Infinite families of congruences for (j, k)-regular overpartitions

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Abstract: Let $\bar{p}_{j,k}(n)$ denotes the number of (j,k)-regular overpartitions of a positive integer n such that none of the parts is congruent to j modulo k. Naika et al.(2021) proved infinite families of congruences modulo powers of 2 for $\bar{p}_{3,6}(n)$, $\bar{p}_{5,10}(n)$ and $\bar{p}_{9,18}(n)$. Saikia et al.(2023) also obtained infinite families of congruences modulo 4, 8, 16, 32 and 64 for $\bar{p}_{4,8}(n)$, modulo 4 and 8 for $\bar{p}_{6,12}(n)$, and modulo 16 for $\bar{p}_{8,16}(n)$. In this paper, we obtain infinite families of congruences modulo 2, 4, 8, 9 for $\bar{p}_{3,3}(n)$, $\bar{p}_{4,4}(n)$, $\bar{p}_{5,5}(n)$ and $\bar{p}_{7,7}(n)$.

MSC2020 **Classification:** Primary $11P83$, Secondary $05A17$.
Keywords and Phrases: (j,k)-regular overpartition, congruence, q-series identities.

1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots \geq \lambda_k = n$. The number of partitions of a natural number n is usually denoted by $p(n)$(with $p(0) = 1$) and the generating function is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where, for any complex number a and q,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \ \ |q| < 1.$$

Throughout the paper, we denote

$$f_k := (q^k;q^k)_\infty,$$

where k is any positive integer. An overpartition introduced by Corteel and Lovejoy[5], of a non-negative integer n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. For example, the 8 overpartitions of 3 are

$$3,\ \bar{3},\ 2+1,\ \bar{2}+1,\ 2+\bar{1},\ \bar{2}+\bar{1},\ 1+1+1,\ \bar{1}+1+1.$$

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Let $\bar{p}(n)$ denotes the number of overpartitions of n with $\bar{p}(0) = 1$. Then, the generating function for $\bar{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \bar{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

Again, for any positive integer l, an l-regular partition of a non-negative integer n is a partition in which no part is divisible by l. If $b_l(n)$ denotes the number of l-regular partitions of n with $(b_l(0) = 1)$, then the generating function of $b_l(n)$ is given by

$$\sum_{n=0}^{\infty} b_l(n)q^n = \frac{f_l}{f_1}.$$

Naika et.al[11] defined a new overpartition function known as (j,k)-regular overpartition. An overpartition of a non-negative integer n is said to be (j,k)-regular overpartition if none of the parts is congruent to $j \pmod k$. If $\bar{p}_{j,k}(n)$ denotes the number of (j,k)-regular overpartition of n (with $\bar{p}_{j,k}(0) = 1$), then its generating function is given by

$$\sum_{n=0}^{\infty} \bar{p}_{j,k}(n)q^n = \frac{(-q;q)_\infty (q^j;q^k)_\infty}{(q;q)_\infty (-q^j;q^k)_\infty}. \tag{1.1}$$

Naika et.al[11] obtained many infinite families of congruences modulo powers of 2 for $\bar{p}_{3,6}(n)$, $\bar{p}_{5,10}(n)$ and $\bar{p}_{9,18}(n)$. Saikia et al[12] obtained infinite families of congruences modulo 4, 8, 16 and 32 for $\bar{p}_{4,8}(n)$, modulo 4 and 8 for $\bar{p}_{6,12}(n)$, and modulo 16 for $\bar{p}_{8,16}(n)$.

In this paper, we obtain infinite families of congruences modulo 2, 4, 8 and 9 for $\bar{p}_{3,3}(n)$, $\bar{p}_{4,4}(n)$, $\bar{p}_{5,5}(n)$ and $\bar{p}_{7,7}(n)$.

2 Preliminaries

In this section, we present some basic terminology and dissection formulas to prove our results.

Ramanujan's general theta function $f(a,b)$ is defined as:

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ \ |ab| < 1 \tag{2.1}$$

$$= (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{2.2}$$

The identity (2.2) is the Jacobi triple product identity in Ramanujan's notation [4, Ch. 16, Entry 19].

It follows from (2.2) and (2.1) that [4, Ch. 16, Entry 22],

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1} \tag{2.3}$$

and

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1. \tag{2.4}$$

Lemma 2.1. *The following* 2*-dissections hold:*

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_{24}}{f_2^4 f_{12}}, \tag{2.5}$$

$$\frac{f_1^2}{f_3^2} = \frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}. \tag{2.6}$$

Equations (2.5) and (2.6) can be found in [15].

Lemma 2.2. *The following two dissection hold:*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \tag{2.7}$$

Equation (2.7) *was proved by Hirschhorn and Sellers[9].*

Lemma 2.3. *The following two dissections hold:*

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}, \tag{2.8}$$

$$\frac{f_1}{f_3} = \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}. \tag{2.9}$$

Xia and Yao[14] gave a proof of (2.8). Replacing q by $-q$ in (2.8) and using the fact that $(-q:-q)_\infty = \frac{f_2^3}{f_1 f_4}$, we get (2.9).

Lemma 2.4. *The following two dissections hold:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{2.10}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}. \tag{2.11}$$

Hirschhorn, Garvan and Borwein[7] have proved equation (2.10) and the equation (2.11) can be found in [3].

Lemma 2.5. *The following dissection hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}. \tag{2.12}$$

The equation can be found in [6].

Lemma 2.6. *The following* 3*-dissection hold:*

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2\frac{f_3 f_{18}^4}{f_6 f_9^2}. \tag{2.13}$$

We can find the equation in [10].

Lemma 2.7. *The following binomial results holds for positive integer k and m;*

$$f_{2k}^m \equiv f_k^{2m} \pmod 2, \tag{2.14}$$
$$f_{2k}^{2m} \equiv f_k^{4m} \pmod 4, \tag{2.15}$$
$$f_k^{8m} \equiv f_{2k}^{4m} \pmod 8, \tag{2.16}$$
$$f_k^{16m} \equiv f_{2k}^{8m} \pmod{16}. \tag{2.17}$$

Lemma 2.8. *The following dissections hold:*

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2} \tag{2.18}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.19}$$

The equations (2.18) and (2.19) can be found in [6].

Lemma 2.9. *For any prime p and positive integer n,*

$$f_1^{p^n} \equiv f_p^{p^{n-1}} \pmod{p^n}. \tag{2.20}$$

The equation can be found in [2].

Lemma 2.10. *We have*

$$f_1 = f_{25}\left(\frac{1}{R(q^5)} - q - q^2 R(q^5)\right), \tag{2.21}$$

where $R(q) = \frac{f(-q,-q^4)}{f(-q^2,-q^3)}$.

We can find the equation (2.21) in [10].

Lemma 2.11. *The following 3-dissections hold:*

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}, \tag{2.22}$$
$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_3^6}. \tag{2.23}$$

Lemma 2.11 was proved by Hirschhorn and Seller[8].

3 Some new congruences for $\bar{p}_{j,k}$

In this section, we present new congruences modulo 2, 4, 8, 9 for $\bar{p}_{3,3}(n)$, $\bar{p}_{4,4}(n)$, $\bar{p}_{5,5}(n)$ and $\bar{p}_{7,7}(n)$.

Theorem 3.1. *For non-negative integers n and the integer α, we have*

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(4n+3)q^n \equiv 6(\psi(q))^6 \pmod 9, \tag{3.1}$$

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(4.2^{\alpha}n + (4.2^{\alpha}-1))q^n \equiv 0 \pmod 9,\ \alpha \geq 1. \tag{3.2}$$

Moreover, for $k = 1, 2, 3, 4, \cdots$ and $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(4.2^{\alpha+1}n + 4k.(2^{\alpha+1}-1)+3)q^n \equiv 0 \pmod 9. \tag{3.3}$$

Proof. Setting $j = k = 3$ in (1.1), we see that

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(n)q^n = \frac{f_3^2 f_2}{f_1^2 f_6}. \tag{3.4}$$

Invoking (2.5) in (3.4), we have

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(n)q^n = \frac{f_4^4 f_{12}^2}{f_2^4 f_8 f_{24}} + 2q\frac{f_4 f_6 f_8 f_{24}}{f_2^3 f_{12}}. \tag{3.5}$$

Equating odd parts of (3.5), we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(2n+1)q^n = 2\frac{f_2 f_3 f_4 f_{12}}{f_1^3 f_6}. \tag{3.6}$$

Employing (2.11) in (3.6), we get

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(2n+1)q^n = 2\frac{f_4^7 f_6^2}{f_2^8 f_{12}} + 6q\frac{f_4^3 f_{12}^3}{f_2^7}. \tag{3.7}$$

Equating the odd parts of equation (3.7), we have

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(4n+3)q^n = 6\frac{f_2^3 f_6^3}{f_1^6}. \tag{3.8}$$

Employing binomial theorem (2.20) in (3.8), we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(4n+3)q^n \equiv 6\Big(\frac{f_2^2}{f_1}\Big)^6 \pmod 9. \tag{3.9}$$

Employing (2.3) in the equation (3.9), we have

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(4n+3)q^n \equiv 6(\psi(q))^6 \pmod 9. \tag{3.10}$$

Collecting the coefficients of q^{2n+1} in the equation (3.8), we get

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(8n+7)q^n \equiv 0 \pmod 9. \tag{3.11}$$

Suppose that the congruence (3.2) is true for all integer $\alpha \geq 1$. Now, collecting the coefficients of q^{2n+1} in (3.2), we have

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(4.2^{\alpha}(2n+1)+(4.2^{\alpha}-1))q^n \equiv 0 \pmod 9. \tag{3.12}$$

This implies

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(4.2^{\alpha+1}n+(4.2^{\alpha+1}-1))q^n \equiv 0 \pmod 9. \tag{3.13}$$

Thus, the result is true for integer $\alpha+1$. Hence, by induction, the result (3.2) is true for all integers $\alpha \geq 1$.
Similarly, we can prove by collecting the coefficients of q^{2n+k} in the equation (3.8) for $k = 2, 3, \cdots$ that

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(4.2^{\alpha+1}n+4k.(2^{\alpha+1}-1)+3)q^n \equiv 0 \pmod 9. \tag{3.14}$$

□

Theorem 3.2. *For positive integer n,*

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}\Big(2^{2\alpha+1}n+\frac{1}{3}(2.4^{\alpha}+1)\Big)q^n \equiv 2\frac{f_3^3}{f_1} \pmod 4, \alpha \geq 0 \tag{3.15}$$

and

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}\Big(2^{2\alpha}n+\frac{1}{3}(2.4^{\alpha-1}+1)\Big)q^n \equiv 2\frac{f_2^3 f_3^2}{f_1^6 f_6} \pmod 4, \alpha \geq 1. \tag{3.16}$$

Proof. Setting $j = k = 3$ in (1.1), we see that

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(n)q^n = \frac{f_3^2 f_2}{f_1^2 f_6}. \tag{3.17}$$

Employing (2.22) in (3.17), we have

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(n)q^n = \frac{f_6^3 f_9^6}{f_3^6 f_{18}^3} + 2q\frac{f_6^2 f_9^3}{f_3^5} + 4q^2\frac{f_6 f_{18}^3}{f_3^4}. \tag{3.18}$$

Equating odd parts in (3.18) and replacing q^3 to q, we have

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(2n+1)q^n = 2\frac{f_2^2 f_3^3}{f_1^5}. \tag{3.19}$$

Using binomial theorem (2.15) in (3.19), we obtain

$$\sum_{n=0}^{\infty}\bar{p}_{3,3}(2n+1)q^n = 2\frac{f_3^3}{f_1} \pmod 4. \tag{3.20}$$

Employing (2.10) in (3.20) and collecting odd parts, we get

$$\sum_{n=0}^{\infty}\bar{p}_{3,3}(4n+3)q^n = 2\frac{f_6^3}{f_2} \pmod 4. \tag{3.21}$$

From (3.21), we have

$$\sum_{n=0}^{\infty}\bar{p}_{3,3}(8n+3)q^n = 2\frac{f_3^3}{f_1} \pmod 4. \tag{3.22}$$

Suppose the congruence (3.15) is true for all integers $\alpha \geq 0$. Using (2.10) in (3.15) and collecting the coefficients of q^{4n+1}, we have

$$\sum_{n=0}^{\infty}\bar{p}_{3,3}\Big(2^{2\alpha+1}(4n+1)+\frac{1}{3}(2.4^{\alpha}+1)\Big)q^n \equiv 2\frac{f_3^3}{f_1} \pmod 4. \tag{3.23}$$

This gives

$$\sum_{n=0}^{\infty}\bar{p}_{3,3}\Big(2^{2(\alpha+1)+1}n+\frac{1}{3}(2.4^{\alpha+1}+1)\Big)q^n \equiv 2\frac{f_3^3}{f_1} \pmod 4. \tag{3.24}$$

The congruence (3.15) is true for integer $\alpha+1$. By induction, the congruence (3.15) is true for all integers $\alpha \geq 0$.
Collecting the even part in (3.15), we have

$$\sum_{n=0}^{\infty}\bar{p}_{3,3}\Big(2^{2\alpha+1}(2n)+\frac{1}{3}(2.4^{\alpha}+1)\Big)q^n \equiv 2\frac{f_2^3f_3^2}{f_1^6f_6} \pmod 4.$$

This gives

$$\sum_{n=0}^{\infty}\bar{p}_{3,3}\Big(2^{2(\alpha+1)}(n)+\frac{1}{3}(2.4^{\alpha}+1)\Big)q^n \equiv 2\frac{f_2^3f_3^2}{f_1^6f_6} \pmod 4.$$

So, the result is true for integer $\alpha+1$, where $\alpha \geq 1$. Thus, the congruence (3.16) is true for all integers $\alpha \geq 1$. □

Corollary 3.1. *For positive integer n and $\alpha \geq 1$,*

$$\sum_{n=0}^{\infty}\bar{p}_{3,3}\Big(4^{\alpha}n+\frac{k}{3}(4^{\alpha+1}-1)\Big)q^n \equiv 0 \pmod 4, \tag{3.25}$$

where $k \not\equiv \pmod 3$, $k > 1$ and

$$\sum_{n=0}^{\infty}\bar{p}_{3,3}(4^{\alpha}n+k(4^{\alpha}-1))q^n \equiv 0 \pmod 4, \tag{3.26}$$

where $k = 1, 2, 3, \cdots$.

Proof. Employing (2.10) in (3.15) and collecting the coefficients of q^{4n+k}, we get

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}\Big(4^{\alpha+1}n + \frac{k}{3}(4^{\alpha+2}-1)\Big)q^n \equiv 0 \pmod 4, \tag{3.27}$$

where $k \not\equiv \pmod 3$, $k > 1$ and

$$\sum_{n=0}^{\infty} \bar{p}_{3,3}(4^{\alpha+1}n + k(4^{\alpha+1}-1))q^n \equiv 0 \pmod 4, \tag{3.28}$$

where $k = 1, 2, 3, \cdots$.
Similar to the proof of Theorem 3.2, using induction, we can prove the congruences (3.25) and (3.26) are true for all integers $\alpha \geq 1$. □

Theorem 3.3. *For non-negative integer n and $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} \bar{p}_{4,4}(4.9^{\alpha}n + 9^{\alpha} + 1)q^n \equiv (-1)^{\alpha}4.f_1^2 f_2^2 \pmod 8, \tag{3.29}$$

$$\sum_{n=0}^{\infty} \bar{p}_{4,4}(4.9^{\alpha+1}n + 2)q^n \equiv (-1)^{\alpha}4.\frac{f_2^2 f_3^8}{f_1^2 f_6^2} \pmod 8, \tag{3.30}$$

$$\sum_{n=0}^{\infty} \bar{p}_{4,4}\Big(4.9^{\alpha+1}n + \frac{1}{2}(9^{\alpha+1}+3)\Big)q^n \equiv 0 \pmod 8, \tag{3.31}$$

$$\sum_{n=0}^{\infty} \bar{p}_{4,4}\Big(4.9^{\alpha+1}n + \frac{1}{2}(3.9^{\alpha+1}+1)\Big)q^n \equiv 0 \pmod 8, \tag{3.32}$$

and

$$\sum_{n=0}^{\infty} \bar{p}_{4,4}(4.9^{\alpha+1}n + 2.9^{\alpha+1})q^n \equiv 0 \pmod 8. \tag{3.33}$$

Proof. Setting $j = k = 4$ in (1.1), we see that

$$\sum_{n=0}^{\infty} \bar{p}_{4,4}(n)q^n = \frac{f_4^2 f_2}{f_1^2 f_8}. \tag{3.34}$$

Making use of (2.12) in equation (3.34) , we get

$$\sum_{n=0}^{\infty} \bar{p}_{4,4}(n)q^n = \frac{f_4^2 f_8^4}{f_2^4 f_{16}^2} + 2q\frac{f_4^4 f_8^2}{f_2^8}. \tag{3.35}$$

Collecting the even parts of the equation (3.35), we have

$$\sum_{n=0}^{\infty} \bar{p}_{4,4}(2n)q^n = \frac{f_2^2 f_4^4}{f_1^4 f_8^2}. \tag{3.36}$$

Now, using (2.19) in (3.36), we obtain

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}(2n)q^n=\frac{f_4^{18}}{f_8^4f_2^{12}}+4q\frac{f_4^6f_8^2}{f_2^8}. \tag{3.37}$$

Collecting the odd parts of equation (3.37), then replacing q^2 to q and using the binomial theorem (2.16), we get

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}(4n+2)q^n=4f_1^2f_2^2 \pmod 8. \tag{3.38}$$

Using (2.13) in above equation (3.38) and then collecting the coefficients of q^{9n+2} from both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}(36n+10)q^n=-4f_1^2f_2^2 \pmod 8. \tag{3.39}$$

Suppose that (3.29) is true for $\alpha\geq 0$. Applying (2.13) in (3.29) and the collecting the coefficients of q^{9n+2}, q^{9n}, q^{9n+1}, q^{9n+3} and q^{9n+4}, we have

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}(4.9^{\alpha}(9n+2)+9^{\alpha}+1)q^n\equiv(-1)^{\alpha+1}4.f_1^2f_2^2 \pmod 8,$$

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}(4.9^{\alpha+1}(9n)+2)q^n\equiv(-1)^{\alpha+1}4.\frac{f_2^2f_3^8}{f_1^2f_6^2} \pmod 8,$$

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}\Big(4.9^{\alpha+1}(9n+1)+\frac{1}{2}(9^{\alpha+1}+3)\Big)q^n\equiv 0 \pmod 8,$$

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}\Big(4.9^{\alpha+1}(9n+3)+\frac{1}{2}(3.9^{\alpha+1}+1)\Big)q^n\equiv 0 \pmod 8$$

and

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}(4.9^{\alpha+1}(9n+4)+2.9^{\alpha+1})q^n\equiv 0 \pmod 8,\ \textit{respectively.}$$

This implies

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}(4.9^{\alpha+1}n+9^{\alpha+1}+1)q^n\equiv(-1)^{\alpha+1}4.f_1^2f_2^2 \pmod 8,$$

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}(4.9^{\alpha+2}n+2)q^n\equiv(-1)^{\alpha+1}4.\frac{f_2^2f_3^8}{f_1^2f_6^2} \pmod 8,$$

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}\Big(4.9^{\alpha+2}n+\frac{1}{2}(9^{\alpha+2}+3)\Big)q^n\equiv 0 \pmod 8,$$

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}\Big(4.9^{\alpha+2}n+\frac{1}{2}(3.9^{\alpha+2}+1)\Big)q^n\equiv 0 \pmod 8,$$

and

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}(4.9^{\alpha+2}n+2.9^{\alpha+2})q^n\equiv 0 \pmod 8, \textit{ respectively.}$$

So, the congruences (3.29), (3.30), (3.31), (3.32) and (3.33) are true for $\alpha+1$. By induction, the congruences (3.29), (3.30), (3.31), (3.32) and (3.33) are true for all integers $\alpha\geq 0$. □

Corollary 3.2. *For positive integer n and $\alpha\geq 1$,*

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}\Big(4.9^{\alpha}n+\frac{k}{2}(9^{\alpha}-1)+2\Big)q^n\equiv 0 \pmod 8, \tag{3.40}$$

where $k=5,6,7,\cdots$.

Proof. From (3.29), collecting the coefficients of q^{9n+k}, we get

$$\sum_{n=0}^{\infty}\bar{p}_{4,4}\Big(4.9^{\alpha+1}n+\frac{k}{2}(9^{\alpha+1}-1)+2\Big)q^n\equiv 0 \pmod 8, \tag{3.41}$$

where $k=5,6,7,\cdots$.
The proof of the Corollary 3.2 is similar to the proof of Theorem 3.3. Using induction, we can prove the congruence (3.40) is true for all integers $\alpha\geq 1$ and $k=5,6,7,\cdots$. □

Theorem 3.4. *For positive integer n and $\alpha\geq 0$,*

$$\sum_{n=0}^{\infty}\bar{p}_{5,5}(4.5^{\alpha}n+5^{\alpha})q^n\equiv 2f_1f_5 \pmod 4, \tag{3.42}$$

$$\sum_{n=0}^{\infty}\bar{p}_{5,5}(4.5^{\alpha}n+1)q^n\equiv 2f_1f_5R(q)^{-1} \pmod 4, \tag{3.43}$$

and

$$\sum_{n=0}^{\infty}\bar{p}_{5,5}(4.5^{\alpha}n+2.5^{\alpha})q^n\equiv 2f_1f_5R(q) \pmod 4, \tag{3.44}$$

where $R(q)=\frac{f(-q,-q^4)}{f(-q^2,-q^3)}$.

Proof. Setting $j=k=5$ in (1.1), we find that

$$\sum_{n=0}^{\infty}\bar{p}_{5,5}(n)q^n=\frac{f_5^2f_2}{f_1^2f_{10}}. \tag{3.45}$$

Applying (2.7) in (3.45), we get

$$\sum_{n=0}^{\infty}\bar{p}_{5,5}(n)q^n=\frac{f_8^2f_{20}^4}{f_{10}f_2^3f_{40}^2}+q^2\frac{f_4^6f_{10}f_{40}^2}{f_2^5f_8^2f_{20}^2}+2q\frac{f_{20}f_4^3}{f_2^4}. \tag{3.46}$$

Collecting the coefficients of q^{2n+1} and using binomial theorem in (3.46), we have

$$\sum_{n=0}^{\infty}\bar{p}_{5,5}(2n+1)q^n\equiv 2f_2f_{10} \pmod 4. \tag{3.47}$$

From the equation (3.47), we have

$$\sum_{n=0}^{\infty} \bar{p}_{5,5}(4n+1)q^n \equiv 2f_1f_5 \pmod 4. \tag{3.48}$$

Applying (2.21) in (3.48), we have

$$\sum_{n=0}^{\infty} \bar{p}_{5,5}(4n+1)q^n \equiv 2f_5\{f_{25}(R(q^5)^{-1} - q - q^2R(q^5))\}. \tag{3.49}$$

Now, collecting the coefficients of q^{5n}, q^{5n+1} and q^{5n+2} from (3.49), we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{5,5}(20n+1)q^n \equiv 2f_1f_5R(q)^{-1} \pmod 4, \tag{3.50}$$

$$\sum_{n=0}^{\infty} \bar{p}_{5,5}(20n+5)q^n \equiv 2f_1f_5 \pmod 4, \tag{3.51}$$

and

$$\sum_{n=0}^{\infty} \bar{p}_{5,5}(20n+5)q^n \equiv 2f_1f_5R(q) \pmod 4, \ \textit{respectively.} \tag{3.52}$$

Suppose that the congruences (3.42), (3.43) and (3.44) are true for $\alpha \geq 0$. Now, applying (2.21) in (3.42) and collecting the coefficients of q^{5n}, q^{5n+1} and q^{5n+2}, we get

$$\sum_{n=0}^{\infty} \bar{p}_{5,5}(4.5^{\alpha}(5n)+1)q^n = \sum_{n=0}^{\infty} \bar{p}_{5,5}(4.5^{\alpha+1}n+1)q^n \equiv 2f_1f_5R(q)^{-1} \pmod 4, \tag{3.53}$$

$$\sum_{n=0}^{\infty} \bar{p}_{5,5}(4.5^{\alpha}(5n+1)+5^{\alpha})q^n = \sum_{n=0}^{\infty} \bar{p}_{5,5}(4.5^{\alpha+1}n+5^{\alpha+1})q^n \equiv 2f_1f_5 \pmod 4, \tag{3.54}$$

and

$$\sum_{n=0}^{\infty} \bar{p}_{5,5}(4.5^{\alpha}(5n+2)+2.5^{\alpha})q^n = \sum_{n=0}^{\infty} \bar{p}_{5,5}(4.5^{\alpha+1}n+2.5^{\alpha+1})q^n \equiv 2f_1f_5R(q) \pmod 4, \tag{3.55}$$

respectively. Therefore, the statements are true for integer $\alpha+1$. Thus, the congruences (3.42), (3.43) and (3.44) are true for all integers $\alpha \geq 0$. □

Corollary 3.3. *For positive integer n and $\alpha \geq 1$,*

$$\sum_{n=0}^{\infty} \bar{p}_{5,5}(4.5^{\alpha}n + k(5^{\alpha}-1)+1)q^n \equiv 0 \pmod 4, \tag{3.56}$$

where $k = 3, 4, 5, 6, 7, \cdots$.

Proof. Employing dissection formula (2.21) in (3.42), then collecting the coefficients of q^{5n+k}, we get

$$\sum_{n=0}^{\infty} \bar{p}_{5,5}(4.5^{\alpha+1}n + k(5^{\alpha+1}-1)+1)q^n \equiv 0 \pmod 4, \tag{3.57}$$

where $k = 3, 4, 5, \cdots$.
The proof of the Corollary 3.3 is similar to the prove of Theorem 3.4. Using induction, we can prove the congruence (3.56) is true for all integers $\alpha \geq 1$ and $k = 3, 4, 5, \cdots$. □

Theorem 3.5. *For non-negative n and $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} \bar{p}_{7,7}(7.9^{\alpha}n)q^n \equiv \frac{f_1^2}{f_2} \pmod 2. \tag{3.58}$$

Proof. Setting $j = k = 7$ in (1.1), we get

$$\sum_{n=0}^{\infty} \bar{p}_{7,7}(n)q^n = \frac{f_7^2 f_2}{f_1^2 f_{14}}. \tag{3.59}$$

Using binomial theorem in equation (3.59), we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{7,7}(n)q^n \equiv \frac{f_7^2}{f_{14}} \pmod 2. \tag{3.60}$$

Comparing the coefficients of q^{7n} in both side of the equation (3.60), we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{7,7}(7n)q^n \equiv \frac{f_1^2}{f_2} \pmod 2. \tag{3.61}$$

Applying the dissection (2.23) in the equation (3.61), we get

$$\sum_{n=0}^{\infty} \bar{p}_{7,7}(7n)q^n \equiv \Big(\frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_3^6}\Big) \pmod 2. \tag{3.62}$$

Collecting the coefficients of q^{9n} in the equation (3.62), we have

$$\sum_{n=0}^{\infty} \bar{p}_{7,7}(7.9n)q^n \equiv \frac{f_1^2}{f_2} \pmod 2.$$

Suppose that (3.58) is true for $\alpha \geq 0$. Applying (2.23) in (3.58), we have

$$\sum_{n=0}^{\infty} \bar{p}_{7,7}(7.9^{\alpha}n)q^n \equiv \Big(\frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_3^6}\Big) \pmod 2$$

and collecting the coefficients of q^{9n}, we get

$$\sum_{n=0}^{\infty} \bar{p}_{7,7}(7.9^{\alpha+1}n)q^n \equiv \frac{f_1^2}{f_2} \pmod 2.$$

Therefore, it is true for integer $\alpha + 1$. Thus, by mathematical induction, it is true for all integer $\alpha \geq 0$. □

Corollary 3.4. *For positive integer n and $\alpha \geq 1$,*

$$\sum_{n=0}^{\infty} \bar{p}_{7,7}\Big(7.9^{\alpha}n + \frac{7k}{8}(9^{\alpha}-1)\Big)q^n \equiv 0 \pmod 2, \tag{3.63}$$

where $k = 1, 2, 3, \cdots$.

Proof. Applying (2.23) in Theorem 3.58 and collecting the coefficients of q^{9n+k}, we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{7,7}\Big(7.9^{\alpha+1}n + \frac{7k}{8}(9^{\alpha+1}-1)\Big)q^n \equiv 0 \pmod 2, \tag{3.64}$$

where $k = 1, 2, 3, \cdots$.
The proof of the Corollary 3.4 is similar to the proof of Theorem 3.5. Using induction, we can prove the congruence (3.63) is true for all integers $\alpha \geq 1$ and $k = 1, 2, 3, \cdots$. □

Acknowledgement. The first author is thankful to GMU, Sambalpur for awarding seed fund research project [Order No.4934/GMU] under which this work has been done.

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