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Didier Clouteau, Denis Aubry, Marie Louise Elhabre, Eric Savin

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# Periodic and Stochastic BEM for Large Structures Embedded in an Elastic Half-Space

## 1. Introduction

Modeling wave propagation around very long structures such as bridges or tunnels is a major issue in the fields of either earthquake engineering or ground borne vibrations induced by car or railway traffic. Indeed despite a seemingly bidimensional or periodic geometry, a true three-dimensional analysis has to be carried out since the loads are fully three-dimensional. Unfortunately usual 3D models are not able to account for such large structures either from the theoretical or the numerical point of view. The development of a periodic approach able to account for 3D loadings is addressed in the paper. Moreover, for such large geometries an accurate knowledge of either the loads or the soil parameters cannot be usually achieved. Consequently the analysis is also carried out in a stochastic sense using the deterministic tools previously defined. The cases of random moving loads and random incident field is studied in detail.

Let us consider a very long structure modeled as an unbounded open set  $\Omega_l$  with given elastic properties and which is embedded in an elastic half-space  $\Omega_s = D - (D \cap \Omega_l)$ ,  $D$  being the full half-space.  $\Omega_g = \Omega_s \cup \Omega_l$  will denote the global domain.  $\Omega_l$  is supposed to be periodic : an elementary bounded cell  $\hat{\Omega}_l$  exists such that  $\Omega_l = \cup_{n=-\infty}^{+\infty} \hat{\Omega}_{ln}$ ,  $\hat{\Omega}_{ln}$  being the translation of  $\hat{\Omega}_l$  of length  $L$  in direction  $d$ . The interfaces between  $\Omega_l$  and  $\Omega_s$  will be denoted  $\Sigma$ . The part of the boundary of each domain  $\Omega_\beta$  on which Neumann boundary conditions are applied will be denoted  $\Gamma_{\sigma\beta}$ .  $u_\beta$  will be the displacement field in each domain  $\Omega_\beta$ ,  $\epsilon(u_\beta)$  and  $\sigma_\beta(u_\beta)$  will be the strain and stress tensors associated to these fields and  $t_\beta(u_\beta)$  the traction vector on the boundary using the outer normal convention (see figure 1). Moreover one will use the following notations,  $\mathbf{a}$  and  $\mathbf{b}$  being any two vectors of  $\mathbb{R}^3$  and  $A$  and  $B$  being any two tensors of  $\mathbb{R}^3 \times \mathbb{R}^3$ ,  $(Div A)_j = \sum_i \partial_i A_{ji}$  is the divergence of the tensor,  $a.b = \sum_i a_i b_i$  is the scalaire product,  $A : B = \sum_{ij} A_{ij} B_{ij}$  is the contraction of two tensors and  $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$  is the tensorial product. The loads are either forces  $f$  applied in  $\Omega_l$  or incident fields  $u_i$  satisfying the Navier equation in  $D$  and the free-surface boundary conditions on  $\partial D$ .

As we assume a linear elastic behaviour for each domain the equations can be written in the frequency domain for a given circular frequency  $\omega$ , and every field will depend implicitly on  $\omega$ . To avoid the definition of proper radiation conditions either in the half-space or in the structure, one will assume that some damping occurs in the materials modeled as a small imaginary part added to the elastic constants being either constant for hysteretic damping or proportional to  $\omega$  for a viscous one. Thanks to this hypothesis, one can work in the usual Sobolev spaces even for the unbounded domains.

## 2. The generalized periodicity

In this section we will assume that the loads are exactly known. In a first step following the framework of Floquet [15] we will show that the analysis can be performed on a set of independent fields defined on the reference cell (see [4] or [21] for more details). On

this cell we apply the classical subdomain approach [2] [7] allowing us to use boundary elements in  $\tilde{\Omega}_s$  and finite elements in  $\tilde{\Omega}_l$ . For these methods the periodic assumption is addressed in detail.

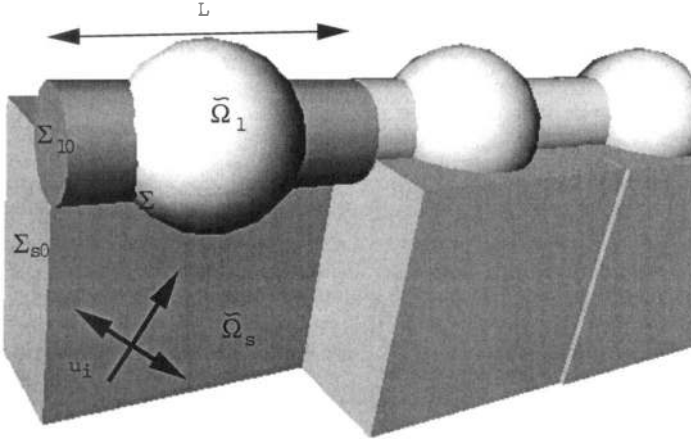


Figure 1: Model layout

## 2.1. The periodic decomposition

Before dealing with 3D periodic structures we first account for the one-dimensional case.

**1D Floquet periodicity** Let us first recall some classical results due to Floquet [15]:

**Definition 1.** A complex valued function  $f$  defined on  $\mathbb{R}$  is periodic of the second kind with period  $L$  and wave number  $\kappa$  if for any  $x$  in  $\mathbb{R}$  :

$$f(x + L) = e^{-i\kappa L} f(x) \quad (1)$$

This means that one can build this function for any  $x$  once it is known on  $]0, L[$ . The following theorem shows that any function can be written as the superposition of a set of periodic functions of the second kind :

**Theorem 1.** Given a function  $f$  defined on  $\mathbb{R}$  and a period  $L$ , its Floquet-transform  $\tilde{f}$  is defined on  $]0, L[\times ]-\pi/L, \pi/L[$  as follows :

$$\tilde{f}(\tilde{x}, \kappa) = \sum_{n=-\infty}^{+\infty} f(\tilde{x} + nL) e^{in\kappa L} \quad (2)$$

$$= \sum_{n=-\infty}^{+\infty} \hat{f}(\kappa + 2n\pi/L) e^{-i(\kappa + 2n\pi/L)\tilde{x}} \quad (3)$$

Where  $\hat{f}$  is the Fourier Transform of  $f$ . For any  $x = \tilde{x} + nL$  and  $k = \kappa + 2n\pi/L$ ,  $f$  and  $\hat{f}$  may be recovered from its Floquet transform  $\tilde{f}$  by :

$$f(x) = \frac{L}{2\pi} \int_{-\pi/L}^{\pi/L} \tilde{f}(\tilde{x}, \kappa) e^{-in\kappa L} d\kappa, \quad \hat{f}(k) = \frac{1}{L} \int_0^L \tilde{f}(\tilde{x}, \kappa) e^{ikx} d\tilde{x} \quad (4)$$

This property is of a great practical importance when dealing with periodic operators defined as follows :

**Definition 2.** A linear differential operator  $A$  with domain  $D(A)$  depending on  $x$  is periodic of period  $L$  if it satisfies for any  $x$  and for any  $u$  in  $D(A)$  :

$$A(x + L)u = A(x)u$$

Then family of operators  $\tilde{A}_\kappa$  defined formally as the restrictions of  $A$  on the reference cell  $]0, L[$  with respect to  $x$  and applied on functions  $\tilde{u}$  being the restriction of functions  $u \in D(A)$  whose traces at the two ends of the reference cell satisfy the periodicity condition  $u(L) = e^{-i\kappa L}u(0)$ .

Indeed one can easily prove that as  $(\widetilde{Au})(\tilde{x}, \kappa) = \tilde{A}_\kappa(\tilde{x})\tilde{u}(\tilde{x}, \kappa)$  for any  $u \in D(A)$  so that the following theorem holds :

**Theorem 2.** Let  $(A, f)$  be a periodic operator and a function. If the following equations have unique solutions  $u$  in  $D(A)$  and  $\tilde{u}$  in  $D(\tilde{A}_\kappa)$  for any  $\kappa \in [-\pi/L, \pi/L[$  then  $\tilde{u} = \tilde{u}$ , the Floquet transform of  $u$  :

$$Au = f, \quad \tilde{A}_\kappa \tilde{u} = \tilde{f}, \quad \tilde{u}(L) = e^{-i\kappa L} \tilde{u}(0) \quad (5)$$

This means that instead of solving the first equation in (5) one can solve the second one on the generic cell for any  $\kappa$  such that  $\tilde{f}$  does not vanish and then build solution  $u$  using the reconstruction formula (4). The key point in using this theorem is that each equation must have a unique solution. In the following this will be achieved as long as damping is accounted for.

**3D domains and fields decomposition** The aforementioned framework can be easily extended to three-dimensional domains having a periodicity  $L$  along direction  $d$  ( $x$  is now a vector in  $\mathbb{R}^3$ ). The analysis can be restricted to the generic domains  $\tilde{\Omega}_\beta = \Omega_\beta \cap S_o$  ( $\beta \in \{l, s\}$ ) where  $S_o = \{x \in \mathbb{R}^3 \text{ such that } 0 < x.d < L\}$ . It is worth noticing that the boundary of  $\tilde{\Omega}_\beta$  not only includes  $\tilde{\Sigma}$  and  $\tilde{\Gamma}_{\sigma\beta}$  the restrictions of  $\Sigma$  and  $\Gamma_{\sigma\beta}$  on the generic cell but also  $\Sigma_{\beta 0}$  and  $\Sigma_{\beta L}$  the additional interfaces between previous and following cells. With the help of the Floquet transform of either the incident field  $\tilde{u}_i(\tilde{x}, \kappa)$  or the force  $\tilde{f}(\tilde{x})$  defined as follows :

$$\tilde{f}(\tilde{x}, \kappa) = \sum_{n=-\infty}^{+\infty} f(\tilde{x} + nLd) e^{in\kappa L}, \quad \tilde{u}_i(\tilde{x}, \kappa) = \sum_{n=-\infty}^{+\infty} u_i(\tilde{x} + nLd) e^{in\kappa L}$$

one has to solve the generic problem for any  $\kappa$ , the global solution  $u(x)$  being recovered using formula (4):

**Problem 1.** Find  $(\tilde{u}_s(\tilde{x}, \kappa), \tilde{u}_l(\tilde{x}, \kappa))$  defined on  $\tilde{\Omega}_s \times \tilde{\Omega}_l$  satisfying :

$$Div \sigma_s(\tilde{u}_s - \tilde{u}_i) = \rho_s \omega^2 (\tilde{u}_s - \tilde{u}_i) \quad \text{in} \quad \tilde{\Omega}_s \quad (6)$$

$$Div \sigma_l(\tilde{u}_l) = \rho_l \omega^2 \tilde{u}_l \quad \text{in} \quad \tilde{\Omega}_l \quad (7)$$

$$\tilde{u}_s = \tilde{u}_l, \quad t_s(\tilde{u}_s) + t_l(\tilde{u}_l) = 0 \quad \text{on} \quad \tilde{\Sigma} \quad (8)$$

$$t_\beta(\tilde{u}_\beta) = 0 \quad \text{on} \quad \tilde{\Gamma}_{\sigma\beta}, \beta \in \{s, l\} \quad (9)$$

$$\tilde{u}_\beta(x) = e^{-i\kappa L} \tilde{u}_\beta(x - L) \quad \text{for} \quad x \in \Sigma_{\beta L}, \beta \in \{s, l\} \quad (10)$$

## 2.2. The subdomain approach

In this section a numerical solution of problem (1) is built using the classical domain decomposition approach [2]. As  $\tilde{\Omega}_l$  is bounded one can decompose the displacement field  $\tilde{u}_l$  on a given basis  $\{\phi_I(\kappa)\}_{I=1, N}$  that has to satisfy the periodic conditions (10).

Moreover let where  $\tilde{u}_{do} + \tilde{u}_i$  and  $\tilde{u}_{dI}$  be fields defined in  $\tilde{\Omega}_s$  satisfying the homogeneous Navier equation (6), the periodicity conditions (10), the homogeneous boundary conditions (9) and the following boundary conditions on  $\tilde{\Sigma}$  :

$$\tilde{u}_{dI} = \phi_I, \quad \tilde{u}_{do} + \tilde{u}_i = 0 \quad \text{on} \quad \tilde{\Sigma} \quad (11)$$

Then one has the following decomposition either in  $\tilde{\Omega}_l$  and in  $\tilde{\Omega}_s$  :

$$\tilde{u}_l(\tilde{x}) = \sum_{I=1}^N c_I \phi_I(\tilde{x}), \quad \tilde{u}_s(\tilde{x}) = \tilde{u}_i(\tilde{x}) + \tilde{u}_{do}(\tilde{x}) + \sum_{I=1}^N c_I \tilde{u}_{dI}(\tilde{x}) \quad (12)$$

At last, using a standard Galerkin approximation procedure in writing the equilibrium of  $\tilde{\Omega}_l$  in a weak sense for any  $\phi_J$  in the basis, one comes up with the following linear system :

$$\{K(\kappa) - \omega^2 M(\kappa) + K_s(\omega, \kappa)\} C(\omega, \kappa) = F(\kappa) + F_s(\omega, \kappa) \quad (13)$$

where :

$$\begin{aligned} K_{IJ} &= \int_{\tilde{\Omega}_l} \sigma_l(\phi_I) : \epsilon(\bar{\phi}_J) dV, \quad M_{IJ} = \int_{\tilde{\Omega}_l} \rho_l \phi_I \cdot \bar{\phi}_J dV, \quad F_J = \int_{\tilde{\Omega}_l} \tilde{f} \cdot \bar{\phi}_J dV \\ K_{sIJ} &= \int_{\tilde{\Sigma}} t_s(\tilde{u}_{dI}) \cdot \bar{\phi}_J dS, \quad F_{sJ} = - \int_{\tilde{\Sigma}} (t_s(\tilde{u}_i) + t_s(\tilde{u}_{d\bullet})) \cdot \bar{\phi}_J dS \end{aligned} \quad (14)$$

In order to solve this equation for any  $\omega$  and  $\kappa$  one has first to compute the unknown traction fields  $t_s(\tilde{u}_{dI})$  and  $t_s(\tilde{u}_{do})$ . The next subsection is devoted to this task using a boundary element technique. Another issue consists in building the basis  $\phi_I(\kappa)$  using a standard Finite Element technique and will be presented in 2.4. The main point is thus that the classical domain decomposition approach is then extended to the case of periodic domain very easily.

### 2.3. Periodic Boundary elements

We propose here to compute the fields  $\tilde{u}_{dI}$  and  $\tilde{u}_{do}$ , solutions of local boundary value problems of the following type :

**Problem 2.** Find  $u$  in  $\tilde{\Omega}_s$  such that :

$$\text{Div}\sigma(u) = -\rho\omega^2u \quad \text{in} \quad \tilde{\Omega}_s \quad (15)$$

$$u = u_o \quad \text{on} \quad \tilde{\Sigma} \quad (16)$$

$$t_s(u) = 0 \quad \text{on} \quad \tilde{\Gamma}_{\sigma s} \quad (17)$$

$$u(x) = e^{-i\kappa L}u(x-L) \quad \text{for} \quad x \in \Sigma_{sL} \quad (18)$$

using an Indirect Boundary Element Method [6] [11]. However as neither the left periodic interface  $\Sigma_{s0}$  nor the right one  $\Sigma_{sL}$  is bounded (see figure 1), a standard BEM cannot be directly used. Moreover, although it may be possible to account for periodic conditions in a BEM framework, this would require some heavy developments. To avoid these drawbacks, let us use the following periodic fundamental solution and integral operators [1][20]:

**Definition 3.**  $U_s^G$  being the Green tensor of the elastic half-space  $D$ , let  $\tilde{U}_s^G$  be the periodic Green Tensor and  $\tilde{U}_s^G$  the associated integral operator defined as follows :

$$\tilde{U}_s^G(\tilde{x}, \tilde{y}) = \sum_{-\infty}^{+\infty} e^{i\kappa n L} U_s^G(\tilde{x}, \tilde{y} + nLd) \quad (19)$$

$$\tilde{U}_s^G(q)(\tilde{y}) = \int_{\tilde{\Sigma}} \tilde{U}_s^G(\tilde{x}, \tilde{y}) \tilde{q}(\tilde{x}) dS(\tilde{x}) \quad (20)$$

From these definitions one can easily remark that  $\tilde{U}_s^G(\tilde{x}, \tilde{y})$  and  $\tilde{U}_s^G(\tilde{q})$  are periodic of the second kind with respect to  $\tilde{y}$  and with a wave number equal to  $\kappa$ . Moreover one can remark that locally  $\tilde{U}_s^G$  has the same singularities as  $U_s^G$ . As a consequence problem (2) is equivalent to the following Boundary integral equation where the integral is only on the bounded interface  $\tilde{\Sigma}$  as periodic boundary conditions are automatically accounted for :

**Problem 3.** Find  $\tilde{q}$  on  $\tilde{\Sigma}$  satisfying for any  $\tilde{y} \in \tilde{\Sigma}$  :

$$\int_{\tilde{\Sigma}} \tilde{U}_s^G(\tilde{x}, \tilde{y}) \tilde{q}(\tilde{x}) dS(\tilde{x}) = u_o(\tilde{y}) \quad (21)$$

The tractions  $t_s(u)$  needed in (14) are then given for any  $\tilde{y} \in \tilde{\Sigma}$  :

$$t_s(u)(\tilde{y}) = -1/2\tilde{q}(\tilde{y}) + \int_{\tilde{\Sigma}} t_s(\tilde{U}_s^G)(\tilde{x}, \tilde{y}) \tilde{q}(\tilde{x}) dS(\tilde{x}) \quad (22)$$

The numerical solution of this integral equation is handled using standard three-dimensional BEM. The only modification consists in computing the periodic Green tensors using formula (19). In this sum the singular terms arising for  $n = 0$  are carried out using the 3D-BEM, the other terms being computed using standard procedures. This sum is truncated when convergence is reached [14].

## 2.4. The periodic structure

When dealing with Finite Elements, the computation of the matrices  $K$  and  $M$  defined in (14) is not straightforward as fields  $\phi_I$  depend explicitly on  $\kappa$  (See [19] for periodic FEM). Using classical dynamic substructuring it is that  $K$  and  $M$  have an explicit dependence on  $\kappa$  by an extension of the Craig-Bampton [12] substructuring technique. It consists in the expansion of the displacement field of the structure  $\tilde{\Omega}_i$  on dynamic eigenmodes  $\phi_{\bullet}$  with a fixed interface and on static modes ( $\omega = 0$  in (7) generated by given displacements of the interface (practically unitary displacements of the nodes belonging to the interface). In the present case the interface consists of three parts  $\tilde{\Sigma}$ ,  $\Sigma_{i0}$  and  $\Sigma_{iL}$ . Let us first call  $\psi_{\beta}$  the static modes that vanish on  $\Sigma_{i0}$  and  $\Sigma_{iL}$ . As a consequence they satisfy the periodic condition (10). Then, as long as the structure is periodic, one can find couples of the remaining static modes  $(\varphi_{0\gamma}, \varphi_{L\gamma})$  satisfying :

$$\varphi_{0\gamma}(\tilde{x}) = \varphi_{L\gamma}(\tilde{x} + Ld) \quad \varphi_{0\gamma}(\tilde{x} + Ld) = \varphi_{L\gamma}(\tilde{x}) = 0 \quad \text{for } \tilde{x} \in \Sigma_{i0} \quad (23)$$

Then one can build new static modes  $\tilde{\varphi}_{\gamma}$  combining these ones such that they also satisfy the periodic condition (10) :

$$\tilde{\varphi}_{\gamma}(\tilde{x}) = e^{i\kappa L/2} \varphi_{0\gamma}(\tilde{x}) + e^{-i\kappa L/2} \varphi_{L\gamma}(\tilde{x}) \quad (24)$$

One is then able to compute easily the stiffness and mass matrices arising in (13) as a function of the FEM stiffness and mass matrices  $K^o$  and  $M^o$ , using a simple projection technique on this new basis. For example, denoting  $\lambda = e^{-i\kappa L/2}$ , the stiffness coefficient for two modes  $\tilde{\varphi}_{\gamma}(\tilde{x})$  and  $\tilde{\varphi}_{\gamma'}(\tilde{x})$  is then given by :

$$K_{\gamma\gamma'} = \lambda^{-2} \varphi_{0\gamma}^T K^o \varphi_{0\gamma'} + (\varphi_{0\gamma}^T K^o \varphi_{L\gamma'} + \varphi_{L\gamma}^T K^o \varphi_{0\gamma'}) + \lambda^2 \varphi_{L\gamma}^T K^o \varphi_{L\gamma'} \quad (25)$$

In this manner it is clearly seen that the stiffness and the mass matrices have an explicit expression with respect to  $\lambda$  and thus need not be computed each time.

## 2.5. The Fundamental solution of the composite domain

Provided with the results of this section one is now able to compute at least numerically either in the Floquet domain  $(\tilde{x}, \kappa)$  or in the physical one the solution in each domain for any loads defined on the domains.

In particular one is able to compute the Green tensor  $U_g^G(x, y)$  of the composite domain  $\Omega_g$  made of the assembling of the individual ones giving the displacement field at any point  $x \in \Omega_g$  created by any punctual forces applied at any point  $y \in \Omega_g$ . In domains modeled using Finite Elements, the punctual force becomes nodal forces when in domains modeled by Boundary Elements one can still account for punctual forces as long as they are applied inside the domain. Indeed, in such domains one can define analytically an incident field  $u_i$  being nothing but the Green function of this local domain having a known singularity around  $y$  and then, computed numerically, the regular diffracted field  $u_d$  in the domain using the classical BEM techniques. This means that in either case one can effectively and accurately use this fundamental solution in either integral representations or integral equations as long as the domain of integration does not include or cross an internal boundary of the composite domain. In this latter case some additional work has to be done. At last let us remark that this procedure has been implicitly used when using the Green functions of the elastic layered media in (21) as described in [3] (see also [10] for BEM and Ray coupling).

### 3. Load variability

The aim of this section is to account for the variability on the load that arises when dealing with very large structures. We will first recall the usefulness of Karhunen-Loeve [22] expansions allowing us to deal with random variables instead of random fields. Then using the classical theory of linear filtering of Gaussian process [17], two particular cases of practical interest are investigated; the first one consists in a random moving source, and the second one in a random incident field.

Given stochastic inputs such as the preceding the response will be characterized by its first and second order moment. We assume here that the input is centered so that the output is centered too. The second moment is embodied by the auto-correlation function and it is the purpose of the following sections to sketch how it may be determined.

#### 3.1. Karhunen-Loeve expansion

Let us consider a second order Gaussian centered random field  $F(x)$  indexed on  $\Omega$ , an open set of  $\mathbb{R}^n$ , with values in  $\mathbb{R}^p$ . It is therefore characterized by its auto-correlation tensor  $c_F(x, x') = E[F(x) \otimes F^*(x')]$ ,  $E[.]$  denoting the mathematical expectation and  $*$ , the conjugate transpose. One can then define the covariance operator  $\mathcal{C}_F$  and its associated Karhunen-Loeve modes  $\Phi_n$  and eigenvalues  $\lambda_n$  as follows ([22]):

**Definition 4.** For any  $v$  defined on  $L_2(\Omega)$  let  $\mathcal{C}_F$  be the hermitian positive definite operator defined by :

$$\mathcal{C}_F(v)(x') = \int_{\Omega} c_F(x, x') \cdot v(x) dV(x) \quad (26)$$

and let  $\Phi_n$  be the normalized eigenfunctions of operator  $\mathcal{C}_F$  and  $\lambda_n^2$  the associated positive eigenvalues. Then any random field  $F$  can be decomposed using the Karhunen-Loeve modes  $\Phi_n$  and its auto-correlation tensor  $c_F(x, x')$  takes the following form :

$$\mathcal{C}_F(x, x') = \sum_{n=1}^{+\infty} \lambda_n^2 \Phi_n(x) \otimes \Phi_n^*(x') \quad (27)$$

For numerical purposes one can truncate the infinite sum in (27) to get an approximation which is controlled by the covariance norm.

#### 3.2. Random loads

As far as Gaussian random loads  $F$  are concerned (either applied forces in the structures or incident fields in the soil) one can use the classical theory of linear filtering of Gaussian random fields to show that the response  $u(y, t)$  at a given point in the domain is also Gaussian. In the following analysis stationary fields with respect to time will be accounted for, a more general development for non-stationary fields can be found in [8]. As a consequence, one can work directly in the frequency domain where the covariance  $C_u$  of the response is given by :

$$C_u(y, y') = \int_{\Omega} \int_{\Omega} H(y, x) \mathcal{C}_F(x, x') H^*(y', x') dV(x) dV(x') \quad (28)$$



where  $H$  is the deterministic transfer function. Using the Karhunen-Loeve expansion the covariance of the load takes a much simpler expression :

$$C_u(y, y') = \sum_{n=1}^{+\infty} \lambda_n^2 H_n(y) \otimes H_n^*(y') \quad , \quad H_n(y) = \int_{\Omega} H(y, x) \Phi_n(x) dV(x) \quad (29)$$

where  $\Phi_n(x)$  appears as a deterministic loading mode,  $H_n$  being the associated deterministic response. As a consequence a stochastic analysis consists on one hand of computing the covariance of the input and on the second hand of computing the transfer function as it is done in the next two examples where the modeling tools defined in the first section will be used.

### 3.3. Random moving sources

Vibrations induced in the ground by cars or trains are mainly due to vertical irregularities of either the wheels or the rolling area. Both of these are added in the following and denoted  $u_o$ . It is assumed that  $u_o$  can be modeled by a centered second-order Gaussian homogeneous random field of known spectral density  $\hat{c}_{u_o}$  and that the force applied under one wheel is vertical and linearly dependent on  $u_o$  at the wheel location. Then the applied force  $f(x_d, t)$  along the road or the rail at point  $x$  by a wheel moving at speed  $v$  along direction  $d$ , is :

$$f(x_d, t) = k_o \delta(x_d - vt) u_o(x_d) \quad (30)$$

where  $x_d = x.d$  is the coordinate of point  $x$  along direction  $d$ ,  $x^\perp = x - x_d$  is the location vector in the plane normal to direction  $d$ ,  $\delta$  is the Dirac distribution and  $k_o$  the local stiffness. Taking the Fourier transform of  $f(x_d, t)$  with respect to time one gets the following autocorrelation function in the frequency domain, denoting  $k_v = \omega/v$  :

$$\begin{aligned} C_f(x_d, x'_d) &= \int_{-\infty}^{+\infty} h_f(x_d, k) h_f^*(x'_d, k) dk \\ h_f(x_d, k) &= k_o e^{ikx_d} \hat{c}_{u_o}^{1/2}(k_v - k) \end{aligned} \quad (31)$$

Splitting the integral in (31) into an infinite series of integrals on  $[(2n-1)\pi/L, (2n+1)\pi/L]$  and taking  $x_d = \tilde{x}_d + n_x L$  and  $x'_d = \tilde{x}'_d + n'_x L$  one gets :

$$\begin{aligned} C_f(x_d, x'_d) &= \int_{-\pi/L}^{\pi/L} e^{i\kappa(n_x - n'_x)L} \sum_{n=-\infty}^{+\infty} \lambda_n^2 \Phi_n(\tilde{x}_d, \kappa) \otimes \Phi_n^*(\tilde{x}'_d, \kappa) d\kappa \\ \Phi_n(\tilde{x}_d, \kappa) &= e^{2in\pi\tilde{x}_d/L} e^{i\kappa\tilde{x}_d} \quad , \quad \lambda_n(\kappa) = \hat{c}_{u_o}^{1/2}(k_v - \kappa - 2n\pi/L) \end{aligned} \quad (32)$$

which is the Karhunen-Loeve expansion of  $\tilde{c}_{u_o}(\tilde{x}_d, \tilde{x}'_d, \kappa)$  on the generic cell for a given  $\kappa$ ,  $\Phi_n(\tilde{x}_d, \kappa)$  being the Karhunen-Loeve modes and  $\lambda_n$  their eigenvalues. The transfer function  $H(y, x_d)$  in the present case is nothing but the Fundamental solution of the global domain including the soil and the structure for a vertical source located on the rail at position  $x^\perp + x_d d$ . Thus  $C_u$  reads :

$$C_u(y, y') = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} H(y, x_d) C_f(x_d, x'_d) H^*(y', x'_d) dx_d dx'_d \quad (33)$$

Noticing that the effective computation of  $H(y, x_d)$  is performed according to section 2 as :

$$H(y, x_d) = \int_{-\pi/L}^{\pi/L} \tilde{H}(\tilde{y}, \tilde{x}_d, \kappa) e^{i\kappa L(n_x - n_y)} d\kappa \quad (34)$$

with  $y = \tilde{y} + n_y L d$ ,  $0 < \tilde{x}_d d < L$  and  $0 < \tilde{y} d < L$ , and splitting the integrals in (33) into pieces of length  $L$ , one can then finally get the auto-covariance of the response :

$$\begin{aligned} C_u(y, y') &= \int_{-\pi/L}^{\pi/L} \sum_{n=-\infty}^{\infty} \lambda_n^2(\kappa) H_n(y, \kappa) \otimes H_n^*(y', \kappa) d\kappa \\ H_n(y, \kappa) &= e^{-i\kappa n_y L} \int_0^L \tilde{H}(\tilde{y}, \tilde{x}_d) \Phi_n(\tilde{x}_d, \kappa) d\tilde{x}_d \end{aligned} \quad (35)$$

Taking this covariance for  $y = y'$  represents the spectral energy emitted by the moving source at point  $y$ . Many simplified expressions may be derived from (35). For example stating that  $v$  is small compared to the wave speed in the soil, and  $y$  is far from the source, the correlation length is small (resp. large) compared to the period  $L$ .

### 3.4. Random incident fields

We now account for a random incident field modeled as a second order homogeneous and stationary random process  $U_i(x, t)$  defined on  $S_o \times \mathbb{R}$  where  $S_o$  is the free surface. It is characterized by its power spectral density  $C_i(x, \omega)$  (see [18] and [13] for either theoretical or experimental expressions). Unfortunately formula (28) cannot be used directly here because although using the inverse transform (4), the modal synthesis (12), and the equation (13) where the force is given by (14), one can come up with a rather complex expression of the response  $u(y)$  at any point  $y$  in the structure. This expression depends linearly on  $\tilde{u}_i$  and  $t_s(\tilde{u}_i)$  on  $\tilde{\Sigma}$  but not depending on  $U_i$  on the free surface as it is given. Thus one has to first find  $\tilde{u}_i$  and its associated covariance  $\tilde{C}_i$  on  $\tilde{\Sigma}$  as a function of  $U_i$  and  $C_i$  on the free surface and then a simple expression of the response as a function of  $\tilde{u}_i$  on  $\tilde{\Sigma}$  that can be practically computed.

**The stochastic deconvolution** The first step is the classical deconvolution process used in earthquake engineering. Given a deterministic ground motion at the free surface one makes some assumptions of a hypothesis on the incident field (usually a vertical incident plane wave) to find its amplitude, this motion being given. The same kind of process can be applied for random incident field [16]. Although the incident field is supposed to be random it is still assumed to satisfy the Navier equation in the half-space  $D$ . Then for an horizontally layered medium one can perform a stochastic deconvolution as follows :

1. compute  $\hat{\Phi}_l(k, z_o)$  the three eigenvectors of  $\tilde{C}_i(k)$  the Fourier transform of  $C_i$  on the free surface  $z = z_o$  with respect to the horizontal space variables,
2. compute  $\hat{\Phi}_l(k, z)$  performing the deterministic deconvolution of  $\hat{\Phi}_l(k, z_o)$  for the given horizontal wavenumber  $k$ ,

3. compute  $\tilde{C}_i(k, z) = \sum_l \hat{\Phi}_l(k, z) \otimes \hat{\Phi}_l^*(k, z')$  and computing  $\tilde{C}_i(\tilde{x}, \tilde{x}', \kappa)$  with  $\tilde{x} = (\tilde{x}_h, z)$ ,  $\tilde{x}' = (\tilde{x}'_h, z')$  on  $\tilde{\Sigma}$  and  $k = (k_\xi, \kappa + 2n\pi/L)$ , using the Fourier to Floquet transform (3) in the periodic horizontal direction and an inverse Fourier transform in the other one.
4. find the Karhunen-Loeve modes  $\Phi_n(\tilde{x}, \kappa)$  of  $\tilde{C}_i(\tilde{x}, \tilde{x}', \kappa)$  on  $\tilde{\Sigma}$  for any  $\kappa$ .

### Covariance on the response

We propose here a simple method to compute the transfer function between the incident field at a given point  $\tilde{x}$  of the soil-structure interface and any point  $y$  in the structure, so that equation (33) can be directly applied. As in the case of a moving load the basic idea is to define the Fundamental solution of the global problem (in the present case for a given  $\kappa$ ), denoted  $\tilde{U}_g^G(x, y)$ , defined and continuous over every domain and which can be computed using section 2 methodology. In addition, let  $\tilde{u}_d$  be the response in the structure for a given incident field  $\tilde{u}_i$  and the diffracted field  $\tilde{u}_s - \tilde{u}_i$  in the soil. This field satisfies all prescribed equations over the different domains except on the interface  $\tilde{\Sigma}$  where it satisfies the following jump relationships:

$$[\tilde{u}_d] = \tilde{u}_i, \quad [\tilde{t}(\tilde{u}_d)] = t_s(\tilde{u}_i) \quad \text{on } \tilde{\Sigma} \quad (36)$$

As a consequence, using the representation theorem one directly gets :

$$\tilde{u}_s(\tilde{y}) = \tilde{u}_d(\tilde{y}) = \int_{\tilde{\Sigma}} \tilde{U}_g^G \cdot t_s(\tilde{u}_i) - t(\tilde{U}_g^G) \cdot \tilde{u}_i dS \quad (37)$$

Now, noticing that either  $\tilde{u}_i$  or any single layer  $\tilde{U}_s^G(\tilde{q}_s)$  defined by (20) satisfies the Navier equation in  $\tilde{\Omega}_s^* = D \setminus \tilde{\Omega}_s$  and thus satisfies the reciprocity theorem in  $\tilde{\Omega}_s^*$ , leads to :

$$\int_{\tilde{\Sigma}} \tilde{U}_s^G(\tilde{q}_s) \cdot t_s(\tilde{u}_i) - t_s(\tilde{U}_s^G(\tilde{q}_s)) \cdot \tilde{u}_i dS = 0 \quad (38)$$

as integrals over the periodic interfaces or the free surface of the half-space vanish and where the traction vectors are defined using the outer normal of  $\tilde{\Omega}_s^*$ . Then using (37), the continuity of  $\tilde{u}_i$ ,  $t_s(\tilde{u}_i)$  and  $\tilde{U}_s^G(\tilde{q}_s)$  across  $\tilde{\Sigma}$  and the classical jump relationship  $[t_s(\tilde{U}_s^G(\tilde{q}_s))] = -\tilde{q}_s$ , one gets :

$$\tilde{u}_s(\tilde{y}) = \tilde{u}_d(\tilde{y}) = \int_{\tilde{\Sigma}} \tilde{Q}_s^g(\tilde{y}, \tilde{x}_d) \cdot \tilde{u}_i(\tilde{x}_d) dS(\tilde{x}_d) \quad (39)$$

where  $\tilde{Q}_s^g$  is the source density on  $\tilde{\Sigma}$  such that  $\tilde{U}_s^G(\tilde{Q}_s^g) = \tilde{U}_g^G$  in  $\tilde{\Omega}_s$  which is computed when solving equation (21). Coming back to the stochastic analysis one then has using the same notation as in (35):

$$\begin{aligned} C_u(y, y') &= \int_{-\pi/L}^{\pi/L} \sum_{n=-\infty}^{\infty} \lambda_n^2(\kappa) H_n(y, \kappa) \otimes H_n^*(y', \kappa) d\kappa \\ H_n(y, \kappa) &= e^{-i\kappa n y} \int_{\tilde{\Sigma}} \tilde{Q}_s^g(\tilde{y}, \tilde{x}_d) \cdot \Phi_n(\tilde{x}_d, \kappa) dS(\tilde{x}_d) \end{aligned} \quad (40)$$

#### 4. Conclusion

We have shown in this paper that deterministic dynamic analyses on long periodic structures embedded in an infinite half-space may be carried out combining Domain Decomposition, BEM and standard FEM even for non periodic loadings (see [14] for numerical results). Moreover it has been shown that these deterministic tools combined with the Karhunen-Loeve expansion technique easily allow a stochastic analysis for various kinds of random loads. One has to remark that this methodology also applies for non periodic cases as presented in [8]. The extension of this analysis to random soil characteristics can be found in [9]. Further developments to account for the coupling between periodic and bounded structures throughout a propagation media can be found in [10] using Boundary Integral techniques together with asymptotic analysis.

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LMSSM, Ecole Centrale de Paris-CNRS/URA 850, 92295 Chatenay-Malabry, France