}_t^{(-i)})^{\theta_i} U_{xx}^{(\alpha_i)}(t, x)} = -\frac{\overline{w}_t^i}{(\tilde{c}_t^{(-i)})^{\theta_i} \alpha_i} x.$$

Since  $V_x(t, -\tilde{V}_y(t, y)) = y$ , we obtain

$$(\tilde{c}_t^{(-i)})^{\theta_i} U_x^{(\alpha_i)}(t, x) = V_x \left( t, \frac{\overline{w}_t^i}{(\tilde{c}_t^{(-i)})^{\theta_i} \alpha_i} x \right), \quad \text{i.e.} \quad (\tilde{c}_t^{(-i)})^{\theta_i} U_x^i \left( t, \frac{(\tilde{c}_t^{(-i)})^{\theta_i} \alpha_i}{\overline{w}_t^i} x \right) = V_x(t, x). \quad (1.28)$$

So integrating leads to

$$V^i(t, x) = \frac{\overline{w}_t^i}{\alpha_i} U^{(\alpha_i)} \left( t, \frac{(\tilde{c}_t^{(-i)})^{\theta_i} \alpha_i}{\overline{w}_t^i} x \right) = \left( \frac{\overline{w}_t^i}{\alpha_i} \right)^{\alpha_i} (\tilde{c}_t^{(-i)})^{\theta_i (1-\alpha_i)} U^{(\alpha_i)}(t, x). \quad (1.29)$$

Thus, utility from consumption  $V^i$  is necessarily of power type, proportional to the utility from wealth  $U^{(\alpha_i)}$ .

**Condition on  $b^{Z^i}(t)$**  - Equation (1.26) gives a condition on the drift of the wealth utility  $U^{(\alpha_i)}$ . In other words, it allows to characterize the drift parameter  $b^{Z^i}$ , as well as the quantity  $\bar{v}_t$ . In fact, comparing the general form of the utility from consumption  $V^i(t, c) = \phi_t^i u^{\alpha_i}(t, c)$  with (1.29) leads:

$$\phi_t^i = Z_t^i \left( \frac{\bar{v}_t^i - (1 - \alpha_i) \theta_i \bar{c}_t^{(-i)}}{\alpha_i} \right)^{\alpha_i} \left( \tilde{c}_t^{(-i)} \right)^{\theta_i(1-\alpha_i)},$$

so that:

$$\bar{v}_t^i = (1 - \alpha_i) \theta_i \bar{c}_t^{(-i)} + \frac{\alpha_i}{(\tilde{c}_t^{(-i)})^{\frac{\theta_i(1-\alpha_i)}{\alpha_i}}} \left( \frac{\phi_t^i}{Z_t^i} \right)^{\frac{1}{\alpha_i}}. \quad (1.30)$$

Finally, the drift condition (1.18) can be rewritten as the following condition on  $b^{Z^i}$

$$\begin{aligned} b^{Z^i}(t) = & (1 - \alpha_i) \delta_B^{Z^i}(t) \theta_i \bar{\sigma} \bar{\pi}_t^{(-i)} - (1 - \alpha_i) f^i(t) + \frac{1}{2} \alpha_i (1 - \alpha_i) \left( g^i(t) - (\nu_i^2 + \sigma_i^2) \pi_t^{i,*} \right) \\ & - (1 - \alpha_i) \theta_i \bar{c}_t^{(-i)} + \frac{\alpha_i}{(\tilde{c}_t^{(-i)})^{\frac{\theta_i(1-\alpha_i)}{\alpha_i}}} \left( \frac{\phi_t^i}{Z_t^i} \right)^{\frac{1}{\alpha_i}}. \end{aligned}$$

□

**Remark 1.2.** The link between  $U^{(\alpha_i)}$  and  $V^i$  pointed out in (1.29) only stands here for CRRA utilities, since it relies on the possibility to express the first and second derivatives of  $U^{(\alpha_i)}$  as functions of itself. In the general case

$$\beta^i(t, x) = -\varphi(t, x) + \frac{1}{2} U_{xx}^i(t, x) x^2 (\nu_i^2 + \sigma_i^2) \left( \pi_t^{i,*} \right)^2 - \tilde{V}^i(t, (\tilde{c}_t^{(-i)})^{\theta_i} U_x^i(t, x)),$$

so that

$$\begin{aligned} (\tilde{c}_t^{(-i)})^{\theta_i} U_{xx}^i(t, x) \tilde{V}_x^i(t, (\tilde{c}_t^{(-i)})^{\theta_i} U_x^i(t, x)) = & -\beta_x^i(t, x) - \varphi_x(t, x) \\ & + \frac{1}{2} (\nu_i^2 + \sigma_i^2) \left( \pi_t^{i,*} \right)^2 (U_{xxx}^i(t, x) x^2 + 2x U_{xx}^i(t, x)). \end{aligned}$$

Denote  $H$  the random field

$$H(t, x) = \frac{-1}{(\tilde{c}_t^{(-i)})^{\theta_i} U_{xx}^i(t, x)} \left( -\beta_x^i(x, t) - \varphi_x(x, t) + \frac{1}{2} (\nu_i^2 + \sigma_i^2) \left( \pi_t^{i,*} \right)^2 (x^2 U_{xxx}^i(x, t) + 2x U_{xx}^i(x, t)) \right)$$

Then, using the identity  $V_x(-\tilde{V}_x(t, x), t) = x$ :

$$\begin{aligned} (\tilde{c}_t^{(-i)})^{\theta_i} U_x^i(t, x) & = V_x \left( -\tilde{V}_x(t, (\tilde{c}_t^{(-i)})^{\theta_i} U_x^i(t, x)) \right) \\ & = V_x^i(H(x, t)). \end{aligned}$$

Consequently, if  $H$  admits an inverse function  $H^{-1}$  with respect to the space variable, we obtain a characterization of the marginal utility from consumption

$$V_x^i(t, x) = (\tilde{c}_t^{(-i)})^{\theta_i} U_x^i(H^{-1}(x, t), t).$$

## 2 The $n$ -agent forward optimization problem

Having characterized forward relative performance processes and the associated optimal strategies, we now investigate the existence of a Nash equilibrium in the many-player asset specialization problem under common noise. Informally, a Nash equilibrium is defined by a set of  $n$  strategies such that no agent can improve their performance being the only one deviating from the equilibrium. In other words, these policies are optimal for every player in the market.

### 2.1 Nash equilibrium for the $n$ -player case under forward relative performance concerns

Let us now introduce the notion of Nash equilibrium in the framework of relative performance processes. In the classical expected utility theory, a Nash equilibrium is characterized as  $n$  strategies  $(\pi_t^{i,*}, c_t^{i,*})_{i=1,\dots,n}$  chosen so that no manager can increase the expected utility of her performance metric by unilateral decision. We can formulate this property using the consistency condition for forward relative performance process, as proposed in [7].

**Definition 2.1.** Let for any  $i \in \{1, \dots, n\}$ ,  $(\pi^{i,*}, c^{i,*}) \in \mathcal{A}_i$  be the optimal strategy for investor  $i$  from Proposition 1.1. Let  $Q^i$  be the progressively measurable random field

$$Q^i(t, x) = U^i(t, x) + \int_0^t V^i(s, \widehat{c}_s^i x) ds,$$

where  $U^i$  and  $V^i$  are progressively measurable random fields satisfying Assumption 1.1. A forward Nash equilibrium consists in  $n$ -triples  $(Q^i, \pi^{i,*}, c^{i,*})$  with  $i = 1, \dots, n$  such that:

- *Time consistency* - Let managers  $j \neq i$  act according  $(\pi^{j,*}, c^{j,*})$  and manager  $i$  with an admissible strategy  $(\pi^i, c^i)$ . Then  $Q^i(t, \widehat{X}_t^i)$  is a (local) supermartingale. Let all managers  $j = 1, \dots, n$  act along their optimal strategy  $(\pi^{j,*}, c^{j,*})$ . Then  $Q^i(t, \widehat{X}_t^{i,*})$  is a (local) martingale.

This definition is coherent with the classical notion of Nash equilibrium. In fact, for any  $i \in \{1, \dots, n\}$ , assuming that every actor  $j \neq i$  use its optimal strategy  $\pi_t^{j,*}$ , we get from the martingale and supermartingale properties of the above definition that

$$\mathbb{E} \left[ Q^i(t, \widehat{X}_t^{i,*}) \right] = Q^i(0, \widehat{X}_0^i) \geq \mathbb{E} \left[ Q^i(t, \widehat{X}_t^i) \right], \quad \text{for all } t \geq 0.$$

Hence no manager can increase her expected utility by being the only one changing her strategy from the Nash equilibrium.

Note that the Nash equilibrium consists in  $n$ -triples  $(Q^i, \pi^{i,*}, c^{i,*})$ , which means that the equilibrium strategies depend on the forward performance process  $Q^i$  and thus on the chosen forward utility  $U^i$ . The investors may also have different type of forward performance process. One may wonder if we can determine an optimal relative performance  $Q^{i,*}$ , or more precisely a  $n$ -tuple  $(Q^{i,*})_{i \in \{1, \dots, n\}}$  in order to maximize expected utility over all other admissible utilities for each agent  $i = 1, \dots, n$

$$Q^{i,*}(x, t) = U^{i,*}(t, x) + \int_0^t V^{i,*}(s, \widehat{c}_s^i x) ds, \quad (2.1)$$

and where  $U^{i,*}$  denotes the optimal forward utility  $U^i$  in the sense that

$$U^{i,*}(t, x) = \operatorname{argsup}_{U^i \in K_{loc}^{2,\delta} \cap C^3} \mathbb{E} \left[ Q^i(t, \widehat{X}_t^{i,*}) | \mathcal{F}_s \right] = \operatorname{argsup}_{U^i \in K_{loc}^{2,\delta} \cap C^3} Q^i(s, \widehat{X}_s^{i,*}), \quad \text{for all } s \leq t.$$

**Remark 2.1.** Assuming that the utilities  $U_i$  solve the SPDEs (1.13), determining the Nash equilibrium portfolio consists in solving the system for  $i = 1, \dots, n$

$$\pi_t^{i,*} = \frac{1}{\nu_i^2 + \sigma_i^2} \left( \sigma_i \theta_i \left( 1 + \frac{U_x^i}{U_{xx}^i \widehat{X}_t^i} \right) \overline{\sigma \pi_t^{(-i)}} - \frac{1}{U_{xx}^i \widehat{X}_t^i} \left( \gamma_{W_x}^i(t, \widehat{X}_t^i) \nu_i + \gamma_{B_x}^i(t, \widehat{X}_t^i) \sigma_i \right) + \mu_i U_x^i \right)$$

The computation in the general case above would result in an expression involving first and second derivatives of every utility  $U^i$ , for  $i = 1, \dots, n$ . However, it is tractable for CRRA utilities.

## 2.2 Nash equilibrium for power form forward utilities: the CRRA case

In this section, we investigate the Nash equilibrium in the  $n$ -player case for forward utilities of CRRA type with separable time and space dependence. The resolution strongly relies on Section 1.3 and the drift constraint (1.25). Consider a family of CRRA forward utilities from wealth and consumption with separable time and space dependence as defined in Section 1.3, that is for  $i = 1, \dots, n$

$$U^{(\alpha_i)}(t, x) = Z_t^i u^{(\alpha_i)}(x), \quad \text{and} \quad V^{(\alpha_i)}(t, c) = \phi_t^i u^{(\alpha_i)}(c), \quad \alpha_i \in (0, 1). \quad (2.2)$$

To state the following results, we need to define the quantities

$$\psi_n = \frac{1}{n-1} \sum_{k=1}^n \frac{\sigma_k^2 \theta_k (1 - \frac{1}{\alpha_k})}{\nu_k^2 + \sigma_k^2 \left( 1 + \frac{\theta_k (1 - \frac{1}{\alpha_k})}{n-1} \right)} \quad \text{and} \quad \phi_n = \frac{1}{n} \sum_{k=1}^n \frac{\frac{\sigma_k}{\alpha_k} \left( \delta^{W, Z^k}(t) \nu_k + \delta^{B, Z^k}(t) \sigma_k + \mu_k \right)}{\nu_k^2 + \sigma_k^2 \left( 1 + \frac{\theta_k (1 - \frac{1}{\alpha_k})}{n-1} \right)}. \quad (2.3)$$

$$\xi_n = \sum_{k=1}^n \frac{\theta_k (1 - \alpha_k)}{(n-1) \alpha_k - (1 - \alpha_k) \theta_k}. \quad (2.4)$$

Moreover, the optimal consumption process in the framework of relative forward performance process relies on the following process

$$A_t^i = \left( \frac{\phi_t^i}{\widetilde{Z}_t^i} \right)^{\frac{n-1}{\alpha_i(n-1) - \theta_i(1-\alpha_i)}} \left( \prod_{k=1}^n \left( \frac{\phi_t^k}{\widetilde{Z}_t^k} \right)^{\frac{n-1}{\alpha_k(n-1) - \theta_k(1-\alpha_k)}} \right)^{\frac{-1}{\xi_n+1} \frac{\theta_i(1-\alpha_i)}{\alpha_i(n-1) - \theta_i(1-\alpha_i)}}, \quad (2.5)$$

where  $\widetilde{Z}_t^i$  is an Itô diffusion with dynamics

$$d\widetilde{Z}_t^i = \widetilde{Z}_t^i \left( b^{\widetilde{Z}^i}(t) dt + \delta^{W, Z^i}(t) dW_t^i + \delta^{B, Z^i}(t) dB_t \right), \quad \widetilde{Z}_0^i = 1, \quad (2.6)$$

with drift parameter  $b^{\widetilde{Z}^i}$  given by

$$b^{\widetilde{Z}^i}(t) = (1 - \alpha_i) \delta_B^{Z^i}(t) \theta_i \overline{\sigma \pi_t^{(-i)}} - (1 - \alpha_i) f^i(t) + \frac{1}{2} \alpha_i (1 - \alpha_i) \left( g^i(t) - (\nu_i^2 + \sigma_i^2) \pi_t^{i,*} \right). \quad (2.7)$$

One can show the process  $A_t^i$  has log-normal dynamics, that we denote

$$dA_t^i = A_t^i \left( b^{A^i}(t) dt + \delta^{A^i}(t) \cdot d\widetilde{W}_t^i \right), \quad (2.8)$$

where  $\widetilde{W}_t^i$  denotes the two-dimensional Brownian motion  $(W_t^i, B_t)$ .



**Theorem 2.1.** *Let the conditions of Proposition 1.1 hold for all agents  $i \in \{1, \dots, n\}$ . Assume moreover that agents have power form forward utility  $U^{(\alpha_i)}(t, x) = Z_t^i u^{(\alpha_i)}(x)$ . If  $\psi_n \neq 1$ , then there exists a unique optimal candidate strategy  $(\pi_t^{i,*}, c_t^{i,*})$  given by*

$$\pi_t^{i,*} = \frac{1}{\nu_i^2 + \sigma_i^2 \left(1 + \frac{\theta_i(1 - \frac{1}{\alpha_i})}{n-1}\right)} \left[ \sigma_i \theta_i \left(1 - \frac{1}{\alpha_i}\right) \frac{n}{n-1} \frac{\phi_n}{1 - \psi_n} + \frac{1}{\alpha_i} \left( \delta^{W, Z^i}(t) \nu_i + \delta^{B, Z^i}(t) \sigma_i + \mu_i \right) \right] \quad (2.9)$$

$$dc_t^{i,*} = c_t^{i,*} \left( (b^{A^i}(t) + c_t^{i,*}) dt + \delta^{A^i}(t) \cdot d\bar{W}_t^i \right). \quad (2.10)$$

*Proof.* The equilibrium portfolio is simply determined with a fix point calculation from (1.14). The characterization of the optimal consumption process is achieved in the same manner but require additional arguments because of consumption dependent terms in the drift condition (1.25).

**Equilibrium portfolio** - For power form forward utilities of the form  $U^{(\alpha_i)}(t, x) = Z_t^i u^{(\alpha_i)}(x)$ , the optimal portfolio strategy (1.14) rewrites as

$$\pi_t^{i,*} = \frac{1}{\nu_i^2 + \sigma_i^2 \left(1 + \frac{\theta_i(1 - \frac{1}{\alpha_i})}{n-1}\right)} \left[ \sigma_i \theta_i \left(1 - \frac{1}{\alpha_i}\right) \frac{n}{n-1} \overline{\sigma \pi}_t + \frac{1}{\alpha_i} \left( \gamma^{W, Z^i}(t) \nu_i + \gamma^{B, Z^i}(t) \sigma_i + \mu_i \right) \right]. \quad (2.11)$$

Multiplying by  $\sigma_i$  and averaging over  $i \in \{1, \dots, n\}$ , we get

$$\overline{\sigma \pi}_t = \overline{\sigma \pi}_t \psi_n + \phi_n,$$

where

$$\psi_n = \frac{1}{n-1} \sum_{k=1}^n \frac{\sigma_k^2 \theta_k \left(1 - \frac{1}{\alpha_k}\right)}{\nu_k^2 + \sigma_k^2 \left(1 + \frac{\theta_k(1 - \frac{1}{\alpha_k})}{n-1}\right)} \quad \text{and} \quad \phi_n = \frac{1}{n} \sum_{k=1}^n \frac{\frac{\sigma_k}{\alpha_k} \left( \gamma^{W, Z^i}(t) \nu_i + \gamma^{B, Z^i}(t) \sigma_i + \mu_k \right)}{\nu_k^2 + \sigma_k^2 \left(1 + \frac{\theta_k(1 - \frac{1}{\alpha_k})}{n-1}\right)}. \quad (2.12)$$

Hence, for  $\psi_n \neq 1$

$$\overline{\sigma \pi}_t = \frac{\phi_n}{1 - \psi_n}. \quad (2.13)$$

Plugging this expression in (2.11) leads to the Nash equilibrium portfolio strategy. If  $\psi_n = 1$ , there exists no Nash equilibrium.

**Equilibrium consumption process** - Let's recall that the optimal consumption process is given by

$$c_t^{i,*} = \frac{(V_x^i)^{-1} \left( t, U_x^i(t, \widehat{X}_t^{i,*}) (\tilde{c}_t^{(-i)}) \theta_i \right) (\tilde{c}_t^{(-i)}) \theta_i}{\widehat{X}_t^{i,*}}. \quad (2.14)$$

From (1.29) and the power form dynamics considered for  $U^{(\alpha_i)}$ , we get

$$V_x^{(\alpha_i)}(t, x) = Z_t^i (\tilde{c}_t^{(-i)})^{\theta_i(1-\alpha_i)} \left( \frac{\alpha_i}{w_t^i} x \right)^{-\alpha_i},$$

so that

$$(V_x^i)^{-1}(t, x) = (Z_t^i)^{\frac{1}{\alpha_i}} \frac{\bar{w}_t^i}{\alpha_i} (\tilde{c}_t^{(-i)})^{\theta_i \frac{1-\alpha_i}{\alpha_i}} x^{-\frac{1}{\alpha_i}}.$$

Then the optimal consumption process writes as

$$\begin{aligned} \tilde{c}_t^{i,*} &= \frac{(\tilde{c}_t^{(-i)})^{\theta_i}}{\widehat{X}_t^{i,*}} (Z_t^i)^{\frac{1}{\alpha_i}} \frac{\overline{w}_t^i}{\alpha_i} (\tilde{c}_t^{(-i)})^{\theta_i \frac{1-\alpha_i}{\alpha_i}} \left( Z_t^i (\widehat{X}_t^{i,*})^{-\alpha_i} (\tilde{c}_t^{(-i)})^{\theta_i} \right)^{-\frac{1}{\alpha_i}} \\ &= \frac{\overline{v}_t^i - (1-\alpha_i)\theta_i \tilde{c}_t^{(-i)}}{\alpha_i}. \end{aligned}$$

Replacing  $\overline{v}_t^i$  with (1.30) leads

$$\tilde{c}_t^{i,*} = \left( \tilde{c}_t^{(-i)} \right)^{-\frac{\theta_i(1-\alpha_i)}{\alpha_i}} \left( \frac{\phi_t^i}{Z_t^i} \right)^{\frac{1}{\alpha_i}}, \quad (2.15)$$

where we recall that  $\tilde{c}_t^{(-i)}$  denotes the geometric average of consumption processes of every agent different from  $i$ , defined in (1.5). Let's modify this last equation to express  $\tilde{c}_t^{i,*}$  as a function of  $\tilde{c}_t = \left( \prod_{k=1}^n \tilde{c}_t^k \right)^{\frac{1}{n}}$

$$\tilde{c}_t^{i,*} = (\tilde{c}_t)^{-\frac{n\theta_i(1-\alpha_i)}{\alpha_i(n-1)-\theta_i(1-\alpha_i)}} \left( \frac{\phi_t^i}{Z_t^i} \right)^{\frac{n-1}{\alpha_i(n-1)-\theta_i(1-\alpha_i)}}. \quad (2.16)$$

Averaging the previous line over  $i = 1, \dots, n$ , we obtain

$$\prod_{k=1}^n \tilde{c}_t^{k,*} = \prod_{k=1}^n \left( \frac{\phi_t^k}{Z_t^k} \right)^{\frac{n-1}{\alpha_k(n-1)-\theta_k(1-\alpha_k)}} (\tilde{c}_t)^{-n \sum_{k=1}^n \frac{\theta_k(1-\alpha_k)}{\alpha_k(n-1)-\theta_k(1-\alpha_k)}}.$$

Then, defining the quantity

$$\xi_n = \sum_{k=1}^n \frac{\theta_k(1-\alpha_k)}{(n-1)\alpha_k - (1-\alpha_k)\theta_k}, \quad (2.17)$$

the geometric average of consumption processes then takes the form

$$\tilde{c}_t = \left( \prod_{k=1}^n \left( \frac{\phi_t^k}{Z_t^k} \right)^{\frac{1}{\alpha_k(n-1)-\theta_k(1-\alpha_k)}} \right)^{\frac{n-1}{n} \frac{1}{\xi_n+1}}. \quad (2.18)$$

Injecting this expression in the optimal consumption (2.16) leads

$$\tilde{c}_t^{i,*} = (\tilde{c}_t)^{-\frac{n\theta_i(1-\alpha_i)}{\alpha_i(n-1)-\theta_i(1-\alpha_i)}} \left( \frac{\phi_t^i}{Z_t^i} \right)^{\frac{n-1}{\alpha_i(n-1)-\theta_i(1-\alpha_i)}}. \quad (2.19)$$

**Nash equilibrium** - We complete the proof by using the optimal policy (2.19) in response to the others players choices and the drift condition (1.25). With  $b^{\tilde{Z}^i}$  given by (2.7), we rewrite the latter as

$$b^{\tilde{Z}^i}(t) = b^{\tilde{Z}^i}(t) - (1-\alpha_i)\theta_i \tilde{c}_t^{(-i),*} - \alpha_i \tilde{c}_t^{i,*}. \quad (2.20)$$

In fact, in order to have an equilibrium, we must solve the coupled system for  $i = 1, \dots, n$  composed of (2.19) and the SDE characterizing the diffusion  $Z^i$ , namely

$$\begin{aligned} \tilde{c}_t^{i,*} &= (\tilde{c}_t)^{-\frac{n\theta_i(1-\alpha_i)}{\alpha_i(n-1)-\theta_i(1-\alpha_i)}} \left( \frac{\phi_t^i}{Z_t^i} \right)^{\frac{n-1}{\alpha_i(n-1)-\theta_i(1-\alpha_i)}} \\ dZ_t^i &= Z_t^i \left( b^{\tilde{Z}^i}(t) dt + \delta^{W, Z^i}(t) dW_t^i + \delta^{B, Z^i}(t) dB_t \right), \quad Z_0^i = 1. \end{aligned} \quad (2.21)$$

We introduce the process  $\tilde{Z}_t$  with dynamics

$$d\tilde{Z}_t^i = \tilde{Z}_t^i \left( \tilde{b}^{Z^i}(t) dt + \delta^{W, Z^i}(t) dW_t^i + \delta^{B, Z^i}(t) dB_t \right). \quad (2.22)$$

This allows to disentangle  $Z_t$  with consumption dependent terms in  $b^{Z^i}$ , writing

$$Z_t^i = \tilde{Z}_t^i \exp \left( -\frac{n}{n-1} (1-\alpha_i) \theta_i \int_0^t \bar{c}_s ds - \frac{(n-1)\alpha_i - (1-\alpha_i)\theta_i}{n-1} \int_0^t c_s^{i,*} ds \right). \quad (2.23)$$

Injecting (2.23) in the optimal consumption (2.10) leads

$$\tilde{c}_t^{i,*} \exp \left( -\int_0^t c_s^{i,*} ds \right) = (\tilde{c}_t)^{-\frac{n\theta_i(1-\alpha_i)}{\alpha_i(n-1)-\theta_i(1-\alpha_i)}} \left( \frac{\phi_t^i}{\tilde{Z}_t^i} \right)^{\frac{n-1}{\alpha_i(n-1)-\theta_i(1-\alpha_i)}} \exp \left( \frac{n(1-\alpha_i)\theta_i}{\alpha_i(n-1)-\theta_i(1-\alpha_i)} \int_0^t \bar{c}_s ds \right) \quad (2.24)$$

Taking the geometric mean of the above over  $i = 1, \dots, n$ , we get

$$\tilde{c}_t \exp \left( -\int_0^t \bar{c}_s ds \right) = (\tilde{c}_t)^{-\xi_n} \exp \left( \xi_n \int_0^t \bar{c}_s ds \right) \prod_{k=1}^n \left( \frac{\phi_t^k}{\tilde{Z}_t^k} \right)^{\frac{n-1}{n} \frac{1}{\alpha_k(n-1)-\theta_k(1-\alpha_k)}}.$$

Thus, taking the inverse

$$(\tilde{c}_t)^{-1} \exp \left( \int_0^t \bar{c}_s ds \right) = \left( \prod_{k=1}^n \left( \frac{\phi_t^k}{\tilde{Z}_t^k} \right)^{\frac{n-1}{n} \frac{1}{\alpha_k(n-1)-\theta_k(1-\alpha_k)}} \right)^{\frac{-1}{\xi_n+1}}. \quad (2.25)$$

Plugging the last expression in (2.24) leads

$$\tilde{c}_t^{i,*} \exp \left( -\int_0^t c_s^{i,*} ds \right) = \left( \frac{\phi_t^i}{\tilde{Z}_t^i} \right)^{\frac{n-1}{\alpha_i(n-1)-\theta_i(1-\alpha_i)}} \left( \prod_{k=1}^n \left( \frac{\phi_t^k}{\tilde{Z}_t^k} \right)^{\frac{n-1}{\alpha_k(n-1)-\theta_k(1-\alpha_k)}} \right)^{\frac{-1}{\xi_n+1} \frac{\theta_i(1-\alpha_i)}{\alpha_i(n-1)-\theta_i(1-\alpha_i)}}. \quad (2.26)$$

Let's denote  $A_t^i$  the process on the right-hand side of (2.26), with log-normal dynamics

$$dA_t^i = A_t^i \left( b^{A^i}(t) dt + \delta^{A^i} \cdot d\bar{W}_t \right).$$

Integrating with respect to time, one can show that equation (2.26) admits a unique solution among continuous Itô processes. Some Itô calculus show that the process  $\tilde{c}_t^{i,*}$  given by

$$d\tilde{c}_t^{i,*} = \tilde{c}_t^{i,*} \left( (b^{A^i}(t) + \tilde{c}_t^{i,*}) dt + \delta^{A^i}(t) \cdot d\bar{W}_t^i \right). \quad (2.27)$$

satisfies (2.26), which concludes the proof.  $\square$

### 2.3 Discussion on admissibility

The candidate equilibrium consumption dynamics (2.10) admits a local solution given by

$$\tilde{c}_t^{i,*} = \frac{\exp \left( \int_0^t (b^{A^i}(s) - \|\delta^{A^i}(s)\|^2) ds + \delta^{A^i}(s) \cdot d\bar{W}_s^i \right)}{1 - \int_0^t \exp \left( \int_0^u (b^{A^i}(s) - \|\delta^{A^i}(s)\|^2) ds + \delta^{A^i}(s) \cdot d\bar{W}_s^i \right) du}, \quad (2.28)$$

defined up to the explosion time  $S$  defined as the first hitting time of 1 of the integrated diffusion process

$$I_t^i = \int_0^t \exp \left( \int_0^u (b^{A^i}(s) - \|\delta^{A^i}(s)\|^2) ds + \delta^{A^i}(s) \cdot d\bar{W}_s^i \right) du.$$

Integrated diffusion processes appear in models with various applications, as for example the simulation of the dynamics of the membrane potential of a neuron or derivatives pricing in finance and risk theory [25], [24]. It has been shown in [8] that the integral of the exponential of Brownian motion with negative drift is distributed as the inverse of a gamma variable, thus supported on  $[0, \infty)$ . This has been generalized in [31] for a general geometric Brownian motion. In the framework of FPP Nash equilibrium, one wants to ensure the law of  $I_\infty$  to be valued only in  $[0, 1)$ . Those results thus ensure that the probability of finite time explosion of (2.28) is strictly positive for constant coefficients  $b^A$  and  $\delta^A$ .

In general, the question of admissibility of the candidate optimal consumption process (2.28) remains open. In Section 3.3, we present two examples with admissible optimal consumption.

### 3 Mean field forward optimization

Consider  $(B_t)_{t \geq 0}$  and  $(W_t)_{t \geq 0}$  be two independent Brownian motions, living in a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ . In order to model the continuum of agent of the mean field optimization problem, we introduce the random type vector  $\zeta$  independent of  $B$  and  $W$ , whose distribution describes the proportion of the population following the corresponding preferences. We will focus on the mean field optimization problem in the framework of relative forward performance process, constructed with separable utilities whose time variation are driven by log normal processes  $Z$  and  $\phi$  solving

$$\begin{aligned} dZ_t &= Z_t (b^Z(t) dt + \delta_W^Z(t) dW_t + \delta_B^Z(t) dB_t), \quad Z_0 = 1, \\ d\phi_t &= \phi_t (b^\phi(t) dt + \delta_W^\phi(t) dW_t + \delta_B^\phi(t) dB_t), \quad \phi_0 = 1. \end{aligned}$$

To allow different preferences across the population, we include the parameters  $\delta_W^Z, \delta_B^Z, b^\phi, \delta_W^\phi$  and  $\delta_B^\phi$  in the random type vector

$$\zeta = \left( \delta_W^Z, \delta_B^Z, b^\phi, \delta_W^\phi, \delta_B^\phi, \xi, \alpha, \theta, m, \nu, \sigma \right)$$

with values on the type space

$$\mathcal{Z} = (\mathcal{L}_{loc}^2(\mathbb{R}^{+,*}))^2 \times \mathcal{L}_{loc}^1(\mathbb{R}) \times (\mathcal{L}_{loc}^2(\mathbb{R}^{+,*}))^2 \times (0, \infty)^2 \times [0, 1] \times (0, \infty)^2 \times [0, \infty)^2.$$

Denote  $(\mathcal{F}_t^{MF})_{t \geq 0}$  the smallest filtration for which  $\zeta$  is  $\mathcal{F}_0^{MF}$  measurable and  $B$  and  $W$  are adapted, and  $(\mathcal{F}_t^B)_{t \geq 0}$  the natural filtration generated by  $B$ . For general study of mean-field games under common noise, we refer the reader to [3], [2].

#### 3.1 Mean field Nash equilibrium under forward relative performance concerns

Under the self-financing condition, the generic agent's wealth process is:

$$dX_t = rX_t + \pi_t X_t (\mu dt + \nu dW_t + \sigma dB_t) - c_t X_t dt, \quad \mu = m - r, \quad X_0 = \xi, \quad (3.1)$$

where  $\pi_t$  stands for the fraction of wealth invested in the risky asset and  $c_t$  is the rate of consumption per unit of wealth. The set of admissible strategies is defined as:

$$\mathcal{A}^{MF} = \left\{ (\pi_t, c_t) \mathcal{F}^{MF} \text{ - progressively measurable process valued in } \mathbb{R} \times \mathbb{R}^{+,*}, \right. \\ \left. \text{such that } \forall t \geq 0, \mathbb{E} \left[ \int_0^t (|\pi_s|^2 + |c_s|^2) ds \right] < \infty \right\} \quad (3.2)$$

**Definition 3.1** (MF CRRA equilibrium). Let  $(\overline{X}_t)_{t \geq 0}$  and  $(\overline{C}_t)_{t \geq 0}$  be  $\mathcal{F}^B$  adapted square integrable stochastic processes representing the geometric average wealth and the geometric average consumption of the continuum of agents. Let  $(\pi_t, c_t)$  be an admissible strategy and  $X^{\pi, c}$  be the wealth process (3.1) following this investment-consumption strategy. Consider a  $\mathcal{F}^{MF}$  progressively measurable random field  $Q$  with dynamics

$$Q(t, x) = U(t, x) + \int_0^t V(s, \widehat{c}_s x) ds, \quad (3.3)$$

where  $\widehat{c}_t = \frac{c_t}{(\overline{C}_t)^\theta}$ , and  $U, V$  are progressively measurable random fields satisfying Assumption 1.1.  $Q$  is a MF forward relative performance process if

1. *Time consistency* - For any admissible strategy  $(\pi_t, c_t)$ ,  $Q\left(t, \frac{X_t^{\pi, c}}{(\overline{X}_t)^\theta}\right)$  is a (local) supermartingale and there exists an admissible strategy  $(\pi_t^*, c_t^*)$ , such that  $Q\left(t, \frac{X_t^{\pi^*, c^*}}{(\overline{X}_t)^\theta}\right)$  is a (local) martingale.
2. *Compatibility* -  $(\pi_t^*, c_t^*) \in \mathcal{A}^{MF}$  is a MF Nash equilibrium if

$$\begin{cases} \overline{X}_t &= \exp(\mathbb{E}[\log X_t^* | \mathcal{F}_t^B]) \\ \overline{C}_t &= \exp(\mathbb{E}[\log c_t^* | \mathcal{F}_t^B]) \end{cases} \quad (3.4)$$

We call *strong* Nash equilibrium a MF Nash equilibrium strategy  $(\pi_t^*, c_t^*)$  measurable with respect to  $\mathcal{F}_0^{MF}$ .

Note that  $\mathbb{E}[\log(\cdot)]$  is the continuous analogue of geometric mean. The last condition above can be understood as a compatibility condition between the generic agent and the continuum. Conditionally on the common noise  $B$ , all agents face i.i.d copies of the same optimization problem. Exploiting the consistency condition among processes  $\overline{X}_t$  satisfying (3.4) and using a fix point calculation, we can determine the MF equilibrium portfolio for the CRRA example. Then, expressing the consumption utility  $V$  as a function of  $U$  as in Proposition 1.2, we can establish the MF equilibrium consumption strategy.

For tractability, we will consider the CRRA utilities example. To state the associated mean field equilibrium theorem, we will need the following assumption, stating the power form of the utility  $U$  considered to be an Itô random field with respect to the 2-dimensional Brownian motion  $(W_t, B_t)_{t \geq 0}$ . We will also make an assumption on the drift of this utility, analogue to Proposition 1.1, sufficient for a random field  $Q$  of type (3.3) to be consistent and thus a FPP.

**Assumption 3.1** (MF CRRA). *Consider a random field  $Q$  with dynamics*

$$Q(t, x) = U^{(\alpha)}(t, x) + \int_0^t V^{(\alpha)}(s, \widehat{c}_s x) ds,$$

where  $U^{(\alpha)}$  and  $V^{(\alpha)}$  are CRRA forward utilities of the form

$$U^{(\alpha)}(t, x) = Z_t u^{(\alpha)}(x), \quad \text{and} \quad V^{(\alpha)}(t, c) = \phi_t u^{(\alpha)}(c), \quad (3.5)$$

where  $u^{(\alpha)}(x) = \frac{x^{1-\alpha}}{1-\alpha}$  is the standard deterministic power utility function with risk aversion  $\alpha \in (0, 1)$ , and processes  $(Z_t)_{t \geq 0}$ ,  $(\phi_t)_{t \geq 0}$  follow log-normal dynamics

$$\begin{aligned} dZ_t &= Z_t (b^Z(t)dt + \delta_W^Z(t)dW_t + \delta_B^Z(t)dB_t), \quad Z_0 = 1. \\ d\phi_t &= \phi_t (b^\phi(t)dt + \delta_W^\phi(t)dW_t + \delta_B^\phi(t)dB_t), \quad \phi_0 = 1. \end{aligned} \quad (3.6)$$

The utility random field  $U^{(\alpha)}$  has local characteristics

$$\beta(t, x) = b^Z(t)U^{(\alpha)}(t, x) \quad \text{and} \quad \gamma(t, x) = (\delta_W^Y(t) \delta_B^Z(t))U^{(\alpha)}(t, x),$$

and is assumed to satisfy the SPDE 1.13 from Proposition 1.1 which is equivalent to

$$\beta(t, z) = -\psi(t, z) + \frac{1}{2}U_{xx}^{(\alpha)}z^2(\nu^2 + \sigma^2)(\pi_t)^2 - \tilde{V}^{(\alpha)}(t, (\bar{C}_t)^\theta U_x), \quad (3.7)$$

where

$$\psi(t, z) = U_x^{(\alpha)}(t, z)z \left( -\theta\overline{\mu\pi_t} + \frac{\theta}{2}\overline{\Sigma\pi_t^2} + \frac{\theta^2}{2}\overline{\sigma\pi_t^2} + \theta\bar{c}_t \right) + \frac{1}{2}U_{xx}^{(\alpha)}(t, z)z^2(\theta\overline{\sigma\pi_t})^2 - \gamma_{B_x}(t, z)\theta\overline{\sigma\pi_t}^{(-i)}z. \quad (3.8)$$

**Proposition 3.1.** Under Assumption 3.1, condition (3.7) takes the form

$$\begin{aligned} b^Z(t) &= (1 - \alpha)\delta_B^Z(t)\theta\overline{\sigma\pi_t} - (1 - \alpha) \left( -\theta\overline{\mu\pi_t} + \frac{\theta}{2}\overline{\Sigma\pi_t^2} + \frac{\theta^2}{2}\overline{\sigma\pi_t^2} \right) \\ &\quad + \frac{1}{2}\alpha(1 - \alpha)((\theta\overline{\sigma\pi_t})^2 - (\nu^2 + \sigma^2)(\pi_t^*)^2) - (1 - \alpha)\theta\bar{c}_t - \frac{\alpha}{(\bar{C}_t)^{\frac{\theta(1-\alpha)}{\alpha}}} \left( \frac{\phi_t}{Z_t} \right)^{\frac{1}{\alpha}} \end{aligned} \quad (3.9)$$

*Proof.* Following the same calculation as in the previous section, one can characterize the utility from consumption for the CRRA case, from the drift condition (3.7)

$$\begin{aligned} b^Z(t)U^{(\alpha)}(t, y) &= -(1 - \alpha)U^{(\alpha)}(t, y) \left( -\theta\overline{\mu\pi_t} + \frac{\theta}{2}\overline{\Sigma\pi_t^2} + \frac{\theta^2}{2}\overline{\sigma\pi_t^2} + \theta\bar{c}_t \right) - \tilde{V}^{(\alpha)}((\bar{C}_t)^\theta U_x(t, y)) \\ &\quad + \frac{1}{2}\alpha(1 - \alpha)U^{(\alpha)}(t, y)(\theta\overline{\sigma\pi_t})^2 - \frac{1}{2}\alpha(1 - \alpha)U^{(\alpha)}(t, y)(\nu^2 + \sigma^2)(\pi_t^*)^2 \\ &\quad + \delta_B^Z(t)\theta\overline{\sigma\pi_t}(1 - \alpha)U^{(\alpha)}(t, x). \end{aligned}$$

The latter implies  $\tilde{V}^{(\alpha)}((\bar{C}_t)^\theta U_x^{(\alpha)}(t, y)) = (\bar{v}_t - (1 - \alpha)\theta\bar{c}_t)U(t, y)$ , where

$$\begin{aligned} \bar{v}_t &= - \left[ b^Z(t) - (1 - \alpha)\delta_B^Z(t)\theta\overline{\sigma\pi_t} + (1 - \alpha) \left( -\theta\overline{\mu\pi_t} + \frac{\theta}{2}\overline{\Sigma\pi_t^2} + \frac{\theta^2}{2}\overline{\sigma\pi_t^2} \right) \right. \\ &\quad \left. - \frac{1}{2}\alpha(1 - \alpha)((\theta\overline{\sigma\pi_t})^2 - (\nu^2 + \sigma^2)(\pi_t^*)^2) \right]. \end{aligned} \quad (3.10)$$

Using the same computation as before, we get

$$V^{(\alpha)}(t, x) = \left( \frac{\bar{v}_t - (1 - \alpha)\theta\bar{c}_t}{\alpha} \right)^\alpha (\bar{C}_t)^{\theta(1-\alpha)} U^{(\alpha)}(t, x). \quad (3.11)$$

Similarly as in Section 1, comparing (3.11) with  $V^{(\alpha)}(t, c) = \phi_t u^{(\alpha)}(t, c)$ , one can express the quantity  $\bar{v}_t$  as

$$\bar{v}_t = (1 - \alpha)\theta\bar{c}_t + \frac{\alpha}{(\bar{C}_t)^{\frac{\theta(1-\alpha)}{\alpha}}} \left( \frac{\phi_t}{Z_t} \right)^{\frac{1}{\alpha}}. \quad (3.12)$$

Finally, combining equation (3.10) with (3.12), we get the following constraint on the drift parameter

$$\begin{aligned} b^Z(t) &= (1 - \alpha)\delta_B^Z(t)\theta\bar{\sigma}\bar{\pi}_t - (1 - \alpha)\left(-\theta\bar{\mu}\bar{\pi}_t + \frac{\theta}{2}\bar{\Sigma}\bar{\pi}_t^2 + \frac{\theta^2}{2}\bar{\sigma}\bar{\pi}_t^2\right) \\ &\quad + \frac{1}{2}\alpha(1 - \alpha)((\theta\bar{\sigma}\bar{\pi}_t)^2 - (\nu^2 + \sigma^2)(\bar{\pi}_t^*)^2) - (1 - \alpha)\theta\bar{c}_t - \frac{\alpha}{(\bar{C}_t)^{\frac{\theta(1-\alpha)}{\alpha}}} \left( \frac{\phi_t}{Z_t} \right)^{\frac{1}{\alpha}} \end{aligned}$$

□

### 3.2 General MF Nash equilibrium

In this section, we investigate the existence of a general Nash equilibrium to the mean field optimization problem associated with a forward relative performance process  $Q$  given by (3.3). We prove the existence of an optimal strategy  $(\pi_t^*, c_t^*) \in \mathcal{F}_t^{MF}$ . We work under Assumption 3.1, so that both utilities are of separable form, respectively driven by two Itô random fields  $\phi_t$  and  $Z_t$ . For the following, let's introduce the quantities

$$\psi^\sigma = \mathbb{E}\left[\theta\left(1 - \frac{1}{\alpha}\right)\frac{\sigma^2}{\nu^2 + \sigma^2}\right], \quad \varphi^\sigma = \mathbb{E}\left[\frac{\sigma}{\alpha}\frac{(\delta_W^Z(t)\nu + \delta_B^Z(t)\sigma + \mu)}{\nu^2 + \sigma^2}\right], \quad K_{\alpha,\theta} = -\frac{\frac{\theta(1-\alpha)}{\alpha}}{1 + \mathbb{E}\left[\frac{\theta(1-\alpha)}{\alpha}\right]}. \quad (3.13)$$

Similarly as in the  $n$ -player case, the equilibrium consumption process relies on an analogue log-normal process  $(A_t)_{t \geq 0}$  defined in the following, with local characteristics

$$\begin{aligned} b^A(t) &= \frac{1}{\alpha}\left(b^\phi(t) - b^{\tilde{Z}}(t) + \frac{1}{2}\left(\frac{1}{\alpha} - 1\right)\|\delta^\phi(t)\|^2 + \frac{1}{2}\left(\frac{1}{\alpha} + 1\right)\|\delta^{\tilde{Z}}(t)\|^2 - \frac{2}{\alpha}\delta^\phi(t) \cdot \delta^{\tilde{Z}}(t)\right) \\ &\quad + K_{\alpha,\theta}\mathbb{E}\left[\frac{1}{\alpha}\left(b^\phi(t) - b^{\tilde{Z}}(t) + \frac{1}{2}(\|\delta^{\tilde{Z}}(t)\|^2 - \|\delta^\phi(t)\|^2)\right)\right] + \frac{1}{2}K_{\alpha,\theta}^2\mathbb{E}\left[\frac{1}{\alpha}(\delta^{\phi,W}(t) - \delta^{\tilde{Z},W}(t))^2\right] \\ &\quad + \frac{K_{\alpha,\theta}}{\alpha}\mathbb{E}\left[\frac{1}{\alpha}(\delta^{\phi,W}(t) - \delta^{\tilde{Z},W}(t))\right](\delta^{\phi,W}(t) - \delta^{\tilde{Z},W}(t)) \end{aligned} \quad (3.14)$$

$$\delta^{A,W}(t) = \frac{1}{\alpha}(\delta^{\phi,W}(t) - \delta^{\tilde{Z},W}(t)) + K_{\alpha,\theta}\mathbb{E}\left[\frac{1}{\alpha}(\delta^{\phi,W}(t) - \delta^{\tilde{Z},W}(t))\right], \quad (3.15)$$

$$\delta^{A,B}(t) = \frac{1}{\alpha}(\delta^{\phi,B}(t) - \delta^{\tilde{Z},B}(t)). \quad (3.16)$$

**Theorem 3.2.** *Under Assumptions 3.1, if  $\psi^\sigma \neq 1$ , there exists a MF Nash equilibrium strategy  $(\pi_t^*, c_t^*) \in \mathcal{F}_t^{MF}$  with optimal portfolio process:*

$$\pi_t^* = \frac{1}{\nu^2 + \sigma^2}\left(\theta\sigma\left(1 - \frac{1}{\alpha}\right)\frac{\varphi^\sigma}{1 - \psi^\sigma} + \frac{1}{\alpha}(\delta_W^Z(t)\nu + \delta_B^Z(t)\sigma + \mu)\right), \quad (3.17)$$

and optimal consumption equilibrium process  $(c_t^*)_{t \geq 0}$  following

$$dc_t^* = c_t^*((b^A(t) + c_t^*)dt + \delta^{A,W}(t)dW_t + \delta^{A,B}(t)dB_t). \quad (3.18)$$

Note that both candidate optimal processes  $\pi_t^*$  and  $c_t^*$  are stochastic, measurable with respect to the general filtration  $\mathcal{F}_t^{MF}$ .

*Proof.* First, from the compatibility condition (3.4), we give the Itô decomposition of  $\overline{X}_t$ . Then, applying Itô-Ventzel's formula to the process  $Q(t, \frac{X^{\pi,c}}{(\overline{X}^{\pi,c})^\theta})$ , we investigate the martingale optimality condition from the second point of Definition 3.1. Then similarly as in Section 1, using Proposition 3.1 and the linear dynamics of utilities leads to an explicit form of the MF Nash equilibrium.

**Average wealth process** - Condition (3.4) of Definition 3.1 allows to restrict ourselves to processes  $(\overline{X}_t)_{t \geq 0}$  satisfying  $\overline{X}_t = \exp \mathbb{E}[\log X_t^\pi | \mathcal{F}_t^B]$ . Consider an admissible strategy  $(\pi_t, c_t) \in \mathcal{F}_t^{MF}$ . Applying Itô's formula

$$\begin{aligned} \overline{X}_t &= \exp \mathbb{E}[\log X_t | \mathcal{F}_t^B] \\ &= \exp \mathbb{E} \left[ \log \xi + \int_0^t (\mu \pi_s - \frac{1}{2} \pi_s^2 (\nu^2 + \sigma^2)) ds + \int_0^t \nu \pi_s dW_s + \int_0^t \sigma \pi_s dB_s - \int_0^t c_s ds | \mathcal{F}_t^B \right] \end{aligned} \quad (3.19)$$

$$= \exp \left( \overline{\log \xi} + \int_0^t (\overline{\mu \pi_s} - \frac{1}{2} \overline{\Sigma \pi_s^2}) ds + \int_0^t \overline{\sigma \pi_s} dB_s - \int_0^t \overline{c_s} ds \right) \quad (3.20)$$

$$= \bar{\xi} + \int_0^t \eta \overline{X}_s ds + \int_0^t \overline{\sigma \pi_s} \overline{X}_s dB_s - \int_0^t \overline{c_s} \overline{X}_s ds, \quad (3.21)$$

where

$$\eta = \overline{\mu \pi_s} - \frac{1}{2} (\overline{\Sigma \pi_s^2} - \overline{\sigma \pi_s^2}), \quad \bar{\xi} = \exp \mathbb{E}[\log \xi], \quad \overline{\mu \pi_s} = \mathbb{E}[\mu \pi_s | \mathcal{F}_s^B], \quad \overline{c_s} = \mathbb{E}[c_s | \mathcal{F}_s^B].$$

In fact, since  $(\pi_t, c_t)$  is admissible, it is progressively measurable with respect to the general filtration  $\mathcal{F}_t^{MF} = \sigma(\zeta, (B_s)_{0 \leq s \leq t}, (W_s)_{0 \leq s \leq t})$ . The admissible strategy at time  $t$  thus writes as a function of  $\zeta$  and the trajectories of the two Brownian motions up to time  $t$ . On the other hand, the filtration  $\mathcal{F}_t^B$  generated by the common noise  $B$  at time  $t$  is the sigma algebra generated by  $\mathcal{F}_s^B$  and the Brownian increments after time  $t$ , namely  $(B_u - B_s)_{s \leq u \leq t}$ . Consequently, for  $0 \leq s \leq t$ , the admissible strategy  $(\pi_s, c_s)$  is a function of  $\zeta, (B_u)_{0 \leq u \leq s}$  and  $(W_u)_{0 \leq u \leq s}$  is thus independent of  $(B_u - B_s)_{s \leq u \leq t}$ , which justifies the equality between (3.19) and (3.20).

For any admissible strategy  $(\pi, c)$ , define  $Y^{\pi,c} = \frac{X^{\pi,c}}{(\overline{X}^{\pi,c})^\theta}$ . From (3.21), the dynamics of  $\overline{X}$  are given by

$$\frac{d\overline{X}_t}{\overline{X}_t} = \bar{\xi} + (\eta - \overline{c_s}) dt + \overline{\sigma \pi_s} dB_t,$$

and an application of Itô's formula leads

$$\begin{aligned} \frac{dY^{\pi,c}}{Y_t^{\pi,c}} &= \left( \mu \pi_t - \theta \overline{\mu \pi_t} + \frac{\theta}{2} \overline{\Sigma \pi_t^2} + \frac{\theta^2}{2} \overline{\sigma \pi_t^2} - \theta \sigma \pi_t \overline{\sigma \pi_t} \right) dt \\ &\quad + \nu \pi_t dW_t + (\sigma \pi_t - \theta \overline{\sigma \pi_t}) dB_t - (c_t - \theta \overline{c_t}) dt, \end{aligned} \quad (3.22)$$

with  $Y_0^{\pi,c} = \frac{\xi}{(\bar{\xi})^\theta}$ .

**Forward performance process** - Applying Itô-Ventzel's formula, we obtain the dynamics of the



forward performance process along the relative wealth process  $Q(t, Y_t^{\pi,c})$ .

$$\begin{aligned}
dQ(t, Y_t^{\pi,c}) &= (\beta(t, Y_t^{\pi,c}) + V(\widehat{c}_t Y_t^{\pi,c}))dt + \gamma_W(t, Y_t^{\pi,c})dW_t + \gamma_B(t, Y_t^{\pi,c})dB_t \\
&+ U_x(Y_t^{\pi,c})dY_t^{\pi,c} + \frac{1}{2}U_{xx}(t, Y_t^{\pi,c})d\langle Y_t^{\pi,c} \rangle + \langle \gamma_{W_x}(t, Y_t^{\pi,c})dW_t + \gamma_{B_x}(t, Y_t^{\pi,c})dB_t \rangle, Y_t^{\pi,c} \\
&= (\beta(t, Y_t^{\pi,c}) + V(\widehat{c}_t Y_t^{\pi,c}, t))dt + \gamma_W(t, Y_t^{\pi,c})dW_t + \gamma_B(t, Y_t^{\pi,c})dB_t \\
&+ U_x(t, Y_t^{\pi,c})Y_t^{\pi,c} \left( \mu\pi_t - \theta\overline{\mu\pi_t} + \frac{\theta}{2}\overline{\Sigma\pi_t^2} + \frac{\theta^2}{2}\overline{\sigma\pi_t^2} - \theta\sigma\pi_t\overline{\sigma\pi_t} - (c_t - \theta\overline{c_t}) \right) dt \\
&+ \frac{1}{2}U_{xx}(t, Y_t^{\pi,c})(Y_t^{\pi,c})^2((\nu\pi_t)^2 + (\sigma\pi_t - \theta\overline{\sigma\pi_t})^2)dt \\
&+ U_x(t, Y_t^{\pi,c})Y_t^{\pi,c}(\nu\pi_t dW_t + (\sigma\pi_t - \theta\overline{\sigma\pi_t})dB_t) \\
&+ (\gamma_{W_x}(t, Y_t^{\pi,c})\nu\pi_t Y_t^{\pi,c} + \gamma_{B_x}(t, Y_t^{\pi,c})(\sigma\pi_t - \theta\overline{\sigma\pi_t}))dt.
\end{aligned}$$

**Best response strategy** - Using the martingale condition on  $Q(t, Y_t^{\pi,c})$  at the optimum, we study the first order condition

$$\begin{aligned}
0 &= U_x(t, Y_t^{\pi,c})(\mu - \theta\sigma\overline{\sigma\pi_t})Y_t^{\pi,c} + \frac{1}{2}U_{xx}(t, Y_t^{\pi,c})(2\pi\nu^2 + 2\pi\sigma^2 - 2\sigma\theta\overline{\sigma\pi_t})(Y_t^{\pi,c})^2 \\
&\quad + \gamma_{W_x}(t, Y_t^{\pi,c})\nu Y_t^{\pi,c} + \gamma_{B_x}(t, Y_t^{\pi,c})\sigma Y_t^{\pi,c} \\
&\Leftrightarrow \pi(\nu^2 + \sigma^2)Y_t^{\pi,c}U_{xx}(t, Y_t^{\pi,c}) = \sigma\theta\overline{\sigma\pi_t}U_{xx}(t, Y_t^{\pi,c}) - U_x(t, Y_t^{\pi,c})(\mu - \theta\sigma\overline{\sigma\pi_t}) \\
&\quad - \gamma_{W_x}(t, Y_t^{\pi,c})\nu - \gamma_{B_x}(t, Y_t^{\pi,c})\sigma \\
&\Leftrightarrow \pi_t^* = \frac{1}{\nu^2 + \sigma^2} \left( \theta\sigma\overline{\sigma\pi_t} - \frac{1}{U_{xx}(t, Y_t^{\pi,c})Y_t^{\pi,c}}(\gamma_{W_x}(t, Y_t^{\pi,c})\nu + \gamma_{B_x}(t, Y_t^{\pi,c})\sigma) + (\mu - \theta\sigma\overline{\sigma\pi_t})U_x(t, Y_t^{\pi,c}) \right). \quad (3.23)
\end{aligned}$$

The first order condition on the consumption process leads

$$\begin{aligned}
0 &= -U_x(t, Y_t^{\pi,c})Y_t^{\pi,c} + \frac{Y_t^{\pi,c}}{(\overline{C}_t)^\theta} V_x(t, \widehat{c}_t Y_t^{\pi,c}) \\
&\Leftrightarrow c_t^* = \frac{V_x^{-1}(t, U_x(t, Y_t^{\pi,c})(\overline{C}_t)^\theta)(\overline{C}_t)^\theta}{Y_t^{\pi,c}}. \quad (3.24)
\end{aligned}$$

Denoting

$$\psi(t, y) = U_x(t, y)y \left( -\theta\overline{\mu\pi_t} + \frac{\theta}{2}\overline{\Sigma\pi_t^2} + \frac{\theta^2}{2}\overline{\sigma\pi_t^2} + \theta\overline{c_t} \right) + \frac{1}{2}U_{xx}(t, y)y^2(\theta\overline{\sigma\pi_t})^2 - \gamma_{B_x}(t, z)\theta\overline{\sigma\pi_t}y,$$

the drift of  $Q(t, Y_t^{\pi,c})$  takes the form

$$\begin{aligned}
\text{drift } Q(t, Y_t^{\pi,c}) &= \beta(t, Y_t^{\pi,c}) + \psi(t, Y_t^{\pi,c}) + \frac{1}{2}U_{xx}(t, Y_t^{\pi,c})(Y_t^{\pi,c})^2(\nu^2 + \sigma^2)(\pi_t^2 - 2\pi_t\pi_t^*) \\
&\quad + \widetilde{V}(t, (\overline{C}_t)^\theta U_x(t, Y_t^{\pi,c})) + V(t, \widehat{c}_t Y_t^{\pi,c}) - c_t Y_t^{\pi,c} U_x(t, Y_t^{\pi,c}) - \widetilde{V}(t, (\overline{C}_t)^\theta U_x(t, Y_t^{\pi,c}))
\end{aligned}$$

The drift condition then writes as

$$\beta(t, y) = -\psi(t, y) + \frac{1}{2}U_{xx}(t, y)y^2(\nu^2 + \sigma^2)(\pi_t)^2 - \widetilde{V}(t, (\overline{C}_t)^\theta U_x(t, y)),$$

which is coherent with (3.7) from Assumption 3.1.

**Optimality of the strategy** - The optimal portfolio (3.23) for forward performance processes of power type writes as:

$$\begin{aligned}\pi_t^* &= \frac{1}{\nu^2 + \sigma^2} \left( \theta \sigma \overline{\sigma \pi_t} + \frac{\delta_W^Z(t) \nu}{\alpha} + \frac{\delta_B^Z(t) \sigma}{\alpha} + \frac{\mu - \theta \sigma \overline{\sigma \pi_t}}{\alpha} \right) \\ &= \frac{1}{\nu^2 + \sigma^2} \left( \theta \sigma \overline{\sigma \pi_t} \left(1 - \frac{1}{\alpha}\right) + \frac{1}{\alpha} (\delta_W^Z(t) \nu + \delta_B^Z(t) \sigma + \mu) \right),\end{aligned}$$

Multiplying both sides by  $\sigma$  and taking the conditional expectation with respect to  $\mathcal{F}_t^B$  leads

$$\begin{aligned}\overline{\sigma \pi_t} &= \theta \left(1 - \frac{1}{\alpha}\right) \frac{\sigma^2}{\nu^2 + \sigma^2} \overline{\sigma \pi_t} + \frac{\sigma}{\alpha} \frac{\delta_W^Z(t) \nu + \delta_B^Z(t) \sigma + \mu}{\nu^2 + \sigma^2} \\ &= \psi^\theta \overline{\sigma \pi_t} + \varphi^\sigma,\end{aligned}$$

where

$$\psi^\sigma = \mathbb{E} \left[ \theta \left(1 - \frac{1}{\alpha}\right) \frac{\sigma^2}{\nu^2 + \sigma^2} \right], \quad \text{and} \quad \varphi^\sigma = \mathbb{E} \left[ \frac{\sigma}{\alpha} \frac{\delta_W^Z(t) \nu + \delta_B^Z(t) \sigma + \mu}{\nu^2 + \sigma^2} \right].$$

In fact, every variable above is measurable with respect to  $\mathcal{F}_0^{MF}$  and thus independent of  $\mathcal{F}_t^B$ . Hence, if  $\psi^\sigma \neq 1$ , there exists a MF portfolio equilibrium  $\pi_t^*$  given by (3.23) with  $\overline{\sigma \pi_t^*} = \frac{\varphi^\sigma}{1 - \psi^\sigma}$ .

**Consumption equilibrium** - According to the first order condition (3.24) and Proposition 3.1, the optimal consumption process takes the form

$$c_t^* = \frac{1}{\alpha} (\overline{v_t} - (1 - \alpha) \theta \overline{c_t}) = \frac{1}{(\overline{C_t})^{\frac{\theta(1-\alpha)}{\alpha}}} \left( \frac{\phi_t}{Z_t} \right)^{\frac{1}{\alpha}}. \quad (3.25)$$

The mean field equilibrium is defined with reference to the continuum of agents processes  $\overline{X_t}, \overline{C_t}$ , with a fixed point identity. The last step to obtain the explicit formula of the optimal consumption process in the mean-field framework is to use the compatibility condition (3.4) in order to express the geometric average consumption of the continuum of agents  $\overline{C_t}$ . Replacing  $c_t^*$  with (3.25) leads

$$\overline{C_t} = \exp \left( \mathbb{E} \left[ \log \left( \frac{1}{(\overline{C_t})^{\frac{\theta(1-\alpha)}{\alpha}}} \left( \frac{\phi_t}{Z_t} \right)^{\frac{1}{\alpha}} \right) \middle| \mathcal{F}_t^B \right] \right).$$

Since the process  $\overline{C_t}$  is  $\mathcal{F}_t^B$ -adapted, we get

$$\overline{C_t} = \exp \left( \mathbb{E} \left[ \frac{1}{\alpha} \log \left( \frac{\phi_t}{Z_t} \right) \middle| \mathcal{F}_t^B \right] \right) \frac{1}{(\overline{C_t})^{\mathbb{E} \left[ \frac{\theta(1-\alpha)}{\alpha} \right]}},$$

so that

$$\overline{C_t} = \exp \left( \mathbb{E} \left[ \frac{1}{\alpha} \log \left( \frac{\phi_t}{Z_t} \right) \middle| \mathcal{F}_t^B \right] \right)^{\frac{1}{1 + \mathbb{E} \left[ \frac{\theta(1-\alpha)}{\alpha} \right]}}.$$

Replacing this expression in (3.25), we get an expression of the candidate optimal consumption process for the mean field problem

$$c_t^* = \exp \left( \mathbb{E} \left[ \frac{1}{\alpha} \log \left( \frac{\phi_t}{Z_t} \right) \middle| \mathcal{F}_t^B \right] \right)^{\frac{-\theta(1-\alpha)}{1 + \mathbb{E} \left[ \frac{\theta(1-\alpha)}{\alpha} \right]}} \left( \frac{\phi_t}{Z_t} \right)^{\frac{1}{\alpha}}. \quad (3.26)$$

**Nash equilibrium** - Following the steps of the proof of Theorem 2.1, we exhibit the equilibrium consumption strategy. Denote

$$\tilde{b}^Z(t) = (1 - \alpha)\delta_B^Z(t)\theta\overline{\sigma\pi}_t - (1 - \alpha)\left(-\theta\overline{\mu\pi}_t + \frac{\theta}{2}\overline{\Sigma\pi}_t^2 + \frac{\theta^2}{2}\overline{\sigma\pi}_t^2\right) + \frac{1}{2}\alpha(1 - \alpha)((\theta\overline{\sigma\pi}_t)^2 - (\nu^2 + \sigma^2)(\pi_t^*)^2),$$

and  $\tilde{Z}_t$  the associated Itô diffusion with drift  $\tilde{b}^Z(t)$

$$d\tilde{Z}_t = \tilde{Z}_t\left(\tilde{b}^Z(t)dt + \delta^{W,Z}dW_t + \delta^{B,Z}dB_t\right). \quad (3.27)$$

Then the linear dynamics of  $Z$  from Assumption 3.1 ensures that

$$Z_t = \tilde{Z}_t \exp\left(- (1 - \alpha)\theta \int_0^t \bar{c}_s ds - \alpha \int_0^t c_s^* ds\right).$$

Replacing the above in the optimal candidate consumption in terms of the average consumption process (3.26) leads

$$c_t^* \exp\left(- \int_0^t c_s^* ds\right) = (\bar{C}_t)^{-\frac{\theta(1-\alpha)}{\alpha}} \left(\frac{\phi_t}{\tilde{Z}_t}\right)^{\frac{1}{\alpha}} \exp\left(\frac{(1-\alpha)\theta}{\alpha} \int_0^t \bar{c}_s ds\right). \quad (3.28)$$

Taking the logarithm, the conditional expectation with respect to  $\mathcal{F}_t^B$  and the exponent of the above expression, we get

$$\bar{C}_t \exp\left(- \int_0^t \bar{c}_s ds\right) = \exp\left(\mathbb{E}\left[\frac{1}{\alpha} \log\left(\frac{\phi_t}{\tilde{Z}_t}\right) \middle| \mathcal{F}_t^B\right]\right) (\bar{C}_t)^{-\mathbb{E}\left[\frac{\theta(1-\alpha)}{\alpha}\right]} \exp\left(\mathbb{E}\left[\frac{(1-\alpha)\theta}{\alpha}\right] \int_0^t \bar{c}_s ds\right).$$

Taking the inverse

$$(\bar{C}_t)^{-1} \exp\left(\int_0^t \bar{c}_s ds\right) = \exp\left(\frac{-1}{1 + \mathbb{E}\left[\frac{\theta(1-\alpha)}{\alpha}\right]} \mathbb{E}\left[\frac{1}{\alpha} \log\left(\frac{\phi_t}{\tilde{Z}_t}\right) \middle| \mathcal{F}_t^B\right]\right),$$

which replaced in (3.28) leads

$$c_t^* \exp\left(- \int_0^t c_s^* ds\right) = \left(\frac{\phi_t}{\tilde{Z}_t}\right)^{\frac{1}{\alpha}} \exp\left(- \frac{\frac{\theta(1-\alpha)}{\alpha}}{1 + \mathbb{E}\left[\frac{\theta(1-\alpha)}{\alpha}\right]} \mathbb{E}\left[\frac{1}{\alpha} \log\left(\frac{\phi_t}{\tilde{Z}_t}\right) \middle| \mathcal{F}_t^B\right]\right). \quad (3.29)$$

Denote  $A_t$  the right-hand side of the above equation and  $K_{\alpha,\theta} = -\frac{\frac{\theta(1-\alpha)}{\alpha}}{1 + \mathbb{E}\left[\frac{\theta(1-\alpha)}{\alpha}\right]}$ . The dynamics of this process are more convenient to write in the mean-field framework. In fact, using Itô's formula

$$d \log\left(\frac{\phi_t}{\tilde{Z}_t}\right) = \left(b^\phi(t) - b^{\tilde{Z}}(t) + \frac{1}{2}\left(\|\delta^{\tilde{Z}}\|^2 - \|\delta^\phi\|^2\right)\right) dt + (\delta^\phi(t) - \delta^{\tilde{Z}}(t)) \cdot d\bar{W}_t, \quad (3.30)$$

The Brownian motion  $B$  being independent from  $W$  and the type vector  $\zeta$ , we get

$$d\mathbb{E}\left[\log\left(\frac{\phi_t}{\tilde{Z}_t}\right) \middle| \mathcal{F}_t^B\right] = \mathbb{E}\left[\frac{1}{\alpha}(b^\phi(t) - b^{\tilde{Z}}(t)) + \frac{1}{2}(\|\delta^{\tilde{Z}}(t)\|^2 - \|\delta^\phi(t)\|^2)\right] dt + \mathbb{E}\left[\frac{1}{\alpha}(\delta^{\phi,W}(t) - \delta^{\tilde{Z},W}(t))\right] dW_t.$$

On the other hand

$$d\left(\frac{\phi_t}{\tilde{Z}_t}\right)^{\frac{1}{\alpha}} = \left(\frac{\phi_t}{\tilde{Z}_t}\right)^{\frac{1}{\alpha}} \left[\left(\frac{1}{\alpha}(b^\phi(t) - b^{\tilde{Z}}(t)) + \frac{1}{2}\left(\frac{1}{\alpha} - 1\right)\|\delta^\phi(t)\|^2 + \frac{1}{2}\left(\frac{1}{\alpha} + 1\right)\|\delta^{\tilde{Z}}(t)\|^2 - \frac{2}{\alpha}\delta^\phi(t) \cdot \delta^{\tilde{Z}}(t)\right) dt + \frac{1}{\alpha}(\delta^\phi(t) - \delta^{\tilde{Z}}(t)) \cdot d\bar{W}_t\right]. \quad (3.31)$$

Finally, Itô's formula leads

$$dA_t = A_t(b^A(t)dt + \delta^{A,W}(t)dW_t + \delta^{A,B}(t)dB_t), \quad (3.32)$$

where

$$\begin{aligned} b^A(t) &= \frac{1}{\alpha} \left( b^\phi(t) - b^{\bar{Z}}(t) + \frac{1}{2} \left( \frac{1}{\alpha} - 1 \right) \left\| \delta^\phi(t) \right\|^2 + \frac{1}{2} \left( \frac{1}{\alpha} + 1 \right) \left\| \delta^{\bar{Z}}(t) \right\|^2 - \frac{2}{\alpha} \delta^\phi(t) \cdot \delta^{\bar{Z}}(t) \right) \\ &\quad + K_{\alpha,\theta} \mathbb{E} \left[ \frac{1}{\alpha} \left( b^\phi(t) - b^{\bar{Z}}(t) + \frac{1}{2} \left( \left\| \delta^{\bar{Z}}(t) \right\|^2 - \left\| \delta^\phi(t) \right\|^2 \right) \right) \right] + \frac{1}{2} K_{\alpha,\theta}^2 \mathbb{E} \left[ \frac{1}{\alpha} (\delta^{\phi,W}(t) - \delta^{\bar{Z},W}(t))^2 \right] \\ &\quad + \frac{K_{\alpha,\theta}}{\alpha} \mathbb{E} \left[ \frac{1}{\alpha} (\delta^{\phi,W}(t) - \delta^{\bar{Z},W}(t)) \right] (\delta^{\phi,W}(t) - \delta^{\bar{Z},W}(t)) \\ \delta^{A,W}(t) &= \frac{1}{\alpha} (\delta^{\phi,W}(t) - \delta^{\bar{Z},W}(t)) + K_{\alpha,\theta} \mathbb{E} \left[ \frac{1}{\alpha} (\delta^{\phi,W}(t) - \delta^{\bar{Z},W}(t)) \right], \\ \delta^{A,B}(t) &= \frac{1}{\alpha} (\delta^{\phi,B}(t) - \delta^{\bar{Z},B}(t)). \end{aligned}$$

Similarly as in the proof of Theorem 2.1, there exists a unique continuous solution  $(c_t^*)_{t \geq 0}$  to (3.29) satisfying

$$dc_t^* = c_t^* ((b^A(t) + c_t^*)dt + \delta^A(t) \cdot d\bar{W}_t). \quad (3.33)$$

□

### 3.3 Examples of MF Nash equilibrium

In this section, we study the equilibrium consumption strategy (3.18) under two simplifying assumptions. First, assuming a proportional relationship between wealth and consumption utility, the optimal strategy is  $\mathcal{F}_0^{MF}$ -measurable. This provides a tractable example of a mean-field Nash equilibrium. If the random time-dependent components of wealth and consumption utilities are non-linearly related through a power function, we recover an equilibrium consumption with logistic-like dynamics.

**Proportional market-risk preference** - Condition (3.9) ensuring time consistency of the considered forward performance process is characterized by the ratio of consumption over wealth utility  $\frac{\phi_t}{Z_t}$ . It is natural to assume this fraction to be known for all times in order to decouple (3.9), with the following assumption.

**Assumption 3.2.**  $\phi_t = k(t)Z_t$ , for some continuous function  $k : \mathbb{R}^+ \rightarrow \mathbb{R}^{+,*}$ .

We add the random variable  $k$  representing the distribution of this proportional factor over the population to the type vector  $\zeta$ , requiring that  $\mathbb{E}[\log(k(t))] < +\infty$  for all  $t \geq 0$ .

**Theorem 3.3.** Under Assumptions 3.1 and 3.2, if  $\psi^\sigma \neq 1$ , there exists a strong MF Nash equilibrium strategy  $(\pi_t^*, c_t^*) \in \mathcal{F}_0^{MF}$ . The optimal portfolio is given by

$$\pi_t^* = \frac{1}{\nu^2 + \sigma^2} \left( \theta \sigma \left( 1 - \frac{1}{\alpha} \right) \frac{\varphi^\sigma}{1 - \psi^\sigma} + \frac{1}{\alpha} (\delta_W^Z(t)\nu + \delta_B^Z(t)\sigma + \mu) \right), \quad (3.34)$$

and the optimal consumption is given by

$$c_t^* = \exp \left( \frac{-\theta(1-\alpha)}{1 + \mathbb{E} \left[ \frac{\theta(1-\alpha)}{\alpha} \right]} \mathbb{E} \left[ \frac{\log(k(t))}{\alpha} \right] \right) k(t)^{\frac{1}{\alpha}}. \quad (3.35)$$

Note that under Assumption 3.2, the drift condition (3.9) is independent of  $Z$ . The existence of a solution to the SDE (3.6) and thus of a forward utility of separable power type is straightforward.

*Proof.* The proof is the same as the one of Theorem 3.2, where the conditional expectation with respect to the common noise are replaced with expectations. Under Assumption 3.2, the ratio  $\frac{\phi_t}{Z_t} = k(t) \in \mathcal{F}_0^{MF}$ , so that the optimal strategy only depends on the type vector  $\zeta$ .  $\square$

The forward relative performance process with non-zero volatility framework thus leads to a strong Nash equilibrium with an optimal portfolio whose time dependence relies on the volatility parameters  $\delta_t^B$  and  $\delta_t^W$ , and an optimal consumption which depends on time through the *market-risk proportional preference parameter*  $k_t$ . In contrast, in the zero volatility forward utility framework, [7] and [21] exhibit a strong equilibrium strategy with constant portfolio and time dependent consumption process, also function of the ratio  $\frac{\phi_t}{Z_t}$ . This justifies Assumption 3.2 when investigating the existence of a strong mean-field Nash equilibrium.

**Remark 3.1.** The mean field equilibrium strategy  $(\pi^*, c^*)$  is in fact the limit as  $n$  goes to infinity of the  $n$  player equilibrium strategies (2.9) and (2.10), presented in the first section.

**Relative market-consumption preference** - One can also consider a non-linear dependence of power type between wealth and consumption utility.

**Assumption 3.3.**  $\phi_t = Z_t^{1-\kappa}$ , where  $\kappa \in \mathbb{R}$  is called the *risk relative consumption preference parameter*.

Injecting this condition in the proof of Theorem 3.2, one can show that (3.28) rewrites as:

$$c_t^* \exp\left(-\kappa \int_0^t c_s^* ds\right) = (\bar{C}_t)^{\frac{1}{\mathbb{E}[\kappa]} \times \frac{-\theta(1-\alpha)}{\alpha} \mathbb{E}[\kappa]} (\tilde{Z}_t)^{\frac{-\kappa}{\alpha}} \exp\left(\frac{\kappa(1-\alpha)\theta}{\alpha} \int_0^t \bar{c}_s ds\right). \quad (3.36)$$

Thus, the decoupling of the above equation is possible assuming that  $\kappa = \mathbb{E}[\kappa]$ . This restrictive assumption also appear in [7] for technical reasons and imposes that every agent in the population share the same preference regarding relative importance of wealth and consumption utility. The risk relative consumption preference parameter  $\kappa$  allows to control the quadratic growth of the Nash equilibrium consumption process.

**Theorem 3.4.** *Suppose that Assumptions 3.1 and 3.3 hold with market risk relative preference parameter  $\kappa$  common to all agents. If  $\psi^\sigma \neq 1$ , there exists a unique candidate MF Nash equilibrium strategy  $(\pi_t^*, c_t^*)$ . The optimal portfolio is given by*

$$\pi_t^* = \frac{1}{\nu^2 + \sigma^2} \left( \theta \sigma \left(1 - \frac{1}{\alpha}\right) \frac{\varphi^\sigma}{1 - \psi^\sigma} + \frac{1}{\alpha} (\delta_W^Z(t) \nu + \delta_B^Z(t) \sigma + \mu) \right), \quad (3.37)$$

and the optimal candidate consumption process  $(c_t^*)_{t \geq 0}$  satisfies

$$dc_t^* = c_t^* ((b^A(t) + \kappa c_t^*) dt + \delta^A(t) \cdot d\bar{W}_t). \quad (3.38)$$

*Proof.* Following the proof of Theorem 3.2 with  $\kappa$  deterministic in (3.36), we obtain

$$c_t^* \exp\left(-\kappa \int_0^t c_s^* ds\right) = \tilde{Z}_t^{-\frac{\kappa}{\alpha}} \exp\left(\frac{\frac{\kappa\theta(1-\alpha)}{\alpha}}{1 + \mathbb{E}\left[\frac{\theta(1-\alpha)}{\alpha}\right]} \mathbb{E}\left[\frac{\log(\tilde{Z}_t)}{\alpha} \middle| \mathcal{F}_t^B\right]\right) := A_t. \quad (3.39)$$

Applying Itô's formula, we compute

$$d(\tilde{Z}_t^{\frac{-\kappa}{\alpha}}) = \tilde{Z}_t^{\frac{-\kappa}{\alpha}} \left[ \left( \frac{1}{2} \frac{\kappa}{\alpha} \left( \frac{\kappa}{\alpha} + 1 \right) \left\| \delta_t^{\tilde{Z}} \right\|^2 - \frac{\kappa}{\alpha} b_t^{\tilde{Z}} \right) dt - \frac{\kappa}{\alpha} \delta_t^{\tilde{Z}} \cdot d\bar{W}_t \right],$$

and,

$$d\mathbb{E} \left[ \log(\tilde{Z}_t) | \mathcal{F}_t^B \right] = \mathbb{E} \left[ b_t^{\tilde{Z}} - \frac{1}{2} \left\| \delta_t^{\tilde{Z}} \right\|^2 \right] dt + \mathbb{E} \left[ \delta_t^{\tilde{Z}, B} \right] dB_t.$$

Finally, the process  $A_t$  has log-normal dynamics where

$$\begin{aligned} b_t^A &= \frac{1}{2} \frac{\kappa}{\alpha} \left( \frac{\kappa}{\alpha} + 1 \right) \left\| \delta_t^{\tilde{Z}} \right\|^2 - \frac{\kappa}{\alpha} b_t^{\tilde{Z}} - \kappa K_{\alpha, \theta} \mathbb{E} \left[ \frac{1}{\alpha} \right] \mathbb{E} \left[ b_t^{\tilde{Z}} - \frac{1}{2} \left\| \delta_t^{\tilde{Z}} \right\|^2 \right] \\ &\quad + \frac{1}{2} \left( \kappa K_{\alpha, \theta} \mathbb{E} \left[ \frac{1}{\alpha} \right] \right)^2 \mathbb{E} \left[ \delta_t^{\tilde{Z}, B} \right]^2 + \frac{\kappa^2}{\alpha} K_{\alpha, \theta} \mathbb{E} \left[ \frac{1}{\alpha} \right] \delta_t^{\tilde{Z}, B} \mathbb{E} \left[ \delta_t^{\tilde{Z}, B} \right], \end{aligned} \quad (3.40)$$

and

$$\delta_t^{A, B} = -\frac{\kappa}{\alpha} \delta_t^{\tilde{Z}, B} - \kappa K_{\alpha, \theta} \mathbb{E} \left[ \frac{1}{\alpha} \right] \mathbb{E} \left[ \delta_t^{\tilde{Z}, B} \right], \quad (3.41)$$

$$\delta_t^{A, W} = -\frac{\kappa}{\alpha} \delta_t^{\tilde{Z}, W}. \quad (3.42)$$

Then, the process  $c_t^*$  with dynamics

$$dc_t^* = c_t^* \left( (b^A(t) + \kappa c_t^*) dt + \delta^A(t) \cdot d\bar{W}_t \right),$$

is solution to equation (3.39).  $\square$

## 4 Discussion of the equilibrium

In the next section, we discuss the equilibrium obtained in the mean field optimization problem in the framework of relative performance processes. We investigate the dependence of the optimal strategy  $(\pi_t^*, c_t^*)$  given in (3.26) on the different parameters of the model.

### 4.1 Investment strategy

The optimal investment strategy in the mean field optimization problem in the CRRA relative performance process framework depends on time only through the volatility parameters of wealth's utility  $U$ , namely  $\delta_W^Z, \delta_B^Z$ . In fact, the optimal investment  $\pi_t^*$  writes as a sum of  $\pi_t^{1,*}$  and  $\pi_t^{2,*}$  where:

$$\pi_t^{1,*} = \frac{1}{\nu^2 + \sigma^2} \theta \sigma \left( 1 - \frac{1}{\alpha} \right) \frac{\varphi^\sigma}{1 - \psi^\sigma} \quad \text{and} \quad \pi_t^{2,*} = \frac{\delta_W^Z(t) \nu + \delta_B^Z(t) \sigma + \mu}{\alpha(\nu^2 + \sigma^2)}. \quad (4.1)$$

In comparison, in the determinist CRRA utility framework of [22], [21] and the forward CRRA utility with null volatility model of [7], the authors provide a time-independent optimal investment strategy.

The effect of competition on the optimal strategy is captured by the negative quantity  $\pi_t^{1,*}$ . Note that, as in [22], the optimal portfolio given by (3.17) is a decreasing function of the competition parameter  $\theta$ . This implies that the more an agent cares about relative performance rather than

absolute, the smaller her optimal portfolio allocation will be. As  $\theta$  goes to zero, the optimal portfolio converges to a limit process  $\pi_t^{2,*}$  which is the optimal strategy in the non-competitive framework [9].

As  $\alpha \in [0, 1]$  decreases to 0, the time dependent quantity  $\pi_t^{2,*}$  increases. Thus, the portfolio allocation tends to be more variable in time for risk seeking agent. Moreover,  $\pi_t^{1,*}$  is also decreasing as  $\alpha$  goes to 0, so that a risk seeking agent will see her optimal investment strategy more influenced by competition.

One major feature of this result is that the equilibrium portfolio for the MF Nash equilibrium does not depend on the relative importance the agents give to utility from wealth compared to consumption. Conversely, we will see in Section 4.2 that the optimal consumption rate heavily depend on this preference, characterized by the ratio  $\frac{\phi_t}{Z_t}$ .

**Single stock** - Assume that  $(\nu, \sigma, \mu)$  are deterministic with  $\nu = 0$ . This corresponds to the situation of a single stock in which all agents trade. The optimal portfolio (3.17) can be rewritten as a function of  $K_{\alpha, \theta}$  defined in (3.13), as

$$\pi_t^* = \frac{1}{\sigma^2} \left( \mathbb{E} \left[ \frac{\delta_B^Z(t)\sigma + \mu}{\alpha} \right] K_{\alpha, \theta} + \frac{\delta_B^Z(t)\sigma + \mu}{\alpha} \right). \quad (4.2)$$

The term  $\pi_t^{*,2} = \frac{\delta_B^Z(t)\sigma + \mu}{\sigma^2 \alpha}$  is the optimal investment when there is no competition, coherent with [9]. This strategy is corrected in presence of competition, with  $\pi_t^{*,2} = \frac{1}{\sigma^2} \mathbb{E} \left[ \frac{\delta_B^Z(t)\sigma + \mu}{\alpha} \right] K_{\alpha, \theta}$ . The magnitude of correction is defined by the coefficient  $K_{\alpha, \theta}$  which we represent in Figure 1. Observe that for a fixed risk aversion, the magnitude of the correction  $K_{\alpha, \theta}$  grows linearly in  $\theta$ .

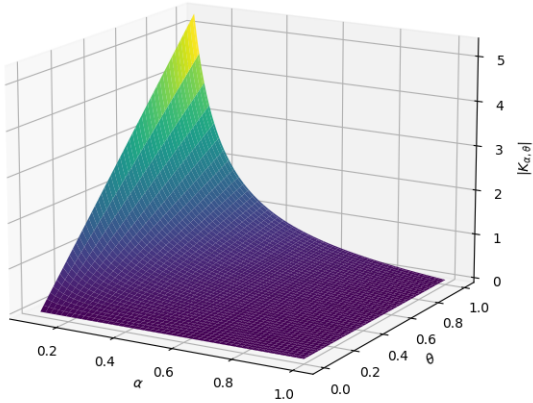


FIGURE 1.  $K_{\alpha, \theta}$  function of risk aversion  $\alpha$  and competition parameter  $\theta$

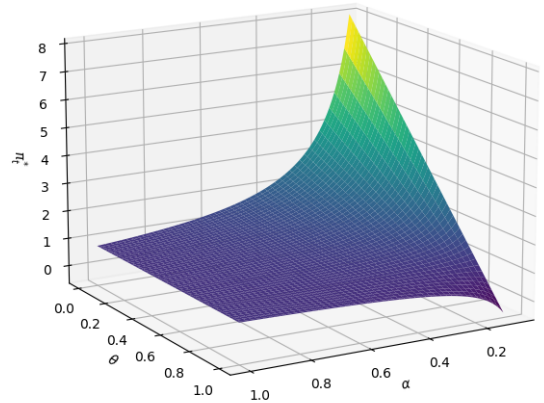


FIGURE 2. Optimal investment for  $\mu = 0.3$ ,  $\sigma = 1$ ,  $\delta_B = \mathbb{E}[\delta_B] = 0.5$ ,  $\mathbb{E}[\frac{1}{\alpha}] = 2$  and  $\mathbb{E}[\theta] = 0.7$ .

We plot the equilibrium portfolio strategy as a function of  $\alpha$  and  $\theta$  in Figure 2. For  $\alpha$  close to 1, the optimal investment is almost constant with respect to  $\theta$ . The optimal strategy of a risk averse agent does not vary much around  $\pi_t^{*,2}$ , when having relative performance concerns. For  $\alpha$  close to 0, the optimal portfolio allocation  $\pi_t^*$  is decreasing in  $\theta$ . Risk seeking agents tend to invest more when with low interest in relative performance, and decrease their investment when competing.

## 4.2 Consumption strategy

Let us now turn our attention to the optimal consumption process  $c_t^*$  given by (3.26). This equilibrium strategy is random, driven by the process  $\frac{\phi_t}{Z_t}$ , which represents the weight the representative agent gives to her utility from consumption, relatively to her wealth utility.

**Proportional market-risk preference equilibrium** - Let's assume that every agent have a utility from consumption proportional to their utility from wealth, that is  $\phi_t = K Z_t$ , for some positive real random variable  $K$  such that  $\mathbb{E}[\log(K)] < +\infty$ , that we add in the type vector  $\zeta$ . This assumption allows to decouple the drift constraints, giving a constant value to the ratio  $\frac{\phi_t}{Z_t}$ . The optimal consumption process  $c_t^*$  is then time independent, function of  $K$ , the competition parameter  $\theta$  and the risk aversion  $\alpha$ . The optimal consumption rate takes the form

$$c_t^* = \exp\left(\frac{\log(K)}{\alpha} + K_{\alpha,\theta} \mathbb{E}\left[\frac{\log(K)}{\alpha}\right]\right), \quad \text{with } K_{\alpha,\theta} = -\frac{\frac{\theta(1-\alpha)}{\alpha}}{1 + \mathbb{E}\left[\frac{\theta(1-\alpha)}{\alpha}\right]}. \quad (4.3)$$

The value of the MF Nash equilibrium consumption thus depends on the difference between the logarithm of the proportional market consumption preference parameter and its geometric average among the population, with specific coefficient representing the risk aversion and the competition between agents. The MF Nash equilibrium when  $\theta = 0$  is given by

$$c_t^{*,NC} = \exp\left(\frac{\log(K)}{\alpha}\right). \quad (4.4)$$

In other words, the optimal consumption rate (4.3) is the optimal rate of consumption  $c_t^{*,NC}$ , corrected in the presence of competition by a term relative to the investment preference of the population  $\mathbb{E}\left[\frac{\log(K)}{\alpha}\right]$ . The magnitude of the correction is determined by the quantity  $K_{\alpha,\theta}$ , represented in Figure 1.

The situation where  $\mathbb{E}[\log(K)] = 0$ , meaning that on average, the continuum of agents prefers neither consumption nor wealth utility coincides with the case with no consumption. Then, if the representative agent prefers consumption to wealth utility that is  $\log(K) > 0$ ,

- $c_t^{*,NC} \xrightarrow{\alpha \rightarrow 0} +\infty$ , meaning that risk seeking agents tends to consume more in this setting,
- $c_t^{*,NC} \xrightarrow{\alpha \rightarrow 1} K$ , so the optimal consumption rate goes to the proportional market consumption preference parameter  $K$  as the risk aversion parameter goes to 1.

Conversely, if the representative agent prefers wealth to consumption, meaning that  $\log(K) < 0$ , then  $c_t^{*,NC} \xrightarrow{\alpha \rightarrow 0} 0$  the optimal consumption of risk seeking agents goes to 0. This reflects that risk seeking agents in this situation prefer to invest more in their portfolio strategy rather than in consumption. The limit behavior with  $\alpha$  of the optimal consumption is the same as before for risk averse agents.

- If  $\mathbb{E}[\log(K)] < 0$ , on average agents in the population prefer wealth utility to consumption, then  $K_{\alpha,\theta} \mathbb{E}\left[\frac{\log(K)}{\alpha}\right] > 0$  and  $c_t^* \geq c_t^{*,NC}$ . In other words, the optimal consumption rate is an increasing function of the competition parameter  $\theta$ .



- If  $\mathbb{E}[\log(K)] > 0$ , in geometric average, the continuum of agents prefers consumption to wealth utility, then  $K_{\alpha,\theta} \mathbb{E}\left[\frac{\log(K)}{\alpha}\right] < 0$  and  $c_t^* \leq c_t^{*,NC}$ . The optimal consumption rate is then a decreasing function of the competition parameter  $\theta$ .

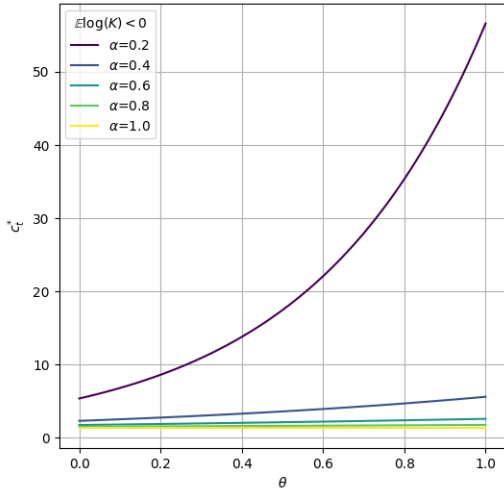


FIGURE 3. Optimal consumption for  $\mathbb{E} \log(K) = -0.5$ ,  $\mathbb{E}[\theta] = 0.7$ ,  $\mathbb{E}\left[\frac{1}{\alpha}\right] = 2$  and  $K = 1.4$ .

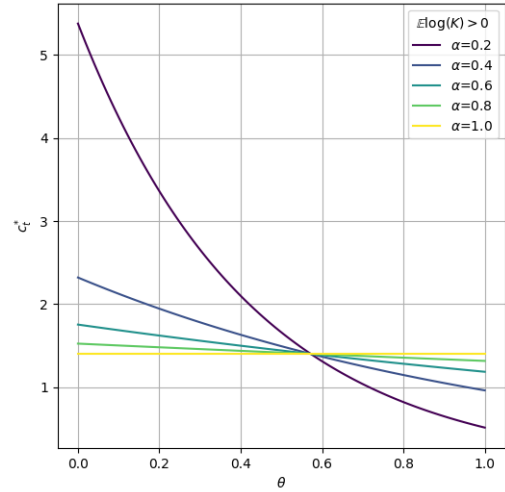


FIGURE 4. Optimal consumption for  $\mathbb{E} \log(K) = 0.5$ ,  $\mathbb{E}[\theta] = 0.7$ ,  $\mathbb{E}\left[\frac{1}{\alpha}\right] = 2$  and  $K = 1.4$ .

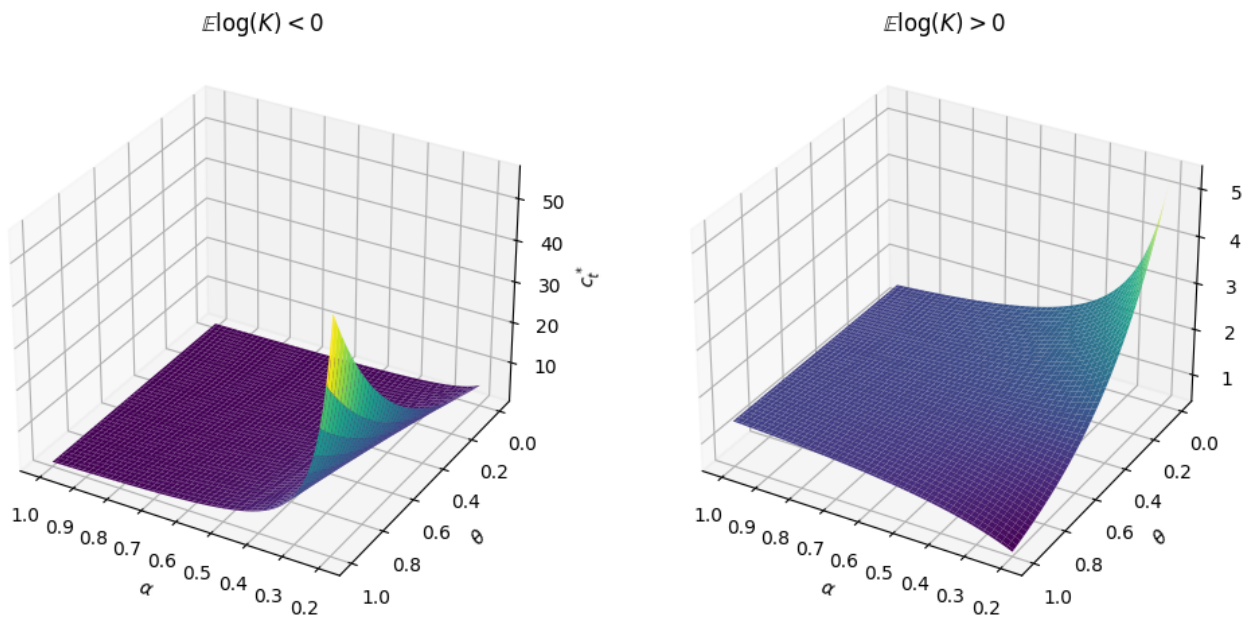


FIGURE 5. Optimal consumption for  $K = 1.4$  and  $|\mathbb{E} \log(K)| = 0.5$

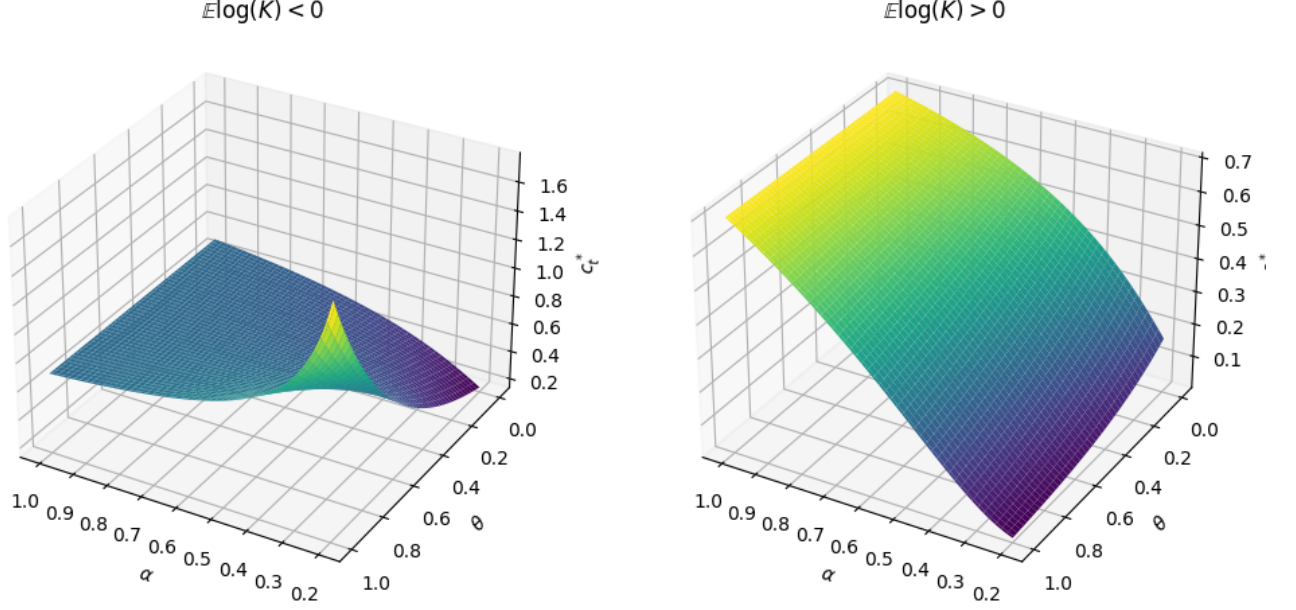


FIGURE 6. Optimal consumption for  $K = 0.7$  and  $|\mathbb{E} \log(K)| = 0.5$

**Relative market-consumption preference parameter** - Taking  $\kappa < 0$  leads to the logistic SDE with global solution, for which results on asymptotic behavior exist (see [18], [13]). In the following, we assume for simplicity that the local characteristics  $b^A$ ,  $\delta^{A,B}$  and  $\delta^{A,W}$  are time independent and denote  $q = \frac{1}{2} - \frac{b^A}{\|\delta^A\|^2}$ .

**Proposition 4.1.** Equation (3.38) admits a strong global solution  $c_t^*$  given by

$$c_t^* = \frac{\exp\left(\left(b^A - \frac{\|\delta^A\|^2}{2}\right)t + \delta^A \cdot d\bar{W}_t\right)}{1 - \kappa \int_0^t \exp\left(\left(b^A - \frac{\|\delta^A\|^2}{2}\right)s + \delta^A \cdot d\bar{W}_s\right) ds} \quad (4.5)$$

- If  $q < 0$ ,  $c_t$  converges in law towards a Gamma distribution with shape parameter  $-2q$  and scale  $\frac{\|\delta^A\|^2}{2|\kappa|}$ .

- If  $q > 0$ , the diffusion goes to 0 almost surely.

With the characteristics given by (3.40), (3.41) and (3.42),  $q < 0$  if and only if  $0 < b^A - \frac{\|\delta^A\|^2}{2}$ , which using  $K_{\alpha,\theta}$  defined in (3.13) is equivalent to

$$0 < \frac{1}{\alpha} \left( b^{\bar{Z}} - \frac{\|\delta^{\bar{Z}}\|^2}{2} \right) + K_{\alpha,\theta} \mathbb{E} \left[ \frac{1}{\alpha} \right] \mathbb{E} \left[ b^{\bar{Z}} - \frac{\|\delta^{\bar{Z}}\|^2}{2} \right]. \quad (4.6)$$

The asymptotic behavior of the equilibrium consumption rate thus depends on the difference between the drift of the logarithm of wealth utility when there is no consumption and its average among the population, with specific coefficients reflecting risk aversion and competition. Moreover, in the situation where the optimal consumption rate converges in law towards a Gamma distribution with parameters

$(-2q, \frac{\|\delta^A\|^2}{2|\kappa|})$ , the asymptotic consumption rate is given by

$$\mathbb{E}[c_\infty] = -2q \frac{\|\delta^A\|^2}{2|\kappa|} = \frac{1}{|\kappa|} \left( \frac{1}{\alpha} \left( b\tilde{Z} - \frac{\|\delta\tilde{Z}\|^2}{2} \right) + K_{\alpha,\theta} \mathbb{E} \left[ \frac{1}{\alpha} \right] \mathbb{E} \left[ b\tilde{Z} - \frac{\|\delta\tilde{Z}\|^2}{2} \right] \right). \quad (4.7)$$

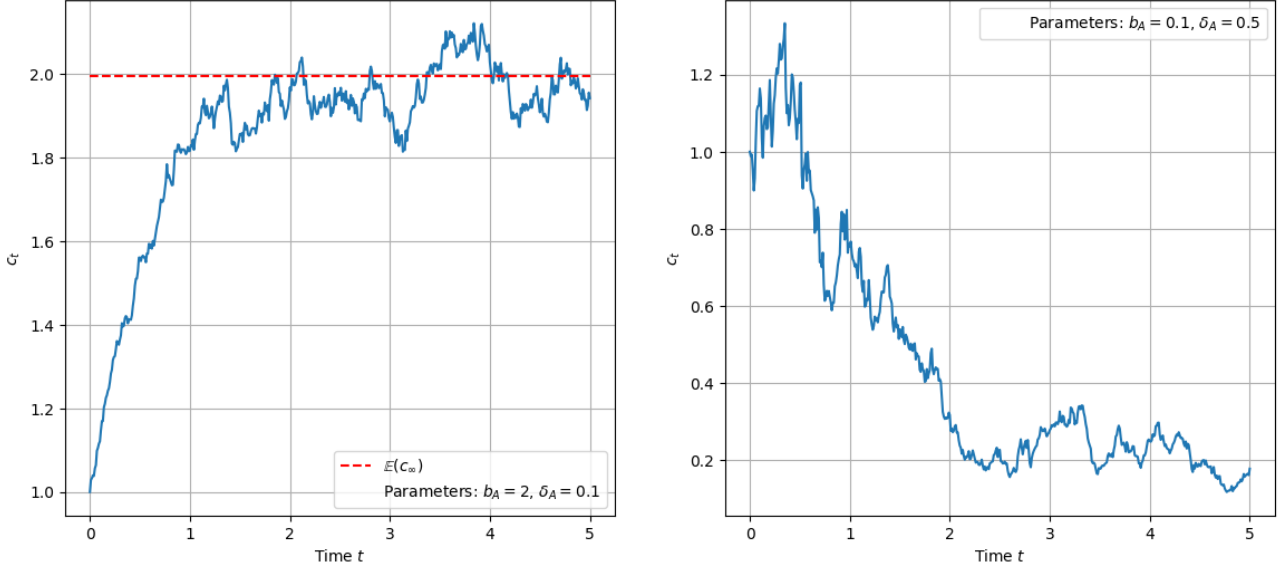


FIGURE 7. Simulated paths of the equilibrium consumption rate process.

Note the similarity between (4.6) and the strong MF equilibrium consumption, where this time the sign of the quantity  $R = b\tilde{Z} - \frac{\|\delta\tilde{Z}\|^2}{2}$  and its expectation reflects the agent and the population's preference regarding consumption or wealth utility. In fact,

- if  $R > 0$  implies that  $\log(\tilde{Z})$  has positive drift so that  $\tilde{Z}$  tends to increase in time. Since  $\tilde{Z}_t$  starts from 1 at time  $t = 0$  and  $\kappa < 0$ , for  $t$  large enough, we will have  $\phi_t = \tilde{Z}_t^{1-\kappa} > \tilde{Z}_t$  almost surely. In this setting, the agent asymptotically prefers consumption to wealth utility.
- Conversely, if  $R < 0$ , the process  $\tilde{Z}_t$  tends to decrease over time so that for  $t$  large enough,  $\phi_t < \tilde{Z}_t$  almost surely. In this case, the agent thus prefers wealth to consumption.

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## A Regular random field spaces

There are several difficulties in the definition of semimartingales  $F(t, x)$  depending on a parameter and their local characteristics  $(\beta, \gamma)$ .

1. The first one is relative to the existence of a continuous modification of the random field  $F$ . According to the works of Kunita [20], such a modification exists when the local characteristics  $(\beta, \gamma)$  are locally  $\delta$ -Hölder, for some  $\delta > 0$  by Kolmogorov's criterion.
2. The second one is relative to differential properties: even if the random field  $F$  and its local characteristics  $(\beta, \gamma)$  are differentiable, it is not enough to ensure that the local characteristics of the derivative random field  $F_x$  are  $(\beta_x, \gamma_x)$ .

In this section, we introduce Sobolev-type seminorms and associated spaces to control, locally or globally, the growth of a random field and its derivatives.

**Definition of seminorms** - Let  $\beta$  be a  $\mathbb{R}^k$ -valued forward random field of class  $C^{m,\delta}([0, +\infty[)$ , with  $m$  a nonnegative integer and  $\delta$  a number in  $(0, 1]$ , i.e  $\beta$  is  $m$  times differentiable in  $x$  and its  $m^{\text{th}}$  derivative is  $\delta$ -Hölder, for any  $t$ , almost surely.

We need to control the asymptotic behavior in 0 and  $\infty$  of  $\beta$ , and the regularity of its Hölder derivatives when they exist. We then introduce the following seminorms: for any  $K \subset ]0, +\infty[$ , we define the family of random Hölder  $K$ -seminorms:

$$\|\beta\|_{m,K}(t, \omega) = \sup_{x \in K} \frac{\|\beta(t, x, \omega)\|}{x} + \sum_{1 \leq j \leq m} \sup_{x \in K} \|\partial_x^j \beta(t, x, \omega)\| \quad (\text{A.1})$$

$$\|\beta\|_{m,\delta,K}(t, \omega) = \|\beta\|_{m,K}(t, \omega) + \sup_{x, y \in K} \frac{\|\partial_x^m \beta(t, x, \omega) - \partial_x^m \beta(t, y, \omega)\|}{|x - y|^\delta} \quad (\text{A.2})$$

These random seminorms allow to keep a control on the function, its partial derivatives and its Hölder regularity, with mainly two purposes:

- the control on the partial derivatives of  $\beta$  up to  $m$  together with the Brownian integrals approximation presented in [17], will allow to differentiate, pass to the limit and commute limit and integral for random fields.
- Hölder regularity is also required so that we can apply Kolmogorov's theorem, in order to work with a continuous modification of  $\beta$ .

**Associated function spaces** - The previous norms are related to the space parameter. We add the temporal dimension by requiring these seminorms (or their square) to be integrable in time with respect to Lebesgue measure on  $[0, T]$ . We then define the following sets:

1.  $K_{\text{loc}}^m$  (resp.  $\overline{K}_{\text{loc}}^m$ ) denotes the set of  $C^m$ -random fields  $\beta$  such that  $\frac{\beta}{x}$  and  $\partial_x^k \beta$  for  $k \leq m$  are  $\mathcal{L}^1$  (resp.  $\mathcal{L}^2$ )-locally bounded, that is for any compact  $K \subset ]0, +\infty[$  and any  $T$ ,

$$\int_0^T \|\beta\|_{m,K}(t, \omega) < \infty, \quad \left( \text{resp.} \int_0^T \|\beta\|_{m,K}^2(t, \omega) < \infty \right)$$

2.  $K_{\text{loc}}^{m,\delta}$  (resp.  $\overline{K}_{\text{loc}}^{m,\delta}$ ) denotes the set of  $C^{m,\delta}$ -random fields such that for any compact  $K \subset ]0, +\infty[$  and any  $T$ ,

$$\int_0^T \|\beta\|_{m,\delta,K}(t, \omega) < \infty, \quad \left( \text{resp. } \int_0^T \|\beta\|_{m,\delta,K}^2(t, \omega) < \infty. \right)$$

3. When these norms are defined on the whole space  $]0, +\infty[$ , the derivatives up to a certain order are bounded in the spatial parameter, with an integrable (resp. square integrable) random bound, so that we use the notation  $K_b^m$ ,  $\overline{K}_b^m$  or  $K_b^{\delta,m}$ ,  $\overline{K}_b^{m,\delta}$ .

**Itô-Ventzel's formula** - The study of the martingale optimality condition of Definition 1.1 in the framework of forward performance process requires the use of Itô-Ventzel's formula. This is an extension of Itô's formula, allowing to compute the dynamics of compounds of random fields.

**Theorem A.1.** *Consider a  $K_{\text{loc}}^2$  semimartingale  $G$  with local characteristics  $(\phi, \psi)$ . For any continuous Itô semimartingale  $X$ , the compound  $G(\cdot, X)$  is a continuous Itô semimartingale satisfying*

$$dG(t, X_t) = \phi(t, X_t)dt + \psi(t, X_t).dW_t + G_x(t, X_t)dX_t + \frac{1}{2}G_{xx}(t, X_t)\langle dX_t \rangle + \langle dG_x(t, x), dX_t \rangle|_{x=X_t}. \quad (\text{A.3})$$

The decomposition of  $G(t, X_t)$  appears to be the sum of three terms. The first one is the time differential of  $G$ , the second one is the usual Itô's formula without differentiation in time and the third one is the infinitesimal covariation between the martingale parts of  $G_x$  and the one of  $X$ . Note that when  $G$  has finite variation, that is  $\psi(t, x) = 0$  and  $\phi(t, x) = G_t(t, x)$ , we recover the classical Itô's formula.

**Differentiability of Itô random fields** The following result characterizes the regularity of an Itô semimartingale random field:

$$F(t, x) = F(0, x) + \int_0^t \beta(s, x)ds + \int_0^t \gamma(s, x)dW_s,$$

with the regularity of its local characteristics  $(\beta, \gamma)$ .

**Theorem A.2.** *(Differential rules) Let  $F$  be an Itô semimartingale random field with local characteristics  $(\beta, \gamma)$  i.e.:*

$$F(x, t) = F(x, 0) + \int_0^t \beta(x, s)ds + \int_0^t \gamma(x, s)dW_s. \quad (\text{A.4})$$

1. *If  $F$  is a  $K_{\text{loc}}^{m,\delta}$ -semimartingale for some  $m > 0$ ,  $\delta \in (0, 1]$ , its local characteristics  $(a, b)$  are of class  $K_{\text{loc}}^{m,\epsilon} \times \overline{K}_{\text{loc}}^{m,\epsilon}$  for any  $\epsilon < \delta$ .*
2. *Conversely, if the local characteristics  $(\beta, \gamma)$  are of class  $K_{\text{loc}}^{m,\delta} \times \overline{K}_{\text{loc}}^{m,\delta}$ , then  $F$  is a  $K_{\text{loc}}^{m,\epsilon}$ -semimartingale, for any  $\delta < \epsilon$ .*
3. *In any case, for  $m \geq 1$ ,  $\delta \in (0, 1]$ , the derivative random field  $F_x$  is an Itô random field with local characteristics  $(\beta_x, \gamma_x)$ .*
4. *Moreover, if  $F$  is a  $K_{\text{loc}}^{1,\delta} \cap C^2$ -semimartingale, then for any Itô process  $X$ ,  $F(X, \cdot)$  is a continuous Itô semimartingale satisfying the Itô-Ventzel formula.*

## B Some formulas of the $n$ -agent problem

For the sake of completeness, we provide explicit formulas for the following quantities:

$$\begin{aligned}\overline{\mu\pi_t^{(-i)}} &= \frac{1}{n-1} \sum_{k \neq i}^n \mu_k \pi_t^k & \overline{\Sigma \pi_t^2^{(-i)}} &= \frac{1}{n-1} \sum_{k \neq i}^n \Sigma_k (\pi_t^k)^2 \\ \overline{\sigma\pi_t^{(-i)}} &= \frac{1}{n-1} \sum_{k \neq i}^n \sigma_k \pi_t^k & \overline{(\nu\pi_t)^2^{(-i)}} &= \frac{1}{n-1} \sum_{k \neq i}^n (\nu_k \pi_t^k)^2.\end{aligned}$$

For  $\overline{\mu\pi_t}$ , multiplying the optimal strategy (2.9) with  $\mu_i$  and averaging over  $i = 1, \dots, n$ , one can write:

$$\overline{\mu\pi_t} = \psi_n^\mu \overline{\sigma\pi_t} + \phi_n^\mu = \psi_n^\mu \frac{\varphi_n^\sigma}{1 - \psi_n^\sigma} + \phi_n^\mu, \quad (\text{B.1})$$

where:

$$\psi_n^\mu = \frac{1}{n-1} \sum_{k=1}^n \frac{\mu_k \sigma_k \theta_k (1 - \frac{1}{\alpha_k})}{\nu_k^2 + \sigma_k^2 \left(1 + \frac{\theta_k (1 - \frac{1}{\alpha_k})}{n-1}\right)} \quad \text{and} \quad \phi_n^\mu = \frac{1}{n} \sum_{k=1}^n \frac{\frac{\mu_k}{\alpha_k} \left(\delta_W^{Z^k}(t) \nu_k + \delta_B^{Z^k}(t) \sigma_k + \mu_k\right)}{\nu_k^2 + \sigma_k^2 \left(1 + \frac{\theta_k (1 - \frac{1}{\alpha_k})}{n-1}\right)}. \quad (\text{B.2})$$

Then, we can write  $\overline{\mu\pi_t^{(-i)}} = \frac{n}{n-1} \overline{\mu\pi_t} - \frac{1}{n-1} \mu_i \pi_t^i$ . For the two other quantities:

$$\overline{(\nu\pi_t)^2} = \frac{1}{n} \sum_{k=1}^n \left( \frac{\nu_k \sigma_k \theta_k (1 - \frac{1}{\alpha_k}) \frac{n}{n-1} \frac{\varphi_n^\sigma}{1 - \psi_n^\sigma} + \frac{\nu_k}{\alpha_k} \left(\delta_W^{Z^k}(t) \nu_k + \delta_B^{Z^k}(t) \sigma_k + \mu_k\right)}{\nu_k^2 + \sigma_k^2 \left(1 + \frac{\theta_k (1 - \frac{1}{\alpha_k})}{n-1}\right)} \right)^2,$$

so that  $\overline{(\nu\pi_t)^2^{(-i)}} = \frac{n}{n-1} \overline{(\nu\pi_t)^2} - \frac{1}{n-1} (\nu_i \pi_t^i)^2$ . Then for  $\overline{\Sigma \pi_t^2} = \overline{(\nu\pi_t)^2} + \overline{(\sigma\pi_t)^2}$  and as before:

$$\overline{(\sigma\pi_t)^2} = \frac{1}{n} \sum_{k=1}^n \left( \frac{\sigma_k^2 \theta_k (1 - \frac{1}{\alpha_k}) \frac{n}{n-1} \frac{\varphi_n^\sigma}{1 - \psi_n^\sigma} + \frac{\sigma_k}{\alpha_k} \left(\delta_W^{Z^k}(t) \nu_k + \delta_B^{Z^k}(t) \sigma_k + \mu_k\right)}{\nu_k^2 + \sigma_k^2 \left(1 + \frac{\theta_k (1 - \frac{1}{\alpha_k})}{n-1}\right)} \right)^2,$$

so that  $\overline{\Sigma \pi_t^2^{(-i)}} = \frac{n}{n-1} \overline{\Sigma \pi_t^2} - \frac{1}{n-1} (\nu_i^2 + \sigma_i^2) (\pi_t^i)^2$ .

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